Mixing With Variable Rates

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MOTIVATION Differential equations allow us to model a wide variety of real-world problems that are easier to describe using rate of change, rather than specific values. Some examples of these types of problems are: population modeling, the physics of falling objects, and mixing tank problems. For this project, we will be focusing solely on mixing problems.

Assume we have a brine solution that flows into a tank at some rate, with some concentration of salt. Then, as soon as the salt enters the tank, it undergoes some instantaneous mixing. Additionally, there is some flow of brine solution flowing out of the tank.

Let x be the amount of salt in the tank. We will let $r(t)$ be the amount of solution flowing *out of* the tank, and multiply it by $\frac{x}{V(t)}$, where $V(t)$ is the volume of the tank, to get the total amount of salt flowing out of the tank. To make the problem easier, we will package the be the amount of salt flowing *into* the tank into one function $i(t)$. Hence, a mixing tank problem can be described by the following differential equation:

$$\frac{dx}{dt} = i(t) - r(t)\frac{x}{V(t)} \tag{1}$$

However, there are some restrictions on t that we must take into account, mainly that $V(t)$ cannot be negative. As such, $t \in [t_0, t_{max}]$ where t_{max} is when the volume of the tank becomes negative. (If it never becomes negative then $t_{max} = \infty$).

If we let v_0 be the initial volume of liquid in the tank, $V(t)$ can be described using the following expression:

$$V(t) = v_0 + \int_{t_0}^{t} (i(u) - r(u))du \tag{2}$$

When $i(t)$ and $r(t)$ are constant *and* equal, the differential equation is fairly easy to solve. It becomes a 1^{st} Order Separable ODE. When $i(t)$ and $r(t)$ are not equal *but still constant*, the differential equation becomes more challenging to solve. It becomes a 1^{st} Order Linear ODE. When $i(t)$ and $r(t)$ are not equal *and* non-constant, the differential equation becomes extremely hard to solve. For some choices of $i(t)$ and $r(t)$, x has no closed form solution, so computer analysis is the only way to get an answer.

This project explores the resulting values of x when given certain functions of $i(t)$ and $r(t)$, along with the values v_0 and t_0 using *MatLab 2019b* and *iode*.

CASE 1 In this first case, we let $i(t) = e^{-0.01t}$ and $r(t) = e^{-0.05t}$, and take the initial volume $v_0 = 100$. Let $V_I(t)$ be the volume function for Case I. Following Eq. (2) with $t_0 = 0$, we can integrate the function like so:

$$\begin{aligned} V_I(t) &= 100 + \int_0^t (e^{-0.01u} - e^{-0.05u})du \\ &= 100 + (-100e^{-0.01t} + 100 + 20e^{-0.05t} - 20) \end{aligned}$$

$$V_I(t) = 180 + 20e^{-0.05t} - 100e^{-0.01t} \tag{3}$$

Taking the limit of this equation, we can see that as $t \to \infty$, both of the exponential terms will approach 0. As a result, the volume inside the tank will approach 180. As the volume of the tank will never become negative, $t_{max} = \infty$. Therefore, $t \in [0, \infty)$

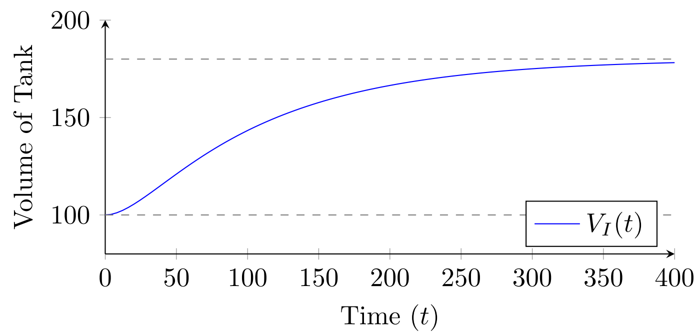

Figure 1: Graph of $V_I(t)$

We now have a function for volume. Now, if we plug in $V_I(t)$ into Eq. (1), we run into a problem. We cannot integrate $V_I(t)$ therefore, there is no closed form solution of x. However, a common way to aid understanding about differential equations is to draw the *slope field* of the equation and use *euler's method* with some initial values to get an approximation of the function. Shown below is an example of the slope field of $\frac{dx}{dt}$.

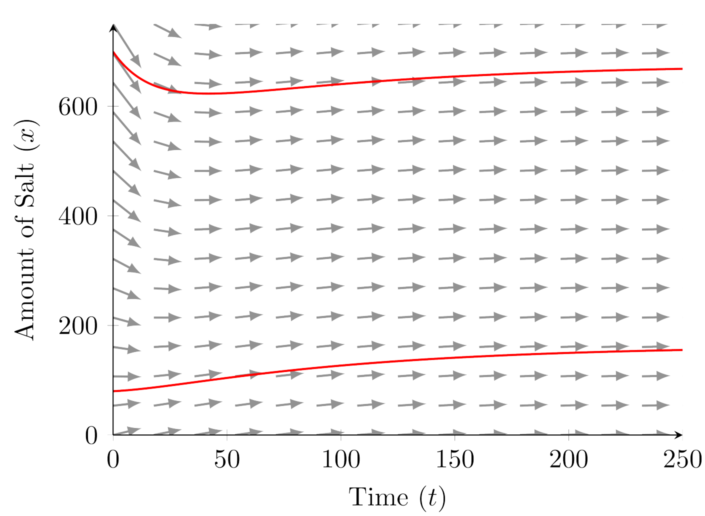

Figure 2: Slope field of $\frac{dx}{dt}$ with $x_0 = 80, 700$

This naive approach doesn't much give way to a better understanding of the problem. Why does the upper approximation dip downward? Does $x(t)$ approach some limiting value? Given a salt content at a certain time, what initial conditions should you select? All of these answers would be much simpler if $x(t)$ had a closed form solution. However, we can still obtain information about the function via different methods.

For instance, let us plug in $i(t)$, $r(t)$, and $V_I(t)$ into Eq. (1). Then, we can take the limit of $\frac{dx}{dt}$ to see whether the function diverges or converges as $t \to \infty$.

Using the fact that $lim_{t\to\infty} e^{ax} = 0$ when $a < 0$, the limit of $i(t)$ and $r(t)$ as $t \to \infty$ equals zero for both functions. Additionally, we already determined that $V_I(t)$ approaches 180 as $t \to \infty$. Hence

$$\begin{aligned}\lim_{t\to\infty} \frac{dx}{dt} &= \lim_{t\to\infty} (e^{-0.01t} - e^{-0.05t} \frac{x}{V_I(t)}) \\ &= 0 - 0 \cdot \frac{x}{180} \\ \lim_{t\to\infty} \frac{dx}{dt} &= 0\end{aligned}$$

Thus, we can say with relative confidence that the function $x(t)$, when given initial conditions, approaches some constant value.

Along a similar line of thinking, while there is not much we can definitively say about the function $x(t)$ itself, we can come up with information about its slope. Suppose we set $\frac{dx}{dt} = c$, where $c \in \mathbb{R}$. If we solve the resulting equation, we can obtain the isocline x_{iso} for $x(t)$ where the slope is equal to c.

$$\begin{aligned}c &= i(t) - r(t)\frac{x_{iso}}{V_I(t)} \\ x_{iso} &= V_I(t)\frac{i(t) - c}{r(t)} \qquad (4)\end{aligned}$$

Note that

$$\begin{aligned}\frac{i(t)}{r(t)} &= \frac{e^{-0.01t}}{e^{-0.05t}} = e^{0.04t} \\ \frac{1}{r(t)} &= \frac{1}{e^{-0.05t}} = e^{0.05t}\end{aligned}$$

Therefore,

$$\begin{aligned}x_{iso} = (180e^{0.04t} + 20e^{-0.01t} - 100e^{0.03t}) \\ -c(180e^{0.05t} - 100e^{0.04t} + 20)\end{aligned} \qquad (5)$$

For a point of reference, we will define the isocline with slope zero as

$$x_{zero} = (180e^{0.04t} + 20e^{-0.01t} - 100e^{0.03t}) \qquad (6)$$

Notice that when c is positive, that x_{iso} will be strictly less than x_{zero} for $t > 0$. This implies that all of the slopes that are positive will be *below* the curve x_{zero}. In a similar fashion, when c is negative, x_{iso} will be strictly greater than x_{zero} for $t < 0$. This implies that the slopes that are negative will be *above* the curve x_{zero}.

With this new information in hand, let us redraw the slope field. This time, we will add more initial values to the slope field, and also plot some isoclines.

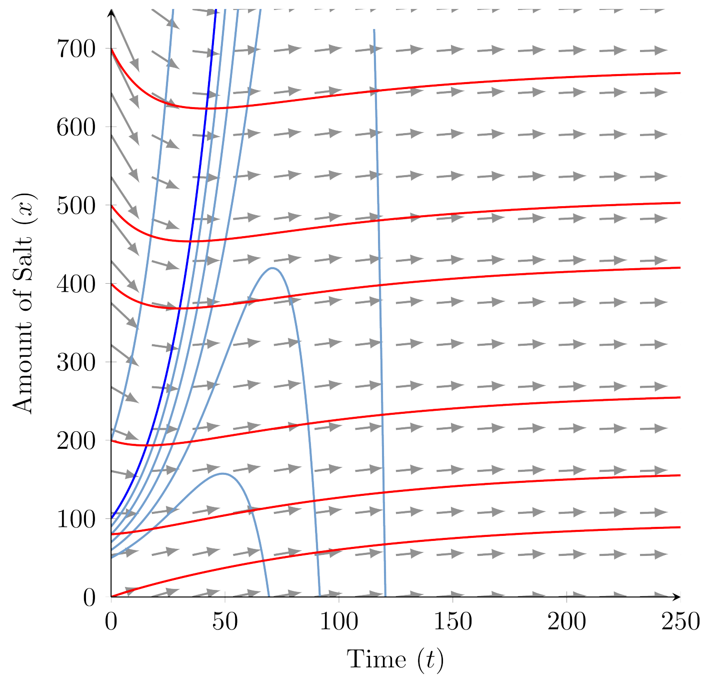

Figure 3: Slope field with isoclines $\frac{dx}{dt} = -1, 0, 0.1, 0.2, 0.3, 0.4, 0.5$ and initial values $x_0 = 0, 80, 200, 400, 500, 700$

This new graph helps us truly understand the nature of $x(t)$. For initial conditions $x(0) > 100$, the function's slope will be negative. As a result it will decrease until it intersects the isocline x_{zero}, at which point it will begin to increase again. For initial conditions $x(0) < 100$, the function's slope will be positive, and only increase. For the initial condition $x(0) = 100$, the function will not cross above x_{zero}, so its behavior is virtually the same as the previous case.

For all of these cases, as we have proved that $\lim_{t\to\infty} \frac{dx}{dt} = 0$, they will all approach some limiting value. Below is a table of Euler's Method applied to $x(t)$, evaluated to $t = 10000$ with step size 0.1. This gives a rough idea of what the limiting values of $x(t)$ with these initial conditions are.

x_0	x_{10000}
0	97.2401
80	163.450
200	262.766
400	428.291
500	511.054
700	676.579

On a final note for Case I, if given that we want the salt content to be 350 at time $t = 50$, what initial salt content x_0 should we select?

The process is much the same as when applying Euler's Method at $t = 0$. However, instead of moving forward in values of t, we must move backward. Applying Euler's Method, starting at $t = 50$ and going to $t = 0$ with a negative step size $h = -0.01$, we find that at time $t = 0$, we should select x_0 to be approximately 373.092.

CASE II For Case II, we let $i(t) = 0.2$ and $r(t) = \frac{1}{5+5t^2}$. Additionally, we take the initial volume to be $v_0 = 100$ and $t_0 = 10$. Let $V_{II}(t)$ ve the volume function for Case II. In the same vein as Case I, we follow Eq. (2) like so:

$$\begin{aligned} V_{II}(t) &= 100 + \int_{10}^{t} (0.2 - \frac{1}{5+5u^2}) du \\ &= 100 + \frac{1}{5}\int_{10}^{t}(1)du - \frac{1}{5}\int_{10}^{t}(\frac{1}{u^2+1})du \\ &= 100 - (\frac{\arctan u - u}{5})|_{10}^{t} \end{aligned}$$

$$V_{II}(t) = -\frac{\arctan t - t - \arctan 10 - 490}{5} \tag{7}$$

Using the fundamental theorem of calculus, we know that via Eq. (2), $\frac{dV_{II}}{dt} = i(t) - r(t)$. We know that $i(t)$ is constant, so the limiting value of $i(t)$ as $t \to \infty = 0.2$. Additionally, we know that $\lim_{t\to\infty} arctan(t) = 0$. As a result, the overall limit of V_{II} as t goes to infinity is 0.2. This means that the volume function, as time goes on, will resemble more and more a function whose slope is a constant 0.2.

Thus, we can say with relative confidence that $\lim_{t\to\infty} V_{II}(t) = \infty$, thus $t_{max} = \infty$, and $t \in [10, \infty)$

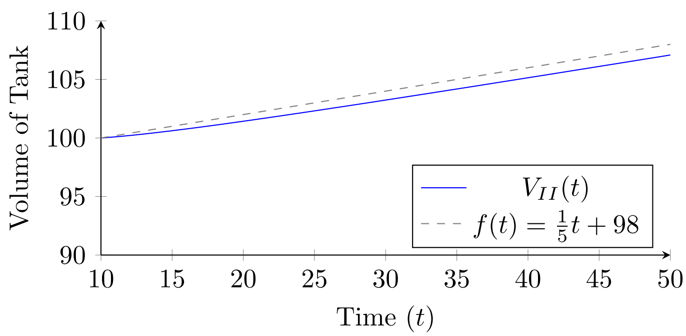

Figure 4: Graph of $V_{II}(t)$ and a similar looking function $f(t) = \frac{1}{5}t + 98$

We now have an equation for volume. Unfortunately, in the same vein as Case I, when we try to plug in $V_{II}(t)$ into Eq. (1), we cannot create a closed form solution. Before we plot the slope field of this differential equation, it would be helpful to obtain some isoclines. Using Eq. (4), we find that the isoclines for Case II follow the general form:

$$x_{iso} = \frac{1}{5}(c - \frac{1}{5})(5 + 5t^2)(\arctan t - t - \arctan 10 - 490) \tag{8}$$

where c is the desired slope value.

This equation is extremely sensitive to values of c when it approaches $\frac{1}{5}$ from below. That is to say, all of the interesting behavior happens around the value $c = 0.2$. Using the same methodology as in Case I, we will now plot the slope field, with some initial values and isoclines. Note that as $t \in [10, \infty)$, we will only plot values of t from 10 to 260.

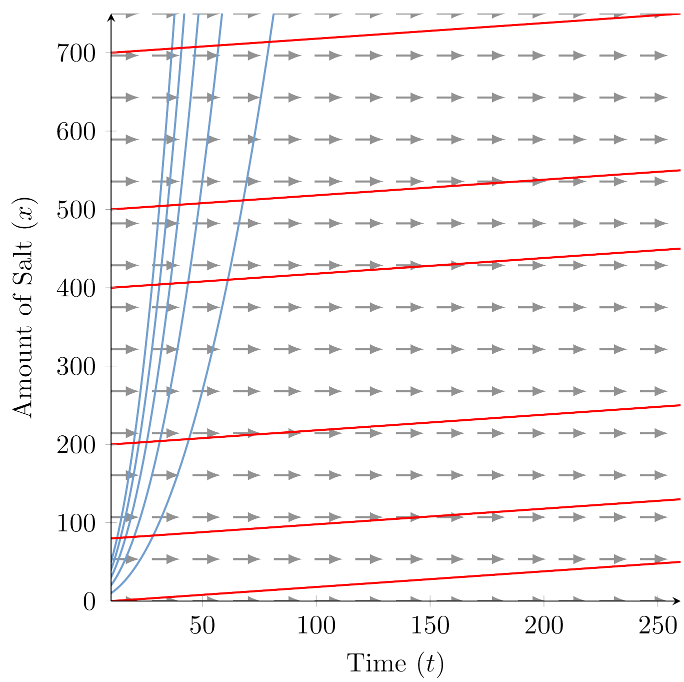

Figure 5:
Slope field with isoclines $\frac{dx}{dt} = 0.199, 0.1992, 0.1994, 0.1996, 0.1998$ and initial values $x_0 = 0, 80, 200, 400, 500, 700$

Recall, as proved earlier, that $\lim_{t\to\infty} V_{II}(t) = \infty$. Therefore, when using Eq. (1) and taking the limit as t goes to infinity, we see that both $r(t)$ and $\frac{1}{V_{II}(t)}$ approaches zero. Therefore, $\lim_{t\to\infty} \frac{dx}{dt} = 0.2$. As a result, there is no finite limiting value for t.

Upon some further analysis, we see that $x(t)$ can be approximated extremely well via a polynomial with degree 1 and slope $\frac{1}{5}$.

Let us assume that as $\frac{dx}{dt}$ approaches $\frac{1}{5}$ rather quickly, that a linear model with a slope of $\frac{1}{5}$ will do. Let $x_{appx}(t)$ be our approximation of $x(t)$. Using point-slope form, we arrive at:

$$\begin{aligned} x_{appx}(t) - x_0 &= m(t - t_0) \\ x_{appx}(t) &= \frac{1}{5}t - \frac{1}{5}t_0 + x_0 \end{aligned} \tag{9}$$

We find that this model is an extremely good approximation of $x(t)$, so much so in fact that after running a least-squares regression model on the Euler's Method data for $x_0 = 200, t_0 = 10$, the resulting approximation was $x = 0.1999t + 197.99$. Running an R^2 Coefficient of Determination test with x_{appx} on the same data yielded $R^2 = 0.999$.

Below is a table of Euler's Method applied to $x(t)$ evaluated to $t = 10000$ with step size 0.1. Additionally, the values for $x_{appx} = \frac{1}{5}t - 2 + x_0$ are shown as well.

x_0	x_{10000}	x_{appx}
0	1998.0	1998
80	2078.0	2078
200	2198.0	2198
400	2398.0	2398
500	2497.9	2498
700	2697.9	2698

It makes sense as to why both the volume function and the function itself is well approximated by a similar type of regression line. Both V_{II} and $x(t)$ involve the same function $r(t)$, which vanishes rather quickly as t begins to grow large. Intuitively, one can think about it as the tank's flow out reduces down to a very small trickle. As a result, the amount of salt accumulating in the tank is determined basically solely by $i(t) = \frac{1}{5}$.

On a final note for Case II, we can use this new approximation to find initial values given a salt content at a specific time. For instance, if we wanted $x(50) = 350$, using point-slope form, we arrive at:

$$x_{appx} = \frac{1}{5}t - \frac{50}{5} + 350$$
$$x_{appx} = \frac{1}{5}t + 340 \tag{10}$$

Plugging in $t = 10$, we arrive at $x_0 = 342$. If we wanted to have $x(50)$ be less than 350, then we would choose x_0 to be less than 342. Additionally, we can verify this using Euler's Method, starting at $t = 50$ and going to $t = 10$ with a negative step size $h = -0.01$, we find that at time $t = 10$, we should select x_0 to be less than 342.000 accurate to 6 significant figures.

EXTRA CREDIT Suppose that we wanted to choose $i(t) > 0$ and $r(t) > 0$ such that yields an output salt content that is periodic. Before we even start selecting any functions, there are some elements that can help guide us in the right direction of choosing $i(t)$ and $r(t)$.

First, let us visualize the physical implications of the problem. We want to make sure that the salt content inside of the tank is periodic. There are only two ways that the amount of salt in the tank can change. Either salt enters the tank via the inflow $i(t)$ or it exits the tank via the outflow $r(t)$. Thus the *only* way that $x(t)$ can be periodic is if i and r are periodic as well. We will choose r and i to be some of the simplest periodic functions that we can deal with, sines and cosines.

Second, i and r must result in a *net* amount of salt going into and out of the tank. An easy way to accomplish this while still maintaining that they are both greater than 0 is to have $r(t) = 2$ and $i(t) = 2 + \cos(\frac{2\pi}{T}t)$, where T is the period of the wave. This ensures that the *net* amount of salt flowing into and out of the tank oscillates between positive and negative values. Additionally, during the process of solving Eq. (1), we will need to construct the integrating factor, which involves calculating $\int_0^t (\frac{r(u)}{V(u)})du$. To make this easier on ourselves, we will have $r(t)$ be a constant value.

Finally, before we can solve Eq. (1) with these choices of $i(t)$ and $r(t)$, we need to find the volume function. Using Eq. (2)

$$V(t) = v_0 + \int_0^t (i(u) - r(u))du$$
$$= v_0 + \int_0^t (2 + \cos(\frac{2\pi}{T}t) - 2)du$$

$$V(t) = v_0 + \frac{T}{2\pi}\sin(\frac{2\pi}{T}t) \tag{11}$$

We can see that as long as $v_0 > 1$, $V(t) > 0$. In order to make our lives easier, we will make some simplifications. First, we will let $\omega = \frac{2\pi}{T}$. Next, we will let $i(t) = 2 + \omega\cos(\omega t)$. This makes the volume function simpler, $V(t) = v_0 + \sin(\omega t)$, which is crucial for the upcoming calculations.

Before we launch into calculating the differential equation, it is a good idea to get a "gut feel" for what we are looking for. If we let $v_0 = 100$ and $\omega = \frac{2\pi}{5}$ we can graph a slope field Eq. (1) with the new functions $i(t)$ and $r(t)$.

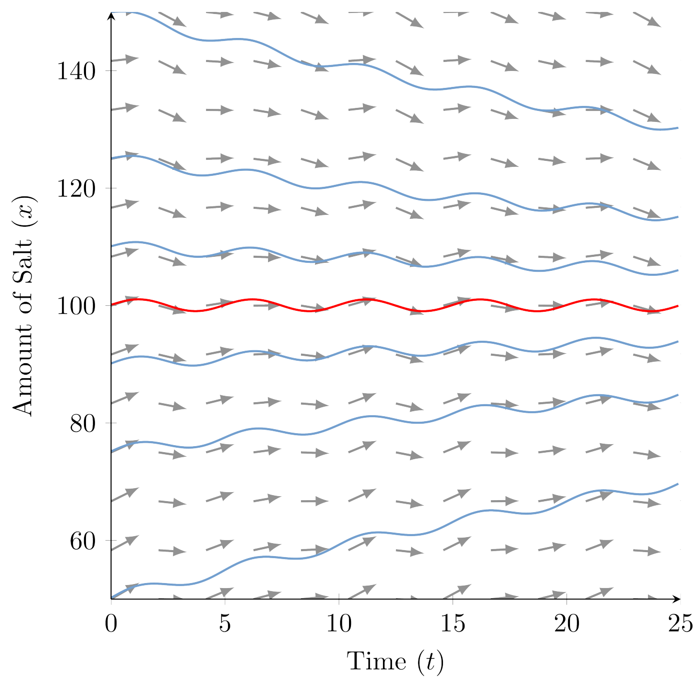

Figure 6: Slope field with initial values $x_0 = 50, 75, 90, 110, 125, 150$ in blue and $x_0 = 100$ in red

Looking at all of the different approximations, it looks like we have begun to achieve our goal. The outputs are vaguely periodic, with a period-like repetition of around 5. Even more promisingly, it looks like when $x(0) = v_0 = 100$, the output is nearly a perfect sine function. From this, what we would like to say is that when $x(0) = v_0$, $x(t) = sin(\omega t) + v_0$. However, we will have to do some calculations to prove this.

Rearranging Eq. (1) and plugging in $r(t)$, $i(t)$, and $V(t)$, we arrive at the differential equation:

$$\frac{dx}{dt} + \frac{2}{v_0 + \sin(\omega t)}x = 2 + \omega\cos(\omega t) \tag{12}$$

First, we compute the integration factor. This taking the integral of $\frac{2}{v_0+\sin(\omega t)}$. Let $m(t)$ be the result of this integral.

$$m(t) = 2\int \frac{1}{v_0 + \sin(\omega t)} dt$$

Substituting $u = \omega t \to \frac{du}{dt} = \omega$

$$m(t) = \frac{2}{\omega}\int \frac{1}{\sin u + v_0} du$$

Use the tangent half-angle substitution $\sin u = \frac{2\tan\frac{u}{2}}{\tan^2\frac{u}{2}+1}$. Then, substitute $v = \tan(\frac{u}{2}) \to \frac{dv}{du} = \frac{\sec^2\frac{u}{2}}{2}$.

$$\begin{aligned} m(t) &= \frac{2}{\omega}\int \frac{1}{\frac{2\tan\frac{u}{2}}{\tan^2\frac{u}{2}+1} + v_0} du \\ &= \frac{4}{\omega}\int \frac{1}{v_0 v^2 + 2v + v_0} dv \\ &= \frac{4}{\omega}\int \frac{1}{(\sqrt{v_0}v + \frac{1}{\sqrt{v_0}})^2 + v_0 - \frac{1}{v_0}} dv \end{aligned}$$

Substituting $w = \frac{v_0 v + 1}{\sqrt{v_0}\sqrt{v_0 - \frac{1}{v_0}}} \to \frac{dw}{dv} = \frac{\sqrt{v_0}}{\sqrt{v_0 - \frac{1}{v_0}}}$

$$\begin{aligned} m(t) &= \frac{4}{\omega}\int \frac{\sqrt{v_0}\sqrt{v_0 - \frac{1}{v_0}}}{v_0(v_0 - \frac{1}{v_0})w^2 + v_0^2 - 1} dw \\ &= \frac{4}{\omega\sqrt{v_0^2 - 1}}\int \frac{1}{w^2+1} dw \\ &= \frac{4}{\omega\sqrt{v_0^2 - 1}} \arctan w \end{aligned}$$

Performing back all of the substitutions, we arrive at

$$m(t) = \frac{4}{\omega\sqrt{v_0^2-1}} \arctan\left(\frac{v_0\tan(\frac{\omega t}{2}) + 1}{\sqrt{v_0^2-1}}\right) \tag{13}$$

Going back around to Eq. (12), we can use the integration factor $e^{m(t)}$ and arrive at

$$\begin{aligned} \frac{d}{dt}(e^{m(t)} \cdot x(t)) &= e^{m(t)}(2 + \omega\cos(\omega t)) \\ e^{m(t)} \cdot x(t) &= \int (e^{m(t)}(2 + \omega\cos(\omega t)))dt \end{aligned}$$

The integral on the right hand side, while extremely difficult, is not impossible to solve. We will now utilize *MatLab* to compute the right-hand side integral, to arrive at

$$\begin{aligned} e^{m(t)} \cdot x(t) &= e^{m(t)}(\sin(\omega t) + v_0) + c \\ x(t) &= \sin(\omega t) + c \cdot e^{-m(t)} + v_0 \end{aligned}$$

Now, if we let $x(0) = v_0$, we arrive at the final solution

$$\begin{aligned} x(0) = v_0 &= 0 + c \cdot e^{-m(t)} + v_0 \\ c &= 0 \end{aligned}$$

Therefore, we have our final solution.

$$\omega = \frac{2\pi}{T}$$
$$i(t) = 2 + \omega\cos(\omega t) \qquad r(t) = 2$$
$$V(t) = v_0 + \sin(\omega t) \qquad x(0) = v_0$$

$$x(t) = \sin(\omega t) + v_0 \tag{14}$$

This confirms our original hypothesis when looking at the vector field earlier. As a final gut-check, this really does make sense. If we set $x(0) = v_0$, we are basically saying that there is a set concentration of 1 at the beginning. As $i(t)$ puts *that same set concentration* into the tank, the amount of salt in the tank follows the same periodicity as $i(t)$.

Now, not only do we have an $i(t)$ and $r(t)$ that makes the salt function periodic, but we can go one step further and actually specify what salt content we want after a given period. If we want an output concentration $x = 10$ at every time interval $\Delta t = 50$, we simply set $v_0 = 10$, and $\omega = \frac{2\pi}{100}$. This yields $x(t) = \sin(\frac{2\pi}{100}t) + 10$, which satisfies the requirements we desire.

CONCLUSION Throughout this project we examined the following:

Given $i(t) = e^{-0.01t}$, $r(t) = e^{-0.05t}$, $v_0 = 100$, and $t_0 = 0$

- $V_I(t) = 180 + 20e^{-0.05t} - 100e^{-0.01t}$, $t \in [0, \infty)$ (Eq. 3)
- There is a limiting value for $x(t)$, as $\frac{dx}{dt}$ approaches 0 as $t \to \infty$, however it is dependent on the initial values of x_0 Fig. (3)
- $x(t)$, when given an initial condition, will decrease until it hits the isocline x_{zero}, and then increase and level off at some value.
- If we want $x(50) = 350$, we would select $x_0 \approx 373.092$

Given $i(t) = 0.2$, $r(t) = \frac{1}{5+5t^2}$, $v_0 = 100$, and $t_0 = 10$

- $V_{II} = -\frac{\arctan t - t - \arctan 10 - 490}{5}$, $t \in [10, \infty)$ Eq.(7)
- There is no finite limiting value for $x(t)$
- When given initial condition x_0, it very closely follows $x_{appx} = \frac{1}{5}t - \frac{1}{5}t_0 + x_0$ Fig. (5) and Eq. (9)
- If we want a concentration less that 50 at time $t = 100$, we must select $x_0 \lessapprox 342.00$. Eq.(10)

Given that $i(t) > 0$ and $r(t) > 0$, and that we want a periodic function for $x(t)$

- If we select $i(t) = 2 + \omega\cos(\omega t)$, $r(t) = 2$, and $x(0) = v_0$, where $\omega = \frac{2\pi}{T}$ and T is the period we desire, then $x(t) = \sin(\omega t) + v_0$ Eq.(14)
- This satisfies the requirements we desire
- If we want a concentration $x = 10$ at every time interval $\Delta t = 50$, then $i(t) = 2 + \frac{2\pi}{100}\cos(\frac{2\pi}{100}t)$, $r(t) = 2$, and $x(0) = 10$ to yield $x(t) = \sin(\frac{2\pi}{100}t) + 10$.