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February 24, 2020

1. Prove that if A' has M rows and k columns, with $M \geq k$, then $A'^T A'$ is invertible. Note this is the A' from the text, and thus the k columns of A' are linearly independent

Based on prior knowledge from the book, we understand that if a vector *v* has k components as stated above, then $v^T A'^T A'v$ is equal to the length of vector $A'v^2$.

Using the contradiction approach, if $A'^T A'$ were to be not invertible, then the matrix columns would find themselves being dependent and we know that there would be a non-zero vector x such that $A'^T A'x$ is the zero vector. This would mean that either $x^T A'^T A'x = 0$ or that the length of vector $A'v^2 = 0$. The length of this vector could only be zero is if the vector itself is zero. If this vector is not the zero vector (as we stated above) then the columns of A' could not be linearly dependent thus we have proved that the columns of A' are linearly independent.

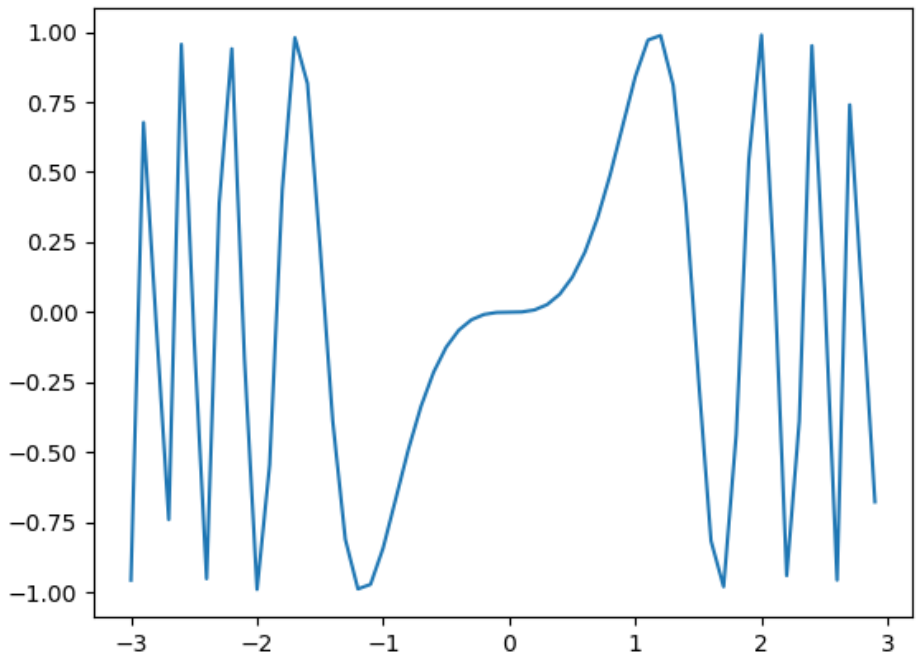
2. For fixed M, find some lower bounds for the size of summation with bounds M, k=1 (N choose k). If M = N = 1000 (which can easily happen for real world problems), how many basic feasible solutions could there be? There are less than 10^{90} sub-atomic objects in the universal (quarks, photons, et cetera). Assume each such object is a supercomputer capable of checking 10^{20} basic solutions a second (this is much faster than current technology!). How many years would be required to check all the basic solutions?

To find a feasible solution we will need the columns to be linearly independent. If we set col = number of columns, we can understand from the information above that 1 <= col <= 1000. For each variation of number of columns, the largest amount of feasible solutions would be 1000 choose col.

We can then take the summation with the bounds c = 1 and 1000 of this 1000 choose col. We can use the binomial theorem to understand this to be $2^{1000} - 1$ (the -1 is arbitrary at this scale but we cannot have zero columns), this gives us a very large number $\sim 10^{31}$. This is far more than the amount of calculations we can process in a second. Based on there being 3.154e+7 seconds in a year, we would need billions of years for this processing.

3. Imagine you want to transmit the shape of the plot f(x) = sin(x 3) on the interval [-3,3]. You have the ability to sample the value of this function for 360 different choices of x. Plot it if you sample uniformly. Is this the best way to sample? How should you sample / choose where to sample?

The plot is shown below alongside the python code used to generate it.
This is not the best way to sample this data. There are high variations excluding the range [-1,1]. If I were to plot this more appropriately then that would be the range of the sample.




```python
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(-3, 3, .1)
y = np.sin(np.power(x, 3))
plt.plot(x, y)
plt.show()
```

4. We say a x is an ordered feasible solution if its non-negative entries are ordered from smallest to largest; thus (1,0,0,4,3,0,0,5,8) is not ordered (as 4 is less than 3) but (1,0,0,3,3,0,0,5,8) is. Prove or disprove: if a canonical linear programming problem has a feasible solution then it has an ordered feasible solution.

If we were to create a vector with all positive and decreasing integers (the worst case for this solution) then there would be no feasible solutions.

5. Give an example of a 4 × 4 matrix such that each entry is positive and all four columns are linearly independent; if you cannot find such a matrix prove that one exists.

A 4x4 matrix with all positive entries and linear independence is one with all ones other than the diagonal.

$$\begin{bmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 \end{bmatrix}$$

* 5 is an arbitrarily number, it could be any non positive number.

6. Redo the previous problem but for an arbitrary N (thus find an N × N matrix where all entries are positive and the N columns are linearly independent).

The logic holds with the above problem when abstracted. Thus the answer is the same with any N x N matrix of being all 1s and a diagonal of some positive integer. This can be shown abstractly below.

$$\begin{bmatrix} X & 1 & 1 & \dots & 1 \\ 1 & X & 1 & \dots & 1 \\ 1 & 1 & X & \dots & 1 \\ 1 & 1 & 1 & \dots & X \end{bmatrix}$$