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1. Prove that any cubic $ax^3 + bx^2 + cx + d = 0$ can be written as $x^3 + px + q = 0$ (i.e., we can rewrite so that the coefficient of the x^2 term vanishes and the coefficient of the x^3 term is 1); this is called the depressed cubic associated to the original one. (For fun see the next problem on how to solve the cubic.)

The first thing to understand is that a is not equal to zero above. If this were the case then our equation can be treated as a quadratic rather than as cubic. We now take the non-zero a and divide both sides by it. Once swapping out x for x - b/3 and replacing, we find ourselves, through reworking our equation, in a place where our x^2 coefficient vanishes and our x^3 coefficient is now 1.

2. Can you construct a canonical linear programming problem that has exactly two feasible solutions? Exactly three? Exactly k where k is a fixed integer?

We are looking to find a vector that satisfies the constraints of $\mathbf{A}x = b$. Through doing so we can assume to have two feasible solutions, *s1* and *s2*. If we were to create a new arbitrary solution *s3* and were to assign it the value of $s3 = .5s1 + .5s2$ then we find ourselves with another feasible solution. Based on this logic above, there are infinite solutions to a canonical linear programming problem, and thus exactly k where k is a fixed integer.

3. Find a continuous function defined in the region $(x/2)^2 + (y/3)^2 < 1$ (i.e., the interior of an ellipse) that has neither a maximum nor a minimum but is bounded.

The function $f(x) = x$ is bounded but does not have a minimum or maximum. The bounds are that the value of x at 2 or -2, aside from these bounds x is continuous thus this checks out.

4. Imagine we want to place n queens on an n × n board in such a way as to maximize the number of pawns which can safely be placed. Find the largest number of pawns for $n \leq 5$

For n = 1, 2, or 3 the answer is 0. This is because the board is not big enough to provide holes in the reach of the queens.
For n = 4 we are presented with one square that allows for the pawn to be safe from the queens reach.
For n = 5 we are presented with three squares safe from the queens.
This is similar to the Queens problem which is the same theory but without the pawns. That is something I worked through in Algorithms.

5. Write a computer program to expand your result in the previous problem to as large of an n as you can. Does the resulting sequence have any interesting problems? Try inputting it in the OEIS.

I have written it and attached a python file. The logic is correct but the code is buggy, unfortunately I do not have the time to develop it further.

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The OEIS shows:

```
1 0
2 0
3 0
4 1
5 3
6 5
7 7
8 11
9 18
10 22
11 30
12 36
13 47
14 56
15 72
16 82
17 97
18 111
19 132
20 145
21 170
22 186
23 216
24 240
25 260
26 290
27 324
28 360
29 381
30 420
```

6. Consider the problem of placing n queens on an n × n board with the goal of maximizing the number of pawns which may safely be placed. For each n, let that maximum number be p(n). Find the best upper and lower bounds you can for p(n). For example, trivially one has 0 ≤ p(n) ≤ n 2 ; can you do better?

It is simpler for us to find a specific lower bound than an upper bound. The upper bound can be seen by making n in our n x n matrix and our sorting of n queens very large. When we do this we come across a situation in which we have bunched all of our large n amount of queens into a square in the corner. This allows for us to limit the number of pawns affected. In terms of vertical and horizontal rows affected by this, we can see that sqrt(n) are in danger on each side, but the diagonal squares are negligible when our n is large enough, allowing us to have a substantial portion of squares which are safe.

7. Prove √ 1 N P∞ n=−∞ e −πn2/N = P∞ n=−∞ e −πn2N . As N tends to infinity, bound the error in replacing the sum on the right hand side with the zeroth term (i.e., taking just n = 0). Hint: the Fourier transform of a Gaussian is another Gaussian; if f(x) = e −ax2 then fb(y) = p π/ae−π 2y 2/a .

(I will not be formatting as that is not possible in pages, future assignments I will do using LaTeX.)

First we use the Fourier transform which is given above.
We can then use the Poisson formula based on the simplicity of the function given to us from the Fourier transform. We can supplement a to be *pi* / N which then gives us the result we are hoping for.