A Uniform Galton Board

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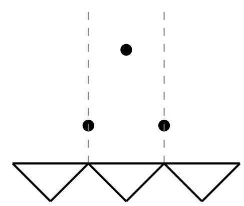
Consider a Galton board with the following arrangement.



As it stands, each peg in the board has an equal chance of sending the ball left or right. The question we focus on in this article, is *how can we weight these pegs such that they produce a uniform distribution in the corresponding bins?*

Supposing we have b bins, we will label the weights of row i (for $i \in \{1, \dots b-1\}$), peg j ($j \in \{1, \dots i\}$) as $w_{i,j}$ where $w_{i,j}$ denotes the probability the ball strikes this peg and moves left. Thus $1 - w_{i,j}$ will be the probability that it moves right. Further, denote the probability we end up in bin k (for $k \in \{1, \dots b\}$) as P_k.

For a 3 bin Galton board as above we have that

$$\begin{aligned}
P_1 &= w_{1,1} \cdot w_{2,1} \\
P_2 &= w_{1,1} \cdot (1 - w_{2,1}) + (1 - w_{1,1}) \cdot w_{2,2} \\
P_3 &= (1 - w_{1,1}) \cdot (1 - w_{2,2})
\end{aligned}$$

For a normal Galton board all $w_{i,j}$ are $\frac{1}{2}$. Which gives

$$\begin{aligned}
P_1 &= \frac{1}{4} \\
P_2 &= \frac{1}{2} \\
P_3 &= \frac{1}{4}
\end{aligned}$$

In general, if we keep all pegs having equal weightings, the probability distribution of these bins tends towards a Gaussian distribution as we increase the number of bins.

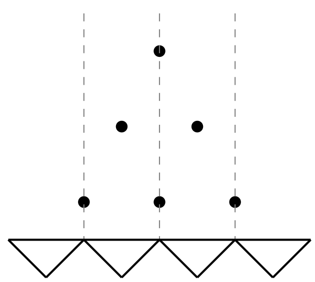
If we instead wanted an equal distribution amongst all bins, we need only set P_k to be $\frac{1}{b}$ which in our case, gives us the following set of equations

$$\begin{aligned}
\frac{1}{3} &= w_{1,1} \cdot w_{2,1} \\
\frac{1}{3} &= w_{1,1} \cdot (1 - w_{2,1}) + (1 - w_{1,1}) \cdot w_{2,2} \\
\frac{1}{3} &= (1 - w_{1,1}) \cdot (1 - w_{2,2})
\end{aligned}$$

It is reasonable to conjecture that the first peg should still be weighted $w_{1,1} = \frac{1}{2}$ since there should be no preference for the initial direction the ball travels. With this we can easily solve the above system and get

$$\begin{aligned}
w_{1,1} &= \frac{1}{2} \\
w_{2,1} &= \frac{2}{3} \\
w_{2,2} &= \frac{1}{3}
\end{aligned}$$

We might now think that this whole calculation needs to be repeated considering every path for 4 bins, but this is not so. Consider a Galton board with 4 bins



If we want our marbles to end up in each bin with even distribution, then we would also want the marbles to hit the last three pegs with even distribution. This is exactly the same as asking what the weights are in order to produce a Galton board with even distribution for 3 bins, the calculation we just performed. Thus we actually already know all the weights for $w_{1,1}, w_{2,1}$, and $w_{2,2}$. With these weights, we can assume that each peg in the last row are hit with a probability of $\frac{1}{3}$ meaning that the probability of falling into the bins is now given by

$$\begin{aligned}
P_1 &= \frac{1}{3} w_{3,1} \\
P_2 &= \frac{1}{3}(1 - w_{3,1}) + \frac{1}{3}(w_{3,2}) \\
P_3 &= \frac{1}{3}(1 - w_{3,2}) + \frac{1}{3}(w_{3,3}) \\
P_4 &= \frac{1}{3}(1 - w_{3,3})
\end{aligned}$$

Setting P_ks equal to $\frac{1}{4}$ gives us a system of equations whose solution is

$$\begin{aligned} w_{3,1} &= \frac{3}{4} \\ w_{3.2} &= \frac{2}{4} \\ w_{3,3} &= \frac{1}{4} \end{aligned}$$

This process can be continued inductively. In particular, we have that for b bins

$$\begin{aligned} P_1 &= \frac{1}{b-1} w_{b-1,1} \\ P_2 &= \frac{1}{b-1}(1 - w_{b-1,1}) + \frac{1}{b-1}(w_{b-1,2}) \\ \dots \\ P_{b-1} &= \frac{1}{b-1}(1 - w_{b-1,b-2}) + \frac{1}{b-1}(w_{b-1,b-1}) \\ P_b &= \frac{1}{b-1}(1 - w_{b-1,b-1}) \end{aligned}$$

Setting each P_k to $\frac{1}{b}$ we arrive at the following weights of the $b-1$th row

$$\begin{aligned} w_{b-1,1} &= \frac{b-1}{b} \\ w_{b-1,2} &= \frac{b-2}{b} \\ &\dots \\ w_{b-1,k-1} &= \frac{1}{b} \end{aligned}$$

Thus, a Galton board with pegs such that

$$w_{i,j} = \frac{i-j+1}{i+1}$$

will produce a uniform distribution amongst all b bins.