FYS3150 oppgavesett 1

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Relevant code can be found at: https://github.com/Jonaproitz/Project_1

PROBLEM 1.

Given

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \tag{1}$$

Inserting $x = 0$

$$u(0) = 1 - (1 - e^{-10}) \cdot 0 - e^{-10 \cdot 0} = 1 - 0 - 1 = 0$$

and $x = 1$

$$u(1) = 1 - (1 - e^{-10}) \cdot 1 - e^{-10 \cdot 1} = 1 - 1 + e^{-10} - e^{-10} = 0$$

furthermore the one-dimensional possion equation can be written

$$-\frac{d^2u}{dx^2} = -\frac{d^2}{dx^2}\left(1 - (1 - e^{-10})x - e^{-10x}\right) = -\frac{d}{dx}\left((1 - e^{-10}) + 10e^{-10x}\right) = 100e^{-10x} = f(x)$$

Hence equation 1 is an exact solution to our problem. ■

PROBLEM 2.

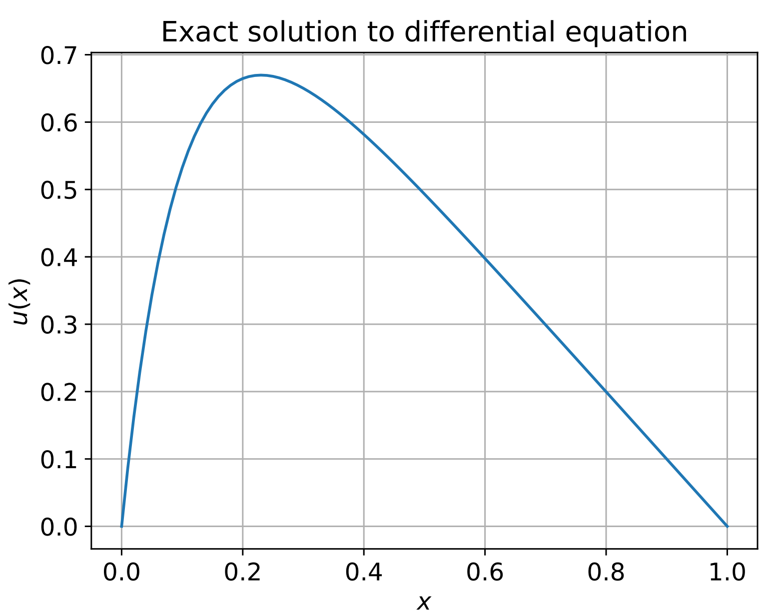

FIG. 1. Plot of equation 1 in the given area $x \in [0, 1]$.

PROBLEM 3.

The one-dimensional poisson equation can be written

$$-\frac{d^2u}{dx^2} = \frac{u(x-h) + 2u(x) - u(x+h)}{h^2} + O(h^2) = f(x)$$

Discretizing x with m values and a given distance h between each distinct value then gives

$$\begin{aligned} x &\rightarrow x_0, x_1, x_2, ..., x_{m-1} \\ u(x) &\rightarrow u_0, u_1, u_2, ..., u_{m-1} \\ f(x) &\rightarrow f_0, f_1, f_2, ..., f_{m-1} \end{aligned}$$

with $u_i = v_i$, such that

$$-\frac{d^2v_i}{dx^2} = -v_{i-1} + 2v_i - v_{i+1} = f_i h^2$$

PROBLEM 4.

The set of equations from problem 3 can be written as

$$\begin{aligned} -v_0 + 2v_1 - v_2 &= h^2 f_1 \\ -v_1 + 2v_2 - v_3 &= h^2 f_2 \\ -v_2 + 2v_3 - v_4 &= h^2 f_3 \\ &\vdots \\ -v_{m-3} + 2v_{m-2} - v_{m-1} &= h^2 f_{m-2} \end{aligned}$$

Wich for

$$\begin{aligned} g_1 =& h^2 f_1 + v_0 \\ g_2 =& h^2 f_2 \\ g_3 =& h^2 f_3 \\ &\vdots \\ g_{m-3} =& h^2 f_{m-3} \\ g_{m-1} =& h^2 f_{m-2} + v_{m-1} \end{aligned}$$

can be written as the matrix equation

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-3} \\ v_{m-2} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{m-3} \\ g_{m-2} \end{pmatrix}$$

on the form $A\vec{v} = \vec{g}$.

PROBLEM 5

a

When finding the matrix, A, in problem 4 it is assumed that v_0 and v_{m-1} are known. Hence theese values are not calculated and

$$n = m - 2$$

b

As discussed above the the vector $\vec{v}$ is equal to $\vec{v}^*$ excluding the first and last element. Hence

$$\vec{v}^* = [v_0, \vec{v}, v_{m-1}] \tag{2}$$

PROBLEM 6

a

A general tridiagonal matrix on the form below, with with each row marked as (R_i)

$$\begin{array}{cccccccc|c}
(R_1) & b_1 & c_1 & 0 & 0 & 0 & \dots & & g_1 \\
(R_2) & a_2 & b_2 & c_2 & 0 & 0 & \dots & & g_2 \\
(R_3) & 0 & a_3 & b_3 & c_3 & 0 & \dots & & g_3 \\
 & & & & \ddots & & & & \vdots \\
(R_n) & 0 & 0 & 0 & 0 & a_n & b_n & & g_n
\end{array} \tag{3}$$

can be solved by first forward substituting, giving

$$R_2 \rightarrow R_2 - \frac{a_2}{b_1} R_1 = 0 \;\; b_2 - \tfrac{a_2}{b_1} c_1 \;\; 0 \;\; 0$$

Continuing this substitution gives the following relation by introducing the variables $\tilde{b}_i$ and $\tilde{g}_i$:

$$\tilde{b}_1 = b_1 \quad \text{and} \quad \tilde{b}_i = b_i - \frac{a_i}{\tilde{b}_{i-1}} c_{i-1}, \qquad \text{For i = 2,3,4....,n}$$
$$\tilde{g}_1 = g_1 \quad \text{and} \quad \tilde{g}_i = g_i - \frac{a_i}{\tilde{g}_{i-1}} \tilde{g}_{i-1}, \qquad \text{For i = 2,3,4....,n}$$

Wich then gives the rowequivalent matrix

$$\begin{array}{ccccccc|c}
\tilde{b} & c_1 & 0 & 0 & 0 & \dots & & \tilde{g}_1 \\
0 & \tilde{b}_2 & c_2 & 0 & 0 & \dots & & \tilde{g}_2 \\
0 & 0 & \tilde{b}_3 & c_3 & 0 & \dots & & \tilde{g}_3 \\
 & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & \tilde{b}_n & & \tilde{g}_n
\end{array}$$

Thus

$$\frac{\tilde{g}_n}{\tilde{b}_n} = v_n$$

And bacwards substitution gives the relation

$$v_i = \frac{\tilde{g}_i - c_i v_{i+1}}{\tilde{b}_i}, \text{For i} = n-1, n-2, n-3, \dots 2, 1$$

This can be interpreted as the following algorithm.

Algorithm 1 General alorithm

$\tilde{b}_1 = b_1$
$\tilde{g}_1 = g_1$
for $i = 2, 3, ..., n$ **do**
    $\gamma = \frac{a_i}{\tilde{b}_{i-1}}$
    $\tilde{b}_i = b_i - \gamma c_{i-1}$
    $\tilde{g}_i = g_i - \gamma \tilde{g}_{i-1}$
$v_n = \frac{\tilde{g}_n}{\tilde{b}_n}$
for $i = n-1, n-2, ..., 1$ **do**
    $v_i = \frac{\tilde{g}_i - c_i v_{i+1}}{\tilde{b}_i}$

b

Notice that the first for loop contains $5(n-1)$ FLOPs while the second for loop contains $3(n-1)$ FLOPs. Hence the total number of FLOPs is roughly $8(n-1)$.

PROBLEM 7

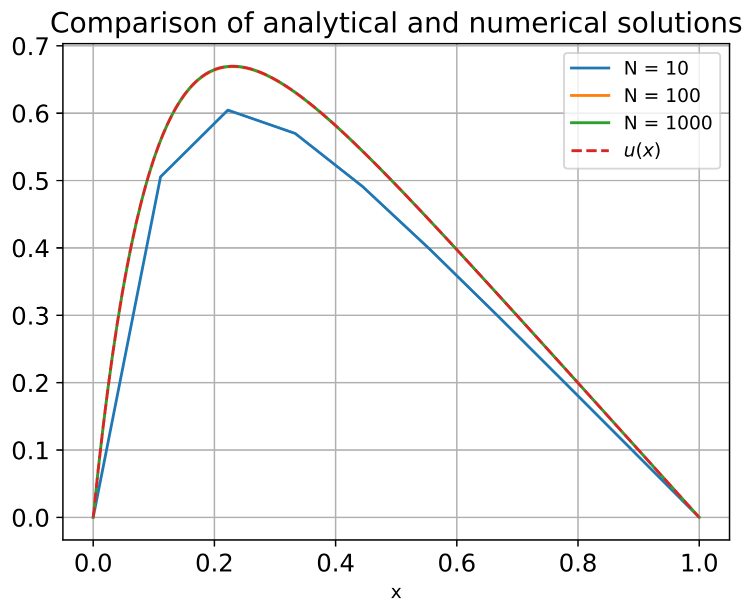

FIG. 2.

PROBLEM 8

PROBLEM 9

a

By making the special algorithm we then get the following results.

$$\tilde{b}_1 = 2$$
$$\tilde{b}_i = 2 - \frac{1}{\tilde{b}_{i-1}}$$
$$\tilde{g}_1 = g_1$$
$$\tilde{g}_i = g_i + \frac{1}{\tilde{b}_{i-1}}\tilde{g}_{i-1}$$
$$v_n = \frac{\tilde{g}_n}{\tilde{b}_n}$$
$$v_i = \frac{\tilde{g}_i + v_{i+1}}{\tilde{b}_i}$$

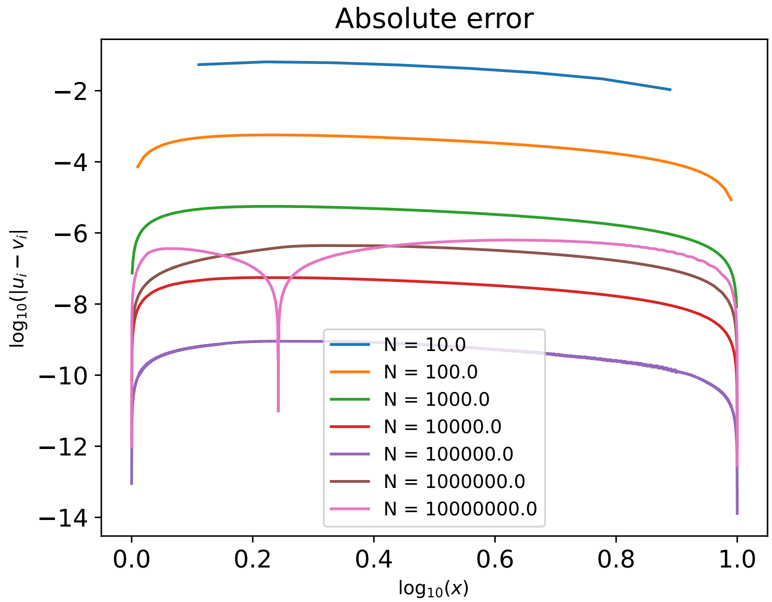

FIG. 3.

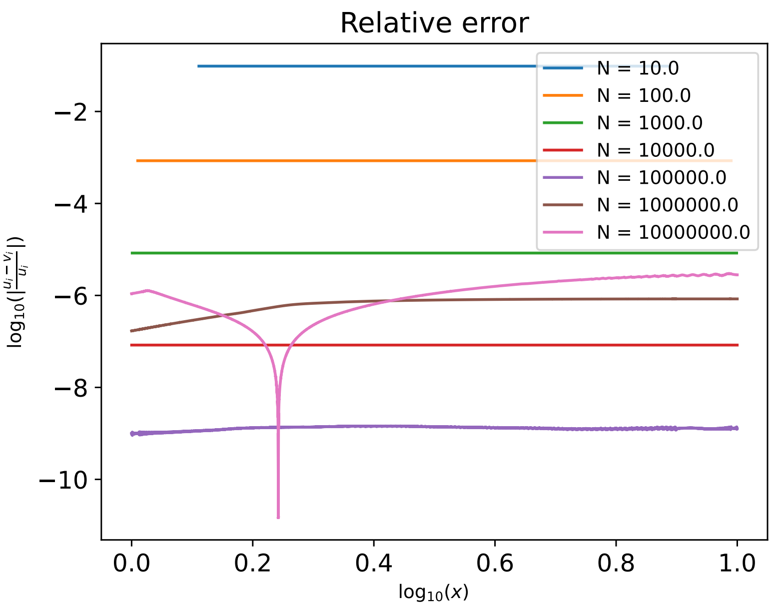

FIG. 4.

b

The number of flops for this special algorithm will then be described as:

$$6(n-1)(+1)$$

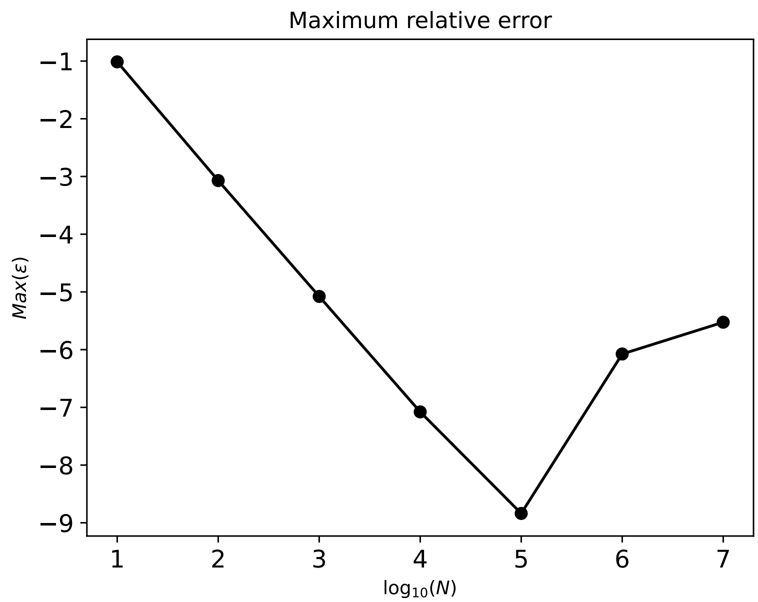

FIG. 5.

c

See the GitHub link

PROBLEM 10

See the GitHub link

PROBLEM 11

ACKNOWLEDGMENTS

Vi ønsker å takke det norske folk for deres støtte og hjelp i dette krevende arbeidet.