FYS3150 oppgavesett 1

Jon Aleksander Prøitz and Marius Torsvoll

Relevant code can be found at: https://github.com/Jonaproitz/Project_1

PROBLEM 1.

Given

$$u(x) = 1 - (1 - e^{-10})x - e^{-10x} \tag{1}$$

Inserting $x = 0$

$$u(0) = 1 - (1 - e^{-10}) \cdot 0 - e^{-10 \cdot 0} = 1 - 0 - 1 = 0$$

and $x = 1$

$$u(1) = 1 - (1 - e^{-10}) \cdot 1 - e^{-10 \cdot 1} = 1 - 1 + e^{-10} - e^{-10} = 0$$

furthermore the one-dimensional possion equation can be written

$$-\frac{d^2u}{dx^2} = -\frac{d^2}{dx^2}\left(1 - (1 - e^{-10})x - e^{-10x}\right) = -\frac{d}{dx}\left((1 - e^{-10}) + 10e^{-10x}\right) = 100e^{-10x} = f(x)$$

Hence equation 1 is an exact solution to our problem. ■

PROBLEM 2.

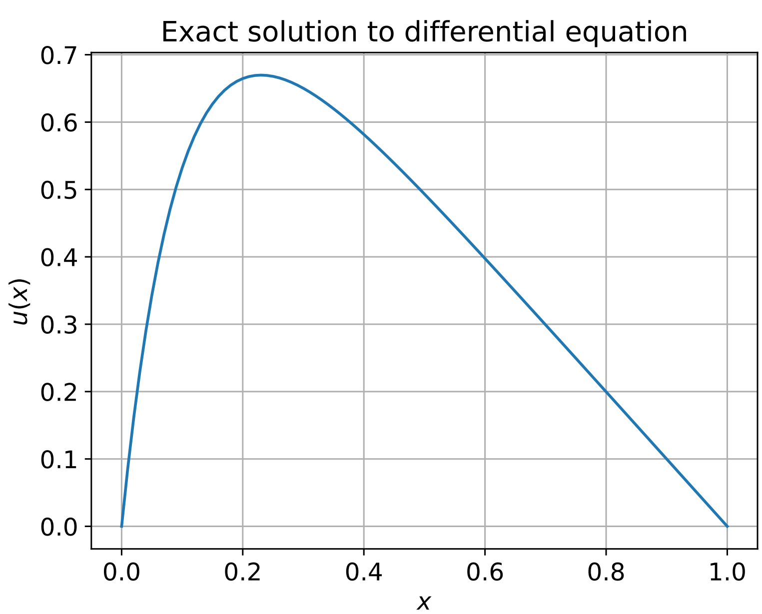


FIG. 1. Plot of equation 1 in the given area $x \in [0, 1]$.

PROBLEM 3.

The one-dimensional poisson equation can be written

$$-\frac{d^2u}{dx^2} = \frac{u(x-h)+2u(x)-u(x+h)}{h^2} + O(h^2) = f(x)$$

Discretizing x with m values and a given distance h between each distinct value then gives

$$\begin{aligned} x &\rightarrow x_0, x_1, x_2, ..., x_{m-1} \\ u(x) &\rightarrow u_0, u_1, u_2, ..., u_{m-1} \\ f(x) &\rightarrow f_0, f_1, f_2, ..., f_{m-1} \end{aligned}$$

with $u_i = v_i$, such that

$$-\frac{d^2v_i}{dx^2} = -v_{i-1} + 2v_i - v_{i+1} = f_i h^2$$

PROBLEM 4.

The set of equations from problem 3 can be written as

$$\begin{aligned} -v_0 + 2v_1 - v_2 &= h^2 f_1 \\ -v_1 + 2v_2 - v_3 &= h^2 f_2 \\ -v_2 + 2v_3 - v_4 &= h^2 f_3 \\ &\vdots \\ -v_{m-3} + 2v_{m-2} - v_{m-1} &= h^2 f_{m-2} \end{aligned}$$

Wich for

$$\begin{aligned} g_1 =& h^2 f_1 + v_0 \\ g_2 =& h^2 f_2 \\ g_3 =& h^2 f_3 \\ &\vdots \\ g_{m-3} =& h^2 f_{m-3} \\ g_{m-1} =& h^2 f_{m-2} + v_{m-1} \end{aligned}$$

can be written as the matrix equation

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & 0 & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{m-3} \\ v_{m-2} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{m-3} \\ g_{m-2} \end{pmatrix}$$

on the form $A\vec{v} = \vec{g}$.

PROBLEM 5

a

When finding the matrix, A, in problem 4 it is assumed that v_0 and v_{m-1} are known. Hence theese values are not calculated and

$$n = m - 2$$

b

As discussed above the the vector $\vec{v}$ is equal to $\vec{v}^*$ excluding the first and last element. Hence

$$\vec{v}^* = [v_0, \vec{v}, v_{m-1}] \tag{2}$$

PROBLEM 6

a

A general tridiagonal matrix is given by the following: syntax:

$$\begin{array}{cccccccc|c} (R_1) & b_1 & c_1 & 0 & 0 & 0 & \dots & & g_1 \\ (R_2) & a_2 & b_2 & c_2 & 0 & 0 & \dots & & g_2 \\ (R_3) & 0 & a_3 & b_3 & c_3 & 0 & \dots & & g_3 \\ & & & & \ddots & & & & \vdots \\ (R_n) & 0 & 0 & 0 & 0 & a_n & b_n & & g_n \end{array} \tag{3}$$

Forward substitution gives us the following relation:

$$R_2 \to R_2 - \frac{a_2}{b_1} R_1 = 0 \;\; b_2 - \tfrac{a_2}{b_1} c_1 \;\; 0 \;\; 0$$

Continuing this substitution with the variables gives us the following results for the variables:

$$\tilde{b}_1 = b_1 \quad \text{and} \quad \tilde{b}_i = b_i - \frac{a_i}{\tilde{b}_{i-1}} c_{i-1}, \qquad \text{For i = 2,3,4....,n}$$
$$\tilde{g}_1 = g_1 \quad \text{and} \quad \tilde{g}_i = g_i - \frac{a_i}{\tilde{g}_{i-1}} \tilde{g}_{i-1}, \qquad \text{For i = 2,3,4....,n}$$

This then gives us the matrix:

$$\begin{array}{cccccc|c} \tilde{b} & c_1 & 0 & 0 & 0 & \dots & \tilde{g}_1 \\ 0 & \tilde{b}_2 & c_2 & 0 & 0 & \dots & \tilde{g}_2 \\ 0 & 0 & \tilde{b}_3 & c_3 & 0 & \dots & \tilde{g}_3 \\ & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & & \tilde{b}_n & \tilde{g}_n \end{array}$$

We then see that:

$$R_n = \frac{R_n}{\tilde{b}_n} = 0 \;\; 0$$

and that:

$$\frac{\tilde{g}_n}{\tilde{b}_n} = v_n$$

Bacwards substitution then gives us the following:

$$v_i = \frac{\tilde{g}_i - c_i v_{i+1}}{\tilde{b}_i}, \text{For i} = n-1, n-2, n-3, \dots 2, 1$$

b

This algorithm is thus:

$$\tilde{b}_1 = b_1$$
$$\tilde{b}_i = b_i - \frac{a_i}{b_{i-1}} c_{i-1}, \text{For i} = 2, 3, 4 \dots n$$
$$\tilde{g}_1 = g_1 \tilde{g}_i = g_i - \frac{a_i}{\tilde{b}_{i-1}} \tilde{g}_{i-1}, \text{For i} = 2, 3, 4 \dots n$$
$$v_n = \frac{\tilde{g}_n}{\tilde{b}_n}$$
$$v_i = \frac{\tilde{g}_i - c_i v_{i+1}}{\tilde{b}_i}, \text{For i} = n-1, n-2, n-3, \dots, 1$$

The total amount of FLOPs will then be described as the sum of FLOPs per n. Thus the FLOPs will then be described as:

$$8(n-1)(+1)$$

TASK 7

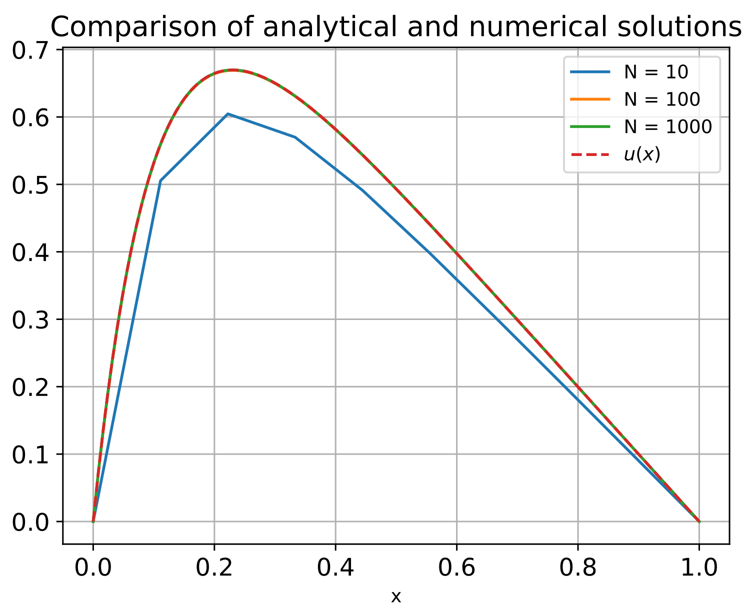
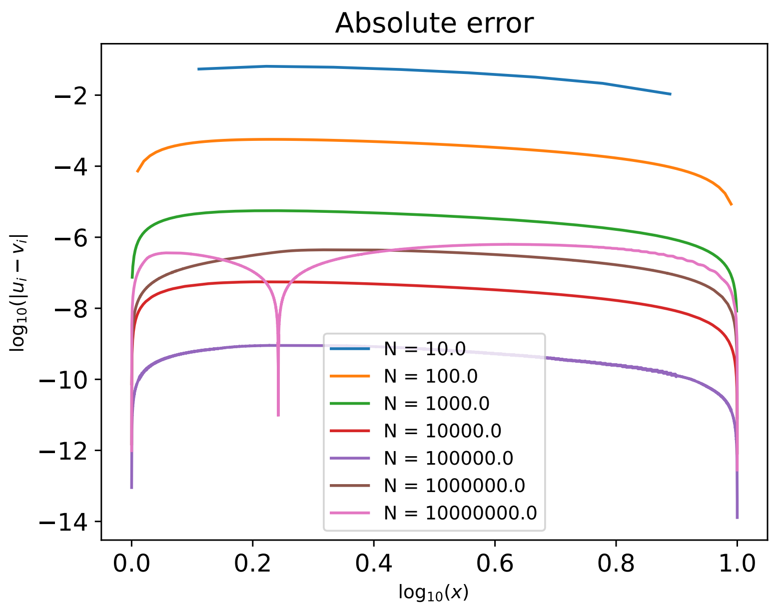
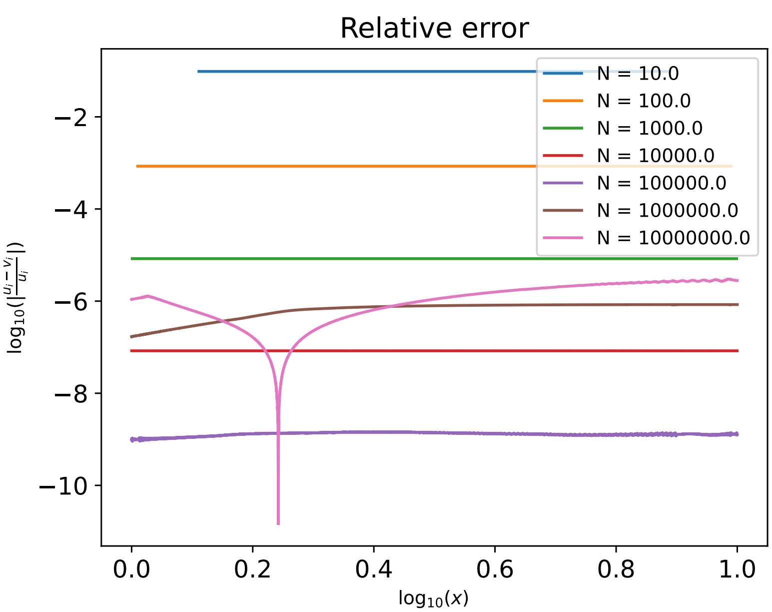


FIG. 2.



Absolute error
−2
−4
−6
−8
−10
−12
−14
$\log_{10}(|u_i - v_i|)$
0.0
0.2
0.4
0.6
0.8
1.0
$\log_{10}(x)$
N = 10.0
N = 100.0
N = 1000.0
N = 10000.0
N = 100000.0
N = 1000000.0
N = 10000000.0

FIG. 3.



Relative error
−2
−4
−6
−8
−10
$\log_{10}(|\frac{u_i - v_i}{u_i}|)$
0.0
0.2
0.4
0.6
0.8
1.0
$\log_{10}(x)$
N = 10.0
N = 100.0
N = 1000.0
N = 10000.0
N = 100000.0
N = 1000000.0
N = 10000000.0

FIG. 4.

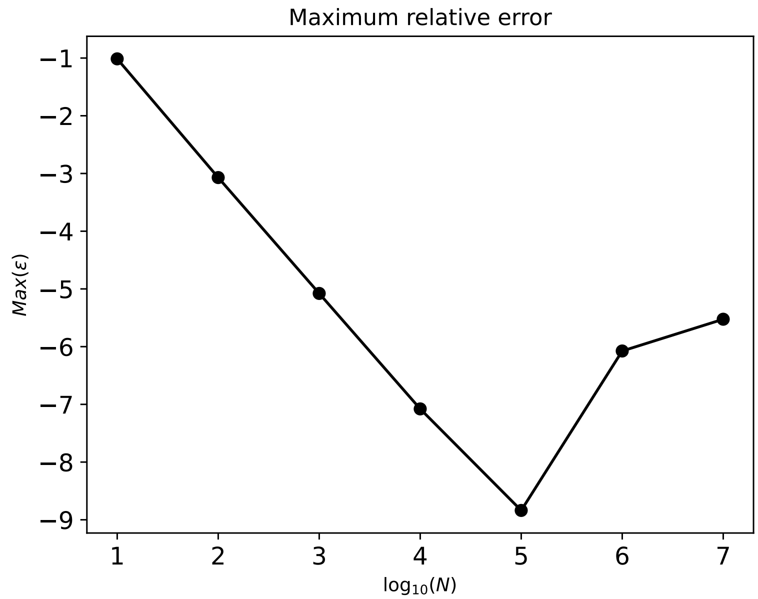


FIG. 5.

TASK 8

TASK 9

a

By making the special algorithm we then get the following results.

$$\tilde{b}_1 = 2$$
$$\tilde{b}_i = 2 - \frac{1}{\tilde{b}_{i-1}}$$
$$\tilde{g}_1 = g_1$$
$$\tilde{g}_i = g_i + \frac{1}{\tilde{b}_{i-1}}\tilde{g}_{i-1}$$
$$v_n = \frac{\tilde{g}_n}{\tilde{b}_n}$$
$$v_i = \frac{\tilde{g}_i + v_{i+1}}{\tilde{b}_i}$$

b

The number of flops for this special algorithm will then be described as:

$$6(n-1)(+1)$$

c

See the GitHub link

TASK 10

See the GitHub link

TASK 11

ACKNOWLEDGMENTS

Vi ønsker å takke det norske folk for deres støtte og hjelp i dette krevende arbeidet.