# Intrinsic Alignment in Cosmology

**Jonas Frugte**[1]

[1]Van Swinderen Institute for Particle Physics and Gravity, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands

E-mail: jonasfrugte@gmail.com

**Abstract.** A brief literature review of intrinsic alignment (IA) in cosmology.

## Contents



### Abstract

EFT: effective field theory

BPs: bias parameters

IA: intrinsic alignment

LIMD: linear in matter density

## 1 Halo Bias and EFT

In this section we introduce the much more established field of galaxy bias and an EFT of galaxy bias. This will directly translate to the EFT used in section 3. We consider a $\Lambda$CDM universe with a perturbed FLRW metric unless stated otherwise.

### 1.1 What is bias

Say we look at a the universe at redshift $z$. Given a perturbation to the matter density $\delta(\mathbf{x}, z)$ we would like to know the perturbation to the density of some other tracer. In this section we will consider the relative number overdensity of dark matter halos, $\delta_h$, but the results are applicable to any tracer in general, notably to galaxy number density. We then model the relation between these two as

$$\delta_h(\mathbf{x}, z) = \sum_{\mathcal{O}} b_{\mathcal{O}}(z)\mathcal{O}(\delta)(\mathbf{x}, z), \tag{1.1}$$

where we sum over some set of operators $\{\mathcal{O}\}$ that map $\delta$ to a new function of position and time and the $b_{\mathcal{O}}$ are the BPs. Crucially, they only depend on z and not on $\mathbf{x}$. 4 questions arise naturally from this model:

1. What is the motivation to use this model? Why is this expected to be accurate up to a (reasonably small) scale cutoff with the right choice of operators and values of BPs?

2. How can we choose a set of operators that is "complete"? I.e. that the model is correct in the sense of the previous question.
3. How do the bias parameters evolves over time?
4. How do we determine bias parameters? Both theoretically and empiricially given a dataset from a real life survey or a simulation.

The rest of this section will be devoted to summarizing the answers to these questions.

### 1.2 Relating bias parameters from different times

We consider a simple case where a population of galaxies instantly formed at some time $\tau_*$. This is sufficient because the evolution equations we obtain will then work as a "Green's function" for the evolution equations of bias parameters in general. We simply integrate over the time evolved bias parameters weighted by the rate of galaxy formation at the integrand time.

We assume that whatever tracer we consider moves with the fluid. This immediately gives the continuity equation:

$$\frac{D}{D\tau}\delta_g = -\theta(1+\delta_g), \tag{1.2}$$

where

$$\frac{D}{D\tau} = \frac{\partial}{\partial\tau} + v^i\frac{\partial}{\partial x^i} \tag{1.3}$$

is the convective time derivative, $v^i$ is the peculiar velocity of the cosmic matter fluid, and $\theta = \partial_i v^i$ is the velocity divergence. To evolve $\delta_g$ we also need evolution equations for $\delta$ and $v^i$. They are given by the continuity equation (again)

$$\frac{D}{D\tau}\delta = -\theta(1+\delta), \tag{1.4}$$

and the Euler equation,

$$\frac{D}{D\tau}\theta = -\mathcal{H}\theta - (\partial^i v^j)^2 - \frac{3}{2}\Omega_m\mathcal{H}^2\delta. \tag{1.5}$$

We proceed by dividing equations 1.4 and 1.2 by $1+\delta$ and $1+\delta_g$, respectively to get

$$\frac{1}{1+\delta_g}\frac{D}{D\tau}\delta_g = \frac{1}{1+\delta}\frac{D}{D\tau}\delta. \tag{1.6}$$

This can be solved by switching to Lagrangian coordinates to get

$$\ln(1+\delta_g(\mathbf{x}(\tau),\tau)) = \ln(1+\delta(\mathbf{x}(\tau),\tau)) + \ln(\frac{1+\delta_g(\mathbf{x}(\tau),\tau)}{1+\delta(\mathbf{x}(\tau),\tau)}), \quad \tau > \tau_*, \tag{1.7}$$

which can be written more simply as

$$1+\delta_g|_\tau = \frac{1+\delta|_\tau}{1+\delta|_{\tau_*}}(1+\delta_g|_{\tau_*}). \tag{1.8}$$

This equation can be made usefull by expanding order by order. Up to second order we get

$$1+\delta_g^{(1)}(\mathbf{x},\tau)+\delta_g^{(2)}(\mathbf{x},\tau)=1+\delta^{(1)}-\delta_*^{(1)}+\delta_{g*}^{(1)}+\delta^{(2)}-\delta(2)_*+\delta_{g*}^{(2)}+[\delta_*^{(1)}]^2-\delta^{(1)}\delta_*^{(1)}+\left[\delta^{(1)}-\delta_*^{(1)}\right]\delta_{g*}^{(1)}. \tag{1.9}$$

Now, the difference between $\mathbf{x}$ and $\mathbf{x}_*$ is first order in perturbations as well. We can Taylor expand around $\mathbf{x}$ and use that the displacement, $\mathbf{x}-\mathbf{x}_*$ evolves according to the growth factor $D$ to get

$$\delta_g^{(1)}(x,\tau)=\left(1+\frac{D_*}{D}\left[b_1^*-1\right]\right)\delta^{(1)}(x,\tau)+\varepsilon^* \tag{1.10}$$

$$\delta_g^{(2)}(x,\tau)=\left\{1+[b_1^*-1]\left(\frac{D_*}{D}\right)^2\right\}\delta^{(2)}+\left\{\frac{D_*}{D}[b_1^*-1]-\left(\frac{D_*}{D}\right)^2[b_1^*-1]+\tfrac{1}{2}b_2^*\left(\frac{D_*}{D}\right)^2\right\}[\delta^{(1)}]^2 \tag{1.11}$$

$$+b_K^{*2}\left(\frac{D_*}{D}\right)^2[K_{ij}^{(1)}]^2+\left(\frac{D_*}{D}-1\right)\frac{D_*}{D}[b_1^*-1]\,s^i_{(1)}\partial_i\delta^{(1)}-\left(\frac{D_*}{D}-1\right)\varepsilon^*\delta^{(1)} \tag{1.12}$$

$$+\left(\frac{D_*}{D}-1\right)s^i_{(1)}\partial_i\varepsilon^*\,. \tag{1.13}$$

Here, $\varepsilon$ is a stochastic parameter added in and $s^i=x^i-x^i_*$ is the displacement. One can then extract the bias parameters at current time (Eularian time) by comparing to the bias expansion at Eularian time and matching terms to get

$$b_1^E=1+\frac{D_*}{D}(b_1^*-1) \tag{1.14}$$

$$b_2^E=b_2^*\left(\frac{D_*}{D}\right)^2+\frac{8}{21}\left(1-\frac{D_*}{D}\right)(b_1^E-1) \tag{1.15}$$

$$b_{K^2}^E=b_{K^2}^*\left(\frac{D_*}{D}\right)^2-\frac{2}{7}\left(1-\frac{D_*}{D}\right)(b_1^E-1) \tag{1.16}$$

Clearly this can be easily generalized to any order.

### 1.3 EFT of bias parameters

Main ref: [1] We would like to find a "basis" of bias parameters up to a given order in perturbation theory. We will assume that (1) gravitation is described by general relativity, (2) that we can neglect the impact of massive neutrinos and dark energy perturbations, (3) that initial conditions are Gaussian and adiabatic.

*We will assume that the only quantities relevant for the formation of galaxies are local on some scale* $R_*$ (which is much larger than the size of a galaxy / whatever tracer we are considering). According to the equivalence principle, the only effects that can be measured by a freely falling observer (i.e. forming galaxy) is the second derivatives of the gravitational potential, $\partial_i\partial_j\Phi$. Higher derivatives will be higher order contributions, because each derivative on a scale of $R_*$ contributes a factor of $k/R_*$, which for large enough scales will be much smaller than one.

Now, the galaxy density, $\delta_g$ will in a very general sense depend on the history of $\partial_i\partial_j\Phi$ along the fluid trajectory. In other words:

$$\partial_g(\mathbf{x},\tau) \supset \int^{\tau} d\tau' f_{\mathcal{O}}(\tau,\tau')\mathcal{O}(\mathbf{x}_{fl}(\tau'),\tau'), \tag{1.17}$$

where $f_O$ is some kernel that we don't need to bother defining. We can then Taylor expand to get

$$\partial_g(\mathbf{x},\tau) \supset \left[\int^{\tau} d\tau' f_{\mathcal{O}}(\tau,\tau')\right]\mathcal{O}(\mathbf{x}_{fl}(\tau),\tau) + \left[\int^{\tau} d\tau'(\tau'-\tau) f_{\mathcal{O}}(\tau,\tau')\right]\frac{D}{D\tau}\mathcal{O}(\mathbf{x}_{fl}(\tau),\tau) + \cdots, \tag{1.18}$$

where $D/D\tau$ is the convective derivative along the fluid flow. In order to define a basis we thus also have to include time derivatives $D/D\tau$ of the second derivatives of the potential. It now seems like we would need an infinite set of operators to form a basis. This can be avoided by noticing at finite order in perturbation theory, only a finite amount of these operators are linearly independent of one another.

In order to show the above statement, we will first switch to Lagrangian coordinates $\mathbf{q} = \mathbf{x}_{fl}(\tau = 0)$ where the convective time derivative simplifies to $\partial/\partial\tau$. We also assume that the $n$-th order growth factor is given by the linear growth factor to the $n$-th power. This is only strictly valid for an EdS (flat matter dominated universe) but is still very accurate for other cosmologies such as $\Lambda$ CDM. Consider some operator in the basis, $O_L(\mathbf{q},\tau)$ which has contributions at various orders, $O_L^i(\mathbf{q},\tau)$, then

$$\left(\frac{D}{D\tau}\right)^n O_L(\mathbf{q},\tau)|^{(n)} = \left(\frac{\partial}{\partial\tau}\right)^n O_L(\mathbf{q},\tau)|^{(n)} = \sum_{i=1}^{n}\left(\frac{d^n}{d\tau^n}D^i(\tau)\right)O_L^i(\mathbf{q},\tau_0). \tag{1.19}$$

Next, although higher order contributions start being linearly dependent on the lower order contributions after some order, as shown above, it is worth noting that these higher order contributions are in general no longer local in $\partial_i\partial_j\phi$. This is immediately obvious if we notice that the convective time derivative contains a spatial derivative that generates higher order $(k/R_*)$ terms. It turns out that, due to symmetry requirements, we only need to consider $\partial_i\partial_j/\nabla^2$ acting on powers of $\partial_k\partial_l\Phi$ in order to get all the new terms created by the extra spatial derivatives.

Constructing a *local* Eularian basis out of $\partial_i\partial_j\Phi(\mathbf{x},\tau)$ and its convective time derivatives can be done explicitly as follows. First define

$$\Pi^{[1]}_{ij}(\mathbf{x},\tau) = \frac{2}{3\Omega_m\mathcal{H}^2}\partial_i\partial_j\Phi(\mathbf{x},\tau) = K_{ij}(\mathbf{x},\tau) + \frac{1}{3}\delta_{ij}\delta(\mathbf{x},\tau) \tag{1.20}$$

and

$$\Pi^{[n]}_{ij} = \frac{1}{(n-1)!}\left[\frac{1}{\mathcal{H}f}\frac{D}{D\tau}\Pi^{[n-1]}_{ij} - (n-1)\Pi^{[n-1]}_{ij}\right], \tag{1.21}$$

as given by ref. [2]. Then take into account that $\mathrm{tr}\Pi^{[n]}$ can be written in terms of lower order operators through the Eularian fluid equations.

To get all terms at order $n$, we then write products of $\Pi^{[k]}$ where $k < n$ and the total order adds up to $n$ or less and take traces of the different possible combinations of each product.

To get the full expansion, we should also take into account higher order spatial derivatives of the gravitational potential. This derivative is taken on the size scale of the tracer, which we will denote by $R_t$. Mathematically, this corresponds to

$$\frac{\partial}{\partial x^i} \to \frac{\partial}{\partial \left(x^i/R_t\right)} = R_t \frac{\partial}{\partial x^i}. \tag{1.22}$$

In Fourier space, every derivative on that scale is thus surpressed by $R_t k$. We see that, if the tracer is small (small $R_t$) compared to the scales that we are looking at $1/k$, then we each order of spatial derivatives is surpressed and we can truncated the expansion up to some order. Under rotation and parity invariance we only get odd orders of spatial derivatives contributing to the bias expansion. (both for scalars and for symmetric tensors as we will discuss later)

## 2 Spin-2 Fields and Their Projections

It is well known how to construct powerspectra from scalar quantities. Constructing them for a spin-2 field requires more care. In particular, we would like to find a way to define powerspectra that is independent of the orientation of the observer.

We model galaxies in 3D space as ellipsoids where we do not care about the overal size. This means they can be described according to the $3 \times 3$ symmetric trace-free tensor $Q_{ij}$ given by

$$Q_{ij} = \mathrm{tf}\left(\frac{\int \mathrm{d}^3 x \rho(\mathbf{x}) x^i x^j}{\int \mathrm{d}^3 x \rho(\mathbf{x})}\right), \tag{2.1}$$

where we assume for simplicity that the galaxy's center of mass is at $x^i = 0$ and $\mathrm{tf}(\cdot)$ means taking the trace free part. The trace corresponds to the overal size of the galaxy and is not of interest to us. This corresponds roughly to $g_{ij}$ from [3]. $Q_{ij}$ at some point can transform under SO(3). Specifically, it transforms under the spin-2 representation, which can immediately be seen because $Q_{ij}$ is a 5 dimensional vector space and there is only one irreducible representation of SO(3) of dimension $2\ell + 1$, with in this case $\ell = 2$. This is the classification theorem. Alternatively, it is clear that under a SO(3) rotation given by $R$, we get $Q_{ij} \to R_{ia} R_{jb} Q_{ab}$. One can compare this to the spin-2 representation of SO(3) to see that they are the same.

We observe shapes as projected onto the sky. If we take $\mathbf{n}$ as our line of sight with associated projection matrix $P^{ij}$, then

$$\gamma_{ij} := \mathrm{tf}_P \left(P^{ik} P^{jl} g_{kl}(\mathbf{x}, z)\right) \tag{2.2}$$

$$= \frac{1}{2}\left(P^{ik} P^{jl} + P^{il} P^{jk} - P^{ij} P^{kl}\right) g_{kl}(\mathbf{x}, z) \tag{2.3}$$

$$=: P^{ijkl} g_{kl}(\mathbf{x}, z). \tag{2.4}$$

In the last line we simply define $P^{ijkl}$ as the combination of $P^{ij}$ from the line above. Going from line 1 to line 2 follows by using that in the projected space, the metric is $P^{ij}$ instead of $\delta^{ij}$ and the trace of an arbitrary tensor $M_{ij}$ thus becomes $P^{ij} M_{ij}$.

$\gamma_{ij}$ is again symmetric, trace-free, and transforms like a spin-2 representation, now of SO(2).

### 2.1 Spherical tensor decomposition

Going back to the non-projected $Q_{ij}$, it's clear that this tensor is defined at each point in space, $\mathbf{x}$. Now look at the Fourier transform, $Q_{ij}(\mathbf{k})$. For a mode in a given direction, $\hat{\mathbf{k}}$, we can decompose $Q_{ij}(\mathbf{k})$ through the spherical tensors $(Y_2^{(m)})(\hat{\mathbf{k}})_{ij}$, with $m = -2, -1, 0, 1, 2$ denoting the helicity. We get

$$(Y_2^{(0)})(\hat{\mathbf{k}})_{ij} = \sqrt{3/2}(\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j - \frac{1}{3}\delta_{ij}) \tag{2.5}$$

$$(Y_2^{(\pm 1)})(\hat{\mathbf{k}})_{ij} = \sqrt{1/2}(\hat{\mathbf{k}}_i\mathbf{e}_j^{\pm} + \hat{\mathbf{k}}_j\mathbf{e}_i^{\pm}) \tag{2.6}$$

$$(Y_2^{(\pm 1)})(\hat{\mathbf{k}})_{ij} = \mathbf{e}_i^{\pm}\mathbf{e}_j^{\pm} \tag{2.7}$$

with $e^{\pm} =$ ...and $i, j = 1, 2, 3$. The unit vectors $e_1$ and $e_2$ are defined by ... For a rotation around $\hat{\mathbf{k}}$ by an angle $\phi$ the spherical tensors transform as

$$(Y_2^{(m)})(\hat{\mathbf{k}})_{ij} \to e^{im\phi}(Y_2^{(m)})(\hat{\mathbf{k}})_{ij}.$$

These spherical tensors have a number of other usefull properties which we will not get into right now.

### 2.2 Defining Power Spectra

We can now define the powerspectrum as a $3 \times 3 \times 3 \times 3$ tensor given by

$$\langle Q_{ij}(\mathbf{k})Q_{ij}(\mathbf{k}') = (2\pi)^3\delta^D(\mathbf{k} + \mathbf{k}')P_{ijkl}^{QQ}(\mathbf{k}). \tag{2.8}$$

This definition is the most obvious, however because $Q_{ij}$ transforms as a tensor under rotations, it leaves $P_{ijkl}^{QQ}(\mathbf{k})$ dependent on the direction of $\mathbf{k}$. We can instead work with the decomposition of $Q_{ij}$ in the spherical tensors:

$$Q_{ij}(\mathbf{k}) = \frac{1}{3}\delta_{ij}Q_0^0(\mathbf{k}) + \sum_{m=-2}^{2} Q_2^m(\mathbf{k})Y_{ij}^{(m)}(\hat{\mathbf{k}}), \tag{2.9}$$

where we now allow $Q_{ij}$ to also have a trace / scalar component $Q_0^0$. We can then define the following powerspectra:

$$\langle S_l^{(m)}(\mathbf{k})S_{l'}^{(m')}(\mathbf{k}')\rangle = (2\pi)^3\delta_{mm'}\delta^D(\mathbf{k} + \mathbf{k}')P_{ll'}^{(m)}(k). \tag{2.10}$$

When the powerspectra are defined in this way, they are scalars. There are 7 independent spectra, however we also have by the properties of the spherical tensors that

$$P_{ll'}^{(m)}(k)^* = P_{ll'}^{(-m)}(k). \tag{2.11}$$

If we also enfore invariance under parity transformations, then we find that the powerspectra are real and thus that

$$P_{ll'}^{(m)}(k) = P_{ll'}^{(-m)}(k). \tag{2.12}$$

This reduces the amount of independent spectra to 5.

## 3 EFT of Galaxy IA

To get the EFT of galaxy IA, we use $\Pi^{[n]}$ again (equation ... ) but instead of making scalar combinations we make symmetric trace free combinations of the form

$$Q_{ij}(\mathbf{x}, \eta) = \sum_{\mathcal{O}} b_{\mathcal{O}}(\eta) \mathcal{O}_{ij}(\mathbf{x}, \eta). \tag{3.1}$$

## References


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