In [ ]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

Problem 1

The given empirical risk function can be rewritten in its matrix form

$$\begin{aligned}\hat{R}(\theta) &= \frac{1}{n}(\mathbf{X}\theta - \mathbf{y})^2 \\ &= \frac{1}{n}(\mathbf{X}\theta - \mathbf{y})^\top(\mathbf{X}\theta - \mathbf{y}) \\ &= \frac{1}{n}(\theta^\top \mathbf{X}^\top \mathbf{X}\theta - 2\mathbf{y}^\top \mathbf{X}\theta + \mathbf{y}^\top \mathbf{y}),\end{aligned}$$

we can compute the gradient of $\hat{R}$ as follows

$$\triangledown \hat{R}(\theta) = \frac{1}{n}(2\mathbf{X}^\top \mathbf{X}\theta - 2\mathbf{y}^\top \mathbf{X}),$$

for the optimal θ, denoted as θ^*, the gradient should be 0, thus we have

$$2\mathbf{X}^\top \mathbf{X}\theta^* = 2\mathbf{y}^\top \mathbf{X}.$$

For gradient descent, we have the following equation

$$\begin{aligned}\theta^{k+1} - \theta^* &= \theta^k - \alpha\triangledown \hat{R}(\theta^k) - \theta^* \\ &= \theta^k - \frac{\alpha}{n}(2\mathbf{X}^\top \mathbf{X}\theta^k - 2\mathbf{y}^\top \mathbf{X}) - \theta^*.\end{aligned}$$

According to eq. (1), we have:

$$\begin{aligned}\theta^{k+1} - \theta^* &= \theta^k - \frac{\alpha}{n}(2\mathbf{X}^\top \mathbf{X}\theta^k - 2\mathbf{X}^\top \mathbf{X}\theta^*) - \theta^* \\ &= \left(\mathbf{I} - \frac{2\alpha}{n}\mathbf{X}^\top \mathbf{X}\right)\left(\theta^k - \theta^*\right),\end{aligned}$$

so the following inequality holds:

$$\left\|\theta^{k+1} - \theta^*\right\|_2 \leq \left\|\mathbf{I} - \frac{2\alpha}{n}\mathbf{X}^\top \mathbf{X}\right\| \left\|\theta^k - \theta^*\right\|_2.$$

A good choice of α should minimize the term $\left\|\mathbf{I} - \frac{2\alpha}{n}\mathbf{X}^\top \mathbf{X}\right\|$. We can justify that $\mathbf{X}^\top \mathbf{X}$ is symmetic and positive definite, thus the eigenvalues and singluar values of the matrix are identical.

Now we perform singular value decomposition on $\mathbf{X} \in \mathbb{R}^{n\times d}$, the resulted sinlugar values are denoted by $\mathbf{\Sigma} = [\sigma_1, , \cdots, \sigma_d]$, with which we have the singluar values of $\mathbf{X}^\top \mathbf{X}$ can be intepreted as $[\sigma_1^2, , \cdots, \sigma_d^2]$.

If we choose the optimal α, the following equation holds

$$\left\|\mathbf{I} - \frac{2\alpha}{n}\mathbf{X}^\top \mathbf{X}\right\| = \max\left(\frac{2\alpha}{n}\sigma_1^2 - 1, 1 - \frac{2\alpha}{n}\sigma_d^2\right),$$

the right hand side is minimized by $\frac{2\alpha}{n} = \frac{2}{\sigma_1 + \sigma_d}$, which means, a good choice of α is given by

$$\alpha^* = \frac{n}{\sigma_1^2 + \sigma_d^2}.$$

Problem 2

1

By definition, for a convex function, given $\lambda \in (0, 1)$ and 2 arbitrary points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$, the following inequality holds

$$\lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) \geq f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2).$$

For the given function, we have

$$\begin{aligned} \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2) &= \lambda \|\mathbf{x}_1 - \mathbf{b}\|_2 + (1-\lambda)\|\mathbf{x}_2 - \mathbf{b}\|_2, \\ f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &= \|\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 - \mathbf{b}\|_2. \end{aligned}$$

The second equation can be rewritten as

$$\begin{aligned} f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &= \|\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 - \lambda \mathbf{b} - (1-\lambda)\mathbf{b}\|_2 \\ &= \|\lambda(\mathbf{x}_1 - b) + (1-\lambda)(\mathbf{x}_2 - b)\|_2, \end{aligned}$$

according to triangle inequality, we have

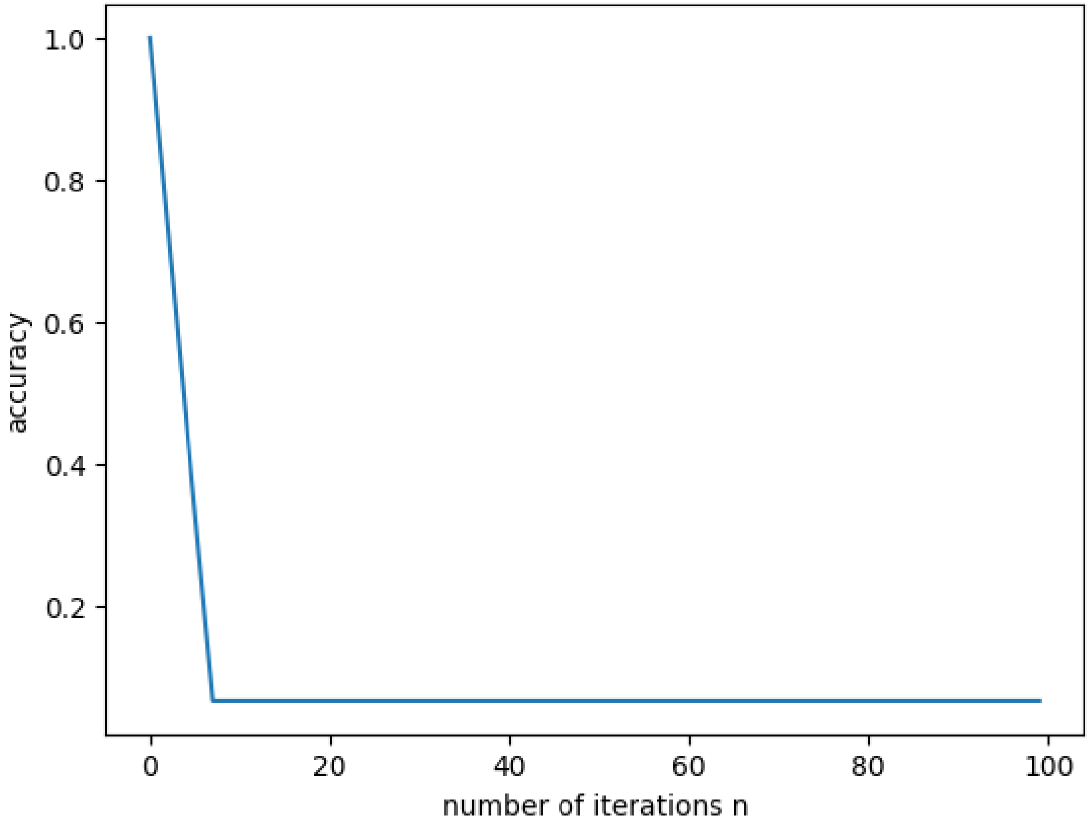
$$\begin{aligned} f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &\leq \|\lambda(\mathbf{x}_1 - b)\| + \|(1-\lambda)(\mathbf{x}_2 - b)\|_2 \\ &= \lambda \|(\mathbf{x}_1 - b)\| + (1-\lambda)\|(\mathbf{x}_2 - b)\|_2 \\ &= \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2). \end{aligned}$$

Therefore the given function is convex.

2

In [ ]:
```
# Define the gradient of the function
def gradient(x):
  b = np.array([4.5, 6])
  return (x - b) / np.linalg.norm(x - b, ord=2)

# Beginning from subtask 5, we have a new gradient
def gradient_2(x):
  b = np.array([4.5, 6])
  return 2 * (x - b)

# Define the accuracy calculation
def accuracy(x, x_star):
  return np.linalg.norm(x_star - x, ord=2) / np.linalg.norm(x_sta
r, ord=2)

# Define the generation of x_k+1
def new_x_gen(x, alpha):
  return x - alpha * gradient(x)

# Define the generation of x_k+1 for subtask 5 and later
def new_x_gen_2(x, alpha):
  return x - alpha * gradient_2(x)

# Define the function
b = np.array([4.5, 6])
f_1 = lambda x : np.linalg.norm(x - b, ord=2)

# Find the minimizer using scipy
initial_guess = np.array([0, 0])
opt_result = minimize(f_1, initial_guess, method='BFGS')
x_star = opt_result.x
print("optimal x:", x_star)

# Define the number of loops
n = np.linspace(0, 99, 100)

# Define the x array
x_2 = np.array([[0, 0]])
accu_2 = np.zeros([n.shape[0], 1])

for i in range(0, n.shape[0]):
  nwe_x_2 = new_x_gen(x_2[i, :], 1)
  x_2 = np.append(x_2, [nwe_x_2], axis=0)
  accu_2[i] = accuracy(x_2[i, :], x_star)
  if accu_2[i, :] <= 0.01:
    print(i)
    break

print("final accuracy:", min(accu_2))
plt.plot(n, accu_2)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
optimal x: [4.5 6. ]
final accuracy: [0.06666667]
```



From the plot we can see that after few iterations, the accuracy stops decaying, which means the newly generated $\mathbf{x}^{k+1}$ starts to oscillate around the optimal $\mathbf{x}^*$

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In [ ]:

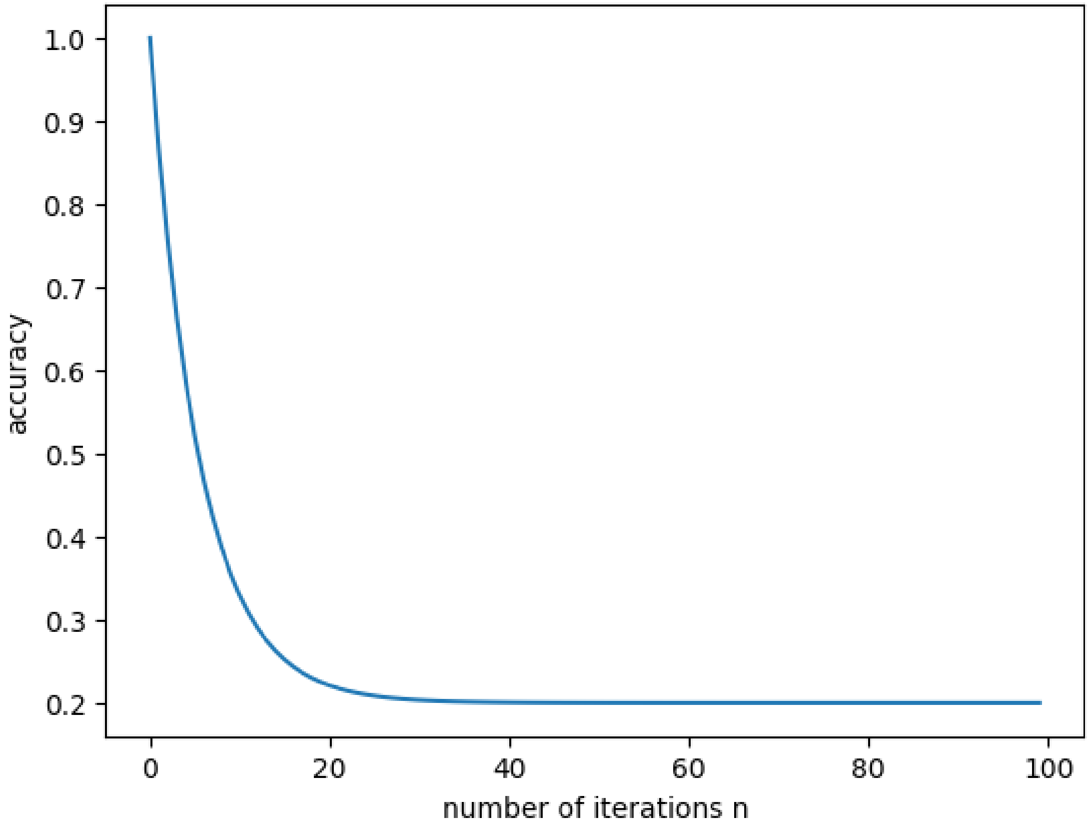
```
# Define the x array
x_3 = np.array([[0, 0]])
accu_3 = np.zeros([n.shape[0], 1])

for i in range(0, n.shape[0]):
  nwe_x_3 = new_x_gen(x_3[i, :], (5/6)**i)
  x_3 = np.append(x_3, [nwe_x_3], axis=0)
  accu_3[i] = accuracy(x_3[i, :], x_star)
  if accu_3[i, :] <= 0.01:
    break

print("final accuracy:", min(accu_3))

plt.plot(n, accu_3)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
final accuracy: [0.20000001]
```



The same happens here, $\mathbf{x}^{k+1}$ oscillates around the optimal $\mathbf{x}^*$ and never converge to the optimal

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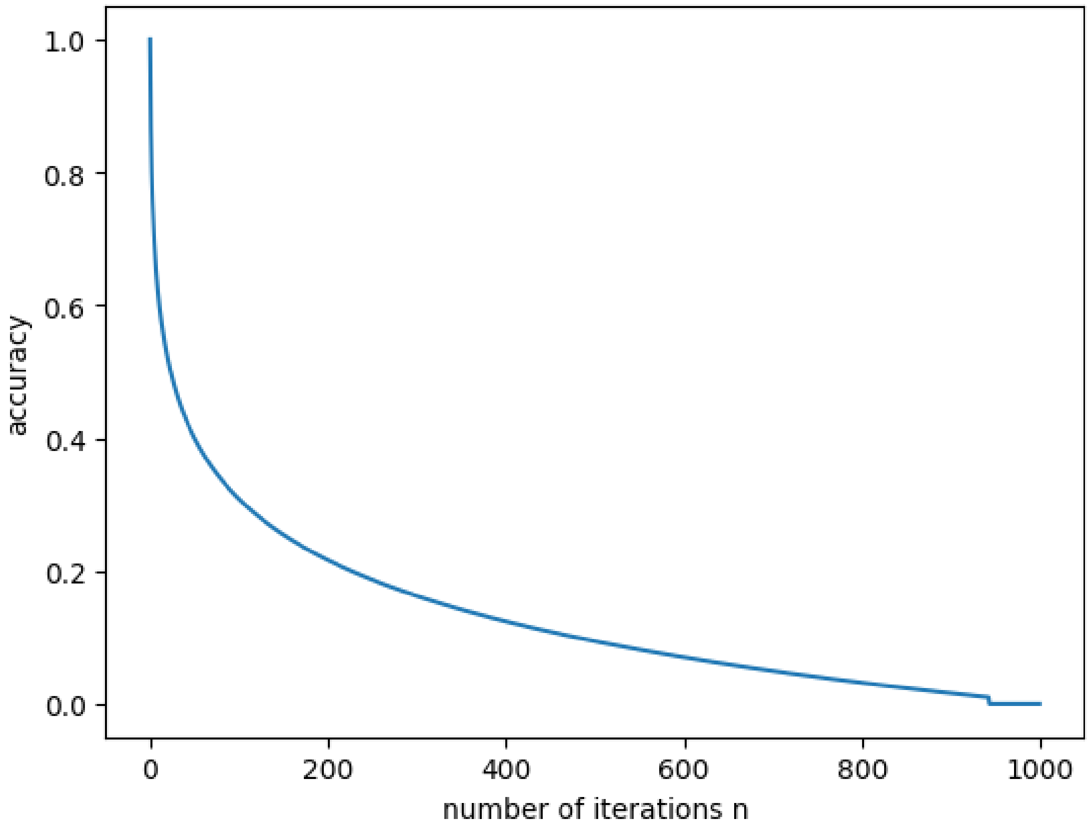
In [ ]:
```
# Define the x array
n_4 = np.linspace(0, 999, 1000)
x_4 = np.array([[0, 0]])
accu_4 = np.zeros([n_4.shape[0], 1])

for i in range(0, n_4.shape[0]):
  nwe_x_4 = new_x_gen(x_4[i, :], 1/(i+1))
  x_4 = np.append(x_4, [nwe_x_4], axis=0)
  accu_4[i] = accuracy(x_4[i, :], x_star)
  if accu_4[i, :] <= 0.01:
    print("number of iterations =",i)
    break

plt.plot(n_4, accu_4)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
number of iterations = 942
```



We see that after 942 iterations the required accuracy is satisfied.

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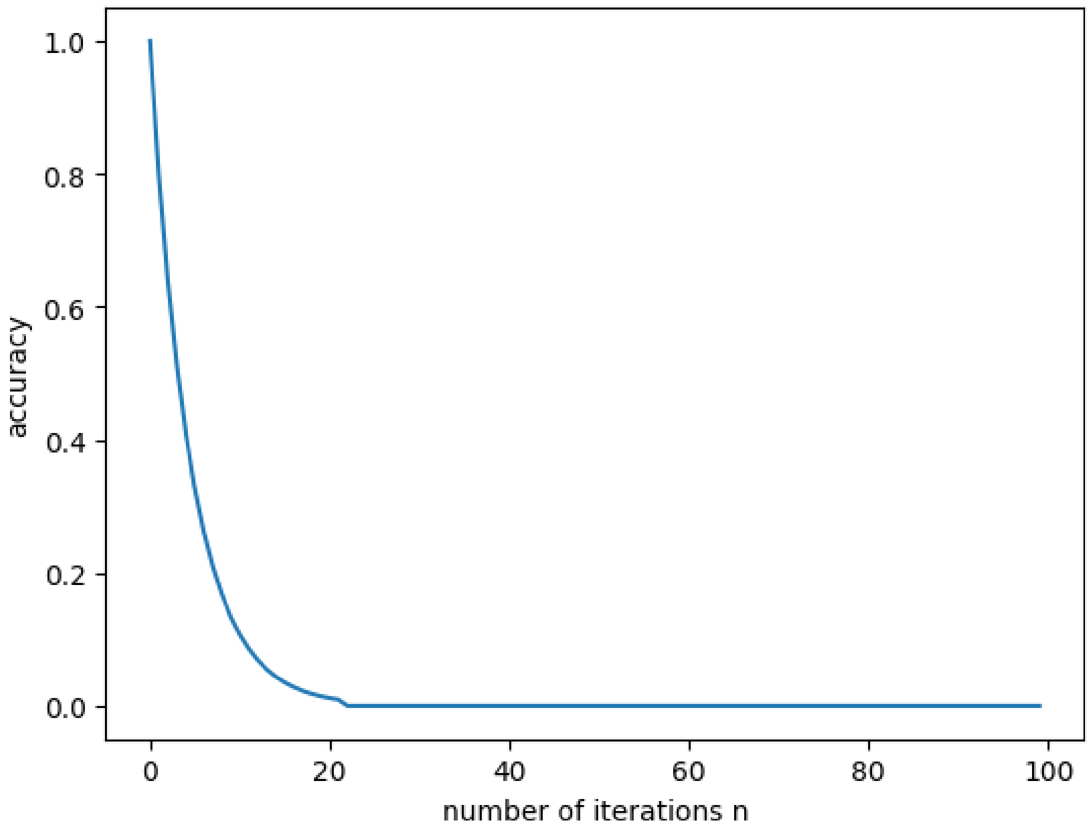
In [ ]:

```
f_2 = lambda x : (np.linalg.norm(x - b, ord=2)) ** 2

# Find the minimizer using scipy
initial_guess = np.array([0, 0])
opt_result_2 = minimize(f_2, initial_guess, method='BFGS')
x_star_2 = opt_result_2.x
print("optimal solution:", x_star_2)

# Define the number of loops
n = np.linspace(0, 99, 100)

# Define the x array
x_5 = np.array([[0, 0]])
accu_5 = np.zeros([n.shape[0], 1])

for i in range(0, n.shape[0]):
  nwe_x_5 = new_x_gen_2(x_5[i, :], 0.1)
  x_5 = np.append(x_5, [nwe_x_5], axis=0)
  accu_5[i] = accuracy(x_5[i, :], x_star_2)
  if accu_5[i, :] <= 0.01:
    print("number of iterations =", i)
    break

plt.plot(n, accu_5)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
optimal solution: [4.4999963  6.00000296]
number of iterations = 21
```



With given $g(\mathbf{x})$ and $\alpha_k = 0.1$, the gradient descent converges to the optimal after 21 iterations

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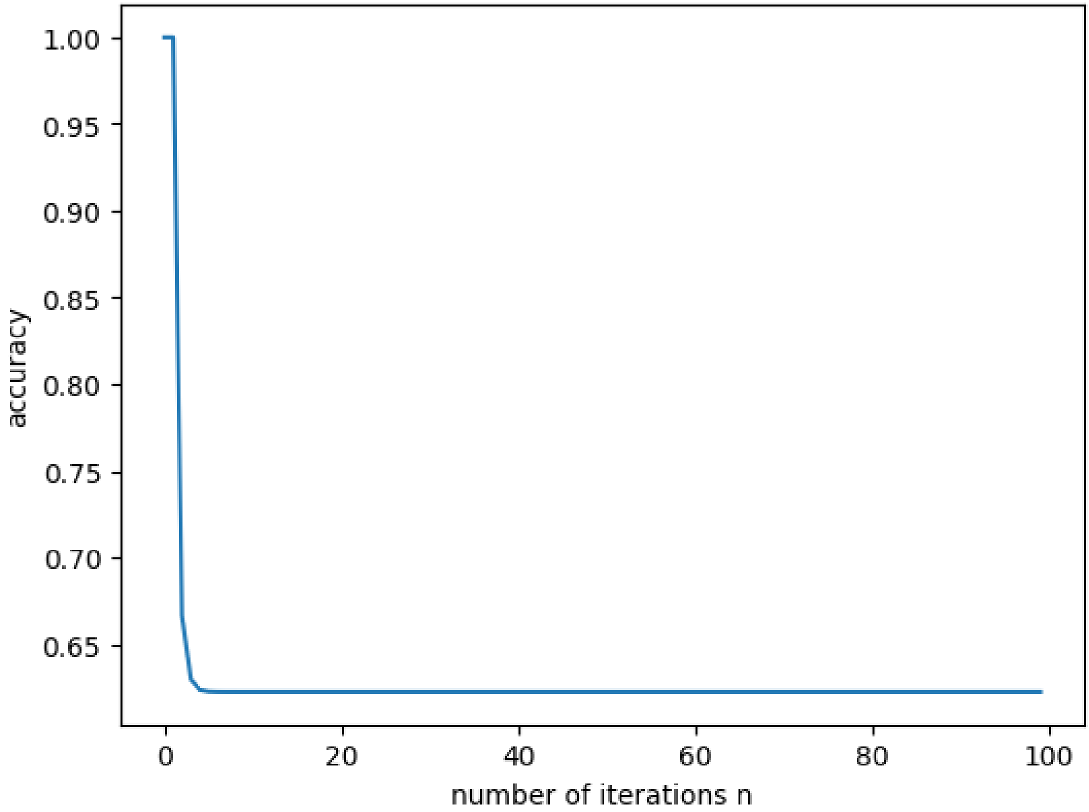
In [ ]:
```
x_6 = np.array([[0, 0]])
accu_6 = np.zeros([n.shape[0], 1])

for i in range(0, n.shape[0]):
  nwe_x_6 = new_x_gen_2(x_6[i, :], (1/6)**i)
  x_6 = np.append(x_6, [nwe_x_6], axis=0)
  accu_6[i] = accuracy(x_6[i, :], x_star_2)
  if accu_6[i, :] <= 0.01:
    print("number of iterations =", i)
    break

print("final accuracy:", min(accu_6))

plt.plot(n, accu_6)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
final accuracy: [0.62264482]
```



As shown, with this choice of stepsize, gradient descent will never converge to optimal

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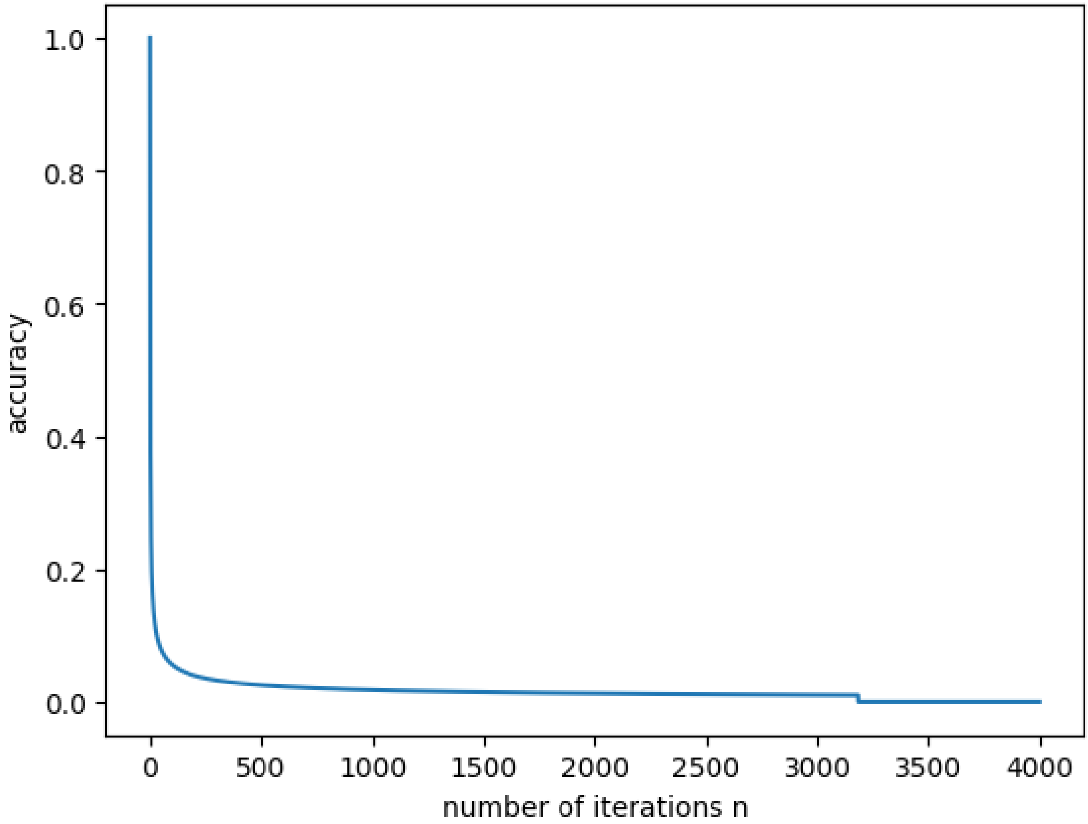
In [ ]:
```
n = np.linspace(0, 3999, 4000)

x_7 = np.array([[0, 0]])
accu_7 = np.zeros([n.shape[0], 1])

for i in range(0, n.shape[0]):
  nwe_x_7 = new_x_gen_2(x_7[i, :], (1/(4*(i+1))))
  x_7 = np.append(x_7, [nwe_x_7], axis=0)
  accu_7[i] = accuracy(x_7[i, :], x_star_2)
  if accu_7[i, :] <= 0.01:
    print("number of iterations =", i)
    break

plt.plot(n, accu_7)
plt.xlabel("number of iterations n")
plt.ylabel("accuracy")
plt.show()
```

```
number of iterations = 3183
```



After 3183 iterations, the required accuracy will be achieved