

$$\text{Beta function: } B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\text{Gamma function: } \Gamma(x) = (x-1)!$$

$$\text{Beta distribution} \Rightarrow \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} = \frac{1}{B(\alpha, \beta)} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \\ = B(\theta | \alpha, \beta)$$

$$\text{Binomial distribution} \Rightarrow p(\theta|x) = \frac{N!}{(N-x)!x!} \cdot \theta^x \cdot (1-\theta)^{N-x} \rightarrow \text{likelihood}$$

* 如果 Prior distribution 和 likelihood function 可使 Posterior distribution 和 Prior distribution 有相同形式 那 Prior distribution 和 likelihood function conjugate

$$\begin{aligned} P(x|\theta) &\propto \text{Prior} * \text{likelihood} = \underbrace{\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}}_{\text{Constant}} \cdot \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \cdot \underbrace{\frac{N!}{(N-x)!x!}}_{\text{Constant}} \cdot \theta^x \cdot (1-\theta)^{N-x} \\ &\propto \theta^{\alpha-1} \cdot (1-\theta)^{\beta-1} \cdot \theta^x \cdot (1-\theta)^{N-x} \cdot \underbrace{C}_{\text{Constant}} \\ &\propto \theta^{\alpha-1+x} \cdot (1-\theta)^{\beta-1+N-x} \cdot \frac{\Gamma(\alpha+\beta+N)}{\Gamma(\alpha+x)\Gamma(\beta+N-x)} \\ &= B(\theta | \alpha+x, \beta-x+N) \end{aligned}$$

Diagram annotations: An arrow points from $P(x|\theta)$ to "Posterior". Another arrow points from "Prior * likelihood" to "etc".