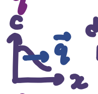
PASSIVE SCALAR

(Inert, neutrally-buoyant)
(Doesn't modify the flow)

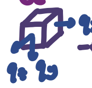
* evolve solution from 1D sequence:
1D → no. of dimension → add advection → boundary condition.

$\vec{q} = -D\nabla C$



downgradient
high to low

$\frac{\partial c}{\partial t} = -\nabla\vec{q}$



q net outflow
$-\nabla\vec{q}$ accumulate.

$\frac{\partial c}{\partial t} = D\nabla^2 C$

C in vicinity larger
$\vec{q}$ flow towards center.

Diffusion Equation ($\vec{q} = -D\nabla C$)

(derive!) (derive!)

Mass continuity
$$\frac{\partial C}{\partial t} = -\nabla \cdot \vec{q}$$

Fick's law
$$\vec{q} = -D\nabla C$$ (mass flux by diffusion)

$$\frac{\partial C}{\partial t} = D\nabla^2 C : \frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right)$$

Solution (1D, 2D, 3D)

if question give M [kg] remember find area [m²] where the mass is spread. and M/A

1D: $$C(x,t) = \frac{M[\text{kg m}^{-2}]}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

where $\sigma = \sqrt{2Dt}$ (spread width) $\frac{d\sigma}{dt} = \frac{D}{\sigma}$ (rate of spread)

2D: $$C(x,y,t) = \frac{M[\text{kg m}^{-1}]}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

3D: $$C(x,y,z,t) = \frac{M[\text{kg}]}{(2\pi)^{3/2}\sigma^3} e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}$$

C decays faster as no. of dimension increases:
1D: $C \sim \sigma^{-1} \sim t^{-1/2}$
2D: $C \sim \sigma^{-2} \sim t^{-1}$
3D: $C \sim \sigma^{-3} \sim t^{-3/2}$

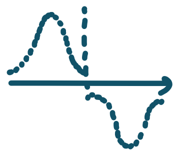
Boundary Condition.

Reflector at ($x=h$): $q_x(x=h) = -D\frac{\partial C}{\partial x}\Big|_{x=h} = 0$

$$C(x,t) = \frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-0)^2}{2\sigma^2}} + \frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-2h)^2}{2\sigma^2}}$$

if source at $x=h$, and $q_x(x=h)=0$, then: $C(x,t) = 2\cdot\frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-h)^2}{2\sigma^2}}$

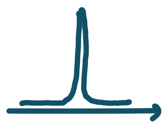
Absorber at ($x=h$): $C(x=h) = 0$



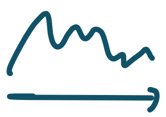
$$C(x,t) = \frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-0)^2}{2\sigma^2}} - \frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-2h)^2}{2\sigma^2}}$$

Initial Condition: $C(t=0)$

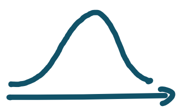
1. $C(t=0) = M\delta(x)$ — dirac's delta function
 kg m⁻³ kg m⁻² m⁻¹ unit: 1/x
 eg. $[\delta(x,y,z)] = m^{-3}$



2. or if arbitrary:
 $C(t=0) = C_0(x) \rightarrow M(x') = C_0(x')\delta x'$
 sol. becomes:
 $$C(x,t) = \int_{-\infty}^{\infty} \frac{C_0(x', t=0)}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-x')^2}{2\sigma^2}} dx'$$



3. or if initially gaussian:
 $$C(t=0) = C_0(x) = \frac{M}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{x^2}{2\sigma_0^2}}$$
 sol. becomes:
 $$C(x,t) = \frac{M}{\sqrt{2\pi}\,\sigma_0} e^{-\frac{x^2}{2\sigma^2}}, \text{ but } \sigma^2 = \sigma_0^2 + 2Dt$$



Advection + Diffusion ($\vec{q} = \vec{u}C$)

Advective mass flux
$$\vec{q} = \underbrace{\vec{u}C}_{\text{advection}} - \underbrace{D\nabla C}_{\text{diffusion}}$$ (sub into $\frac{\partial C}{\partial t} = -\nabla\vec{q}$)

$$\frac{\partial C}{\partial t} + \underbrace{\vec{u}\cdot\nabla C}_{\text{advection}} = \underbrace{D\nabla^2 C}_{\text{diffusion}} : \frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right)$$

Solution : $X = x - ut,\ Y = y - vt,\ Z = z - wt$

eg. $\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = 0$, $C_0(x,y) = \frac{M}{2\pi\sigma_0^2} e^{-\frac{x^2+y^2}{2\sigma_0^2}}$ (σ_0 constant)

sol. $C(x,y,t) = C_0(X,Y,t)$
$= \frac{M}{2\pi\sigma_0^2} e^{-(X^2+Y^2)/2\sigma_0^2}$
$= \frac{M}{2\pi\sigma_0^2} e^{-\frac{[(x-ut)^2 + (y-vt)^2]}{2\sigma^2}}$

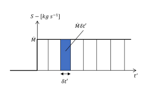
this example is pure advection!
if advection-diffusion:
- solve diffusion first
- then replace $X = x - ut$, $Y = y - vt$, $Z = z - wt$

Péclet Number:
$$Pe = \frac{\text{advection}}{\text{diffusion}} = \frac{U_0 L_0}{D}$$, if Pe is big: advection dominate (can ignore diffusion)
if Pe is small: diffusion dominate (can ignore advection)

Continuous Release.

$$\frac{\partial C}{\partial t} + \vec{u}\nabla C = D\nabla^2 C + \dot{M}\delta(x,y,z)$$

sum of all the individual instantaneous release. (individual release at $t = t'$)



↓ Solution $m = \dot{m}dt'$, $t: t - t'$ $\sigma = \sqrt{2D(t-t')}$

Diffusion only:
3D: $$C(x,y,z,t) = \int_0^t \frac{\dot{M}}{(2\pi)^{3/2}\sigma^3} e^{-\frac{r^2}{2\sigma^2}} dt' = \frac{\dot{M}}{4\pi D r}\,\mathrm{erfc}\left(\frac{r}{\sqrt{4Dt}}\right)$$

3D steady ($t\to\infty$): $C(x,y,z) = \frac{\dot{M}}{4\pi D r}$ (steady: not time dependent)

Diffusion-advection:
3D steady ($t\to\infty$): $$C(x,y,z) = \frac{\dot{M}}{4\pi D r} e^{-\frac{U}{2D}(r-x)}$$

solved from full eqn:
$$\cancel{\frac{\partial C}{\partial t}}_{0} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \dot{M}\delta(x,y,z)$$

3D steady + far field ($t\to\infty$, $x\to\infty$):
$$C(x,y,z) = \frac{\dot{M}}{2\pi U\sigma^2} e^{-\frac{y^2+z^2}{2\sigma^2}}, \quad \sigma^2 = \frac{2Dx}{U}$$
(derived from 2D instantaneous release. sub $x = Ut$)

solved from simplified eqn:
$$\cancel{\frac{\partial C}{\partial t}}_{0 \text{ steady}} + u\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) + \dot{M}\delta(x,y,z)$$
- advection only in x
- diffusion only in y, z

2D steady + far field ($t\to\infty$, $x\to\infty$):
$$C(x,y) = \frac{\dot{M}}{\sqrt{2\pi}\,hU\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad \sigma^2 = \frac{2Dx}{U}$$

(far field means only advection in x, no diffusion)
(if high Pe number – fast moving river: we also ignore diffusion in x-direction!)

Turbulence Dispersion ($\vec{q} = \overline{u'c'}$)

from advection + diffusion:
$$\vec{q} = \underbrace{\vec{u}C}_{\text{advection}} - \underbrace{D\nabla C}_{\text{diffusion}}$$

Reynold's averaging the $\vec{u}$ and C (from advection)
$u(t) = \bar{u} + u'(t)$; $c(t) = \bar{c} + c'(t)$
$\overline{uc} = \overline{(\bar{u}+u')(\bar{c}+c')} = \overline{\bar{u}\bar{c}} + \cancel{\overline{u'\bar{c}}} + \cancel{\overline{\bar{u}c'}} + \overline{u'c'}$
$= \overline{\bar{u}\bar{c}} + \overline{u'c'}$

$$\vec{q} = \underbrace{\bar{u}\bar{C}}_{\text{advection}} + \underbrace{\overline{u'c'}}_{\text{turb.}} - \underbrace{D\nabla\bar{C}}_{\text{diffusion.}}$$ (sub $\vec{q}$ into $\frac{\partial \bar{C}}{\partial t} = -\nabla\vec{q}$)

$\vec{q}_{turb} = \overline{u'c'} = -\mathcal{U}\frac{\partial\bar{c}}{\partial x} = -D_T\frac{\partial\bar{c}}{\partial x}$ — $\mathcal{U}$ is eddy diffusivity
(derivation pls remember pg 13)

$$\frac{\partial\bar{C}}{\partial t} + \underbrace{\bar{\vec{u}}\cdot\nabla\bar{C}}_{\text{advection}} = \underbrace{\cancel{D\nabla^2\bar{C}}}_{0 \text{ as } D_T \gg D} + \underbrace{D_T\nabla^2\bar{C}}_{\text{turbulence.}}$$

$\bar{\vec{u}} = \begin{pmatrix}\bar{u}\\ \bar{v}\\ \bar{w}\end{pmatrix}$

(basically if turbulent, replace u with $\bar{u}$, C with $\bar{c}$ and D with D_T)

↓ solution : replace D with D_T

$$\bar{C}(x,y,z,t) = \frac{M}{(2\pi)^{3/2}\sigma^3} e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}, \quad \sigma = \sqrt{2D_T t}$$

if turbulence anisotropic:
$D_T = [D_{Tx}, D_{Ty}, D_{Tz}]$
$$\frac{\partial\bar{C}}{\partial t} + \bar{\vec{u}}\cdot\nabla C = D_{Tx}\frac{\partial^2 C}{\partial x^2} + D_{Ty}\frac{\partial^2 C}{\partial y^2} + D_{Tz}\frac{\partial^2 C}{\partial z^2}$$
$$\bar{C}(x,y,z,t) = \frac{M}{(2\pi)^{3/2}\sigma_x\sigma_y\sigma_z} e^{-\left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} + \frac{z^2}{2\sigma_z^2}\right)}$$

anisotropic:
$\frac{\partial C}{\partial t} = D_{Tx}\frac{\partial^2 C}{\partial x^2} + D_{Ty}\frac{\partial^2 C}{\partial y^2} + \ldots$ $\overline{u'u'} \neq \overline{v'v'} \neq \overline{w'w'}$, $\frac{\partial}{\partial x}(\overline{u'u'}) = \frac{\partial}{\partial y}(\overline{v'v'}) = \ldots$

heterogeneous:
$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \ldots$ $\overline{u'u'} = \overline{v'v'} = \ldots$, $\frac{\partial}{\partial x}(\overline{u'u'}) \neq \frac{\partial}{\partial y}(\overline{v'v'}) \neq \ldots$

both:
$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D_{Tx}\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(D_{Ty}\frac{\partial C}{\partial y}\right) + \ldots$ $\overline{u'u'} \neq \overline{v'v'} \neq \ldots$, $\frac{\partial}{\partial x}(\overline{u'u'}) \neq \frac{\partial}{\partial y}(\overline{v'v'}) \neq \ldots$

Shear Dispersion ($\vec{q} = \langle\tilde{u}\tilde{c}\rangle$) ALWAYS FOR 1D MODEL.

from advection + diffusion:
$$\vec{q} = \underbrace{\vec{u}C}_{\text{advection}} - \underbrace{D\nabla C}_{\text{diffusion}}$$

Reynold's and Depth averaging the u and C:
$u = U + \tilde{u} + u'$
$C = \langle\bar{C}\rangle + \tilde{c} + c'$

$$\vec{q} = \underbrace{U\langle C\rangle}_{\text{adv.}} + \underbrace{\langle\tilde{u}\tilde{c}\rangle}_{\text{shear}} + \underbrace{\langle\overline{u'c'}\rangle}_{\text{turb.}} - \underbrace{D\frac{\partial\langle C\rangle}{\partial x}}_{\text{dif}}$$ (sub $\vec{q}$ into $\frac{\partial\langle C\rangle}{\partial t} = -\nabla\vec{q}$)

$\vec{q}_{shear} = \langle\tilde{u}\tilde{c}\rangle = -D_s\frac{\partial\langle C\rangle}{\partial x}$

$$\frac{\partial\langle C\rangle}{\partial t} + U\nabla\langle C\rangle = \underbrace{\cancel{D\nabla^2 C} + \cancel{D_T\nabla^2 C}}_{0 \text{ as } D_s \gg D_T \gg D} + D_s\frac{\partial^2\langle C\rangle}{\partial x^2}$$

↓ solution: replace D with D_s

$$\langle C\rangle(x,t) = \frac{M}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}, \quad \sigma = \sqrt{2D_s t}$$

this M is per unit Area: $\frac{M}{A}$!

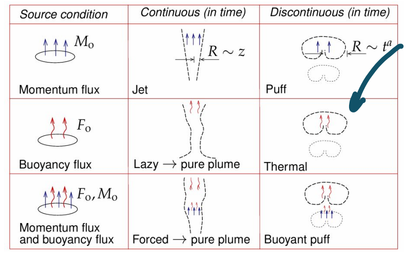
The Boussineq Approximation

Definition of buoyancy: Buoyancy is the reduced gravity acceleration acting on a fluid parcel.

$$b=\frac{\rho_h-\rho}{\rho_0}g \approx \frac{T-T_h}{T_0}g$$ (note that T_0 must be expressed in Kelvin!)

$\rho(x)$ is density of the parcel of fluid at location x; $\rho_h(z)$ is the density of surrounding at height z; ρ_0 is a suitable reference density (eg. $\rho_0=\rho_h(z=0)$ is often used)

In Part 1: Passive Scalar, we kept on discussing about concentration, C. In Part 2: Active Scalar, our main focus are F, Q and b. However, it is still related to Part 1 when $C=\dot{m}/Q$ where $\dot{m}$ is mass flux ($kg\,s^{-1}$)



ACTIVE SCALAR
(Induce buoyancy-driven flow)
(Modify the flow)

Emptying Boxes (Simple case)

replenish source modelled as F (if there's any), no stratification

Part 1: What Drives a flow?

1. Temperature Difference creates Density Differences (Boussineq)

According to the eqn of state (or known as The Boussineq Approx.), fluid density depends on temperature:

$$b=\frac{\rho_0-\rho}{\rho_0}g \approx \frac{T-T_0}{T_0}g$$ (taking reference ρ and T as equal to surrounding)

$$\rho=\rho_0-\rho_0\left(\frac{T-T_0}{T_0}\right) \rightarrow \rho=\rho_0(1-\beta(T-T_0)) \rightarrow$$ hotter fluid ($T>T_0$) is lighter ($\rho<\rho_0$)

2. Density Differences create Hydrostatic Pressure Differences. (Hydrostatic)

When the fluid is at rest (but not necessary in equilibrium), pressure is determined by the weight of fluid column above it.

$\frac{dp}{dz}=-\rho g$ { high density: pressure decreases rapidly with height; low density: pressure decreases slowly with height.

(if question starts by asking how temp. dif. drives flow?)

$\frac{dp}{dz}=-\rho_0 g+\frac{\rho_0}{T_0}(T-T_0)g$ → pressure changes is proportional to $T-T_0$

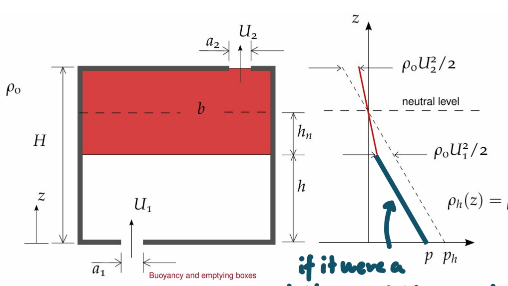
Because pressure drops at different rates outside and inside (or left and right), a pressure difference (Δp) develops between it at any given height (except at neutral level)

3. Pressure differences drives flow. (Bernoulli)

Fluids move from areas of high pressure to area of low pressure. The magnitude of this pressure difference determines the velocity of the air through the opening.

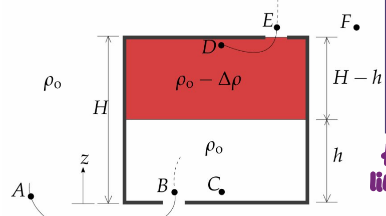
$\Delta p=\frac{1}{2}\rho U^2$ → potential energy (pressure) converts into kinetic energy (velocity)

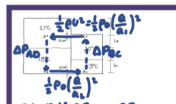
Part 2: Sketch the Pressure Distribution.



if it were a left room right room qtn, this line's gradient (ρ) can be different from surrounding (ρ_0)

Part 3: Calculate Flow Rate (Q)





$\frac{1}{2}\rho U^2=\frac{1}{2}\rho_0\left(\frac{Q}{a_1}\right)^2$, ΔP_{AD}, ΔP_{BC}, $\frac{1}{2}\rho_0\left(\frac{Q}{a_2}\right)^2$

A to B if $\Delta P_{AD}>\Delta P_{BC}$

driving pressure = sum of pressure lost

$|\Delta P_{AD}-\Delta P_{BC}|=\frac{1}{2}\rho_0\left(\frac{Q}{a_1}\right)^2+\frac{1}{2}\rho_0\left(\frac{Q}{a_2}\right)^2$

this mean like $H_A=H_B$

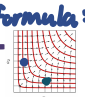
Bernoulli's: $\underbrace{\rho_0\frac{U^2}{2}}_{\text{kinetic}}+\underbrace{p}_{\text{pressure}}+\underbrace{\int_0^z\rho(z)g\,dz}_{\text{potential}}=\text{constant}$

1. A-B, D-E: $\rho_0\frac{U^2}{2}+p=\text{constant}$ (ρgz constant → same height)
2. C-D: $p+\int_0^z\rho(z)g\,dz=\text{constant}$ ($\frac{1}{2}\rho U^2$ constant → $U\approx 0$ in the box)
3. B-C doesn't follow bernoulli's as there is energy lost (due to vena contracta). Hence:

B-C: $p=\text{constant}$ (pressure balance assumption)

By solving all these equations, we get the formula:

$$Q=A^*\sqrt{2b(H-h)}\ ,\ A^*=\sqrt{\frac{a_1^2a_2^2}{a_1^2+a_2^2}}$$

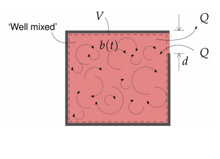


$A\uparrow\circ:\frac{\partial A^*}{\partial a_1}>\frac{\partial A^*}{\partial a_2}$

$A\uparrow\circ:\frac{\partial A^*}{\partial a_2}>\frac{\partial A^*}{\partial a_1}$

F (buoyancy flux) = Qb

* note that if F is in Watt: convert to m^4s^{-3} by $\times 2.8\times10^{-5}$



for one opening (well-mixed box):

$$Q=\frac{k}{3}\omega\sqrt{bd^3}$$ { $k\approx0.5$ vertical opening; $k\approx0.25$ horizontal opening.

Part 4: Transient vs Steady State.

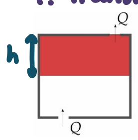
Transient: no sources replenishing the 'heated' layer

Steady State: there's F replenishing source of buoyancy. $F_{replenish}=F_{out}=Qb$

1. Transient: How long to empty the box?

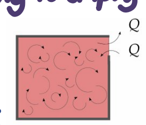
$\frac{dV}{dt}=Q_{plume}(h)-Q_{ent}$

$\frac{d(bV)}{dt}=F_{source}-Q_{ent}\cdot b$



$\frac{dh}{dt}=\frac{Q(t)}{S}$

$Q=A^*\sqrt{2b(H-h)}$



$\frac{db}{dt}=-\frac{b(t)\cdot Q(t)}{V}$

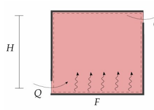
$Q=\frac{k}{3}\omega\sqrt{bd^3}$

solution: $\frac{h}{H}=1-\left(1-\frac{t}{t_e}\right)^2$, $t_e=\left(\frac{2S}{A^*}\right)\left(\frac{H}{2b}\right)^{1/2}$

solution: $\frac{b}{b_0}=\left(1+\frac{t}{\tau}\right)^{-2}$, $\tau=\frac{6V}{k\omega}(b_0d)^{-1/2}$

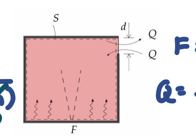
2. Steady: $F_{replenish}=F_{out}-F_{in}=Qb(z=H)-\cancel{Qb(z=0)}$ $b=\frac{T_0-T_0}{T_0}g$

(F, Q, b given 1, find the other 2)



$F=Qb$

$Q=A^*\sqrt{2b(H-\cancel{h})}$



$F=Qb$

$Q=\frac{k}{3}\omega\sqrt{bd^3}$ } solve simultaneous equation.

eg. if given F, solve for Q and b: (treat F as a constant)

$Q=(2A^{*2}FH)^{1/3}$

$b=\left(\frac{F^2}{2A^{*2}H}\right)^{1/3}$

$b=\left(\frac{3F}{k\omega d^{3/2}}\right)^{2/3}$

$Q=\frac{k}{3}\omega\sqrt{bd^3}$

Introduction to Dimensional Analysis.

using thermal as example (cause it is simple)

Every phenomenon has a driver and obey conservation laws.

For thermal, the driver is the discontinuous source of buoyancy flux, F_0

And total buoyancy, $B=bV$ is conserved.
(even though $b(t)$ and $V(t)$ changes with time)

Question: How $R(t)$, $b(t)$, and $z(t)$ changes with time?

$[R]=L$, $[b]=LT^{-2}$, $[B]=L^4T^{-2}$, $[t]=T$

→ $R(t)\sim B^{1/4}t^{1/2}$, $b(t)\sim B^{1/4}t^{-3/2}$, $z\sim B^{1/4}t^{1/2}$ (and if wind blow horizontally: $X\sim Ut$, $z\sim\sqrt{z}$)

Turbulent Jets vs Plumes

Part 1: Turbulent Jets vs. Turbulent Plumes.

* Characteristic quantity: $w_m=\frac{M}{Q}$, $r_m=\frac{Q}{\sqrt{\pi M}}$, $b_m=\frac{F}{Q}$ (plumes only)

Turbulent Jets

1. Driver

A continuous source of momentum (M_0) with zero buoyancy ($b=0$)

2. Physics

Driven by inertia (initial kick)
Scale with Momentum flux (M)

3. Conservation Laws

Volume flux INCREASED ($\frac{dQ}{dz}>0$) due to entrainment.

Momentum flux CONSERVED ($\boxed{\frac{dM}{dz}=0}$). No external forces acting on it.

Buoyancy Flux (F) is zero. No buoyancy ($b=0$)

Turbulent Plumes

1. Driver

A continuous source of Buoyancy flux (F_0) (eg. heat source, fire, fresh water salt water)

2. Physics

Driven by gravity acting on density differences ($b=\frac{\rho_h-\rho}{\rho_0}$)
Fluid accelerates as it rises.

3. Conservation Laws

Volume flux INCREASED ($\frac{dQ}{dz}>0$) due to entrainment.

Momentum flux INCREASED ($\frac{dM}{dz}>0$). The buoyancy force creates acceleration, constantly adding momentum to the flow.

Buoyancy flux CONSERVED ($\boxed{\frac{dF}{dz}=0}$). Heat is neither created nor destroyed, just spread out.

(everything that is not conserved can be find as a function of time by using dimensional analysis)

Part 2: Dimensional Analysis

far field behaviour, supposedly:

$Q=f(z,M_0,\cancel{Q_0})$, $Q=f(z,\cancel{M_0},\cancel{Q_0},F_0)$

The momentum flux $M(z)$ is much greater than the source momentum flux M_0. (plume)

The volume flux $Q(z)$ is much greater than the source volume flux Q_0. (jet, plume)

Turbulent Jets cause M_0 is constant

The only quantity that are known: $[M]=L^4T^{-2}$, $[z]=L$

We can hence solve for:

1. Volume flux, Q: $[Q]=[M]^{1/2}[z]^1$ → $\boxed{Q\propto M_0^{1/2}z}$
2. Velocity, w_m: $[w]=[M]^{1/2}[z]^{-1}$ → $w_m\propto M_0^{1/2}z^{-1}$
3. Radius, r_m: $[r]=[Q][M]^{-1/2}$ → $r_m\propto z$

Turbulent Plumes cause F_0 is constant

The only quantity that are known: $[F]=L^4T^{-3}$, $[z]=L$

We can hence solve for:

1. Volume flux: $[Q]=[F]^{1/3}[z]^{5/3}$ → $\boxed{Q\propto F_0^{1/3}z^{5/3}}$
2. Momentum flux: $[M]=[F]^{2/3}[z]^{4/3}$ → $\boxed{M\propto F_0^{2/3}z^{4/3}}$
3. Velocity, w_m: $[w]=[F]^{1/3}[z]^{-1/3}$ → $w_m\propto F_0^{1/3}z^{-1/3}$
4. Buoyancy, b_m: $[b]=[F]^{2/3}[z]^{-5/3}$ → $b_m\propto F_0^{2/3}z^{-5/3}$

(although w_m, r_m, b_m can be found using dimensional analysis, in exam it is more pratical to calculate Q (and M for plume) and then use characteristic eqn.)

(Note that if it is $\underbrace{\text{planar jets / plume}}_{\text{line source}}$: Q, M, F are all per unit length, eg. $[Q]=L^2T^{-1}$)

(all the highlighted 'blue' eqn related to governing eqn, sometimes adding α) entrainment

$\frac{dQ}{dz}=2\pi^{\frac{1}{2}}\alpha M^{\frac{1}{2}}$ (used for both jets and plumes) solved for jet: $Q=2\alpha\pi^{1/2}M^{1/2}z+Q_0$

$\frac{dM}{dz}=\frac{FQ}{M}$ (used for plume cause for jet: $\frac{dM}{dz}=0\rightarrow M(z)=M_0$)

$\frac{dF}{dz}=-N^2Q$ (only for plume, jet: $F(z)=0$)

$Q(z)=\frac{6\alpha}{5}\left(\frac{9\pi^2}{10}\alpha F\right)^{1/3}z^{5/3}$

$M(z)=\left(\frac{9\pi^{\frac{1}{2}}}{10}\alpha F\right)^{2/3}z^{4/3}$

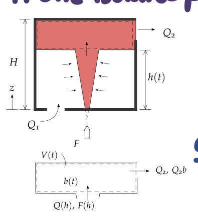
$F(z)=F_0$

} all solved for plume.

$\alpha\approx0.1$ if not given

Part 3: Emptying Filling Box (turbulent plume sources)

1. One isolated point source of buoyancy (Turbulent Plume)



$$\frac{dV}{dt}=\underbrace{cF^{1/3}h^{5/3}}_{Q_{plume}(h)}-\underbrace{A^*\sqrt{2b(H-h)}}_{Q_{ent}}=0$$ STEADY STATE

$$\frac{d(bV)}{dt}=\underbrace{F}_{F_{source}}-\underbrace{bA^*\sqrt{2b(H-h)}}_{Q_{ent}\cdot b}=0$$

Trying to solve for: What h gives steady-state?

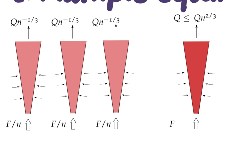
unknown: F, b, A^*, h, H (A^* and H are constants)

→ 3 unknowns, 2 eqn...

→ remember characteristic eqn: $b=\frac{F}{Q_{plume}}=\frac{F}{cF^{1/3}h^{5/3}}=\frac{F^{2/3}}{ch^{5/3}}$

solution: $\frac{\sqrt{2}A^*}{H^2c^{3/2}}=\left(\frac{\xi^5}{1-\xi}\right)^{1/2}$ (use numerical method) $\xi_{n+1}=(1-\xi_n)^{1/5}$, $\xi_1=0.5$ ($A^*\uparrow,\xi\uparrow$)

2. Multiple equal isolated point source



Single turbulent plume: $Q\propto F^{1/3}$

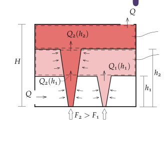
n small turbulent plume: $Q\propto n\left(\frac{F}{n}\right)^{1/3}$ (total buoyancy flux still equal to F)

$Q\propto n^{2/3}F^{1/3}$

solution: $\frac{\sqrt{2}A^*}{nH^2c^{3/2}}=\left(\frac{\xi^5}{1-\xi}\right)^{1/2}$

it's weird that there is just n without any power that's because $Q_{ent}\propto(A^*)^{2/3}$ ∴ $n^{2/3}\propto(A^*)^{2/3}$

3. Multiple unequal isolated point source



$Q=A^*\sqrt{2b_1(h_2-h_1)+2b_2(H-h_2)}$

$F_1=Q_1(h_1)\cdot b_1$

$F_2=Q_2(h_2)\cdot b_2-Q_1(h_1)\cdot b_1$

{ can be used even if no turbulent plume point source eg. transient state emptying box replace Q in $\frac{dh}{dt}=\frac{Q}{S}$ if it is multiple layer.

* If stratified instead of layers: $Q=A^*\left(2\int_0^H b(z)dz\right)^{1/2}$