Imperial College London

CIVE50005
Fluid Mechanics 2

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Autumn Term

Syllabus for Spring Term

1. Introduction
2. Real Fluids
    - Concept of Viscosity
    - Governing Equations
    - Applying the Governing Equations
    - Laminar and Turbulent Flow
3. Pipes
    - Steady Pipe Flow
    - Flow Profiles
    - Pipe Systems
4. Gradually Varied Open Channel Flow
    - Introduction
    - Uniform Channel Flow
    - Varied Channel Flow
    - Gradually Varied Flow
    - Water Surface synthesis in Channel flow

Real Fluids

- Concept of Viscosity
- Conservation of mass
  - ↓ derivation.
  - Continuity eqn
- Governing Equations:
  - → Euler Eqn (ideal fluid, no viscosity)
  - → Navier Stokes Eqn (real fluid, can be used for both laminar and turbulent)
    - } reynold decomposition
  - → Reynold's Average Navier Stokes Equation (for real fluid, better to describe turbulent, but laminar can use too)
    - reynold's stress term ≈0

1 Introduction - What is Fluid Mechanics?

The description of the behaviour of fluids at rest and in motion.

Properties

- density [M/L^3]
- bulk modulus, $V\frac{\partial p}{\partial V}$ [F/L^2]
- viscosity [M/LT]

We work only with <u>incompressible</u> fluids – fluids that do not change volume/density when squeezed, typically valid at Mach number, $M \leq\approx 0.3$

Concepts / Understanding

- pressure
- buoyancy
- compressibility
- viscosity
- stream lines
- potential lines
- flow nets
- continuity
- momentum
- laminar/turbulent flow

Equations

- mass continuity
- momentum conservation
- force balance
- shear stress
- Bernoulli
- Euler equations

Only a comprehensive understanding of properties + concepts + equations will allow for the correct modelling of fluid behaviour. It is important to cross-link existing knowledge and develop an understanding for 'the bigger picture'.

A Bigger Picture: Summary

- Fluid Mechanics is a **challenging** and **extremely exciting** subject
- The governing equations describing a general fluid flow are highly complex and cannot be solved analytically
- No matter how complicated any given equation may appear (e.g. Euler) it might still be missing important terms (viscosity in this case)
- Often equations appear complex and intimidating. However, with time they become more familiar and are readily employed (e.g. Bernoulli)
- The 'Art of Fluid Mechanics' is to make the **appropriate simplifications** and **assumptions**
- Many disciplines of fluid mechanics have well established **solution patterns**

This term links important concepts and enables application of this knowledge to real word civil engineering problems.

- **Exploit mathematical descriptions of physical problems**
- **Exploit observations and the understanding these engender**

2 Real Fluids

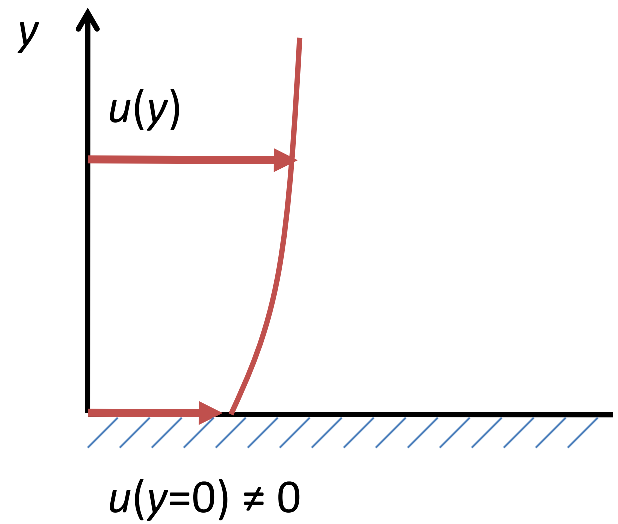
2.1 Concept of Viscosity

Would you expect these two fluids to have **identical dynamic behaviour**?

Ideal and Real Fluids

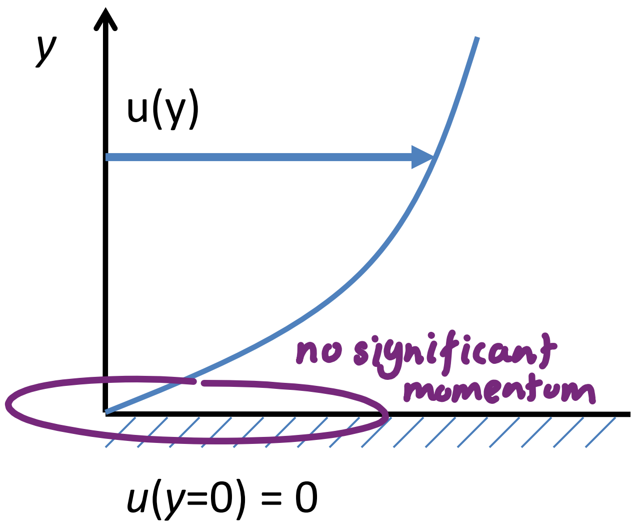
IDEAL Fluid
- No shear stress
- Only normal stress
- **SLIP** at solid boundary

$$\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx} \ldots = 0$$

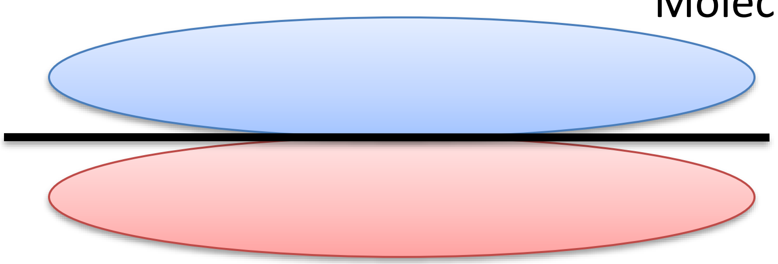


REAL Fluid
- Shear stress is supported
- Normal stress is supported
- **NO SLIP** at solid boundary

$$\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx} \ldots \neq 0$$



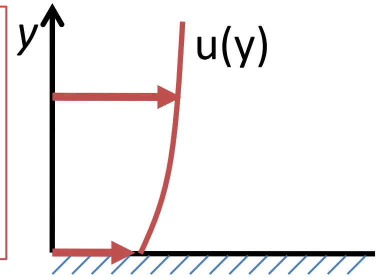
Solids and Fluids



- **NO SLIP** at the boundary is result of molecular scale interactions at the wall. Solid molecules at wall interact with fluid molecules immediately adjacent to the wall, so that fluid at wall moves together with the solid wall
- In a **SOLID**, all stress components (tension, compression and shear) are supported in static equilibrium. Each results in a small elastic deformation
- In a **FLUID**, response to stress components is characteristically different
  - **tension** is **NOT** supported (true unless you're a plant scientist!) → absolute pressure can't be negative!
  - **compression** is supported, and results in a small elastic deformation
  - **shear** is supported, but results in **FLOW**

Ideal and Real Fluids – some context…

IDEAL Fluid
- No shear stress
- Only normal stress
- **SLIP** at solid boundary



Potential flow

Bernoulli (1730's)

Laplace (1740's)

Euler (1750's)

Newton (mid 1600's)

Navier (1820's)

Stokes (1820's)

Potential flow linked to 'inner regions' e.g BLs

Hagen (1840's)

Poiseuille (1840's)

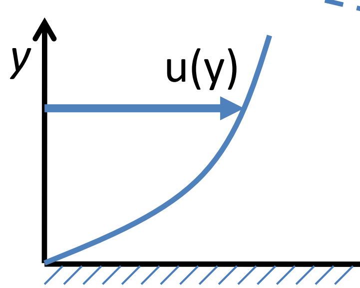
Darcy (1850's)

Slow viscous flow

Empirical 'rules' and **increasingly** specialised solutions

Turbulent flows

REAL Fluid
- Shear stress is supported
- Normal stress is supported
- **NO SLIP** at solid boundary

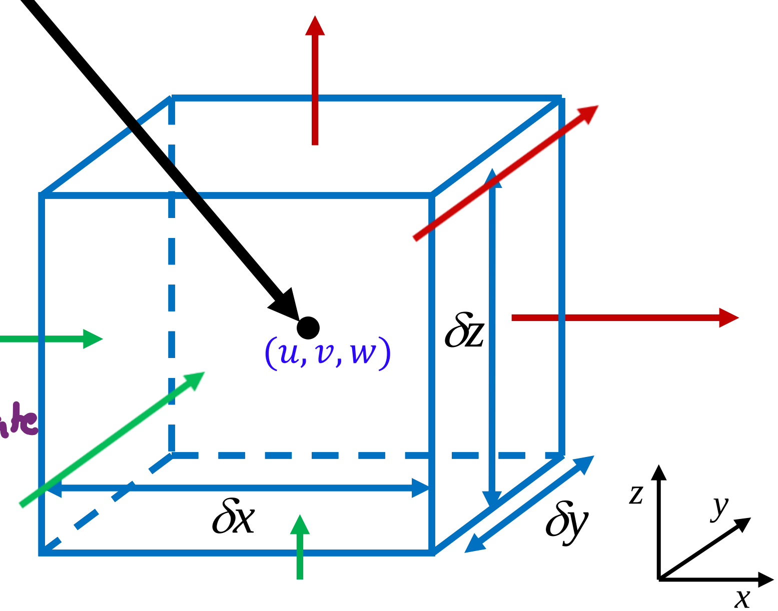


2.2a Conservation of Mass, applies for all fluids, i.e. Ideal Fluids and Real Fluids

Mass cannot be created not destroyed without significant energy implications

- $E = mc^2$
- This is relevant within the Sun and inside nuclear reactors, otherwise…

At a point in space, fluid is moving with velocity components (u, v, w)



Mass must be conserved

in civ eng, fluid mechanics, fluid → incompressible

Incompressibility is a powerful assumption:

- Consider an element of fluid
- When the fluid is incompressible we cannot squeeze more/less fluid into our element:
  - Therefore what goes in
  - Must come out.

treat fluid incompressible → cannot accumulate in the CV

Now let's express that via maths…

Element analysis and Taylor series expansions…

For illustration, take a 2D slice through our fluid element…

Due to incompressibility, fluid cannot accumulate within the element,
But that does NOT mean the amount of fluid entering in one direction will leave in the same direction.
Just consider the velocity of the fluid in the $\boldsymbol{x}$-direction:

$$\boldsymbol{u}\left(\boldsymbol{x}-\frac{\boldsymbol{\delta x}}{\mathbf{2}}\right) \neq \boldsymbol{u}\left(\boldsymbol{x}+\frac{\boldsymbol{\delta x}}{\mathbf{2}}\right)$$

Remember, here:

$$\boldsymbol{u}\left(\boldsymbol{x}+\frac{\boldsymbol{\delta x}}{\mathbf{2}}\right) \neq u \times \left(x+\frac{\delta x}{2}\right)$$

It is notation for:

'$\boldsymbol{u}$' evaluated at: $\boldsymbol{x}+\dfrac{\boldsymbol{\delta x}}{\mathbf{2}}$

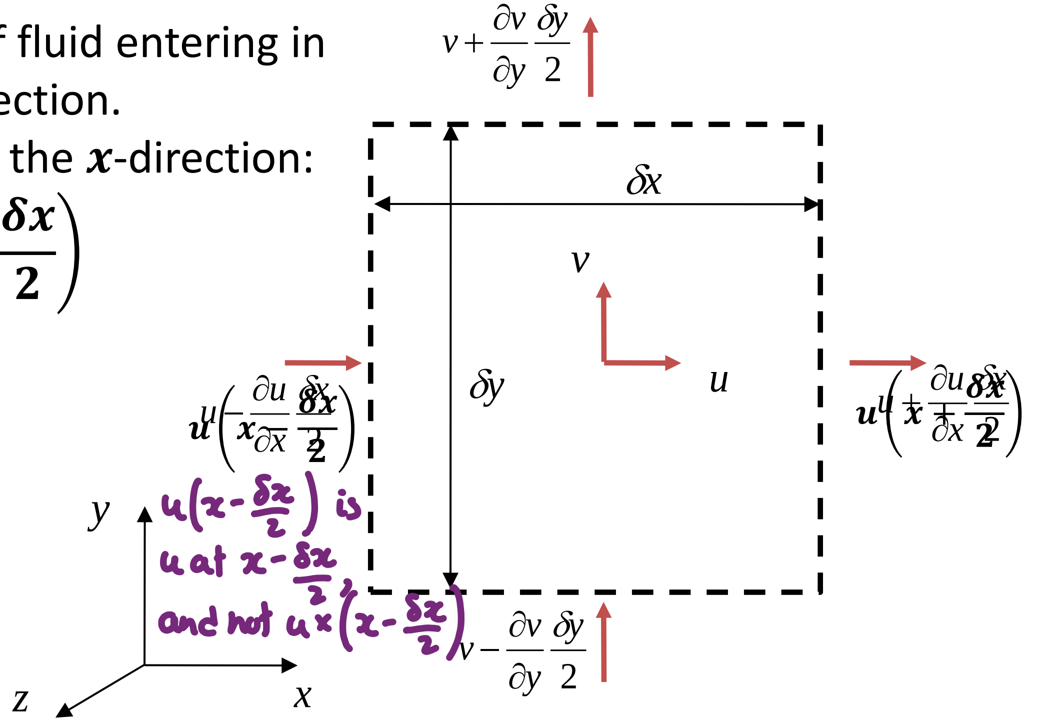
Remember your Taylor series expansion, e.g.

$$u\left(x+\frac{\delta x}{2}\right) = u(x) + \frac{\partial u}{\partial x}\frac{\delta x}{2} + \frac{\partial^2 u}{\partial x^2}\frac{\delta x^2}{2!\,2^2} + \ldots$$

first two term is accurate enough

Retain leading order terms:

$$\boldsymbol{u}\left(\boldsymbol{x}+\frac{\boldsymbol{\delta x}}{\mathbf{2}}\right) = \boldsymbol{u}(\boldsymbol{x}) + \frac{\boldsymbol{\partial u}}{\boldsymbol{\partial x}}\frac{\boldsymbol{\delta x}}{\mathbf{2}}$$



$u\left(x-\frac{\delta x}{2}\right)$ is u at $x-\frac{\delta x}{2}$, and not $u \times \left(x-\frac{\delta x}{2}\right)$

IMPORTANT

2.2a Conservation of Mass, applies for all fluids, i.e. Ideal Fluids and Real Fluids

Now apply this to our 3D element $\underbrace{\left(u \text{ at } x+\frac{\delta x}{2}\right)\times \text{Area}} = \left(u+\frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$

The flux leaving in the x-direction is: $\left(u+\frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$, and

The balance in the x-direction is: $\left(\left[u+\frac{\partial u}{\partial x}\frac{\delta x}{2}\right]-\left[u-\frac{\partial u}{\partial x}\frac{\delta x}{2}\right]\right)\delta y\,\delta z=\frac{\partial u}{\partial x}\delta x\,\delta y\,\delta z$,

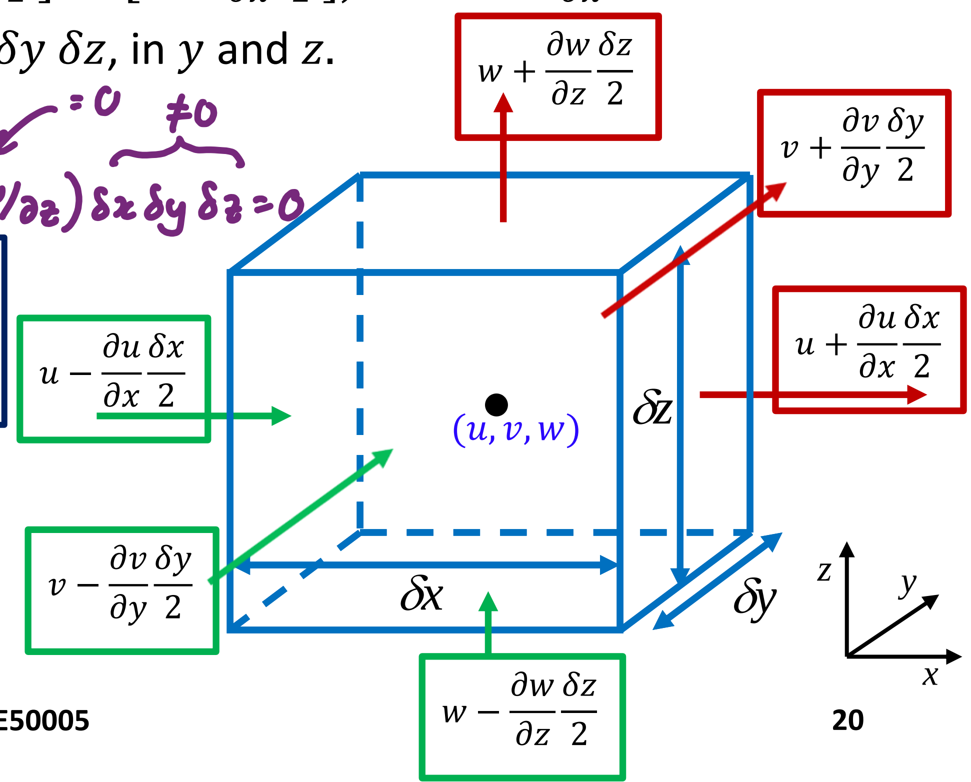
with equivalents: $\frac{\partial v}{\partial y}\delta x\,\delta y\,\delta z$, and $\frac{\partial w}{\partial z}\delta x\,\delta y\,\delta z$, in y and z.

In total the balances must sum to zero in order to conserve mass. $\left(\partial u/\partial x+\partial v/\partial y+\partial w/\partial z\right)^{=0}\overbrace{\delta x\,\delta y\,\delta z}^{\neq 0}=0$

This is mass conservation or the famous continuity equation

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

EXTREMELY IMPORTANT



2.2b The Governing Equations for Ideal Fluids: The Euler Equation

We are now going to derive the Euler equation which is Newton's second law applied to Ideal Fluids.
Remember: Ideal Fluids only support normal stresses, i.e. no shear stresses
Consider our fluid element again...

Now carry out a force balance:

- **Pressure**

gives forces

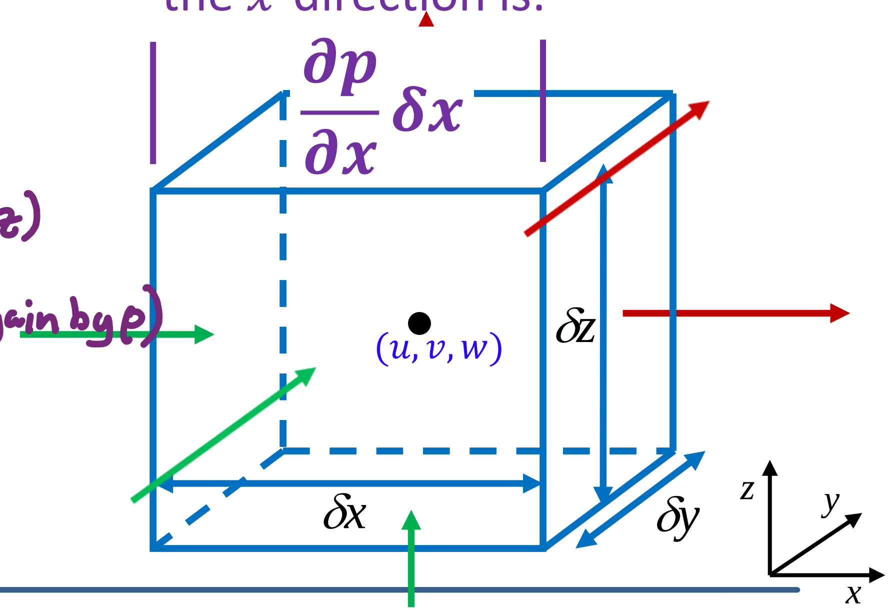
$\left(\frac{\partial p}{\partial x}\delta x\right)\delta y\delta z$, $\left(\frac{\partial p}{\partial y}\delta y\right)\delta x\delta z$, and $\left(\frac{\partial p}{\partial z}\delta z\right)\delta x\delta y$

Per unit volume: $\frac{\partial p}{\partial x}$, $\frac{\partial p}{\partial y}$, and $\frac{\partial p}{\partial z}$ (divide by $\delta x \delta y \delta z$)

Or per unit mass: $\frac{1}{\rho}\frac{\partial p}{\partial x}$, $\frac{1}{\rho}\frac{\partial p}{\partial y}$, and $\frac{1}{\rho}\frac{\partial p}{\partial z}$ (divide again by ρ)

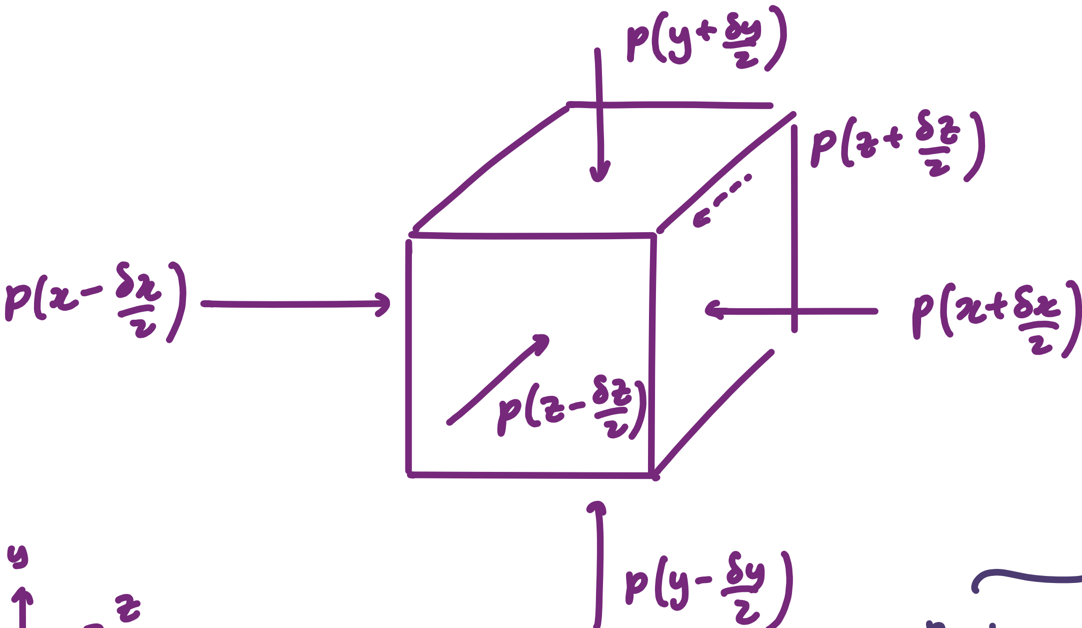
Note: fluid is driven from high to low pressure, i.e. down gradient, so in the equations we will see the terms appear with negative signs

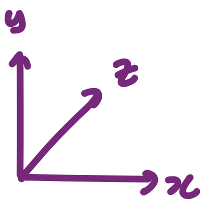
The pressure difference across the element in the x-direction is:



☆ IMPORTANT

Actually you can do the same like how we did for u!





$F = ma$

$F_x = ma_x$

$F_y = ma_y$

$F_z = ma_z$

eg. $$\overbrace{\overbrace{\left[p\left(x-\frac{\delta x}{2}\right) - p\left(x+\frac{\delta x}{2}\right)\right]}^{P_{net}} \overbrace{\cancel{\delta y \delta z}}^{A}}^{F_{net}} = \overbrace{\rho\, \delta x \cancel{\delta y \delta z}}^{m} \times a_x$$

$F = PA$, $m = V\rho$:

$$\cancel{p} - \frac{\delta p}{\delta x}\frac{\delta x}{2} - \left(\cancel{p} + \frac{\delta p}{\delta x}\frac{\delta x}{2}\right) = \rho\, \delta x \times a_x$$

The main difference is here:

$P_{before} - P_{after}$
(left - right)

in continuity eqn we do

$u_{out} - u_{in}$
(right - left)

$$-\frac{\delta p}{\delta x}\cancel{\delta x} = \rho \cancel{\delta x} \times a_x$$

$$-\frac{1}{\rho}\frac{\delta p}{\delta x} = a_x$$

2.2b The Governing Equations for Ideal Fluids: The Euler Equation

F=ma

We are now going to derive the Euler equation which is Newton's second law applied to Ideal Fluids.
Remember: Ideal Fluids only support normal stresses, i.e. no shear stresses
Consider our fluid element again…

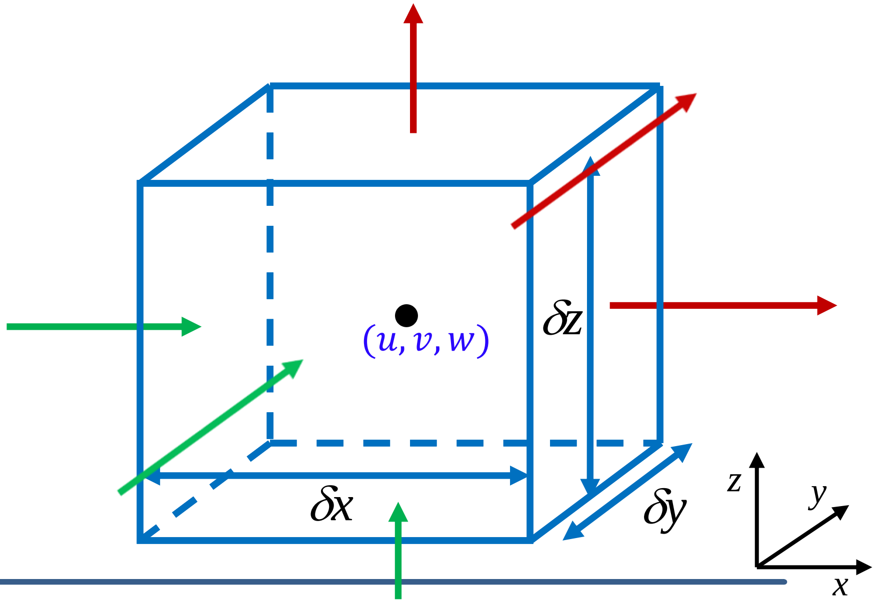
Now carry out a force balance:
- Body forces = all external forces – we'll denote these

Per unit volume as: F_x, F_y, and F_z

Or per unit mass (accelerations): g_x, g_y, and g_z

Newton taught us that the net of these forces result in acceleration (change in momentum)

In 3D we need 3 (orthogonal) components…



Expressing accelerations for fluids…

For example, let's denote the acceleration in the x-direction as a_x.

By definition $a_x = \frac{\mathrm{d}u}{\mathrm{d}t}$: take care to note the total derivative.

$$
\begin{array}{l}
u \to t \\
u \to x \to t \\
u \to y \to t \\
u \to z \to t
\end{array}
$$

☆ u is dependent on time and space !

Now, in general, $u = u(x, y, z, t)$, so:

$$a_x = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

Remember: $\frac{\partial x}{\partial t} = u$, $\frac{\partial y}{\partial t} = v$, and $\frac{\partial z}{\partial t} = w$, giving:

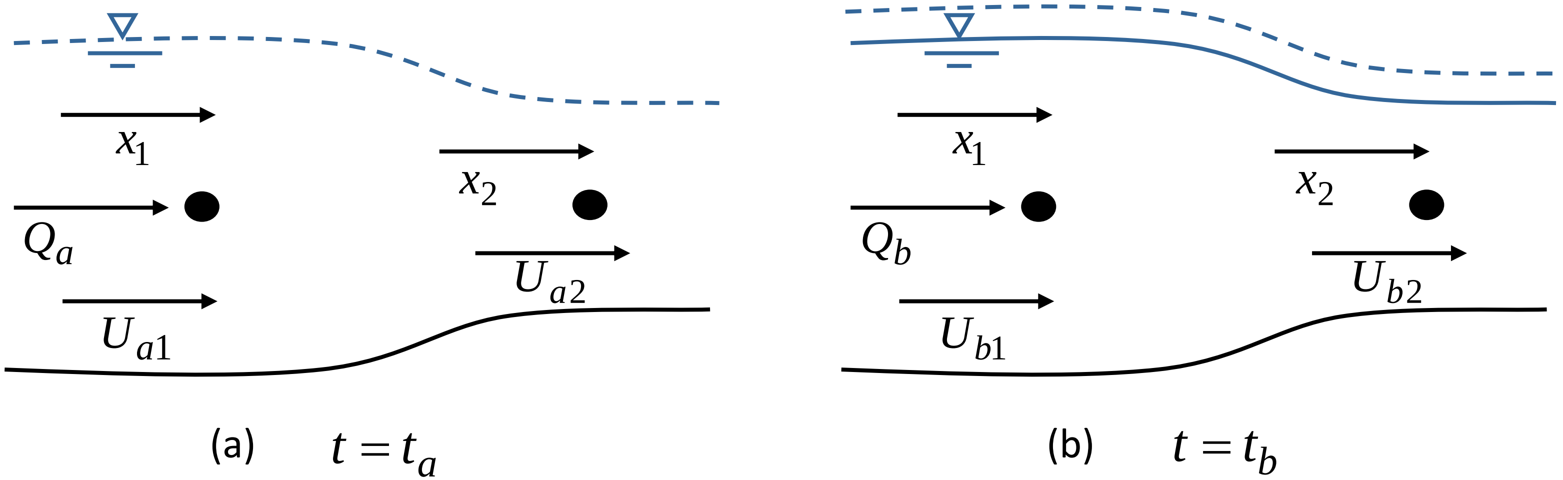
$$a_x = \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial t} + \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}^{\text{due to Eulerian's view (fixed control volume) (frame of reference)}}$$

We describe terms of the form: $\frac{\partial u}{\partial t}$ as 'unsteady' acceleration terms, and

those of the form: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$ as 'convective' acceleration terms.

Let's look at a simple physical example of fluid acceleration…

Accelerations for fluids: a 1D example



Unsteady acceleration (at $x = x_1$):

$$\frac{\partial u_1}{\partial t} = \frac{U_{b1} - U_{a1}}{(t_b - t_a)} \qquad x = x_1$$

Convective acceleration (at $t = t_a$):

$$u\frac{\partial u}{\partial x} = \frac{(U_{a2} - U_{a1})}{x_2 - x_1} U_{a1} \qquad t = t_a$$

<u>**Now we have the net forces and the accelerations, let's put them together**</u>

The Euler Equation (per unit mass)

negative cause fluid flow from high pressure to low pressure.

Newton's second law for Ideal Fluids: $F = m \times a$ or: $a = F \div m$

Pressure terms

Body force terms

Acceleration terms

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z$$

These equations are referred to as the **Euler Equations**. They describe the **dynamics** of an **inviscid** and **incompressible** flow.

→ no viscosity (eg. free surface small amplitude waves)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Four unknowns (u, v, w, p)
- Four differential equations (3 x Euler + mass continuity)
- **Viscosity** is **NOT** included ☆

(IDEAL FLUID)
NO SHEAR

The Euler Equation (per unit volume)

Newton's second law for Ideal Fluids: $F = m \times a$ or: $a = F \div m$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + F_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + F_y$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + F_z$$

These equations are referred to as the **Euler Equations**. They describe the **dynamics** of an **inviscid** and **incompressible** flow.

- Four unknowns (*u*, *v*, *w*, *p*)
- Four differential equations (3 x Euler + mass continuity)
- **Viscosity** is **NOT** included

2.2c Toward Real Fluids: Viscosity

Velocity and Shear stress (or Viscous stress)

Newton postulated that shear stress:

$$\tau = \frac{F}{A} \propto \partial u / \partial y$$

" $\frac{\partial}{\partial y}$ " " u "

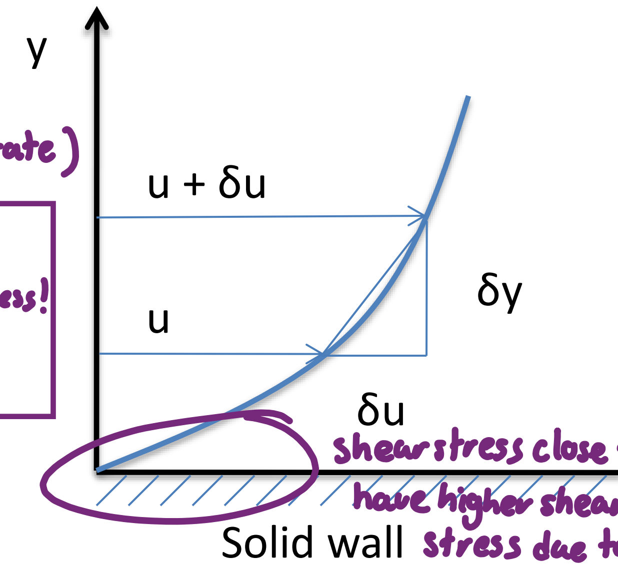
spatial gradient of velocity (strain rate)

or

$$\tau = \mu \frac{\partial u}{\partial y}$$

if fluid doesn't change in velocity, no shear stress!

where μ is a constant factor of proportionality, which is a FLUID PROPERTY



shear stress close to wall have higher shear stress due to 'no-slip condition'

- μ = viscosity = dynamic viscosity $\quad [\mu] = \frac{\text{kg}}{\text{ms}}$
- $\nu = \mu/\rho$ = kinematic viscosity $\quad [\nu] = \frac{\text{m}^2}{\text{s}}$

μ independent of $\frac{\partial u}{\partial y}$! ($\mu$ constant) (€293)

For Newtonian fluids the viscosity is independent of the rate of shear

Real fluids - equations of State

- **Every** fluid property (ρ, μ, ...) represents the integral consequences of molecular activity
- The level of molecular activity is dependent on the *thermodynamic state variables*:
  - pressure, p
  - temperature, T
  - chemical composition, say concentration C
- **Changes** in p, T and/or C may lead to changes in ρ, μ, ...
- The relationships $\rho(p, T, C)$, $\mu(p, T, C)$, ... are called **Equations of State**
- For **fresh water** at standard temperatures and pressures:
  - $\rho(p, T, C) \approx \rho(T)$ and
  - $\mu(p, T, C) \approx \mu(T)$

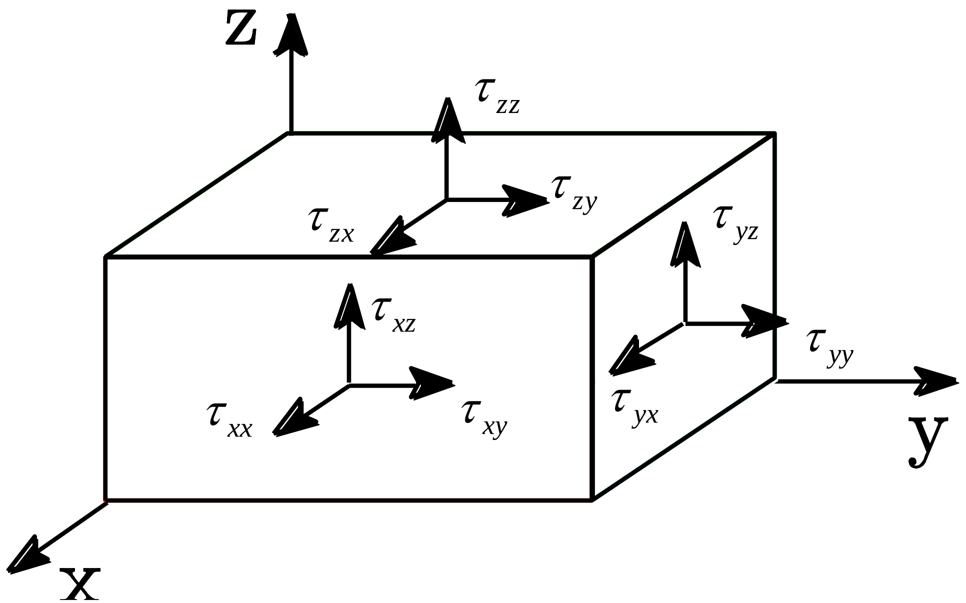
Temperature is the most dominant one

T [° C]	ρ [kg/m³]	μ [kg/ms]	ν [m²/s]
5	1000	1.52×10^{-3}	1.52×10^{-6}
10	999.7	1.31×10^{-3}	1.31×10^{-6}
20	998.2	1.005×10^{-3}	1.007×10^{-6}
30	995.7	0.801×10^{-3}	0.804×10^{-6}
Approximate	1000	10^{-3}	10^{-6}

Dynamic and Kinematic Viscosity

Material	T [° C]	μ [10^{-3} kg/ms]	ν [10^{-6} m²/s]
Air	0	0.0172	13
	20	0.0182	15
	100	0.0218	23
Water Vapour	100	0.0172	29
	250	0.0184	44
Helium	20	0.0220	123.2
Ethanol	20	1.16	1.47
Water	0	1.792	1.794
	20	1.005	1.007
	100	0.284	0.296
Motor oil	20	20 ... 10 000	20 ... 10 0000
Laboratory Glass	700	10^8	0.4×10^8
Tar Pitch	20	3×10^{10}	2.5×10^{10}

Local Stress Components



Normal Stress: $\tau_{xx}, \tau_{yy}, \tau_{zz}$ (or $\sigma_x, \sigma_y, \sigma_z$)

Shear Stress: $\tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zx} \ldots$

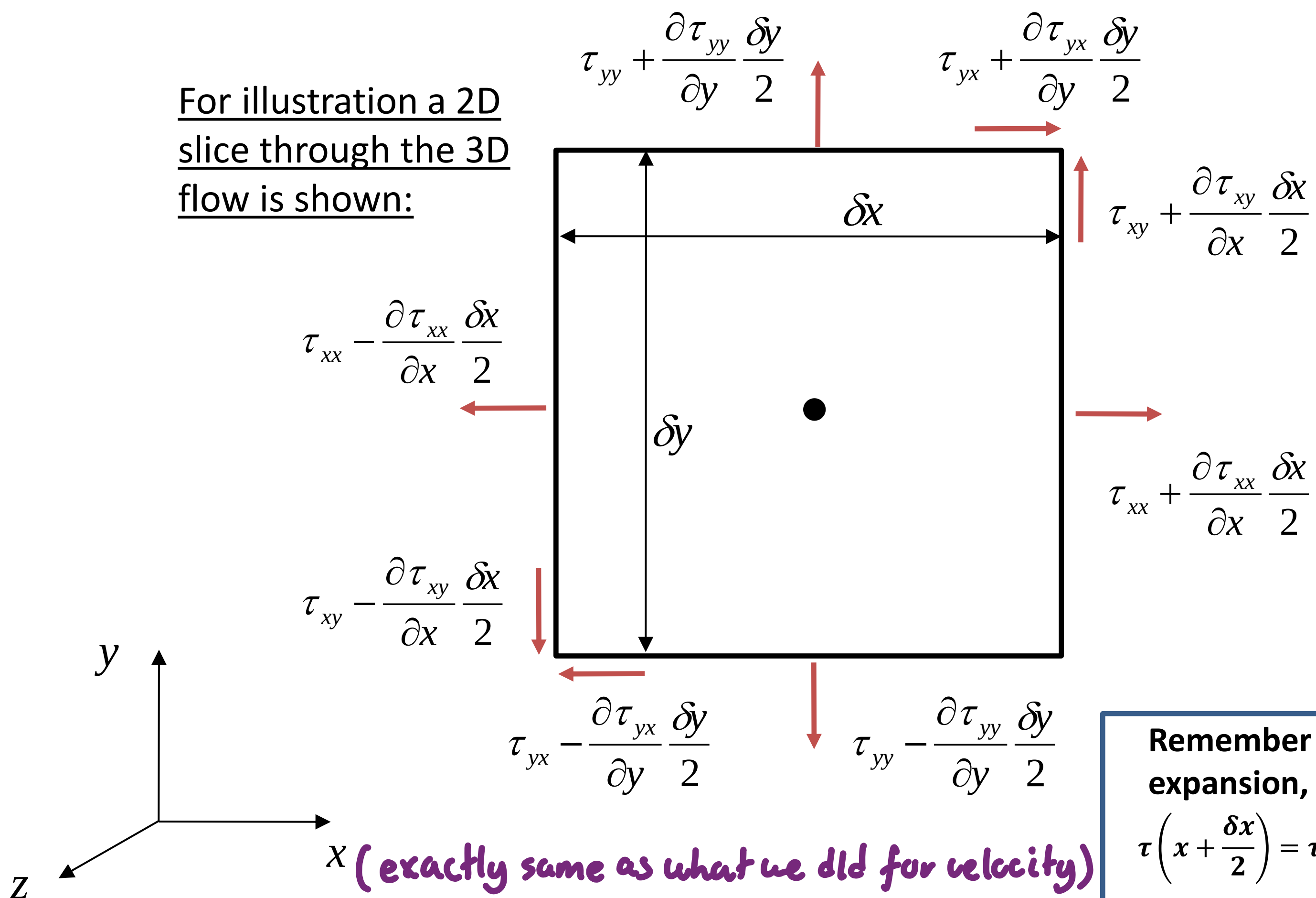
$$\tau_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix}$$

- First index: index *i* expresses that the stress acts on a plane normal to axis *i*
- Second index: index *j* denotes the direction *j* in which the stress acts

Stresses on a fluid element

In **viscous** flow the **shear stresses** are no longer zero. Consider the **shear stress** acting on a **fluid particle** (NB pressure is already included in the Euler equations).

For illustration a 2D slice through the 3D flow is shown:



(exactly same as what we did for velocity)

Remember your Taylor series expansion, for example:

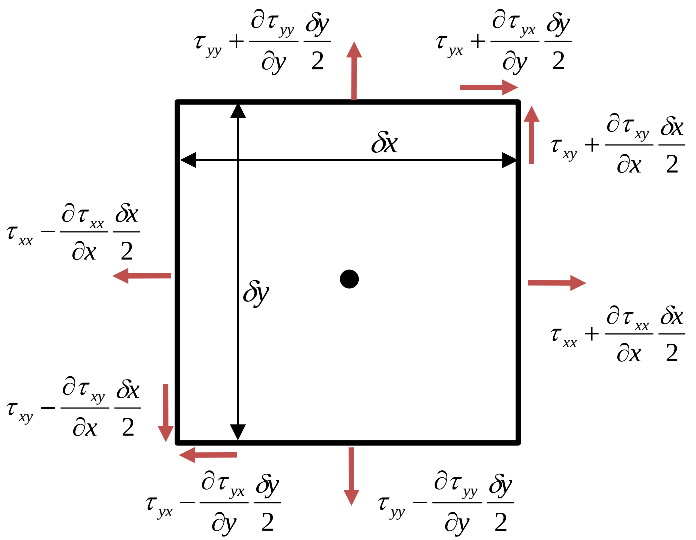
$$\boldsymbol{\tau}\left(\boldsymbol{x} + \frac{\boldsymbol{\delta x}}{\mathbf{2}}\right) = \boldsymbol{\tau}(\boldsymbol{x}) + \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{x}}\frac{\boldsymbol{\delta x}}{\mathbf{2}} + \frac{\partial^2 \tau}{\partial x^2}\frac{\delta x^2}{8} + \ldots$$

Stresses on a fluid element

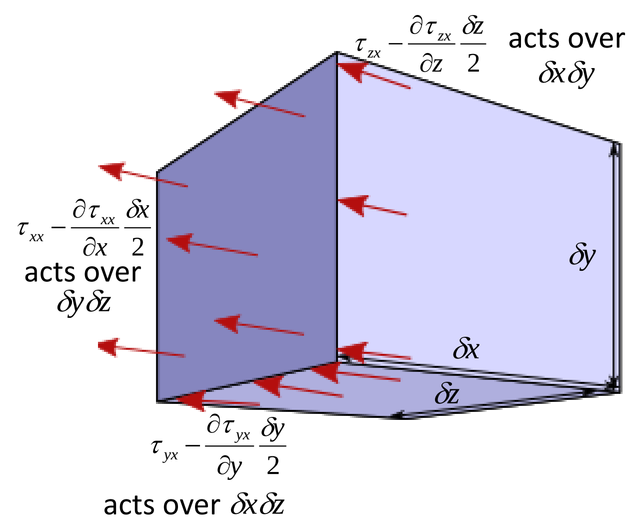
Each of the stresses is associated with:

- The plane on which it acts (**first <u>stress</u> index is the axis normal to the plane**)
- The direction in which it acts (**second <u>stress</u> index**)

For illustration a 2D slice through the 3D flow is shown:



For illustration a 3D schematic **only** showing the stresses in the ***x*-direction** at the distance, $x-\frac{\delta x}{2}$, are shown:



Stress is related to force by the area over which the stress acts…

Stresses on a fluid element

$\tau = \frac{F}{A}$

$F = \tau A \rightarrow \delta F = \tau \delta A$

Considering the **stresses** as given**,** the force due to the **normal and shear stresses** acting in the ***x*-direction**, $\delta F_{x,\tau}$, is:

all "τ" terms will be cancelled out

$$\delta F_{x,\tau} = -\left(\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z + \left(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$
$$-\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z$$
$$-\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y$$

Illustrated in the 2D slice

Infer by symmetry (or otherwise)

The three above terms are on the 3-D schematic

Similarly, the **force** in the ***y*-direction** and the ***z*-direction** can be written down.

2.2d The Governing Equations for Real Fluids: The Navier Stokes Equation

Evaluating the viscous stress force component in the x-direction, after some simplification, yields:

$$\delta F_{x,\tau} = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \delta y \delta z$$

→ per unit volume form $\frac{\delta F}{\partial x \partial y \partial z}$

Adding the viscous forces to the Euler equations (slide 18) gives:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

☆☆☆ The old euler eqn we look at are only for ideal fluid, since it doesn't include shear stresses.

← now by adding this VISCOUS STRESS TERM, we can use "F=ma" (navier-stokes eqn) for REAL FLUIDS.
(fluid that has shear stresses)

- Ten dependent variables: $u, v, w, p, \tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy} (=\tau_{yx}), \tau_{xz} (=\tau_{zx}), \tau_{yz} (=\tau_{zy})$
- Four equations (3 equations of motion, one per spatial dimension, + mass continuity)
- We need to link the viscous stress terms to the velocity field

Fluid Stress and Strain

μ and $\frac{du}{dy}$ independent.

so we are assuming newtonian!
i.e. $\tau = \mu \frac{du}{dy}$ (linear)

$\tau = \mu \frac{du}{dy}$ (1D)

Stress in **real fluids** is related to the **rate of strain**. For a **Newtonian fluid** this relationship will be **linear**. In two dimensions, the **stress-strain relationships** are given by (this will be shown more formally in 4th year):

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right);\ \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right);\ \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right);\ \tau_{xx} = 2\mu\frac{\partial u}{\partial x};\ \tau_{yy} = 2\mu\frac{\partial v}{\partial y};\ \tau_{zz} = 2\mu\frac{\partial w}{\partial z}.$$

Considering the viscous terms in the *x*-direction gives

y3 fluids

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial y \partial x} + \mu\frac{\partial^2 u}{\partial z^2} + \mu\frac{\partial^2 w}{\partial z \partial x}$$

Assuming constant viscosity and interchanging the order of differentiation for the last term yields:

$$\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\cancelto{0}{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}\right)$$

Zero due to mass continuity

Fluid Stress and Strain (cont.)

Substituting this result (and equivalent derivations for the *y*- and z-directions) into the equations of motion, yields:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

VISCOUS STRESS TERM now is in term of u v w !

(only if $\nabla^2 u$ is big, else this term is actually insignificant)
like when we are far away from boundary,

$\nabla^2 u$ is small enough to ignore viscous stress term

- Four dependent variables: *u, v, w, p*
- Four equations (3 equations of motion, one per spatial dimension, + mass continuity)
- Mathematically closed

Navier Stokes, general equations of motion for fluids (per unit volume)

For the general case in three dimensions:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Or in vector form:

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\equiv\rho\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)=-\nabla p+F_{body}+\mu\nabla^2\mathbf{u}$$

Navier Stokes, general equations of motion for fluids (per unit mass)

For the general case in three dimensions:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+g_x+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+g_y+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+g_z+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Or in vector form:

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\equiv\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=-\frac{1}{\rho}\nabla p+\boldsymbol{g}+\nu\nabla^2\mathbf{u}$$

$$\mathbf{u}=\begin{pmatrix}u\\v\\w\end{pmatrix},\quad \nabla^2\mathbf{u}=\begin{pmatrix}\frac{\partial^2}{\partial x^2}\\ \frac{\partial^2}{\partial y^2}\\ \frac{\partial^2}{\partial z^2}\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\\ \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\end{pmatrix},$$

$$\mathbf{u}\cdot\nabla\mathbf{u}=\begin{pmatrix}u\\v\\w\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial u}{\partial x}&\frac{\partial u}{\partial y}&\frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x}&\frac{\partial v}{\partial y}&\frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x}&\frac{\partial w}{\partial y}&\frac{\partial w}{\partial z}\end{pmatrix}=\begin{pmatrix}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\end{pmatrix}$$

You can now complete:
Tutorial 1, Q1-Q3

Navier-Stokes Equations

These equations are referred to as the **Navier-Stokes Equation**. They describe the **dynamics** of **viscous** and **incompressible** flow. In the form presented here, the fluid is assumed to be Newtonian (due to the linear stress-strain assumption).

$$\boxed{\underbrace{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \equiv \rho\left(\overbrace{\frac{\partial \mathbf{u}}{\partial t}}^{\text{unsteady}} + \overbrace{\mathbf{u}\cdot\nabla\mathbf{u}}^{\text{convective}}\right)}_{\text{acceleration term}} = \underbrace{-\nabla p + F_{body} + \overbrace{\mu\nabla^2\mathbf{u}}^{\text{viscous stress term}}}_{\text{forces per unit volume.}}}$$

(why called F/v? eg. $F_{body} = \rho g$
$\rho = m/v$ $\rho g = mg/v$ (F/v!))

☆ this form can be used to describe both turbulent and laminar, although by Reynold's decomposition, another better one for turbulent flow is obtainable

- Four unknowns (u, v, w, p)
- Four differential equations (3 x Navier-Stokes + mass continuity)
- For μ=0 (inviscid flow) the Navier-Stokes Equations reduce to the Euler Equations
- Equations are a statement of Newton's second law for the rate of change of momentum, *cf*. "F=ma" or "ma=F"!
- Viscosity affects the momentum only via the second spatial derivative of velocity – one can expect viscosity is important only where large changes in velocity gradients are supported, e.g. near (no-slip) walls... - Basis for Boundary Layer theory, slide 64...

2.3 Application of Navier-Stokes

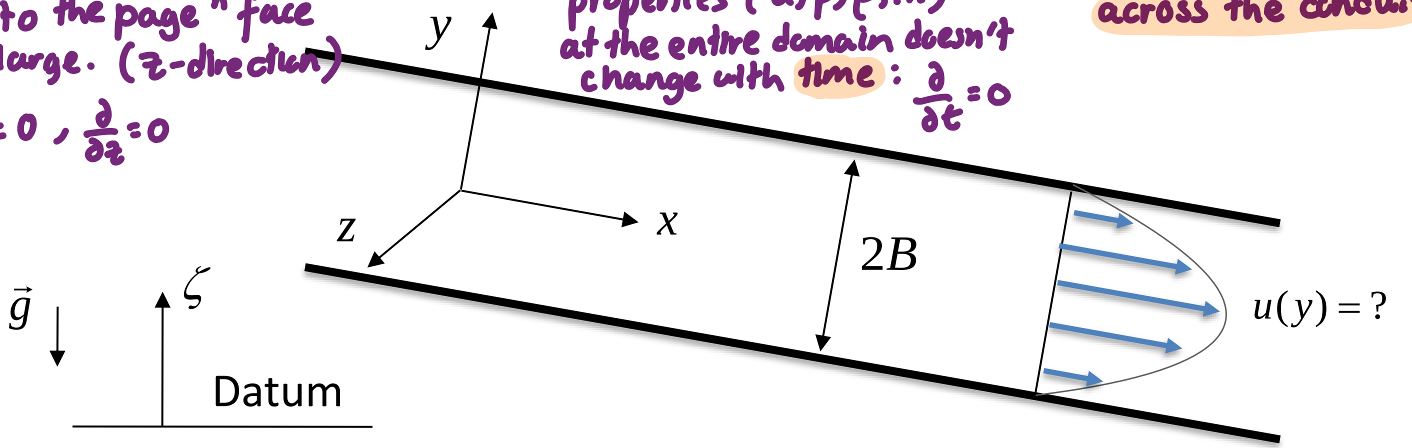
The Navier-Stokes equations can be used to describe **turbulent** fluid flows. For simplicity, we will consider the **steady**, **laminar** and **fully developed** flow between two **very large** parallel plates.

very large means the "into the page" face is large. (z-direction)

$w = 0$, $\frac{\partial}{\partial z} = 0$

steady flow means fluid's properties ($u, p, \rho, \ldots$) at the entire domain doesn't change with time : $\frac{\partial}{\partial t} = 0$

fully developed means velocity profile is constant across the conduit : $\frac{\partial u}{\partial x} = 0$



i. **Think** it through and specify all the boundary conditions
ii. **Simplify** Navier-Stokes
iii. **Solve** for $u(y)$
iv. **Discuss** the result

i. Think

- The flow is assumed to be steady. All time derivatives will be zero $\boxed{\partial / \partial t = 0}$

- The flow is assumed to be fully developed. As a result, the velocity component in the x-direction will remain constant → basically anything $\frac{\partial}{\partial x}$ $\boxed{\partial u / \partial x = 0}$

- The two plates are very large. The effect of any boundaries in the z-directions can be neglected. We will also assume that there is $\underbrace{\text{no flow in the } z\text{-direction}}_{\text{cause it is normal to gravity!}}$ $\boxed{w = 0}$ $\boxed{\partial / \partial z = 0}$

- Considering the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow$$

$\partial u / \partial x = 0$, $\partial / \partial z = 0$: $0 + \frac{\partial v}{\partial y} + 0 = 0$

$\partial v / \partial y = 0$

$\boxed{\frac{\partial v}{\partial y} = 0}$

- Moreover, the wall can be assumed to be impermeable $\boxed{v\big|_{y=B} = 0}$

v at y=B, =0
$\frac{\partial v}{\partial y} = 0$
v at all y = 0

- Since the gradient of v is zero, and v is zero at the boundary, v must be zero everywhere (which is what we would expect from laminar flow!) $\boxed{v = 0}$

ii. Simplify

v=0, not $\partial u / \partial y = 0$!

The complete formulation of the Navier-Stokes equations

$\frac{\partial v}{\partial y} = 0$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$v=0$, $\partial v / \partial y = 0$
$\rightarrow \partial^2 v / \partial y^2 = 0$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

simplifies to

all $\frac{\partial}{\partial t} = 0$ (steady)

$\frac{\partial u}{\partial z} = 0$

w=0 everywhere $\rightarrow \partial w / \partial y = 0$

w=0

$\partial / \partial x = 0$

w=0 everywhere $\rightarrow \partial w / \partial y = 0 \rightarrow \partial^2 w / \partial y^2 = 0$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x + \mu\frac{d^2 u}{dy^2} \qquad \underbrace{u(x,y,z) \rightarrow u(y)}$$

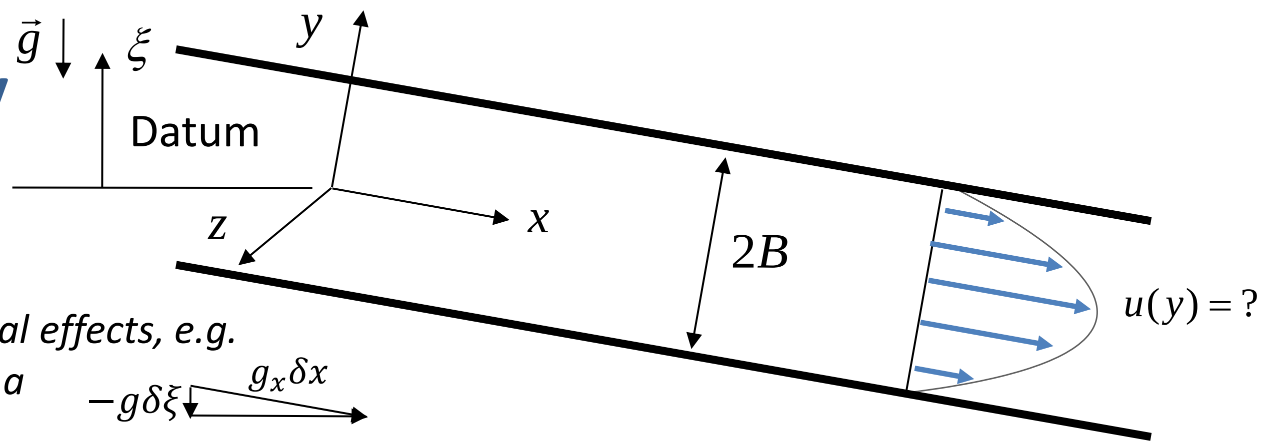
u(x,y,z) gets simplified into u(y)
→ can be solved analytically!

$$0 = -\frac{\partial p}{\partial y} + \rho g_y$$

$$\boxed{0 = -\frac{\partial p}{\partial z} + \rho g_z}$$

Unlike the Navier-Stokes equations, the **simplified** version can be solved **analytically**!

ii. Simplify



Think about how gravitational effects, e.g. potential energy, change for a small move in x direction:

The components of gravity in the *x*, *y* and *z*-directions are

$$g_x = -g\frac{\partial \xi}{\partial x} \qquad g_y = g\frac{\partial \xi}{\partial y} \qquad g_z = 0 = \frac{\partial \xi}{\partial z}$$

Substituting this result into the governing equations gives

$$0 = -\frac{\partial(p+\rho g\xi)}{\partial x} + \mu\frac{d^2u}{dy^2} \qquad 0 = \frac{\partial(p+\rho g\xi)}{\partial y} \qquad 0 = -\frac{\partial(p+\rho g\xi)}{\partial z}$$

The expression $p+\rho g\xi$ is clearly a function of *x* alone, as is *u* of *y* (see earlier). All that remains from our original set of equations (Navier-Stokes) is

$$\boxed{\frac{d}{dx}(p+\rho g\xi) = \mu\frac{d^2u}{dy^2}}$$

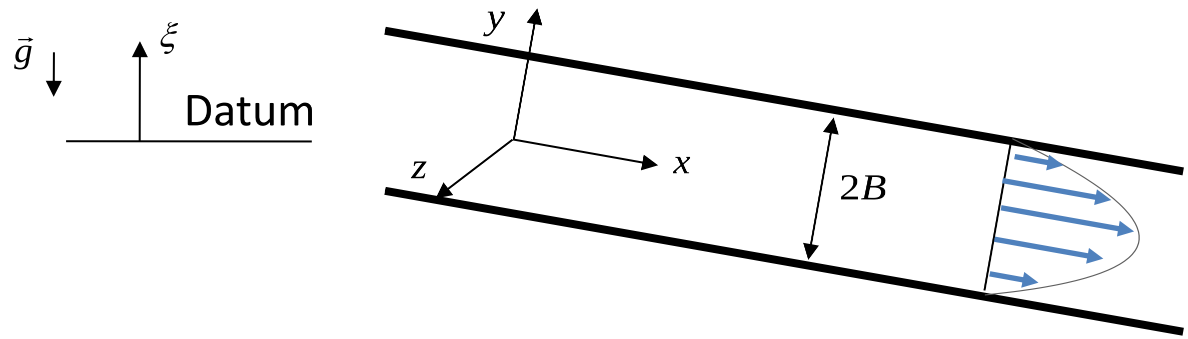
$\partial^2 u/\partial x^2 = 0$
$\partial^2 u/\partial y^2 = 0$
$\to u = g(y)$

these two eqn shows that
$p+\rho g\xi = h(x)$

Note the **total** derivative!

since $p+\rho g\xi = h(x)$, can now convert $\partial/\partial x$ to d/dx

iii. Solve



We want to **solve:**

$$\frac{d}{dx}(p+\rho g\xi)=\mu\frac{d^2u}{dy^2}$$

Since the left hand side of this equation is a function of *x* only and the right hand side is a function of *y* only, both sides must be equal to an identical constant.

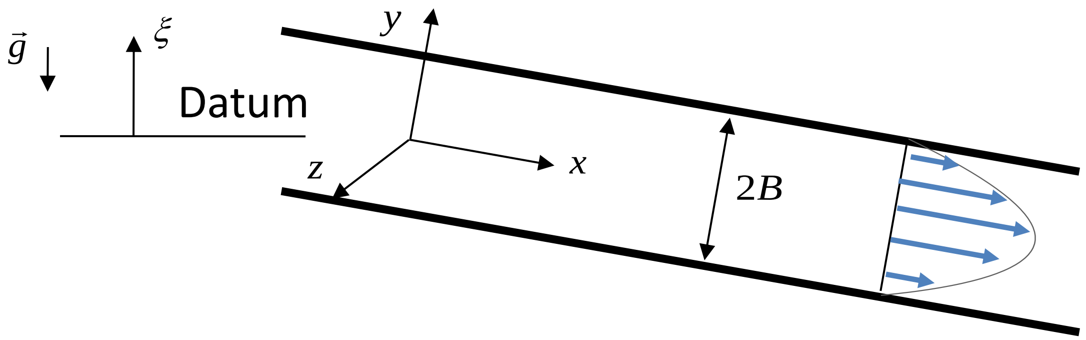
Hence: LHS = *f*(*x*), RHS = *f*(*y*). Introduce **constant (akin to a separation constant)**, say ***G***

$$\frac{d}{dx}(p+\rho g\xi)=G=\mu\frac{d^2u}{dy^2}$$

Integrating the RHS twice with respect to *y* gives

$$u(y)=\frac{G}{2\mu}y^2+C_1y+C_2$$

iii. Solve



Substituting the two **boundary conditions** at $y = \pm B$

$$u(y=B)=0=\frac{G}{2\mu}B^2+C_1B+C_2 \qquad u(y=-B)=0=\frac{G}{2\mu}B^2-C_1B+C_2$$

Comparing (or adding or subtracting or substituting) the two conditions gives

$$C_2=-\frac{G}{2\mu}B^2 \qquad C_1=0 \qquad \frac{d}{dx}(p+\rho g\xi)=G \qquad u(y)=\frac{G\,B^2}{2\mu}\left[\left(\frac{y}{B}\right)^2-1\right]$$

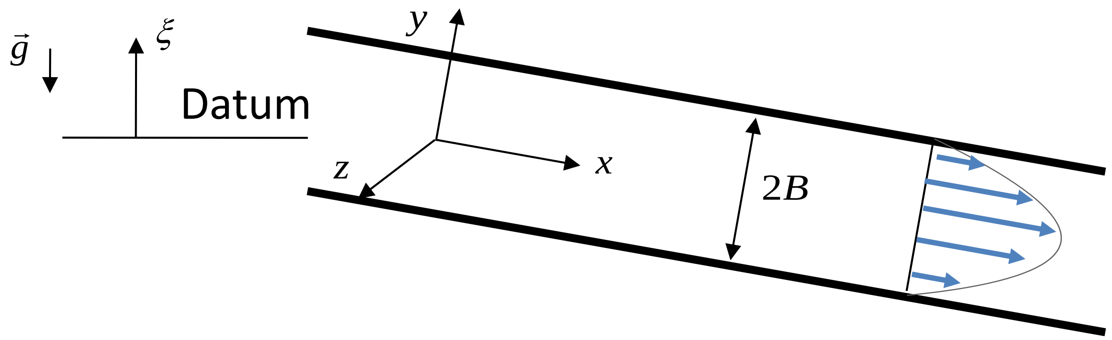
Hence

$$u(y)=\frac{-B^2}{2\mu}\frac{\mathrm{d}}{\mathrm{d}x}(p+\rho g\,\zeta)\left[1-\left(\frac{y}{B}\right)^2\right]$$

$$u(y)=u_{\max}\left[1-\left(\frac{y}{B}\right)^2\right] \qquad u_{\max}=\frac{-B^2}{2\mu}\frac{d(p+\rho g\xi)}{dx}$$

You can now complete: Tutorial 1, Q4

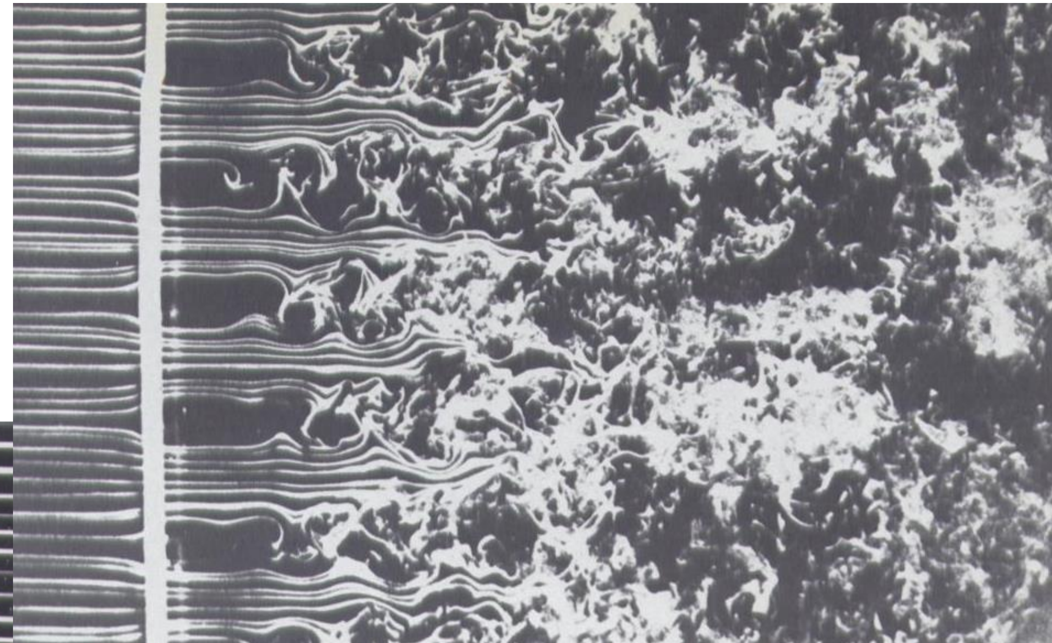
iv. Discuss the Result

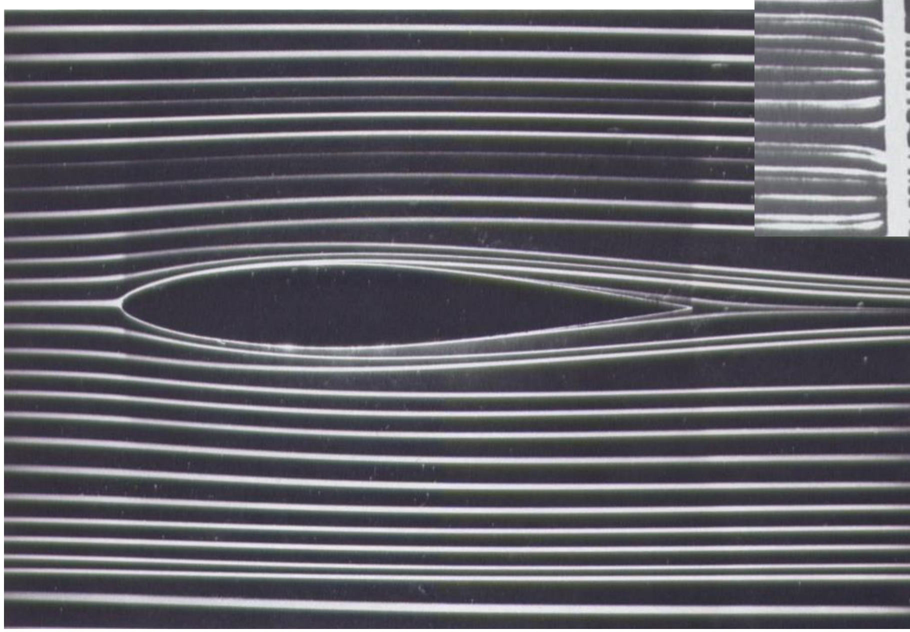


$$u(y) = u_{\text{max}}\left[1-\left(\frac{y}{B}\right)^2\right] \qquad u_{\text{max}} = \frac{-B^2}{2\mu}\frac{d(p+\rho g\xi)}{dx}$$

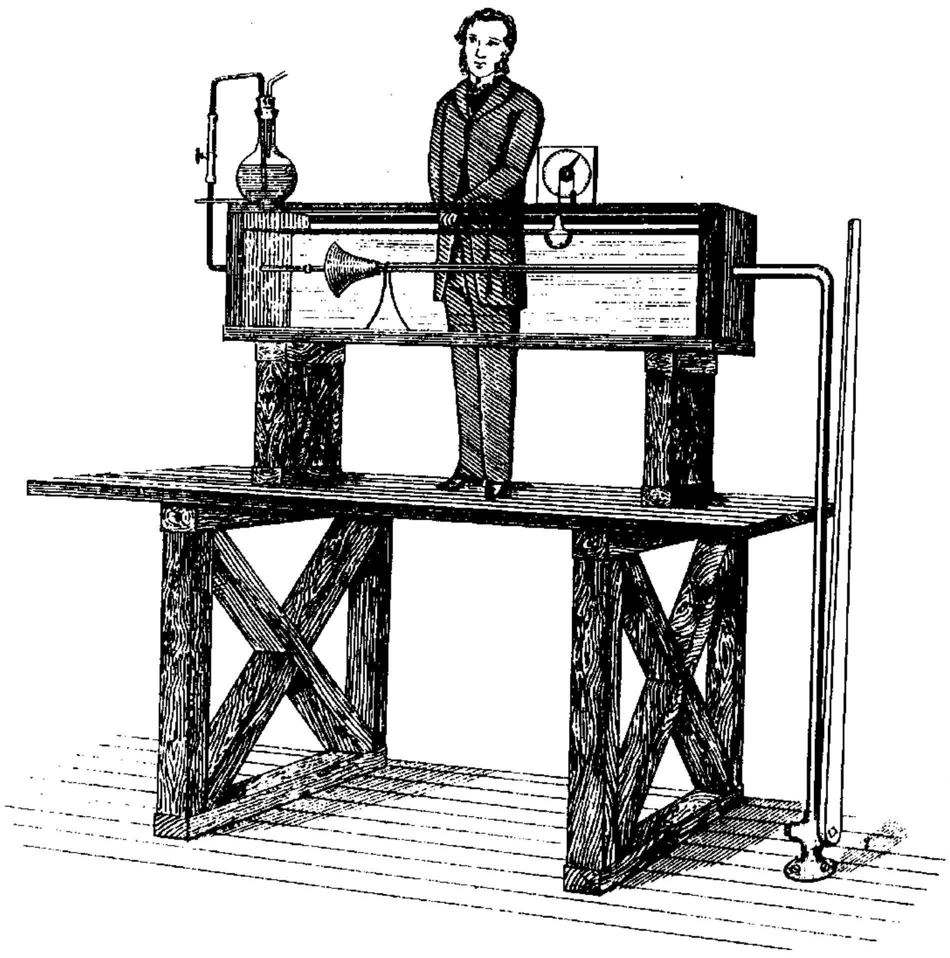
- **Parabolic** velocity distribution
- At *y*=*B*, ***u*(±*B*) = 0**. This is as expected from a **no slip boundary condition**
- The maximum **velocity decreases** for larger values of μ (**higher viscosity**)
- The maximum **velocity increases** for a **wider spacing** *B* (flow less affected by the wall)
- Larger pressure gradient and slope increase the maximum velocity

2.4 Laminar and Turbulent Flow





Reynolds Experiment (1883)



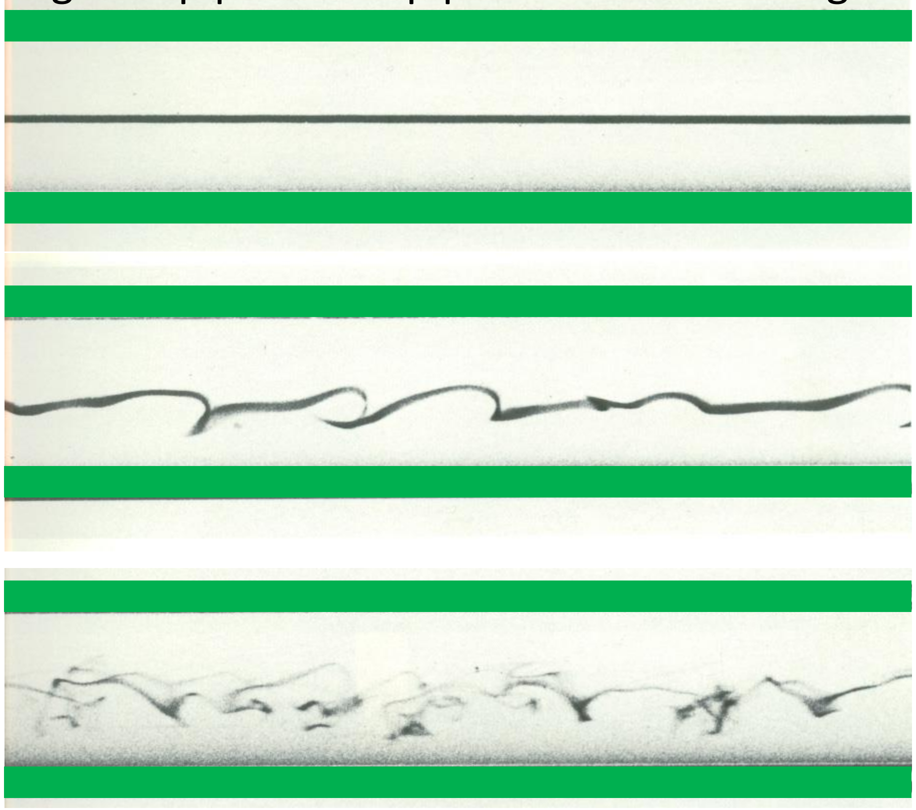
if viscous forces dominates → laminar.
if inertial forces dominates → turbulent.

Reynolds Experiment - Observation

$$\mathrm{Re} = \frac{\text{Inertial Forces}}{\text{Viscous Forces}} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

- U – velocity scale
- L – length scale

Images of pipe flow – pipe walls overlaid in green



(a) Laminar at low Re

Dye streak (dark line) remains thin straight line

(b) Transition

Dye streak wavers but does not break up

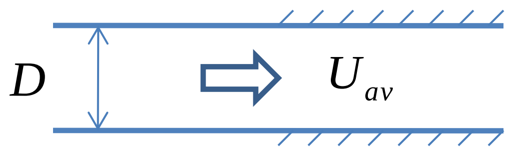
(c) Turbulent at high velocity

Dye streak breaks up, and mixes across entire pipe

Critical Reynolds Number

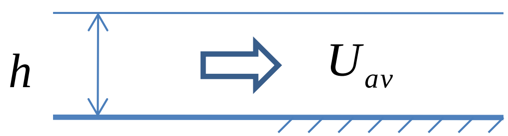
- Common to consider critical Reynolds number value: $\mathbf{Re_{crit}}$
- For $\mathbf{Re < Re_{crit}}$ laminar flow is expected
- For $\mathbf{Re > Re_{crit}}$ Turbulent flow (likely transitional close to $\mathbf{Re_{crit}}$)
- ***Most*** natural and **civil engineering** flows are highly **turbulent**

Pipe



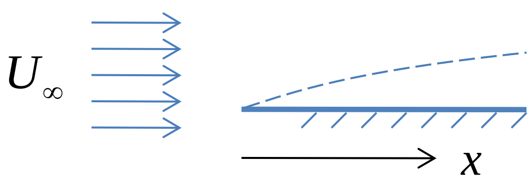
$$\mathrm{Re} = \frac{U_{av} D}{\nu} \qquad \mathrm{Re}_{\text{crit}} \approx (2.3 - 4) \times 10^3$$

Channel



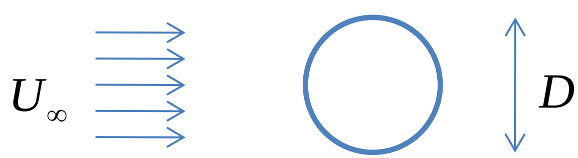
$$\mathrm{Re} = \frac{U_{av} h}{\nu} \qquad \mathrm{Re}_{\text{crit}} \approx 600$$

Flat Plate



$$\mathrm{Re} = \frac{U_\infty x}{\nu} \qquad \mathrm{Re}_{\text{crit}} \approx 5\text{x}10^5$$

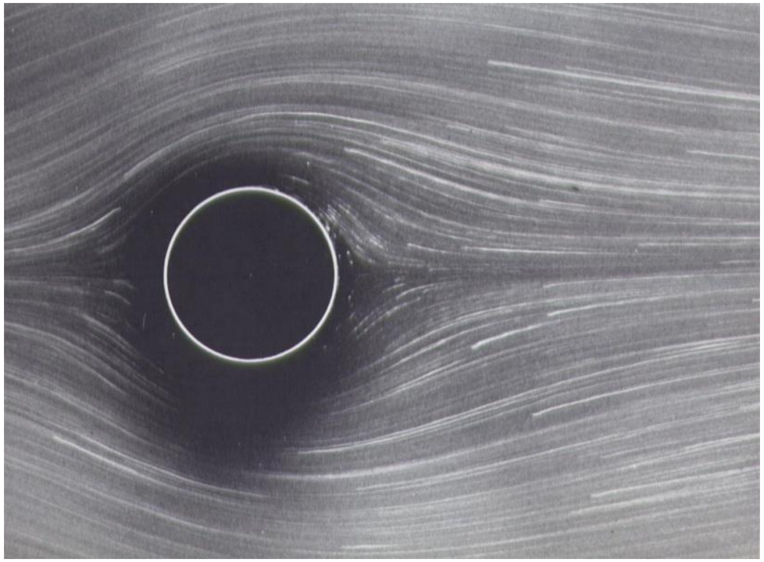
Sphere

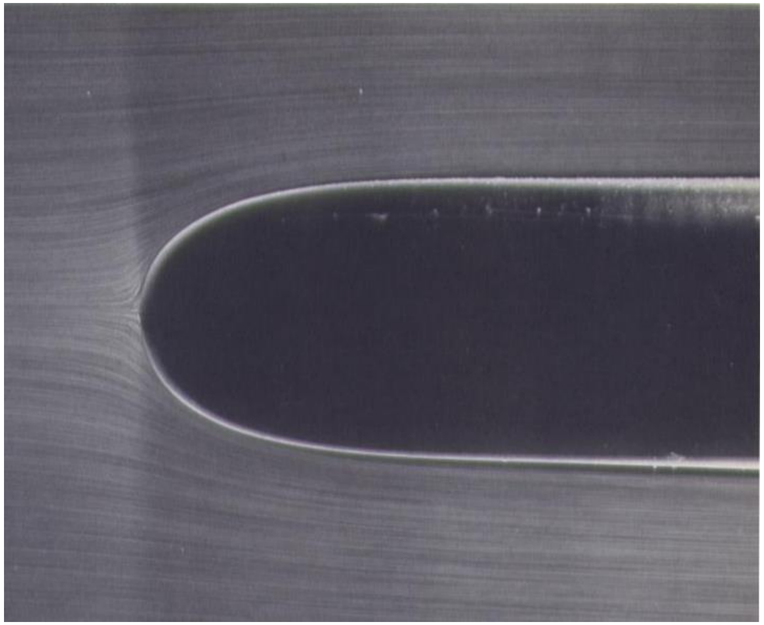


$$\mathrm{Re} = \frac{U_\infty D}{\nu} \qquad \mathrm{Re}_{\text{crit}} \approx 1$$

Laminar Flow

- Imagine **flow in layers**
- Individual particles follow paths (streamlines) that do not cross those of their neighbouring particles
- **No velocity** component **transverse to streamlines**
- **Viscous forces dominate** over inertial forces (low Reynolds number)
- **Laws** of laminar flow are **well understood**
- **Analytical solutions** for simple boundary conditions possible
- In **pipe** and **open channel flow**, the **flow** is nearly always **turbulent** (large velocities)
- In such cases a thin laminar layer exists in proximity of a solid boundary





Determination of Dynamic Viscosity Problem:

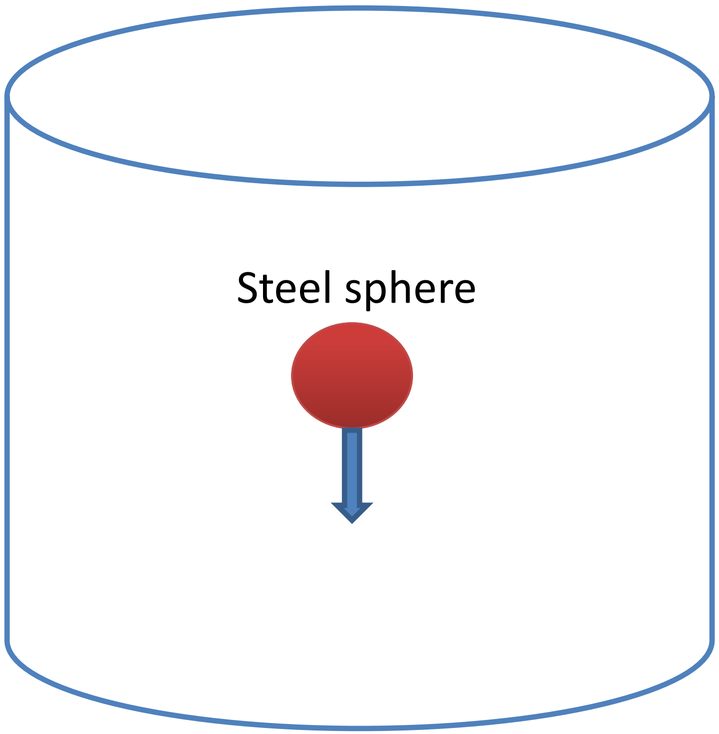
The dynamic viscosity μ of a fluid may be determined by measuring the constant velocity of a sphere sinking within the fluid.

so use high viscosity fluid (eg. Honey!)

Assumption: The flow around the sphere must be laminar ($Re < 0.5$) and the oil filled cylinder is infinitely large ($\approx 100\ D_{sphere}$ for very precise experiments – but let's see how we go with a pint glass!).

We would like to prove this using oil (μ=0.8 kg/ms, ρ=900 kg/m^3) and a steel sphere (ρ=7700 kg/m^3).

What is the maximum diameter of the sphere?



Large oil filled cylinder

Determination of Dynamic Viscosity
Solution:

viscous force = constant × dynamic viscosity × velocity × length scale.

(i) Stokes Law (drag for slow, creeping, viscous flow) $F_S = 3\pi \mu u D$ ☆ new!

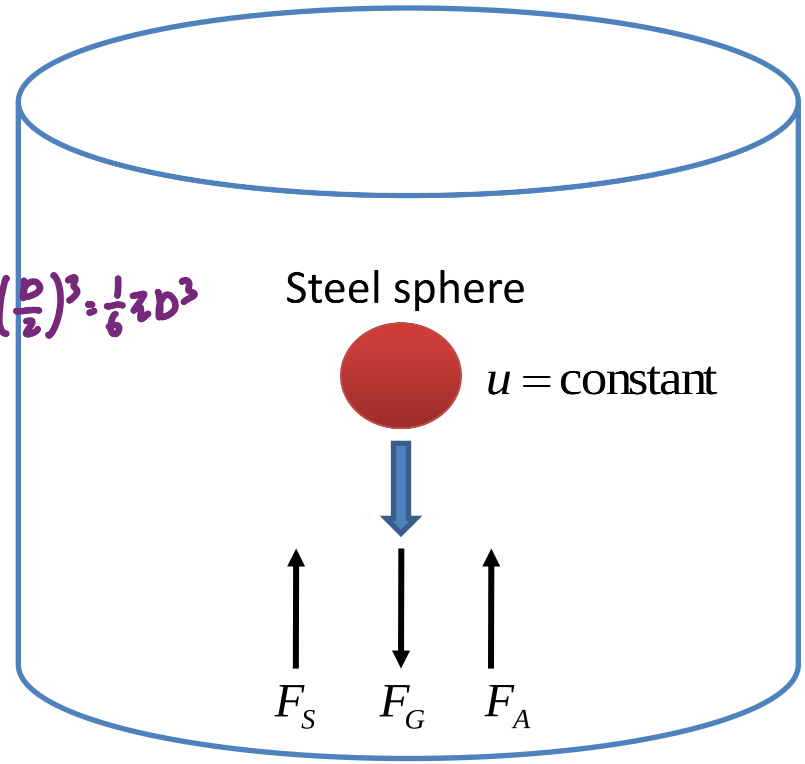
(ii) Buoyancy force $F_B = (\rho_{Sphere} - \rho_{fl})Vg$

- 'Archimedes' Force
  $F_A = \rho_{fl} Vg$
- 'Gravity' Force
  $$F_G = \rho_{Sphere} Vg = \rho_{Sphere} \overbrace{(1/6)\pi D^3}^{\frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{D}{2}\right)^3 = \frac{1}{6}\pi D^3} g$$

(i) Force Balance

$$F_S = F_B \Rightarrow 3\pi \mu u D = (\rho_{Sphere} - \rho_{fl})(1/6)\pi D^3 g$$

⇒ $D = 4.6mm$ Using $\mathrm{Re} = \mathrm{Re}_{crit}$



Large oil filled cylinder

Tutorial 2!

You can now complete: Tutorial 2

Laminar Pipe Flow: Example

The **wave basin** in our Hydrodynamics Laboratory has a **capacity of 300 m^3**. The basin is to be filled within an hour → volume flow rate of 0.083 m^3/s (or 83 litres/s)

Calculate the **required pipe diameter** so that the **flow** remains **laminar**. The dynamic viscosity of water may be assumed as 0.001 kg/ms.

The critical Reynolds number for pipe flows is given by 2300

$$\text{Re} = \frac{\rho U_{av} D}{\mu} < 2300$$

The average fluid velocity is directly linked to the volume flow rate

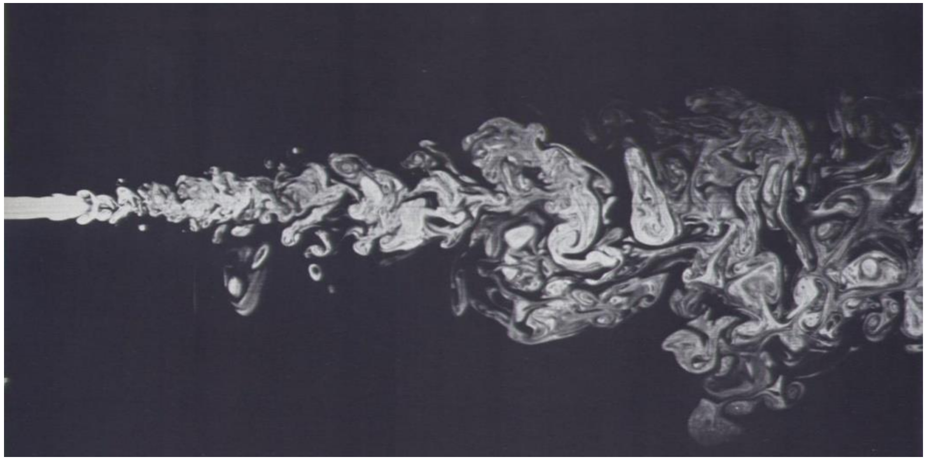
$$U_{av} = \frac{Q}{A} = \frac{Q}{\pi D^2 / 4} \qquad D_{\text{crit}} = \frac{\rho Q}{(\pi / 4)\mu \,\text{Re}_{\text{crit}}}$$

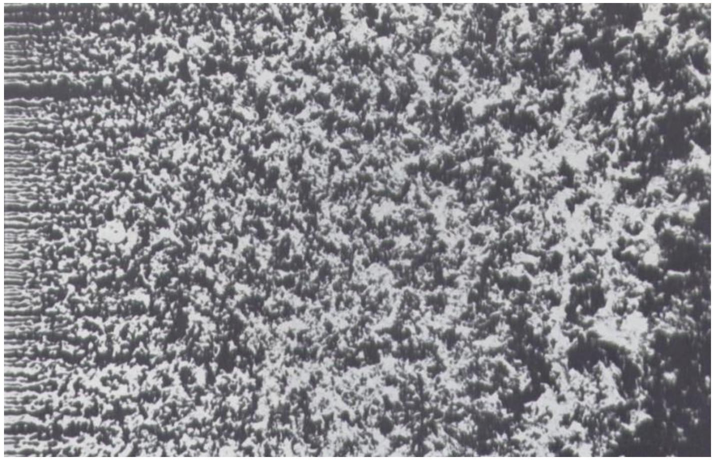
$$\boxed{D_{crit} \approx 46\text{m and } U_{av} \approx 50\mu\text{m/s}}$$

This is clearly impractical! Most civil engineering flows are turbulent!

Turbulent Flow (mixing momentum)

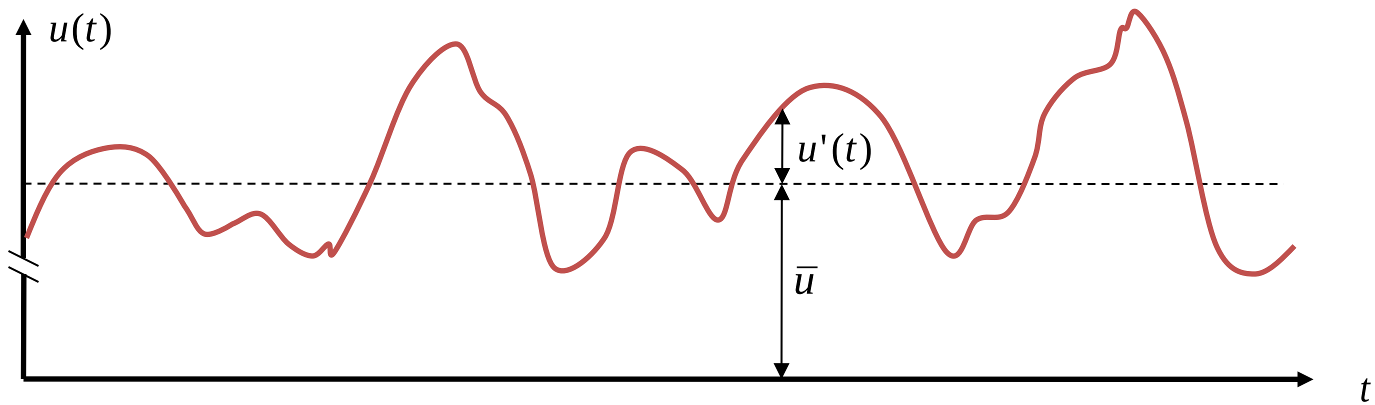
- In turbulent flows, an irregular fluctuation is superimposed on the main flow
- The effect of the turbulent mixing motion is as if the viscosity were increased by a large factor (hundred, ten thousand or even more)
- Small turbulent fluctuations have a significant influence on the mean flow
- Turbulent stress often dominate over viscous stress
- Modelling of turbulent flows usually considers time averaged motions (due to the complexity of the flow)
- Much of the work is based on experimental data





Observations of Turbulence

A typical time history in a turbulent flow:



$$u(t) = \overline{u} + u'(t) \qquad \overline{u} = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} u(t)dt$$

NB: the prime notation, e.g u**'**, does not denote any derivative here – it is only denoting a fluctuation about a mean!

- Similar traces for $v(t)$, $w(t)$, $p(t)$
- Separation into time-averaged velocity and turbulent fluctuation

Schlichting, H.: '*In other words the presence of fluctuations manifests itself in an apparent increase in the viscosity of the fundamental flow. This increased apparent viscosity of the mean stream forms the central concept of all theoretical considerations of turbulent motion.*'

How do we account for the **increased apparent viscosity**?

$$\frac{\partial u^2}{\partial x} = \frac{\partial (u^2)}{\partial u} \times \frac{\partial u}{\partial x} = 2u\frac{\partial u}{\partial x} \quad \rightarrow \quad \frac{\partial u^2}{\partial x} - u\frac{\partial u}{\partial x} = u\frac{\partial u}{\partial x}$$

Reynolds decomposition (averaging)

Focusing just on the x-direction of a 3-D flow:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Note (using the chain rule) that

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial\, u^2}{\partial x} + \frac{\partial\, uv}{\partial y} + \frac{\partial\, uw}{\partial z} - u\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)}_{0}$$

Zero due to mass continuity

Averaging (integrating) over some time interval …

Writing: $u = \overline{u} + u', v = \overline{v} + v'\ w = \overline{w} + w', p = \overline{p} + p'$, and noting

$$\overline{\frac{\partial \blacksquare}{\partial t}} = 0, \qquad \overline{\blacksquare'} = 0, \qquad \overline{\frac{\partial \overline{\blacksquare}}{\partial \alpha}} = \frac{\partial \overline{\blacksquare}}{\partial \alpha}, \qquad \overline{\frac{\partial \blacksquare'}{\partial \alpha}} = 0, \qquad \overline{\overline{\blacksquare} \cdot \boldsymbol{\alpha}} = \overline{\blacksquare} \cdot \overline{\boldsymbol{\alpha}}$$

Gives

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + g_x + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right) - \left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g_x + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + g_y + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Reynolds decomposition (averaging)

Per unit volume

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

Thinking in terms stresses

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\right)=-\frac{\partial \bar{p}}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)+\left(\frac{\partial \tau'_{xx}}{\partial x}+\frac{\partial \tau'_{xy}}{\partial y}+\frac{\partial \tau'_{xz}}{\partial z}\right)$$

Where

$$\begin{pmatrix} \tau'_{xx} & \tau'_{xy} & \tau'_{xz} \\ \tau'_{yx} & \tau'_{yy} & \tau'_{yz} \\ \tau'_{zx} & \tau'_{zy} & \tau'_{zz} \end{pmatrix}=-\rho\begin{pmatrix} \overline{u'^2} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'^2} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'^2} \end{pmatrix}$$

What does it all mean for us and civil engineering?

$$\rho\left(\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}+\overline{w}\frac{\partial \overline{u}}{\partial z}\right)=-\frac{\partial \overline{p}}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 \overline{u}}{\partial x^2}+\frac{\partial^2 \overline{u}}{\partial y^2}+\frac{\partial^2 \overline{u}}{\partial z^2}\right)-\rho\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)$$

The additional terms are referred to as **apparent** or **turbulent shear stress** or **Reynolds Stresses** and represent an additional resistance (like the viscous terms)

- Think frictional force due to fluctuations in velocity

Reynolds Stress

due to turbulent transport of momentum from high momentum regions to low momentum region.

usually alot bigger than

so both p and u,v,w are now separated into mean and fluctuating contribution.

Viscous Stress Terms

Reynolds stress terms!

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+g_x+\boxed{\nu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}+\frac{\partial^2 \bar{u}}{\partial z^2}\right)}\boxed{-\left(\frac{\partial \overline{u'^2}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}\right)}$$

This equation is used usually for turbulent (laminar also valid, but reynold's stress term ≈ 0)

- Reynolds Stress represents the interaction of turbulence with mean flow
- Reynolds Stress is a tensor with **9 components** (6 variables)
- Predicting these terms is a **major challenge** in Fluid Dynamics
- In pipe and channel flow, the **turbulence model is indirect**, being a **low-order model** or sometimes an **empirical equation** for the boundary shear
- In **turbulent** flows, **Reynolds Stress** is often **dominant** over viscous stress