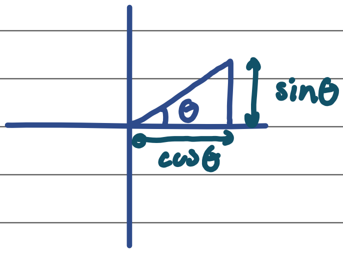
# Year 1 Mathematics Notes

# c1. Revision.

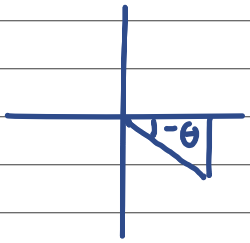
## 1.1 Trigonometry Trick

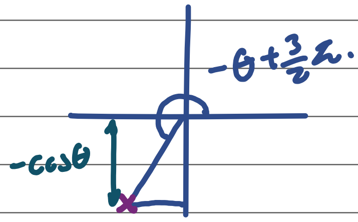
eg1. find $\sin\left(\frac{3}{2}\pi-\theta\right)$




we know that:
x-coord on a unit circle is always equal to cos and y-coord is always equal to sin.

$\sin(-\theta)$ $\qquad\qquad$ $\sin\left(-\theta+\frac{3}{2}\pi\right)$

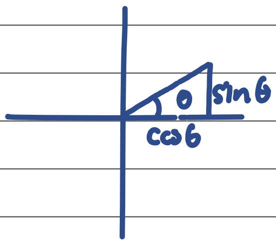


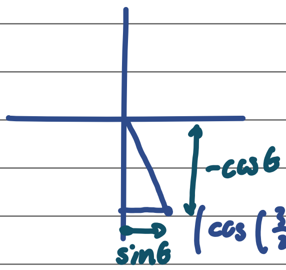




since we want to find. $\sin\left(-\theta+\frac{3}{2}\pi\right)$ it is the y-coord of the end of triangle.

$\therefore \ \sin\left(\frac{3}{2}\pi-\theta\right)=-\cos\theta$

eg2. $\tan\left(\frac{3}{2}\pi+\theta\right)$




$\left(\cos\left(\frac{3}{2}\pi+\theta\right), \sin\left(\frac{3}{2}\pi+\theta\right)\right)$

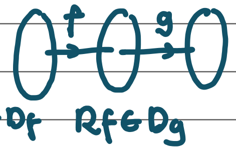
$\sin\left(\frac{3}{2}\pi+\theta\right)=-\cos\theta$

$\cos\left(\frac{3}{2}\pi+\theta\right)=\sin\theta$

$\tan\left(\frac{3}{2}\pi+\theta\right)=-\cot\theta$

## 1.2 Domain of a composite function.

$gf(x)$, $D_{gf}: \{x\in\mathbb{R} : x\in D_f \cap f(x)\in D_g\}$




eg. $f(x)=\sqrt{x}$ , $g(x)=\frac{1}{x^2-1}$ , find $D_{fg}$

$D_f=\{x\in\mathbb{R}: x\geq 0\}$, $D_g=\{x\in\mathbb{R}: x\neq 1 \cap x\neq -1\}$

$$D_{fg}=\{x\in\mathbb{R}: \overbrace{x\neq 1 \cap x\neq -1}^{x\in D_g} \cap \overbrace{g(x)\geq 0}^{R_g\in D_f}\}$$

$$=\{x\in\mathbb{R}: x\neq 1 \cap x\neq -1 \cap \frac{1}{x^2-1}\geq 0\}$$

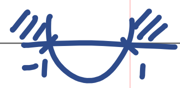
$$=\{x\in\mathbb{R}: (x\neq 1 \cap x\neq -1)\cap(x<-1 \cup x>1)\}$$

$$=\{x\in\mathbb{R}: x<-1 \cup x>1\}$$

$\infty \geq \frac{1}{x^2-1} \geq 0$

$0\leq x^2-1 \leq \infty$

$0 \leq (x+1)(x-1)$




$x<-1 \cup x>1$

## ☆ 1.3 University not-ignored details:

1. $\sqrt{a^2}=|a|$ not $a$! , $(\sqrt{a})^2=a$

# C2. Limits, Continuity

## 2.1 Definition. (for reference only, not important)

If $f(x)$ tends to a limit $L$ as $x$ approaches $a$, then for any number $\varepsilon$ (however small) it must be possible to find a number $\delta$, such that $|f(x) - L| < \varepsilon$ when $|x - a| < \delta$.

This definition translates in simpler terms as: $f(x)$ can be made as close as desired to $L$ by making the independent variable $x$ close enough, but not equal, to the value $a$.

basically $\lim_{x \to a} f(x) = L$

when $x \to a$ (but $x \neq a$), $f(x) \to L$ (but $f(x) \neq L$)

$$\forall \varepsilon > 0\ \exists \delta > 0\ (\forall x \in D_f): [0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

### 2.1.2 Epsilon - Delta proof (FOR REFERENCE ONLY)

→ useful for multivariable limits.

$$\forall \varepsilon > 0\ \exists \delta > 0\ (\forall x \in D_f): [0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$

eg. $\lim_{x \to 4} \sqrt{2x+1} = \sqrt{2(4)+1} = \sqrt{9} = 3$

1. given $\varepsilon > 0$
2. choose $\delta = \frac{\varepsilon}{2}$ (filled in the end)
3. suppose $0 < |x-4| < \delta$
4. check $\left|\sqrt{2x+1} - 3\right|$

$$\left|\frac{(\sqrt{2x+1} - 3)(\sqrt{2x+1} + 3)}{\sqrt{2x+1} + 3}\right| = \left|\frac{2x+1-3^2}{\sqrt{2x+1}+3}\right| = \frac{2|x-4|}{\sqrt{2x+1}+3}$$

$\sqrt{2x+1} > 0$ for $\forall x \in \mathbb{R}$, $3 > 0$, $\sqrt{2x+1} + 3 > 0$ → no need modulus.

$$\frac{2|x-4|}{\sqrt{2x+1}+3} < 2|x-4|$$

$\sqrt{2x+1} > 0$

$\sqrt{2x+1} + 3 > 3$

$\sqrt{2x+1} + 3 > 1$

$$\left|\sqrt{2x+1} - 3\right| < 2|x-4| < 2\delta$$

$$\left|\sqrt{2x+1} - 3\right| < 2\delta \rightarrow \left|\sqrt{2x+1} - 3\right| < \varepsilon \quad \#$$

$\therefore \delta = \frac{\varepsilon}{2}$

## 2·2 Basic Limit .

ALTHOUGH SOME LIMIT WE SUBSTITUTE $a$ into $f(x)$ getting $f(a)$ but remember by DEFINITION :

$$\lim_{x \to a} f(x) \neq f(a)$$

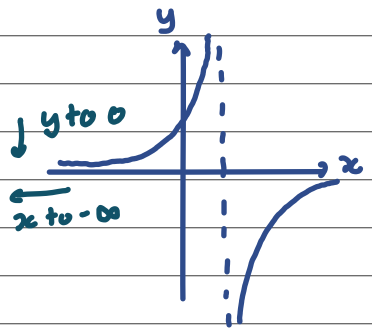
eg1. $\lim_{x \to 2} (x^2 - 2x - 3) = 2^2 - 2(2) - 3 = -3$ //

eg2. $\lim_{x \to \infty} \left( \frac{x-3}{2x+1} \right) = \lim_{x \to \infty} \left( \frac{\frac{x}{x} - \frac{3}{x}}{\frac{2x}{x} + \frac{1}{x}} \right) = \lim_{x \to \infty} \left( \frac{1 - \frac{3}{x}}{2 + \frac{1}{x}} \right) = \frac{1}{2}$

only when approaches infinity can use this trick .

some limit have to use graphical :

eg . $\lim_{x \to -\infty} \left( - \frac{1}{x-1} \right) = 0$




Some common standard limit:

$\lim_{x \to 0} \left( \frac{\sin x}{x} \right) = 1$

$\lim_{x \to a} \left( \frac{x^r - a^r}{x - a} \right) = ra^{r-1}$

$\lim_{x \to \pm\infty} \left( x \sin\left( \frac{c}{x} \right) \right) = c$

$\lim_{x \to +\infty} \left( 1 + \frac{k}{x} \right)^x = e^k$

## 2.3 Properties of Limits.

$$\lim_{x\to a}\left(f(x)\pm g(x)\right) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left(f(x)\cdot g(x)\right) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$\lim_{x\to a} f^n(x) \text{ or } \lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n \quad , n\in\mathbb{Z}^+$$

☆ $$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] \text{ or } \lim_{x\to a}[f(x)]^{\frac{1}{n}} = \sqrt[n]{\lim_{x\to a} f(x)} \text{ or } \left(\lim_{x\to a} f(x)\right)^{\frac{1}{n}}$$ , IF n is EVEN! and $f(x)>0$

## 2.4 Concept of Dominance.

Constant < Logaritmic < Root < Polynomial (including linear) < Exponential < Facturial < Tetration (and higher hyperoperations)

eg1. $\lim_{x\to+\infty}\left(\frac{x^5}{e^x}\right) = 0$

exponent > polynomial

eg2. $\lim_{x\to+\infty}\left(\frac{x!}{\ln x}\right) = \infty$

factorial > logarithmic.

## 2.5 Squeeze Theorem.

to find $\lim_{x \to a} f(x)$, if we can find two function such that:

$g(x) \leq f(x) \leq h(x)$, when $x \in ]m, n[$, then:

$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$, for $a \in ]m, n[$

☆ remember, $g(x)$ and $h(x)$ don't have to be always $g(x) \leq f(x) \leq h(x)$ for $x \in \mathbb{R}$, only need to be for $m \leq x \leq n$ where $m \leq a \leq n$

eg. $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right)$

$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$, $x \in \mathbb{R}$

$-x^2 \leq \sin\left(\frac{1}{x^2}\right) \leq x^2$, $x \in \mathbb{R}$

$\lim_{x \to 0} (x^2) = 0$

$\lim_{x \to 0} (-x^2) = 0$

we just need $g(0) \leq f(0) \leq h(0)$
but ofc if can more it's better.

$\therefore \lim_{x \to 0} \left(\sin\left(\frac{1}{x^2}\right)\right) = 0$ //

## 2.6 L'Hopital's Rule.

'$\frac{+\infty}{-\infty}$, $\frac{-\infty}{+\infty}$'

if $\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ or $\left(\frac{\pm\infty}{\pm\infty}\right)$, then.

$a \in \mathbb{R}$ (finite or infinite).

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a} \left(\frac{f'(x)}{g'(x)}\right)$$

this can be repeated a lot of time as long as $\frac{f'(x)}{g'(x)}$ after limit still $\left(\frac{0}{0}\right)$ or etc...

## ☆ 2.6 Lateral Limits.

$\lim_{x \to a^-} f(x) = L_1$ , $\lim_{x \to a^+} f(x) = L_2$

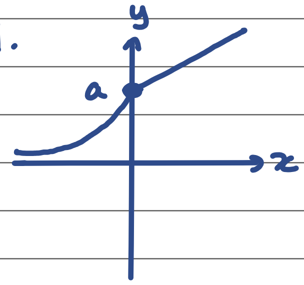
If $L_1 = L_2 = L$ :

$\lim_{x \to a} f(x) = L$ (limit exist)

**IMPORTANT!**

1. Limit exist when both lateral limit approaches same <u>finite</u> value

eg1.

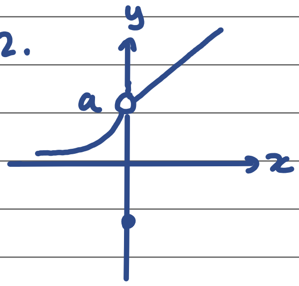



$\lim_{x \to 0^+} f(x) = a$

$\lim_{x \to 0^-} f(x) = a$

$\lim_{x \to 0} f(x) = a$

eg2.




$\lim_{x \to 0^+} f(x) = a$

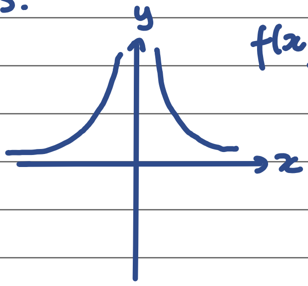
$\lim_{x \to 0^-} f(x) = a$

$\lim_{x \to 0} f(x) = a$ (limit EXIST! even though $f(0) \neq a$!)

limit can exist when :

$\lim_{x \to a} f(x) \neq f(a)$

eg3.




$f(x) = \frac{1}{x^2}$

$\lim_{x \to 0^-} f(x) = +\infty$

$\lim_{x \to 0^+} f(x) = +\infty$

BUT LIMIT DOES NOT EXIST !

once one of the lateral limit approaches infinity (+ or −) it's not finite so LIMIT DOESN'T EXIST!

## 2.7 Continuity.

just need to remember this!

$f(x)$ is continuous at $x=a$ if $\underline{\lim_{x\to a} f(x) = f(a)}$

this mean three things:

$f(x)$ is defined at $x=a$ (no vertical asymptote)

$\lim_{x\to a} f(x)$ exists and

$\lim_{x\to a} f(x) = f(a)$

## 2.8 Differentiation and Continuity. (more on c3)

$f(x)$ is said to be differentiable at $x=a$ if $f'(a)$ exists.

and $f'(a)$ exist only when:

$$f'(a^+) = \lim_{x\to a^+}\left(\frac{f(x)-f(a)}{x-a}\right)$$ is equal to

'right hand' derivative.

$$f'(a^-) = \lim_{x\to a^-}\left(\frac{f(x)-f(a)}{x-a}\right)$$

'left-hand' derivative.

if $f$ is differentiable at $x=a$, then it is continuous at $x=a$

if $f$ is not continuous at $x=a$, then it is not differentiable at $x=a$.

eg $f(x) = \begin{cases} x^2-1, & x<2 \\ 4x-5, & x\geq 2 \end{cases}$ assess whether continuous or differentiable at $x=2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(x^2-1) = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x-5) = 3$

$\lim_{x\to 2} f(x) = 3$

$f(2) = 4(2)-5 = 3$

$\therefore \lim_{x\to 2} f(x) = f(2)$, $f(x)$ is continuous at $x=2$

$$f'(2^-) = \lim_{x \to 2^-} \left( \frac{f(x)-f(2)}{x-2} \right) = \lim_{x \to 2^-} \left( \frac{x^2-1-3}{x-2} \right) = \lim_{x \to 2^-} \left( \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} \right)$$

$$= \lim_{x \to 2^-} (x+2) = 4$$

$$f'(2^+) = \lim_{x \to 2^+} \left( \frac{f(x)-f(2)}{x-2} \right) = \lim_{x \to 2^+} \left( \frac{4x-5-3}{x-2} \right) = \lim_{x \to 2^+} \left( \frac{4\cancel{(x-2)}}{\cancel{x-2}} \right) = 4$$

since $f'(2^-) = f'(2^+) = f'(2)$, $f(x)$ is differentiable and continuous at $x=2$ //

## 2.9 5 Theorem of Continuity (not really important)

**Theorem 1**

If $f$ and $g$ are continuous at $x = a$, then so are the following functions

$$f \pm g$$

$$f \cdot g$$

$$f/g \quad (except\ where\ g(x) = 0)$$

$$f^n$$

$$\sqrt[n]{f}\ (if\ n\ is\ even, f(x) \geq 0)$$

**Theorem 2**

If $g$ is continuous at $x = a$ and $f$ is continuous at $b = g(a)$, then $(f \circ g)$ is continuous at $x = a$.

$g(x)$ continuous at $x=a$ $g(a)$
$f(x)$ continuous at $x=g(a)$ $fg(a)$
$fg(x)$ continuous at $x=a$. $fg(a)$

**Theorem 3**

The following functions are continuous:

(a) constant function $(f(x) = c)$

(b) identity function $(f(x) = x)$

(c) polynomial functions $(f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0)$

(d) exponential functions $(f(x) = a^x, a > 0)$

(e) logarithmic functions $(f(x) = log_a(x))$

(f) trigonometric functions $(f(x) = sin(x), f(x) = cos(x), etc.)$

**Theorem 4 – Intermediate value theorem (also known as Bolzano's theorem or Bolzano-Cauchy's theorem)**

If $f$ is a real-valued continuous function on the interval $[a, b]$ and $u$ is a number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$, such that $f(c) = u$.

T4 indicates continuity for $x \in [a,b]$
useful for sign change rule
finding interval of roots.

**Theorem 5 – Extreme value theorem (also known as Weierstrass's theorem or Bolzano-Cauchy's theorem)**

If $f$ is a real-valued continuous function on the closed interval $[a, b]$, then f must attain its maximum $(M)$ and minimum $(m)$ values, each at least once. That is, there are numbers $c$ and $d$ in $[a, b]$ such that

$$f(c) \geq f(x) \geq f(d), \qquad for\ all\ x \in [a, b]$$

T5 says that between a close interval, a continuous function must have max and min of that interval.

# C3 Differentiation

## 3.1 Definition

The derivative of a function $y = f(x)$ at the point $x$ is defined as

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

known as first principle of derivative.

where $\Delta f$ is the change in $f(x)$ due to the change $\Delta x$ in $x$. Two possible notations are used without preference in this module for the derivative:

$$\textit{Leibniz's notation:} \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

$$\textit{Lagrange's notation:} \quad f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

## 3.2 Alternative expression for first principle.

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

, let $x = a + \Delta x$, where $x = a$ is the point where we want to find the derivative at:

$$\lim_{x-a \to 0} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \quad //$$

☆ why $a$ not $a^-$? cause there is no such thing as $f(a^-)$!

## 3.3 Lateral Derivative.

'Left-hand' Derivative $\quad f'(a^-) = \lim_{x \to a^-} \left( \frac{f(x)-f(a)}{x-a} \right)$ where $f(x)$ is to the left of $x = a$

'Right-hand' Derivative $\quad f'(a^+) = \lim_{x \to a^+} \left( \frac{f(x)-f(a)}{x-a} \right)$ where $f(x)$ is to the right of $x = a$

why $a$ not $a^+$? $x-a$ and $x-a^+$ are the same.

**REMEMBER!**

when finding $g'(a^+)$ or $g'(a^-)$, do not straight away find $g'(x)$ and substitute $x = a^+$ or $x = a^-$!

## 3·4 Differentiation Techniques.

### 3·4·1 Elementary Differentiation Rules.

$(a\cdot f(x))' = a\cdot f'(x)$

$(f(x) \pm g(x))' = f'(x) \pm g'(x)$

$(f(x)\cdot g(x))' = f(x)g'(x) + f'(x)g(x)$ (product rule)

• $(f(x)\cdot g(x)\cdot h(x))' = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x)$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ (quotient rule)

$[f[g(x)]]' = f'(g(x))\cdot g'(x)$ (chain rule)

### 3·4·2 Logarithmic Differentiation

for any kind of function that can be simplified through logarithm,
we can use logarithmic differentiation to make calculation more simple.

eg. $f = \frac{u\cdot v}{w}$

$\ln f = \ln\left(\frac{u\cdot v}{w}\right)$

$\ln f = \ln u + \ln v - \ln w$

$[\ln f]' = [\ln u]' + [\ln v]' - [\ln w]'$

$\frac{f'}{f} = \frac{u'}{u} + \frac{v'}{v} - \frac{w'}{w}$

$f' = f\cdot\left(\frac{u'}{u} + \frac{v'}{v} + \frac{w'}{w}\right)$

eg2. question marked with • at 3·4·1 can use logarithmic differentiation.

$\ln f = \ln(uvw)$
$\ln f = \ln u + \ln v + \ln w \ldots$

## 3.4.3 Lebniz's Rule

$$\left(f(x)\cdot g(x)\right)^{(n)} = \sum_{r=0}^{n} {}^nC_r \cdot f^{(n-r)}(x)\cdot g^{(r)}(x)$$

eg. $f(x) = x^4 \cdot a^x$, find $f'(x)$

let $u = x^4$ $\quad v = a^x$

$u' = 4x^3 \quad v' = \ln a \cdot a^x$

$u'' = 12x^2 \quad v'' = (\ln a)^2 \cdot a^x$

$u''' = 24x \quad v''' = (\ln a)^3 \cdot a^x$

$$\begin{array}{ccccccc} & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 \end{array}$$

$$f^{(3)}(x) = u'''v + 3u''v' + 3u'v'' + uv'''$$

$$= 24x(a^x) + 3(12x^2)\cdot \ln a \cdot a^x + 3(4x^3)\cdot(\ln a)^2 \cdot a^x + x^4 \cdot (\ln a)^3 \cdot a^x$$

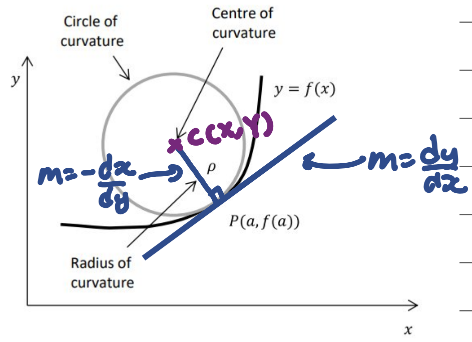
$$= xa^x\left[24 + 36x\ln a + 12x^2(\ln a)^2 + x^3(\ln a)^3\right]$$

## 3.5 Curvesketching. (more detail at 'Further Pure Mathematics 1')

| | |
|---|---|
| (a) | **Domain of $f$:** determine all $x$ for which $f(x)$ is defined. |
| (b) | **Continuity of $f$:** determine if $f$ is continuous in its domain and, if not, characterise the discontinuities. |
| (c) | **Intersection with axes:** solve $f(x) = 0$ to calculate the intersections with the $x$-axis and calculate $f(0)$ to determine the intersection with the $y$-axis. |
| (d) | **Symmetry:** determine $f(-x)$. If $f(-x) = f(x)$ the function is even and its graph is symmetric with respect to the $y$-axis. If $f(-x) = -f(x)$ the function is odd and its graph is symmetric with respect to the origin. |
| (e) | **Critical points and local extrema:** determine $f'(x)$ and calculate the critical points (i.e. either $f'(x) = 0$ or $f'(x)$ does not exist). Use the first derivative to search for local extrema. Use the sign of $f'(x)$ to determine intervals where the $f(x)$ is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$). Determine whether sharp points exist in the graph. |
| (f) | **Concavity and inflexion points:** determine $f''(x)$ and use the second derivative test when adequate. If $f''(x) > 0$ the graph is concave upward and if $f''(x) < 0$ the graph is concave downward. If $f$ is continuous at a point $x = c$ and if $f''(x)$ changes sign at $x = c$ then $P(c, f(c))$ is an inflexion point. |
| (g) | **Asymptotes:** If $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line defined by $y = L$ is a horizontal asymptote. If $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$, then the line defined by $x = a$ is a vertical asymptote. |

# 3·6 Applications.

## 3·6·1 Radius of Curvature. (and tangent & normal)




The radius of curvature of a curve at a point $P(a, f(a))$ is given by:

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$$ (will be given in formula booklet)

while the coordinates of the centre of curvature, $C(X, Y)$ are calculated using:

$$X = a - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}, \quad Y = f(a) + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

## 3·6·2 Increments.

$$f'(a) = \lim_{x \to a} \left(\frac{f(x)-f(a)}{x-a}\right)$$

when $x \to a$:

$$f'(a) \approx \frac{f(x)-f(a)}{x-a}$$ or can be written as : $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$

$$\Delta y \approx \frac{dy}{dx} \cdot \Delta x$$

$$\underbrace{f(x) = f(a) + f'(a)\cdot(x-a)}$$

also known as linear approximation.

new $y$ = old $y$ + gradient · change in $x$.

eg. estimate $\tan(33^\circ)$

$$\Delta y \approx \frac{dy}{dx} \times \Delta x \rightarrow \underbrace{\tan(33^\circ) - \tan(30^\circ)}^{y_{new} - y_{old}} = \sec^2 x \times \frac{3\pi}{180}.$$

let $y = \tan x$, $\Delta x = \frac{3\pi}{180}$

$\frac{dy}{dx} = \sec^2 x$

$$\tan(33^\circ) = \tan(30^\circ) + \sec^2(30) \times \frac{3\pi}{180}.$$

$$= \frac{\sqrt{3}}{3} + \frac{4}{3}\left(\frac{3\pi}{180}\right) \; //$$

## 3.7 Parametric Equation

### 3.7.1 Introduction

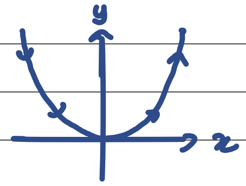
$x = f(t)$ , $y = g(t)$ , $t$ is known as parameter.

most simplest way to obtain a parametric eqn:

$x = t$ , $y = f(t)$ eg. $y = x^2 \rightarrow x = t$ , $y = t^2$

orientation of a parametrised curve (indicated by arrowhead) is the direction determined by increasing values of parameter.

eg. $x = t$ , $y = t^2$




$\Longrightarrow$ NOTE: that some parametric eqn only give part of the graph:

eg. $y = x^3 \rightarrow x = \sin t$ , $y = \sin^3 t$ ; $t \in \mathbb{R}$

$-1 \leq \sin t \leq 1$ for $t \in \mathbb{R}$

### 3.7.2 First and second derivative of a parametric eqn.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad , \text{ provided } \frac{dx}{dt} \neq 0$$

hence, $\frac{dy}{dt} = 0 \rightarrow \frac{dy}{dx} = 0 \rightarrow$ tangent parallel to $x$-axis.

$\frac{dx}{dt} = 0 \rightarrow \frac{dy}{dx} \rightarrow \infty \rightarrow$ tangent parallel to $y$-axis.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y') = \frac{\frac{d}{dt}(y')}{\frac{d}{dt}(x)} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

better idea. $\hookrightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$. (differentiate $t$ instead of $x$)

## 3.7.3 Area of parametric Equation / Definite Integral

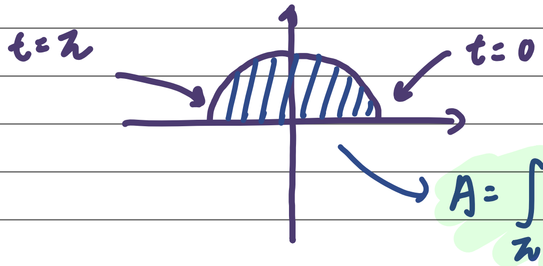
let $y=f(x)$ $\qquad$ $x=g(t)$, $y=h(t)$

$dx=g'(t)\,dt$

$$A=\int y\,dx$$

$$=\int_{\text{limit }1}^{\text{limit }2} h(t)\,g'(t)\,dt$$

Limit 2 must yield more 'right' than limit 1:




$A=\int_{\pi}^{0} h(t)g'(t)\,dt$ instead of $\int_{0}^{\pi} h(t)g'(t)\,dt$

# C4. Complex Number.

Complex number is split into two parts:
4.1 Basic , 4.2 Advanced.

## 4.1 Basic Complex Number.

### 4.1.1 Argand Diagram. (always fix the arg(z), ie $\theta$ to the correct quadrant!)

eg1. $z = -2\sqrt{2} - 2\sqrt{2}i$

$r = |z| = \sqrt{(-2\sqrt{2})^2 + (-2\sqrt{2})^2} = 4$

$\theta = \tan^{-1}\left(\frac{-2\sqrt{2}}{-2\sqrt{2}}\right) = \frac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$

$\theta = -\frac{3}{4}\pi + 2k\pi, \ k \in \mathbb{Z}$

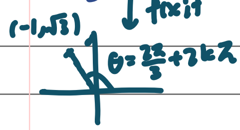
$(Q_3)$ ← it is always better to write final answer in principle value!

eg2. $z = -1 + \sqrt{3}i$ , find $\theta$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3} + k\pi$

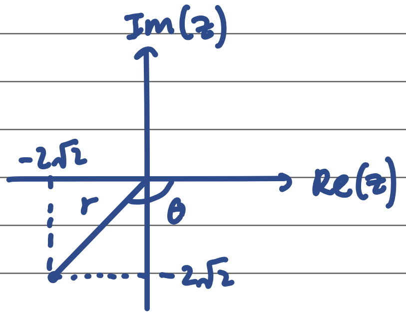
↓ fix it

$= \frac{2\pi}{3} + 2k\pi, \ k \in \mathbb{Z}$ #




\# Arg(z) is principle value where (different from arg(z), where arg(z) can be any angle)

$-\pi \le \text{Arg}(z) \le \pi$




### 4.1.2 Logarithm of a complex number.

$z = re^{i(\theta + 2k\pi)}, \ k \in \mathbb{Z}$

$\ln(z) = \ln(re^{i(\theta+2k\pi)})$
$\ln(z) = \ln(r) + \ln(e^{i(\theta+2k\pi)})$
$\ln(z) = \ln(r) + i(\theta + 2k\pi)\ln e$

$\therefore \ \ln(z) = \ln(r) + \underbrace{i(\theta + 2k\pi)}_{\text{this indicates there are infinite no. of solutions.}}, \ k \in \mathbb{Z}$

# 4·2 Advanced Complex Number

## 4·2·1 De Moivre's Theorem

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$-$ positive.

$(\cos\theta - i\sin\theta)^n = (\cos(-\theta) + i\sin(-\theta))^n$
$= \cos n(-\theta) + i\sin n(-\theta)$
$= \cos n\theta - i\sin n\theta$

## 4·2·2 Multiple to Power (binomial expansion and De Moivre's Theorem)

for $\cos n\theta$ or $\sin n\theta$

$(\cos\theta + i\sin\theta)^n = {}^nC_0 \times (\cos\theta)^n (i\sin\theta)^0 + {}^nC_1 \times (\cos\theta)^{n-1}(i\sin\theta)^1 + \ldots$

$\cos n\theta + i\sin n\theta = \ldots$

* compare Re, Im

## 4·2·3 Power to Multiple. (binomial and grouping)

$z + \frac{1}{z} = 2\cos\theta$
$z - \frac{1}{z} = 2i\sin\theta$ } start with these

$z^n + \frac{1}{z^n} = 2\cos n\theta$
$z^n - \frac{1}{z^n} = 2i\sin n\theta$ } use for reducing

where $z = \cos\theta + i\sin\theta = e^{i\theta}$

eg. $\sin^n\theta$,

$(z - \frac{1}{z})^n = z^n + {}^nC_1 z^{n-1}(-\frac{1}{z})^1 + {}^nC_2 z^{n-2}(-\frac{1}{z})^2$

$2i\sin^n\theta = z^n + \frac{1}{z^n} + \ldots$

$\sin^n\theta = \frac{1}{2i}[\ldots]$

## 4·2·4 Summation

$z = \cos\theta + i\sin\theta,\ \bar{z} = \cos\theta - i\sin\theta$
$z = e^{i\theta},\ \bar{z} = e^{-i\theta}$
$z = a + be^{i\theta},\ \bar{z} = a + be^{-i\theta}$
} to be use when reducing

to find summation of $\sin f(r)\theta$ or $\cos f(r)\theta$

consider: $\sum_{r=a}^{b} \cos f(r)\theta + i\sin f(r)\theta = \sum_{r=a}^{b} e^{if(r)\theta}$ ← expand with GP and reduce it.

compare Im or Re part in the end.

eg find $\sin 3\theta + \sin 5\theta + \sin 7\theta + \ldots$

$\mathrm{Im}\left\{\sum_{r=1}^{\infty} \cos(2r+1)\theta + i\sin(2r+1)\theta\right\}$

$= \mathrm{Im}\left\{\sum_{r=1}^{\infty} e^{(2r+1)\theta}\right\}$

$= \mathrm{Im}\left\{\frac{e^{3\theta}}{1-e^{2\theta}}\right\}$

## 4.2.5 Root of unity (or of a complex number)

$z^n = 1$ $\qquad 1 = re^{i\theta}$ , $r=1, \theta = 0^\circ$

$= 1e^{i(0+2\pi k)}$ , $k = 0,1,2,3,4,\dots$

$= e^{i(2\pi k)}$ , $k=0,1,2,3,4,\dots$

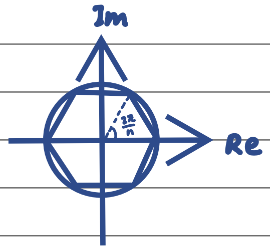
$z = e^{i\left(\frac{2\pi k}{n}\right)}$ , $k = \underbrace{0,1,2,3}_{n \text{ term only}}$

$z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$

$z^n = a+bi$

$= re^{i(\theta + 2\pi k)}$ , $k = 0,1,2,3,4,\dots$

$z = r^{\frac{1}{n}} e^{i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)}$ , $k = \underbrace{0,1,2,3,\dots}_{n \text{ term only.}}$




for unity, all roots lie on unit circle (for complex no. all roots lie on circle with modulus $r$)

for unity and complex roots, all roots have $\frac{2\pi}{n}$ rad. spacing between them. forming a regular polygon.

for $n^{th}$ root of unity extra formula:

$z^n = 1$

$z^n - 1 = 0$ , since $z=1$ is a known real solution...

$$\begin{array}{r|l} & z^{n-1} + z^{n-2} + \dots + z^3 + z^2 + z + 1 \\ z-1 & z^n - 1 \\ & \underline{z^n - z^{n-1}} \\ & z^{n-1} - 1 \\ & \underline{z^{n-1} - z^{n-2}} \\ & z^{n-2} \\ & \vdots \end{array}$$

$\therefore \quad (z-1)(1+z+z^2+z^3+\dots+z^{n-1}) = 0$

## 4.2.1 Multiple to Power (Binomial Expansion) (always come with polynomial)

Express $\cos 3\theta$ in term of $\cos\theta$

Express $\sin 3\theta$ in term of $\sin\theta$

Express $\tan 3\theta$ in term of $\tan\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(\cos\theta + i\sin\theta)^3 = {}^3C_0 \cos^3\theta + {}^3C_1 \cos^2\theta(i\sin\theta) + {}^3C_2 \cos\theta(i\sin\theta)^2 + {}^3C_3 (i\sin\theta)^3$

$\cos 3\theta + i\sin 3\theta = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \\ &= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

$$\begin{aligned}\tan 3\theta &= \frac{\sin 3\theta}{\cos 3\theta} \\ &= \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} \quad \begin{matrix}\div\cos^3\theta \\ \div\cos^3\theta\end{matrix} \\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

eg1. Express $\cos 4\theta$ in term of $\cos\theta$, hence expand $\sin 4\theta$ and $\tan 4\theta$

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + {}^4C_1\cos^3\theta(i\sin\theta) - {}^4C_2\cos^2\theta\sin^2\theta - {}^4C_3\cos\theta(i\sin^3\theta) + {}^4C_4\sin^4\theta$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta\sin\theta i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta i + \sin^4\theta$$

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2 \\ &= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta \\ &= 8\cos^4\theta - 8\cos^2\theta + 1\end{aligned}$$

$$\begin{aligned}\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\ &= 4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta)\end{aligned}$$

$$\begin{aligned}\tan 4\theta &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \quad \begin{matrix}\div\cos^4\theta \\ \div\cos^4\theta\end{matrix} \\ &= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}\end{aligned}$$

☆ Hence, solve $16x^4 - 16x^2 + 1 = 0$, leave answer in term of $\cos 4\theta$

$$16x^4 - 16x^2 + 1 = 0$$
$$8x^4 - 8x^2 + \frac{1}{2} = 0$$
$$8x^4 - 8x^2 + 1 = \frac{1}{2}$$

power 4 → 4 answer (distinct)

let $x = \cos\theta$

$\cos\frac{\pi}{12}$, $\cos\frac{5\pi}{12}$, $\cos\frac{7\pi}{12}$, $\cos\frac{11\pi}{12}$

$$8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$$
$$\cos 4\theta = \frac{1}{2}$$

$$\text{r.A.} = \frac{\pi}{3}$$

when check repetition, check with this

write more since need to check whether is it distinct.

$$\rightarrow 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}, \ldots$$

$$\theta = \underbrace{\frac{\pi}{12}, \frac{5\pi}{12}}_{Q_I}, \underbrace{\frac{7\pi}{12}, \frac{11\pi}{12}}_{Q_{II}}, \underbrace{\frac{13\pi}{12}, \frac{17\pi}{12}}_{Q_{III}}, \ldots$$

write till $Q_{IV}$ to be sure.

$$\underbrace{\cos 4\theta}_{\text{from DMT}} = k = \underbrace{8\cos^4\theta - 8\cos^2\theta + 1}_{\text{modified or original eqn.}}$$

→ get sets of solution for $\theta$ ; since both of LHS and RHS satisfy the same eqn. solution for $\theta$ from either side must satisfy each other

↓ infinite solution ↓ four distinct solution.

pick four distinct sol. →

continue from last qtn...

find the value of $\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} + \cos\frac{11\pi}{12} + \cos\frac{7\pi}{12}$

$SOR = -\frac{b}{a} = -\frac{0}{16} = 0$

Find the exact value of $\cos^2\frac{\pi}{12}$, and hence determine the exact value of $\cos\frac{\pi}{12}$

if determine exact value → quadratic eqn.

$16x^4 - 16x^2 + 1 = 0$

let $y = x^2$

$16y^2 - 16y + 1 = 0$

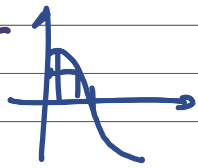
$y = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(16)(1)}}{2(16)}$

$= \frac{1}{2} \pm \frac{1}{4}\sqrt{3}$

since $\cos^2\frac{\pi}{12} = \cos^2\frac{11\pi}{12}$

$\cos^2\frac{5\pi}{12} = \cos^2\frac{7\pi}{12}$

$\cos\frac{\pi}{12} > \cos\frac{5\pi}{12}$



This bring to this

$\cos^2\frac{\pi}{12} = \frac{1}{2} + \frac{1}{4}\sqrt{3}$

$\cos\frac{\pi}{12} = \sqrt{\frac{1}{2} + \frac{1}{4}\sqrt{3}}$ , reject -ve since $\cos\frac{\pi}{12} > 0$

eg2. From DMT, $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$,
Obtain the roots for the following eqn, in the form of $\cos q\pi$

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0 \quad \div 2$$

we want to solve this ... → $64x^7 - 112x^5 + 56x^3 - 7x = -\frac{1}{2}$

... by substitution → let $x = \cos\theta$

the complex equation can be simplified → $64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta = -\frac{1}{2}$

... into this → $\cos 7\theta = -\frac{1}{2}$

$R.A. : \pi/3$

$7\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3},$

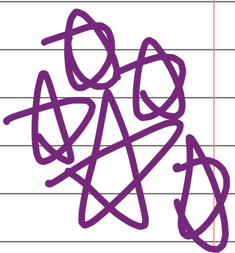
$\frac{20\pi}{3}, \frac{22\pi}{3}, \frac{26\pi}{3}, \frac{28\pi}{3}$

$\theta = \frac{2\pi}{21}, \frac{4\pi}{21}, \frac{8\pi}{21}, \frac{10\pi}{21}, \frac{14\pi}{21}, \frac{16\pi}{21},$

$\frac{20\pi}{21}, \frac{22\pi}{21}, \frac{26\pi}{21}, \frac{28\pi}{21}$

since $x = \cos\theta$,
check every $\theta$ with $\cos\theta$
to eliminate repetition.

$x = \cos\left(\frac{2\pi}{21}, \frac{4\pi}{21}, \frac{8\pi}{21}, \frac{10\pi}{21}, \frac{14\pi}{21}, \frac{16\pi}{21}, \frac{20\pi}{21}\right)$



IF $x = \operatorname{cosec}\theta$ check with $\sin\theta$
$x = \sec\theta$, check with $\cos\theta$
$x = \tan\theta$, check with $\tan\theta$

eg3. using the result of $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, show that the roots of the eqn.

$$8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0 \qquad (\cos 4\theta = -\cos 3\theta)$$

are $\cos\frac{1}{7}\pi$, $\cos\frac{3}{7}\pi$, $\cos\frac{5}{7}\pi$, $-1$ ; and hence deduce that $\cos\frac{1}{7}\pi + \cos\frac{3}{7}\pi + \cos\frac{5}{7}\pi = \frac{1}{2}$

$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$$

$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta\sin\theta\, i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin\theta\, i + \sin^4\theta$$

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

$$\cos 4\theta = -\cos 3\theta$$
$$8\cos^4\theta - 8\cos^2\theta + 1 = -4\cos^3\theta + 3\cos\theta$$
$$8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$$

$$8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0$$
let $c = \cos\theta$
$$8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$$

important.
$$\cos 4\theta + \cos 3\theta = 0$$
$$2\cos\left(\frac{4\theta+3\theta}{2}\right)\cos\left(\frac{4\theta-3\theta}{2}\right) = 0$$
$$2\cos\left(\frac{7}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right) = 0$$

$$\cos\left(\frac{7}{2}\theta\right) = 0 \qquad \cos\left(\frac{1}{2}\theta\right) = 0$$

$$\frac{7}{2}\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots \qquad \frac{1}{2}\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

$$\theta = \frac{1}{7}\pi, \frac{3}{7}\pi, \frac{5}{7}\pi, \pi, \ldots \qquad \theta = \pi, 3\pi, 5\pi, 7\pi \ldots$$

$$c = \cos\left(\frac{1}{7}\pi\right), \cos\left(\frac{3}{7}\pi\right), \cos\left(\frac{5}{7}\pi\right), \cos\pi$$

$$S.O.R = -\frac{b}{a}$$
$$\cos\left(\frac{1}{7}\pi\right) + \cos\left(\frac{3}{7}\pi\right) + \cos\left(\frac{5}{7}\pi\right) - 1 = -\frac{4}{8}$$

$$\cos\left(\frac{1}{7}\pi\right) + \cos\left(\frac{3}{7}\pi\right) + \cos\left(\frac{5}{7}\pi\right) = \frac{1}{2}$$

when finding exact value
→ quadratic formula.

eg 4. given $\cos 5\theta = \cos\theta(16\sin^4\theta - 12\sin^2\theta + 1)$

by considering $\cos 5\theta = 0$, show that the exact value of $\sin^2(\frac{1}{10}\pi)$ is $\frac{3-\sqrt{5}}{8}$

① we always solve from DMT since it is simple. →

$\cos 5\theta = 0$

p.v. $= \frac{\pi}{2}$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}$

② Check which ans fit? look at which eqn we solving. ie. $16x^4 - 12x^2 + 1 = 0$ →

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \cancel{\frac{5\pi}{10}}, \cancel{\frac{7\pi}{10}}, \cancel{\frac{9\pi}{10}}, \frac{11\pi}{10}, \frac{13\pi}{10}$

since it is for $\cos\theta = 0$

same as $\frac{3\pi}{10}, \frac{\pi}{10}$

$x = \sin\theta$

$x = \sin\frac{\pi}{10}, \sin\frac{3\pi}{10}, \sin\frac{11\pi}{10}, \sin\frac{13\pi}{10}$

③ (4 solution)

$16x^4 - 12x^2 + 1 = 0$

$0 = \cos\theta(16\sin^4\theta - 12\sin^2\theta + 1)$

$\cos\theta = 0$ , $16\sin^4\theta - 12\sin^2\theta + 1 = 0$

let $y = \sin^2\theta$

$16y^2 - 12y + 1 = 0$

$y = \frac{-(-12) \pm \sqrt{144 - 4(16)(1)}}{2(16)} = \frac{3}{8} \pm \frac{\sqrt{5}}{8}$

$\sin^2\theta = \frac{3}{8} \pm \frac{\sqrt{5}}{8}$

$\sin\theta = \pm\sqrt{\frac{3}{8} \pm \frac{\sqrt{5}}{8}}$

$\sin\frac{\pi}{10} = -\sin\frac{11\pi}{10}$ $\quad$ $\sin\frac{3\pi}{10} = -\sin\frac{13\pi}{10}$

$\sin^2\frac{\pi}{10} = \sin^2\frac{11\pi}{10}$ $\quad$ $\sin^2\frac{3\pi}{10} = \sin^2\frac{13\pi}{10}$

⑤ 2nd ±

choose +

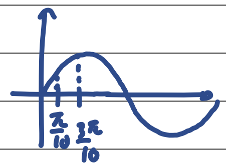
because $\sin\frac{\pi}{10} > 0$

$\sin\frac{11\pi}{10}$, its conjugate $< 0$

④ 1st ±

choose −

because: $\sin^2\frac{\pi}{10} < \sin^2\frac{13\pi}{10}$




$\sin\frac{\pi}{10} < \sin\frac{13\pi}{10}$

$\sin\frac{\pi}{10} = \sqrt{\frac{3-\sqrt{5}}{8}}$

## 4.2.2 Power to Multiple (always come with integration)

| | |
|---|---|
| $z = \cos\theta + i\sin\theta$ <br> $\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$ <br> $= \cos(-\theta) + i\sin(-\theta)$ <br> $= \cos\theta - i\sin\theta$ | $z^n = (\cos\theta + i\sin\theta)^n$ <br> $= \cos n\theta + i\sin n\theta$ <br> $\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n}$ <br> $= \cos(-n\theta) + i\sin(-n\theta)$ <br> $= \cos n\theta - i\sin n\theta$ |

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta) = 2i\sin n\theta$$

☆

$$z + \frac{1}{z} = 2\cos\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

eg1. $\sin^4\theta$

$$\left(z-\frac{1}{z}\right)^4 = {}^4C_0 z^4\left(-\frac{1}{z}\right)^0 + {}^4C_1 z^3\left(-\frac{1}{z}\right) + {}^4C_2 z^2\left(-\frac{1}{z}\right)^2 + {}^4C_3 (z)\left(-\frac{1}{z}\right)^3 + {}^4C_4 (z)^0\left(-\frac{1}{z}\right)^4$$

$$= z^4 - 4z^2 + 6 - 4\left(\frac{1}{z^2}\right) + \frac{1}{z^4}$$

$$= z^4 + \frac{1}{z^4} - 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$(2i\sin\theta)^4 = 2\cos 4\theta - 4(2\cos 2\theta) + 6$$

$$16i^4\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$$

$$\sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

eg2. $\cos^4\theta$

$$\left(z+\frac{1}{z}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$= z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

$$(2\cos\theta)^4 = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

eg3. $\sin^6\theta$

$$\left(z-\frac{1}{z}\right)^6 = z^6 + 6z^5\left(-\frac{1}{z}\right) + 15z^4\left(-\frac{1}{z}\right)^2 + 20z^3\left(-\frac{1}{z}\right)^3 + 15z^2\left(-\frac{1}{z}\right)^4 + 6z\left(-\frac{1}{z}\right)^5 + \left(-\frac{1}{z}\right)^6$$

$$(2i\sin\theta)^6 = z^6 + \frac{1}{z^6} - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$64i^4i^2\sin^6\theta = 2\cos6\theta - 6(2\cos4\theta) + 15(2\cos2\theta) - 20$$

$$-64\sin^6\theta = 2\cos6\theta - 12\cos4\theta + 30\cos2\theta - 20$$

$$\sin^6\theta = \frac{1}{32}\left(10 - 15\cos2\theta + 6\cos4\theta - \cos6\theta\right)$$

**3.** Expand $\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2$ and, by substituting $z=\cos\theta+\mathrm{i}\sin\theta$, find integers $p, q, r, s$ such that

$$64\sin^2\theta\cos^4\theta = p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta. \qquad [6]$$

Using the substitution $x = 2\cos\theta$, show that

$$\int_1^2 x^4\sqrt{(4-x^2)}\,\mathrm{d}x = \tfrac{4}{3}\pi + \sqrt{3}. \qquad [4]$$

Ans : p = 4, q = 2, r = -4, s = -2

$$\left(z+\frac{1}{z}\right)^4\left(z-\frac{1}{z}\right)^2 = \left(z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}\right)\left(z^2-2+\frac{1}{z^2}\right)$$

$$= z^6-2z^4+z^2+4z^4-8z^2+4+6z^2-12+\frac{6}{z^2}+4-\frac{8}{z^2}+\frac{4}{z^4}$$

$$-\frac{1}{z^2}-\frac{2}{z^4}+\frac{1}{z^6}$$

$$(2\cos\theta)^4\times(2i\sin\theta)^2 = \left(z^6+\frac{1}{z^6}\right)+2\left(z^4+\frac{1}{z^4}\right)-\left(z^2+\frac{1}{z^2}\right)-4$$

$$16\cos^4\theta(4i^2\sin^2\theta) = 2\cos 6\theta + 2(2\cos 4\theta) - 2\cos 2\theta - 4$$

$$-64\sin^2\theta\cos^4\theta = 2\cos 6\theta + 4\cos 4\theta - 2\cos 2\theta - 4$$

$$64\sin^2\theta\cos^4\theta = 4 + 2\cos 2\theta - 4\cos 4\theta - 2\cos 6\theta$$

$$\int_1^2 x^4\sqrt{4-x^2}\,dx$$

$x = 2\cos\theta$, $dx = -2\sin\theta\, d\theta$

$x = 2$, $2\cos\theta = 2$, $\cos\theta = 1$, $\theta = 0$

$x = 1$, $2\cos\theta = 1$, $\cos\theta = 1/2$, $\theta = \pi/3$

$$= \int_{\pi/3}^{0} (2\cos\theta)^4\sqrt{4-4\cos^2\theta}\,(-2\sin\theta\, d\theta)$$

$$= \int_{\pi/3}^{0} -64\cos^4\theta\sin^2\theta\, d\theta$$

$$= \int_{\pi/3}^{0} (-4-2\cos 2\theta+4\cos 4\theta+2\cos 6\theta)\, d\theta$$

$$= \left[-4\theta - \frac{2\sin 2\theta}{2} + \frac{4\sin 4\theta}{4} + \frac{2\sin 6\theta}{6}\right]_{\pi/3}^{0}$$

$$= 0 - \left[-\frac{4\pi}{3} - \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} + \frac{1}{3}\sin\frac{6\pi}{3}\right]$$

$$= 0 - \left[-\frac{4\pi}{3} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + 0\right]$$

$$= \frac{4\pi}{3} + \sqrt{3}$$

## p4.2.3 Conjugate in exponential form.

$$z = a + be^{i\theta}$$
$$= a + b(\cos\theta + i\sin\theta)$$
$$= a + b\cos\theta + ib\sin\theta$$

$$\bar{z} = a + b\cos\theta - i\sin\theta$$
$$= a + b(\cos\theta - i\sin\theta)$$
$$= a + b(\cos(-\theta) + i\sin(-\theta))$$
$$= a + be^{-i\theta}$$

eg. $z = 2 - e^{-3i\theta}$, $\bar{z} = 2 - e^{3i\theta}$

$z = 4 + e^{i\frac{n\theta}{2}}$, $\bar{z} = 4 + e^{-i\frac{n\theta}{2}}$

## p4.2.3(2) Reducing

eg1.
$$\frac{2 - e^{i\theta}}{1 + e^{i\theta}}$$
$$= \frac{(2 - e^{i\theta})(1 + e^{-i\theta})}{(1 + e^{i\theta})(1 + e^{-i\theta})}$$
$$= \frac{2 + 2e^{-i\theta} - e^{i\theta} - e^{0}}{1 + e^{-i\theta} + e^{i\theta} + e^{0}}$$
$$= \frac{2 + 2(\cos\theta - i\sin\theta) - (\cos\theta + i\sin\theta) - 1}{1 + \cos\theta - i\sin\theta + \cos\theta + i\sin\theta + 1}$$
$$= \frac{\cos\theta - 3i\sin\theta + 1}{2 + 2\cos\theta}$$
$$= \frac{\cos\theta + 1}{2 + 2\cos\theta} - \frac{3\sin\theta}{2 + 2\cos\theta} i$$

eg2.
$$\frac{2e^{i\theta} + 7}{3 - e^{i\theta}} \times \frac{3 - e^{-i\theta}}{3 - e^{-i\theta}}$$
$$= \frac{6e^{i\theta} - 2e^{0} + 21 - 7e^{-i\theta}}{9 - 3e^{-i\theta} - 3e^{i\theta} + e^{0}}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$= \frac{6e^{i\theta} - 7e^{-i\theta} + 19}{-3e^{i\theta} - 3e^{-i\theta} + 10}$$

$- i\sin\theta$

$$= \frac{6(\cos\theta + i\sin\theta) - 7(\cos(-\theta) + i\sin(-\theta)) + 19}{-3(\cos\theta + i\sin\theta) - 3(\cos(-\theta) + i\sin(-\theta)) + 10}$$
$$= \frac{6\cos\theta + 6i\sin\theta - 7\cos\theta + 7i\sin\theta + 19}{-3\cos\theta - \cancel{3i\sin\theta} - 3\cos\theta + \cancel{3i\sin\theta} + 10}$$
$$= \frac{-\cos\theta + 13i\sin\theta + 19}{-6\cos\theta + 10}$$
$$= \frac{19 - \cos\theta}{10 - 6\cos\theta} + \frac{13\sin\theta}{10 - 6\cos\theta} i$$

#

Alternative way for same coeff. (reducing)

$$\frac{2-e^{i\theta}}{1+e^{i\theta}} = \frac{(2-\cos\theta - i\sin\theta)(1+\cos\theta - i\sin\theta)}{(1+\cos\theta + i\sin\theta)(1+\cos\theta - i\sin\theta)}$$

same coeff.

$$= \frac{2-e^{i\theta}}{e^{\frac{i\theta}{2}}[e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}]}$$

$$= \frac{e^{-\frac{i\theta}{2}}(2-e^{i\theta})}{e^{-\frac{i\theta}{2}}+e^{\frac{i\theta}{2}}}$$

$$= \frac{2e^{-\frac{i\theta}{2}}-e^{\frac{i\theta}{2}}}{\cos\left(-\frac{\theta}{2}\right)+i\sin\left(-\frac{\theta}{2}\right)+\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}$$

$$= \frac{2\left[\cos\left(-\frac{\theta}{2}\right)+i\sin\left(-\frac{\theta}{2}\right)\right]-\left[\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right]}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}$$

$$= \frac{2\cos\frac{\theta}{2} - 2i\sin\frac{\theta}{2}-\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

### 4.2.3 Summation.

eg1. find $\sin\theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n+1)\theta$

$$\sum_{r=0}^{n} \sin(2r+1)\theta$$

$$\sum_{r=0}^{n} \left[\cos(2r+1)\theta + i\sin(2r+1)\theta\right]$$

$$= \sum_{r=0}^{n} e^{i(2r+1)\theta}$$

$$= e^{i\theta} + e^{i3\theta} + e^{i5\theta} + \dots + e^{i(2n+1)\theta}$$

gp: $a = e^{i\theta}$

$r: \frac{e^{i3\theta}}{e^{i\theta}} = e^{i2\theta}$

$r: n-0+1 = n+1$

$$S_{n+1} = \frac{a(1-r^{n+1})}{1-r}$$

$$= \frac{e^{i\theta}(1-e^{i2\theta(n+1)})}{1-e^{i2\theta}}$$

$$= \frac{\cancel{e^{i\theta}}(1-e^{i2(n+1)\theta})}{\cancel{e^{i\theta}}(e^{-i\theta}-e^{i\theta})}$$

$$= \frac{1-\cos 2(n+1)\theta - i\sin 2(n+1)\theta}{\cos\theta - i\sin\theta - \cos\theta - i\sin\theta} \quad \begin{matrix}\times i\\ \times i\end{matrix}$$

$$= \frac{(1-\cos 2(n+1)\theta)i - i^2\sin 2(n+1)\theta}{-2i^2\sin\theta}$$

$$= \frac{1-\cos 2(n+1)\theta}{2\sin\theta}i + \frac{\sin 2(n+1)\theta}{2\sin\theta}$$

$$\sum_{r=0}^{n} \sin(2r+1)\theta = \operatorname{Im}\left\{\sum \cos(2r+1)\theta + i\sin(2r+1)\theta\right\}$$

$$= \operatorname{Im}\left\{\frac{1-\cos 2(n+1)\theta}{2\sin\theta}i + \frac{\sin 2(n+1)\theta}{2\sin\theta}\right\}$$

$$= \frac{1-\cos(2n+1)\theta}{2\sin\theta}$$

**1.** By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = e^{i\theta}$, show that

$$\sum_{n=1}^{N} \cos(2n-1)\theta = \frac{\sin(2N\theta)}{2\sin\theta},$$

where $\sin\theta \neq 0$. [6]

Deduce that

$$\sum_{n=1}^{N} (2n-1)\sin\left[\frac{(2n-1)\pi}{N}\right] = -N\operatorname{cosec}\frac{\pi}{N}.$$ [4]

$$\sum_{n=1}^{N} \cos(2n-1)\theta + i\sin(2n-1)\theta$$

$$= \sum_{n=1}^{N} e^{i(2n-1)\theta}$$

$$= e^{i\theta} + e^{i3\theta} + e^{i5\theta} + \dots + e^{i(2N-1)\theta}$$

$$S_N = \frac{e^{i\theta}(1 - e^{i2\theta(N)})}{1 - e^{i2\theta}} \quad \text{or} \quad \times \frac{1 - e^{-i2\theta}}{1 - e^{-i2\theta}}$$

$$= \frac{\cancel{e^{i\theta}}(1 - e^{i2\theta(N)})}{\cancel{e^{i\theta}}(e^{-i\theta} - e^{i\theta})}$$

$$= \dots$$

$$= \frac{(1 - \cos 2N\theta)}{2\sin\theta} i + \frac{\sin 2N\theta}{2\sin\theta}$$

$$\therefore \sum_{n=1}^{N} \cos(2n-1)\theta = \operatorname{Re}\left\{ \sum \right.$$

## 4.2.4 $n^{th}$ roots of unity or $n^{th}$ roots of complex number.

$\sqrt[3]{1}$

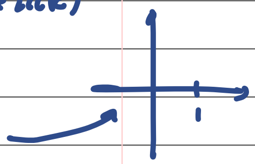
let $z=\sqrt[3]{1}$

$z^3=1$ $\qquad re^{i\theta}$

$r=1,\ \theta=0, 2\pi, 4\pi, 6\pi, \ldots$
$= 2\pi k$

$z^3=\underbrace{1}_{}=1e^{i(0+2\pi k)}$

$r=1,\ \theta=0$



$z^3=1e^{i(2\pi k)}$

$z=e^{i\frac{2\pi k}{3}}$

$k=0,1,2,3,\ldots$
$k=0,-1,-2,-3,\ldots$
$k=0,\pm1,\pm2,\pm3,\ldots$

exp. form $\quad z=e^0,\ e^{\frac{2\pi}{3}i},\ e^{\frac{4\pi}{3}i}$

rcis form. $\quad =1,\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$

normal form. $\quad =1,\ -\frac{1}{2}+i\frac{\sqrt{3}}{2},\ -\frac{1}{2}-i\frac{\sqrt{3}}{2}$

$z^3-1=0$

$1+e^{\frac{2\pi}{3}i}+e^{\frac{4\pi}{3}i}=$ s.o.r $=-\frac{b}{a}$
$=\frac{0}{1}$
$=0$

$1(e^{\frac{2\pi}{3}i})(e^{\frac{4\pi}{3}i})=$ p.o.r. $=-\frac{d}{a}$
$=\frac{-(-1)}{1}$
$=1$

eg 1. $\quad \sqrt[5]{1}$

$z=\sqrt[5]{1}$

$z^5=1$

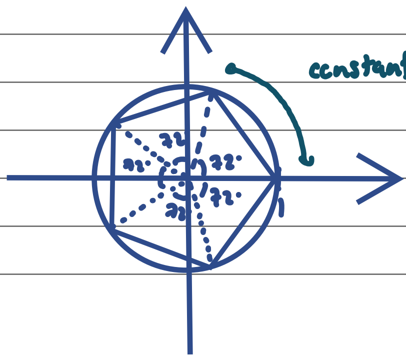
$z^5=re^{i\theta}$

$=1e^{i(0+2\pi k)}$ $\qquad$ where $k=0,1,2,3,4,5,\ldots$

radius of graph lies on circle with $r=1$

$=e^{i2\pi k}$ $\qquad$ where $k=0,1,2,3,4,5,\ldots$

$z=e^{i\frac{2\pi k}{5}}$ $\qquad$ where $k=0,1,2,3,4$ ; or $k=1,2,3,4,5$ ; or $k=0,\pm1,\pm2$




constant spacing since $\theta+\underbrace{2\pi k}_{\text{constant addition of } 2\pi}$

causing a polygon.

**4.** **(a)** Express in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$:

**(i)** $4(1 + i\sqrt{3})$; **(ii)** $4(1 - i\sqrt{3})$.

**(b)** The complex number $z$ satisfies the equation

$$(z^3 - 4)^2 = -48$$

Show that $z^3 = 4 \pm 4\sqrt{3}\,i$.

**(c)** **(i)** Solve the equation

$$(z^3 - 4)^2 = -48$$

giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$.

**(ii)** Illustrate the roots on an Argand diagram.

**(d)** **(i)** Explain why the sum of the roots of the equation

$$(z^3 - 4)^2 = -48$$

is zero.

**(ii)** Deduce that $\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}$.

(a) (i) $4+4\sqrt{3}i$

$r = \sqrt{4^2+(4\sqrt{3})^2}$

$= 8$

$\theta = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right) = \frac{\pi}{3}$

$8e^{i\frac{\pi}{3}}$

(ii) $r = \sqrt{4^2+(-4\sqrt{3})^2}$

$= 8$

$\theta = -\frac{\pi}{3}$

(b) $z^3 - 4 = \pm\sqrt{-48}$

$z^3 = 4 \pm 4\sqrt{3}i$

(c) (i) $(z^3-4)^2 = -48$

$z^3 = 4 \pm 4\sqrt{3}i$

$z^3 = 8e^{i(\frac{\pi}{3}+2\pi k)}$, $k = 0, \pm1, \pm2, \dots$ $\quad$ $z^3 = 8e^{i(-\frac{\pi}{3}+2\pi k)}$

$z = 2e^{i(\frac{\pi}{9}+\frac{2\pi k}{3})}$, $k = 0, \pm1$ $\quad$ $z = 2e^{i(-\frac{\pi}{9}+\frac{2\pi k}{3})}$

$z = \underbrace{2e^{i\frac{\pi}{9}},\ 2e^{i\frac{7\pi}{9}},\ 2e^{i(-\frac{5\pi}{9})}}_{\text{This three form regular polygon}},\ \underbrace{2e^{i(-\frac{\pi}{9})},\ 2e^{i\frac{5\pi}{9}},\ 2e^{i(-\frac{7\pi}{9})}}_{\text{This three too.}}$

but this six together aren't regular hexagon.

eg4. $\sqrt[3]{\sqrt{3}-\sqrt{3}i}$

fixed.

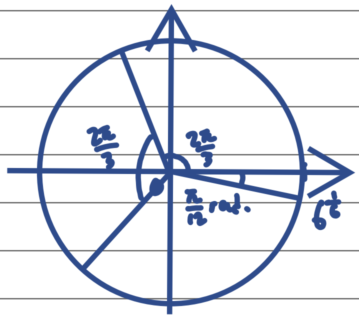
$z^3 = \sqrt{3}-\sqrt{3}i$ $\qquad$ $r = \text{sqrt}\left(\sqrt{3}^2+\sqrt{3}^2\right) = \sqrt{6}$ $\qquad$ $\theta = -\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = -\frac{\pi}{4}$

$= re^{i\theta}$

$= \sqrt{6}e^{i\left(-\frac{\pi}{4}+2\pi k\right)}$ $\qquad$ $k = 0, 1, 2, 3, 4, 5, \ldots$

$z = \left(6^{\frac{1}{2}}\right)^{\frac{1}{3}} e^{i\left(-\frac{\pi}{12}+\frac{2\pi k}{3}\right)}$

$= 6^{\frac{1}{6}} e^{i\left(-\frac{\pi}{12}+\frac{2\pi k}{3}\right)}$ $\qquad$ where $k = 0, 1, 2$



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**(i)** Write down, in any form, all the complex roots of the equation

$$w^{12} = 1. \quad [2]$$

**(ii)** Explain why the equation

$$(z+2)^{12} = z^{12} \quad (*)$$

has exactly 10 non-real roots and show that they may be expressed in the form

$$-1 - \mathrm{i}\cot(\tfrac{1}{12}k\pi),$$

where $k = \pm1, \pm2, \pm3, \pm4, \pm5$. [6]

**(iii)** Show that

$$\{-1 - \mathrm{i}\cot(\tfrac{1}{12}k\pi)\}\{-1 + \mathrm{i}\cot(\tfrac{1}{12}k\pi)\} = \operatorname{cosec}^2(\tfrac{1}{12}k\pi). \quad [1]$$

$(a-b)\quad(a+b) = a^2-b^2 = (-1)^2 - [i\cot(\frac{k\pi}{12})]^2 = 1 - i^2\cot^2\frac{k\pi}{12} = 1+\cot^2\frac{1}{12}k\pi$
$= \operatorname{cosec}^2\frac{1}{12}k\pi$

P.O.R $= -\frac{b}{a}, \frac{c}{a}, -\frac{d}{a}, \ldots \frac{j}{a}; \frac{c}{a}$

**(iv)** Given that the product of the roots of (*) is $-\frac{512}{3}$, find the value of

$$\sin^2(\tfrac{1}{12}\pi)\sin^2(\tfrac{2}{12}\pi)\sin^2(\tfrac{3}{12}\pi)\sin^2(\tfrac{4}{12}\pi)\sin^2(\tfrac{5}{12}\pi). \quad [2]$$

(i) $w^{12} = e^{i2\pi k}$, $k = 0, \pm1, \pm2, \pm3, \pm4, \ldots$

$w = e^{i\frac{\pi k}{6}}$, $k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, 6$

(ii) $(z+2)^{12} = z^{12}$

$\cancel{z^{12}} + 12z^{11}(2) + {}^{12}C_2 z^{10}(2)^2 + \ldots = \cancel{z^{12}}$

$44z^{11} + 264z^{10} + \ldots = 0$

polynomial of degree 11 have 11 roots.
Since all the coefficient of the eqn are real ...
Complex roots occur in conjugate pairs.

$(z+2)^{12} = z^{12}$

$\frac{(z+2)^{12}}{z^{12}} = 1$

$\left(\frac{z+2}{z}\right)^{12} = 1$

$\frac{z+2}{z} = e^{\frac{i\pi k}{6}}$, where $k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, 6$

$z+2 = ze^{\frac{i\pi k}{6}}$

$z(e^{\frac{i\pi k}{6}} - 1) = 2$

$z = \frac{2}{e^{\frac{i\pi k}{6}} - 1} = \frac{2}{e^{\frac{i\pi k}{12}}(e^{\frac{i\pi k}{12}} - e^{-\frac{i\pi k}{12}})}$

(iv) $-1 - i\cot(\frac{\pi}{12}), -1 - i\cot(-\frac{\pi}{12}) = -1 + i\cot(\frac{\pi}{12})$

$(-1)(-1 - i\cot\frac{\pi}{12})(-1 - i\cot(-\frac{\pi}{12}))$
$(-1 - i\cot(\frac{2\pi}{12}))(-1 - i\cot(-\frac{2\pi}{12}))\ldots$
$(-1 - i\cot(\frac{5\pi}{12}))(-1 - i\cot(-\frac{5\pi}{12}))$

$= (-1)(-1 - i\cot\frac{\pi}{12})(-1 + i\cot\frac{\pi}{12})(-1 - i\cot\frac{2\pi}{12})$
$(-1 + i\cot\frac{2\pi}{12})\ldots(-1 - i\cot\frac{5\pi}{12})(-1 + i\cot\frac{5\pi}{12})$

$-\frac{512}{3} = -\operatorname{cosec}^2\frac{\pi}{12}\operatorname{cosec}^2\frac{2\pi}{12}\operatorname{cosec}^2\frac{3\pi}{12}\ldots\operatorname{cosec}^2\frac{5\pi}{12}$

$= \frac{1}{\sin^2\frac{\pi}{12}}\left(\frac{1}{\sin^2\frac{2\pi}{12}}\right)\ldots\left(\frac{1}{\sin^2\frac{5\pi}{12}}\right)$

$\sin^2\frac{\pi}{12}\sin^2\frac{2\pi}{12}\ldots\sin^2\frac{5\pi}{12} = \frac{3}{512}$

$$= \frac{2e^{-i\frac{\pi k}{12}}}{\cancel{\cos\frac{\pi k}{12}} + i\sin\frac{\pi k}{12} - \left(\cancel{\cos\frac{\pi k}{12}} - i\sin\frac{\pi k}{12}\right)}$$

$$= \frac{\cancel{2}\left(\cos\frac{\pi k}{12} - i\sin\frac{\pi k}{12}\right) \times i}{\cancel{2}i\sin\frac{\pi k}{12} \quad \times i}$$

$$= \frac{\cos\frac{\pi k}{12}i + \sin\frac{\pi k}{12}}{-\sin\frac{\pi k}{12}}$$

$$= -\cot\frac{\pi k}{12}i - 1$$

$$= -1 - \cot\frac{\pi}{12}k$$

when $k=0$, $\cot\frac{\pi}{12}k \to \infty$ (reject)

when $k=6$, $\cot\frac{\pi}{12}k = \cot\frac{\pi}{2} = 0$ $\quad z = -1-i0 = -1$ (real roots)

$(z+2)^{12} = z^{12}$ has one real roots and 10 non-real roots

which is $-1-i\cot\frac{k\pi}{12}$, $k = \pm1, \pm2, \pm3, \pm4, \pm5$

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**(i)** Write down the five fifth roots of unity. [2]

**(ii)** Hence find all the roots of the equation

$$z^5 + 16 + (16\sqrt{3})\mathrm{i} = 0,$$

giving answers in the form $re^{\mathrm{i}q\pi}$, where $r > 0$ and $q$ is a rational number. Show these roots on an Argand diagram. [4]

Let $w$ be a root of the equation in part **(ii)**.

**(iii)** Show that

$$\sum_{k=0}^{4}\left(\frac{w}{2}\right)^k = \frac{3+\mathrm{i}\sqrt{3}}{2-w}.$$ [3]

**(iv)** Identify the root for which $|2-w|$ is least. [2]

(i) $z^5 = 1$

$z^5 = e^{i2\pi k}$ , $k = 0, \pm1, \pm2, \pm3, \dots$

$z = e^{i\frac{2}{5}\pi k}$ , $k = 0, \pm1, \pm2$

(ii) $z^5 = -16 - 16\sqrt{3}i$

do this only if qtn relate roots of complex no. to roots of unity

$= (-16-16\sqrt{3}i)(1)$

$= \left(32e^{i\left(-\frac{2\pi}{3}\right)}\right)\left(e^{i2\pi k}\right)$ , $k = 0, \pm1, \pm2, \pm3, \dots$

$= \sqrt[5]{32}\, e^{-i\frac{2\pi}{15}} e^{i\frac{2\pi k}{5}}$ , $k = 0, \pm1, \pm2$

$= 2\left(e^{-i\frac{2\pi}{15} + \frac{6\pi k}{15}}\right)$ , $k = 0, \pm1, \pm2$

list it out

$r = \sqrt{(-16)^2 + (-16\sqrt{3})^2} = 32$

$\arg(-16-16\sqrt{3}i) = -\left(\pi - \tan^{-1}\left(\frac{16\sqrt{3}}{16}\right)\right)$

$= -\frac{2\pi}{3}$

(iii) $\sum_{k=0}^{4}\left(\frac{w}{2}\right)^k = \left(\frac{w}{2}\right)^0 + \left(\frac{w}{2}\right)^1 + \left(\frac{w}{2}\right)^2 + \left(\frac{w}{2}\right)^3 + \left(\frac{w}{2}\right)^4$

$= 1 + \frac{w}{2} + \frac{w^2}{2^2} + \frac{w^3}{2^3} + \frac{w^4}{2^4}$

$= \dfrac{1\left(1 - \frac{w}{2}^5\right)}{1 - \frac{w}{2}}$

since $w$ is the root of $z^5 + 16 + i16\sqrt{3} = 0$

$w^5 = -16 - i16\sqrt{3}$

$= \dfrac{1 - \frac{w^5}{32}}{1 - \frac{w}{2}}$ $\times 2$ / $\times 2$

$= \dfrac{2 - \frac{w^5}{16}}{2 - w}$

$= \dfrac{2 - \frac{(-16 - i16\sqrt{3})}{16}}{2 - w} = \dfrac{2 + 1 + i\sqrt{3}}{2 - w}$

# C5. Sequences, Series and Power Series

## 5.1 Definition.

$a_n$ is a sequence.
$S_n$ is sequence of partial sum.
$S_\infty$ is series / infinite series (exact same thing)

## 5.2 Sequence.

$a_n$ is a sequence.
$a_1$ is the first term, $a_2$ is the second term and so on...

sequence CONVERGES when : $\lim\limits_{n\to\infty} a_n = L$

and DIVERGES when : $\lim\limits_{n\to\infty} a_n = \pm\infty$ or Does Not Exist

* when doing limit in sequence (or series)

$\lim\limits_{n\to\infty} a_n = \ldots$ is alright, but when you can't do it directly and require differentiation (like using L'Hopital's Rule) or integration, we need to convert $a_n$ into a continuous function $f(x)$

$$\boxed{\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) \quad , \text{ where } a_n = f(n)}$$

E1. $a_n = \frac{5n}{e^{2n}}$

$$\lim_{n\to\infty} \frac{5n}{e^{2n}} = \lim_{x\to\infty} \frac{5x}{e^{2x}} = \lim_{x\to\infty} \left(\frac{5}{2e^{2x}}\right) = 0$$

→ $a_n$ is convergent with limit 0

5.3 Divergence Test, Both Positive Term Series and Alternating Series.
(To test a series is conv. or dive. we usually start with a quick & short divergence test)

$\sum_{n=1}^{\infty} a_n$ : if $\lim_{n\to\infty} a_n \neq 0$, the series is divergent.

$\lim_{n\to\infty} a_n = 0$ DOESN'T MEAN IT IS CONFIRMED CONVERGENT!
(we need to do more test!)

5.4 Convergence Test, Positive Term Series.

1. Geometric Series.

$$\sum_{n=1}^{\infty} ar^{n-1}$$

converges, with sum $S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$, IF $|r|<1$
diverges IF $|r| \geq 1$

2. p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges, if $p>1$
diverges, if $p \leq 1$

3. Comparison Test

if we have $\sum_{n=1}^{\infty} a_n$, and we can figure out a similar series where
$\sum_{n=1}^{\infty} b_n$, and $b_n \geq a_n > 0$ (positive term series only)
$\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ (bigger than $a_n$ one) converges.

or if $b_n \leq a_n$,

$\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ (smaller than $a_n$ one) diverges.

4. Limit comparison test

if we have $\sum_{n=1}^{\infty} a_n$ and we figure out a similar series where $\sum_{n=1}^{\infty} b_n$, ($a_n$, $b_n$ which is bigger / smaller not important)

and if $a_n > 0$, $b_n > 0$ (positive term series only),

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is positive and finite.

then either both series converge or diverge.

5. Ratio test.

(positive term series only!)

$\sum_{n=1}^{\infty} a_n$

$-\infty$ is not possible since $a_n$, $\underline{n \in \mathbb{N}}$

if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ (or $\infty$), then the series

converges ABSOLUTELY, if $L < 1$
diverges, if $L > 1$ (inc. $\infty$)
inconclusive, if $L = 1$

6. Root test (useful for series with power n)

(positive term series only!)

$\sum_{n=1}^{\infty} a_n$

$-\infty$ is not possible since $a_n$, $\underline{n \in \mathbb{N}}$

if $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ (or $\infty$)

converges, if $L < 1$
diverges, if $L > 1$
inconclusive, if $L = 1$

## 7. Integral Test

$\sum_{n=1}^{\infty} a_n$ , $f(n) = a_n$ :

if $\int_1^{+\infty} f(x)\,dx$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

if $\int_1^{+\infty} f(x)\,dx$ diverges, $\sum_{n=1}^{\infty} a_n$ diverges.

## 5.5 Convergence Test, Alternating Series.

$\sum_{n=1}^{\infty} (-1)^n a_n$ , $a_n > 0$

$\bigstar$ alt. series can converge when its original series diverge, but original series cannot converge if its alt. series diverge

converge, if (i) $\lim_{n \to \infty} a_n = 0$

(ii) $\underline{a_k > a_{k+1}}$ ( $> 0$ ) positive term series.

should always check what kind of convergence.

and is...

absolutely convergent if $\sum_{n=1}^{\infty} a_n$ is convergent.

or

conditionally convergent if $\sum_{n=1}^{\infty} a_n$ is divergent.

[ check $a_k > a_{k+1}$ ]

$\bigstar$ always check $a_n$ diverge or not, then check whether $\sum_{n=1}^{\infty} (-1)^n a^n$ is converging or not, then check $\sum_{n=1}^{\infty} a^n$ with all the test whether is it absolutely convergent.

## 5.6 Power Series.

☆ power series always have radius of convergence!

$\sum_{x=1}^{\infty} a_n(x-c)^n$ if this is a power series,

then one of the following will be true:

(i) the series only converges when $x=c$ (diverge everywhere, except $x=c$)
(ii) the series ABSOLUTELY CONVERGE for every $x$ (converge everywhere)
(iii) ABSOLUTELY CONVERGE when $|x-c| < R$
diverge when $|x-c| > R$

☆ need to check for $x=c+R$ and $x=c-R$ seperately.
usually $\sum a_n(c\pm R)^n$ is one positive term series and the other alternating series

# 5.2 Sequence.

$a_n$ is a sequence.
$a_1$ is the first term, $a_2$ is the second term and so on...

sequence CONVERGES when : $\lim\limits_{n\to\infty} a_n = L$

and DIVERGES when : $\lim\limits_{n\to\infty} a_n = \pm\infty$ or <u>Does Not Exist</u>

76. Determine whether the following sequences converge or diverge, and if they converge, find the limit:

(a) $\left\{\frac{(-1)^n \ln n}{n}\right\}$

(b) $\left\{\frac{4n^4+1}{2n^2-1}\right\}$

(c) $\left\{n^{\frac{1}{n}}\right\}$

(d) $\left\{\frac{n^2}{2n-1} - \frac{n^2}{2n+1}\right\}$

(a) $\lim\limits_{n\to\infty}\left[\frac{(-1)^n \ln n}{n}\right] = (-1)^n \lim\limits_{n\to\infty}\left[\frac{\ln n}{n}\right] = (-1)^n \lim\limits_{n\to\infty}\left(\frac{\frac{1}{n}}{1}\right)$

$= (-1)^n(0) = 0$ # $a_n$ converge with limit 0

(c) let $y = \lim\limits_{n\to\infty}\left(n^{\frac{1}{n}}\right)$

$\ln y = \ln \lim\limits_{n\to\infty}\left(n^{\frac{1}{n}}\right)$

$\ln y = \lim\limits_{n\to\infty}\left(\ln n^{\frac{1}{n}}\right)$

$\ln y = \lim\limits_{n\to\infty}\left(\frac{1}{n}\ln n\right)$

$\ln y = \lim\limits_{n\to\infty}\left(\frac{\frac{1}{n}}{1}\right)$

$\ln y = 0$

$y = e^0 = 1$

$a_n$ converges with limit 1

# 5.3 Positive Term Divergence Test (also known as $n^{th}$ term test)

( To test a series is conv. or dive. we usually start with a quick & short divergence test )

$\sum_{n=1}^{\infty} a_n$ , if $\lim_{n\to\infty} a_n \neq 0$ , the series is divergent.

$\lim_{n\to\infty} a_n = 0$ DOESN'T MEAN IT IS CONFIRMED CONVERGENT !

( we need to do more test ! )

78. Use the nth-term test to determine whether the following series diverge or need further investigation:

(a) $\sum_{n=1}^{\infty} \frac{3n}{5n-1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ (c) $\sum_{n=1}^{\infty} \frac{n}{\ln(n+1)}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{n^2+3}\right) = 0$ the series need further investigation #

(c) $\sum_{n=1}^{\infty} \frac{n}{\ln(n+1)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n}{\ln(n+1)}\right) = \lim_{n\to\infty} \left(\frac{1}{\frac{1}{n+1}}\right) = \lim_{n\to\infty} (n+1) = \infty$

$\sum_{n=1}^{\infty} a_n$ diverges #

# 5.4 Positive Term Convergence Test.

## 5.4.1. Geometric Series.

$$\sum_{n=1}^{\infty} ar^{n-1}$$

converges, with sum $S = \lim_{n\to\infty} S_n = \frac{a}{1-r}$, IF $|r|<1$

diverges IF $|r| \geq 1$

77. Determine whether the following geometric series converge or diverge. If a series is convergent, calculate its sum.

(a) $3+\frac{3}{4}+\cdots+\frac{3}{4^{n-1}}+\cdots$ (b) $0.37+0.0037+\cdots+\frac{37}{100^n}+\cdots$ (c) $\sum_{n=1}^{\infty} 2^{-n}3^{n-1}$

(b) $a_n = \frac{37}{100^n} = 0.37\left(\frac{1}{100}\right)^{n-1}$

since $|r|<1$

$$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r} = \frac{0.37}{1-\frac{1}{100}} = \frac{37}{99}$$

(c) $\sum_{n=1}^{\infty} 2^{-n} \cdot 3^{n-1} = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{3^n}{3} = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{3}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{3} \times \frac{3}{2}\left(\frac{3}{2}\right)^{n-1}$

$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-1}$ $a=\frac{1}{2}$ $r=\frac{3}{2}$

$|r|>1$, $\sum_{n=1}^{\infty} a_n$ diverges.

## 5.4.2 p-series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges, if $p > 1$

diverges, if $p \leq 1$

## 5.4.3. Comparison Test

if we have $\sum_{n=1}^{\infty} a_n$, and we can figure out a similar series where $\sum_{n=1}^{\infty} b_n$, and $b_n \geq a_n > 0$ (positive term series only)

$\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ (bigger than $a_n$ one) converges.

or if $b_n \leq a_n$,

$\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ (smaller than $a_n$ one) diverges.

## 5.4.4. Limit comparison test

if we have $\sum_{n=1}^{\infty} a_n$ and we figure out a similar series where $\sum_{n=1}^{\infty} b_n$, ($a_n, b_n$ which is bigger / smaller not important)

and if $a_n > 0$, $b_n > 0$ (positive term series only),

if $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $c$ is positive and finite.

then either both series converge or diverge.

79. Use the first comparison test to determine whether the following series converge or diverge:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^4 + n^2 + 1}$ (b) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$

(a) $b_n = \frac{1}{n^4} > a_n = \frac{1}{n^4+n^2+1}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^4}$ this series converges.

hence $\sum_{n=1}^{\infty} a_n$ converges.

## 5.4.5 Ratio test

(positive term series only!)

$$\sum_{n=1}^{\infty} a_n$$

$-\infty$ is not possible since $a_n$, $n \in \mathbb{N}$

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ (or $\infty$), then the series

converges ABSOLUTELY, if $L < 1$

diverges, if $L > 1$ (inc. $\infty$)

inconclusive, if $L = 1$

80. Use the ratio test to determine whether the following series diverge or need further investigation:

(a) $\sum_{n=1}^{\infty} \frac{5^n}{n(3^{n+1})}$ (b) $\sum_{n=1}^{\infty} \frac{n+3}{n^2+2n+5}$ (c) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$

(a) $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{(n+1)(3^{n+2})}}{\frac{5^n}{n(3^{n+1})}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{5^n} \cdot 5}{\cancel{5^n}} \cdot \frac{n(\cancel{3^n} \cdot \cancel{3})}{(n+1)(\cancel{3^n} \cdot \cancel{3} \cdot 3)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5n}{3n+3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5}{3+\frac{3}{n}} \right|$$

$$= \frac{5}{3}$$

the series diverges.

(basically alt series easier to converge)
alt. series can converge when its original series diverge,
& but original series cannot converge if its alt. series diverge (can only diverge)

## 5.4.6 Alternating series test

$\sum_{n=1}^{\infty} (-1)^n a_n \quad , a_n > 0$

converge, if (i) $\lim_{n\to\infty} a_n = 0$

(ii) $a_k > a_{k+1}$ ($> 0$) positive term series. $\forall k \in S$

for (ii) part.
① $\frac{a_{k+1}}{a_k} < 1$ ② $a_{k+1} - a_k < 0$ ③ $a_n = f(n) \to f(x)$ & $f'(x) < 0$

should always check what kind of convergence.

and is ...
absolutely convergent if $\sum_{n=1}^{\infty} a_n$ is convergent.

or
conditionally convergent if $\sum_{n=1}^{\infty} a_n$ is divergent.

always check $a_n$ diverge or not, then check whether $\sum_{n=1}^{\infty} (-1)^n a^n$ is converging or not [check $a_k > a_{k+1}$], then check $\sum_{n=1}^{\infty} a^n$ with all the test whether is it absolutely convergent.

81. Determine whether the series satisfy the conditions of the alternating series test and conclude on their convergence or divergence:

(a) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2+7}$ (b) $\sum_{n=1}^{\infty} (-1)^n (1+e^{-n})$

(b) $a_n = 1+e^{-n}$ (satisfy the conditions of alternating series test)

since $n > 0$
$-n < 0$
$e^{-n} > 0$
$1+e^{-n} > 0$
$a_n > 0$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+e^{-n}) = 1$ the series diverges.

(we already know it's diverge, but if want to test $a_{k+1} > a_k > 0$)

$a_{k+1} = 1+e^{-(k+1)} = 1+e^{-k} \cdot e^{-1} = 1 + \frac{1}{e} \cdot e^{-k}$

$a_k = 1+e^{-k}$

method ②: $a_{k+1} - a_k = 1 + \frac{1}{e} e^{-k} - 1 - e^{-k}$
$= (\frac{1}{e} - 1) e^{-k}$

since $e > 1$ $a_{k+1} - a_k < 0$ #
$\frac{1}{e} < 1$
$\frac{1}{e} - 1 < 0$

method ③:
let $f(x) = 1+e^{-x}$
$f'(x) = -e^{-x}$

$e^{-x} > 0$ $\therefore a_k > a_{k+1} > 0$
$-e^{-x} < 0$
$f'(x) < 0$

82. Assess whether the following series are absolutely convergent, conditionally convergent or divergent:

(a) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)}$ (b) $\sum_{n=1}^{\infty}(-1)^{n}\frac{5}{(n^3+1)}$

(c) $\sum_{n=3}^{+\infty}(-1)^{n}\left(\frac{n}{n^2-3}\right)$ (d) $\sum_{n=2}^{\infty}(-1)^{n}\frac{n}{\ln n}$

(c) $a_n = \frac{n}{n^2-3}$ $\lim_{n\to\infty}\left(\frac{n}{n^2-3}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1-\frac{3}{n^2}}\right) = 0$

let $f(x) = \frac{x}{x^2-3}$, $f'(x) = \frac{(x^2-3)-x(2x)}{(x^2-3)^2} = \frac{x^2-2x^2-3}{(x^2-3)^2} = \frac{-x^2-3}{(x^2-3)^2}$

$x^2 > 0$ $(x^2-3)^2 > 0$

$-x^2 < 0$ $\frac{-x^2-3}{(x^2-3)^2} < 0$

$-x^2-3 < -3$

$f'(x) < 0$

$\therefore a_k > a_{k+1} > 0$

→ till this point it's conditionally convergent. (need to check whether it's absolutely convergent)

we will use ratio test here:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(n+1)}{(n+1)^2-3}}{\frac{n}{n^2-3}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{n^2-3}{n^2+2n-2}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^3+n^2-3n-3}{n^3+2n^2-2n}\right|$$

$= 1$ (inconclusive) → need to use other test.

we will use limit comparison test:

$b_n = \frac{n}{n^2} = \frac{1}{n}$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n}$, this series diverges.

$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n}{n^2-3}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{n^2-3}\right| = 1$ (1 is finite and positive)

and since $b_n$ diverges, $a_n$ also diverges #

$\therefore$ Hence, $\sum_{n=3}^{\infty}(-1)^n\left(\frac{n}{n^2-3}\right)$ is conditionally convergent.

(d) $\sum_{n=2}^{\infty}(-1)^n \frac{n}{\ln n}$

$a_n = \frac{n}{\ln n}$

$\lim_{n\to\infty}\left(\frac{n}{\ln n}\right) = \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n}}\right) = \lim_{n\to\infty}(n) = \infty$

the series diverges.

83. A rubber ball is dropped from a height of 10 metres. If it rebounds approximately one-half the distance after each fall, use a geometric series to approximate the total distance the ball travels before coming to rest.

$a = 5$ , $r = \frac{1}{2}$ $\qquad a_n = 5\left(\frac{1}{2}\right)^{n-1}$

$\sum_{n=1}^{\infty} a_n = \frac{5}{1-\frac{1}{2}} = 10$

total distance $= 10 + 2(10) = 30$

5.5 Power Series. $\rightarrow$ usually find with ratio test.

* power series always have radius of convergence!

$$\sum_{x=1}^{\infty} a_n (x-c)^n$$ if this is a power series,

then one of the following will be true:

(i) the series only converges when $x = c$ (diverge everywhere, except $x=c$)

(ii) the series ABSOLUTELY CONVERGE for every $x$ (converge everywhere)

(iii) ABSOLUTELY CONVERGE when $|x-c| < R$

diverge when $|x-c| > R$

need to check for $x = c+R$ and $x = c-R$ seperately.

usually $\sum a_n (c \pm R)^n$ is one positive term series and the other alternating series

84. Determine the interval of convergence of the following power series:

(a) $\sum_{n=0}^{\infty} \frac{1}{n+1} x^n$ (b) $\sum_{n=0}^{\infty} \frac{n^2}{2^n} x^n$

(c) $\sum_{n=0}^{\infty} (-1)^n \frac{n^n}{n+1} (x-3)^n$ (d) $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1} (x-2)^n$

(e) $\sum_{n=0}^{\infty} \frac{n!}{100^n} x^n$ (f) $\sum_{n=0}^{\infty} \frac{2^n}{(2n)!} x^{2n}$

(g) $\sum_{n=0}^{+\infty} \frac{n^2}{2^{3n}} x^n$ (h) $\sum_{n=0}^{\infty} \frac{x^n}{2^{2n} \cdot (2n+1)}$

(b) $$\sum_{n=0}^{\infty} n^2 \left(\frac{x}{2}\right)^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(n+1)^2 \left(\frac{x}{2}\right)^{n+1}}{n^2 \left(\frac{x}{2}\right)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{x}{2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{2} \cdot \frac{n^2+2n+1}{n^2} \right|$$

$$\lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| = 1$$

$$= \left| \frac{x}{2} \cdot 1 \right| = \left| \frac{x}{2} \right|$$

$\sum_{n=0}^{\infty} n^2 \left(\frac{x}{2}\right)^n$ converges when $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\rightarrow \left|\frac{x}{2}\right| < 1$$

$$\frac{x}{2} < 1 \vee \frac{x}{2} > -1$$

$$x < 2 \vee x > -2$$

when $x = 2$, $\sum_{n=0}^{\infty} n^2 (1)^n = \sum_{n=0}^{\infty} n^2$ (diverges) when $x=-2$, $\sum_{n=0}^{\infty} n^2(-1)^n$ (diverges)

$$\lim_{n\to\infty} n^2 = \infty$$

$\therefore$ interval of convergence : $x = ]-2,2[$ ,
radius of convergence = 2

(c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n^n}{n+1}(x-3)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(n+1)^{n+1}}{n+2}(x-3)^{n+1}}{(-1)^n\frac{n^n}{n+1}(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\left|-(x-3)\cdot\frac{(n+1)^{n+1}}{n+2}\cdot\frac{n+1}{n^n}\right|$$

$$= \lim_{n\to\infty}\left|-(x-3)\cdot\frac{\cancel{n^n}\cdot n}{\cancel{n}}\cdot\frac{\cancel{n}}{\cancel{n^n}}\right|$$

$$= \lim_{n\to\infty}\left|-(x-3)\cdot n\right|$$ when n is on the numerator, radius of convergence is usually 0.

series only converges when $x=3$ :
interval of convergence : $x=3$,
radius of convergence = 0

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}(x-2)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3^{2(n+1)}}{n+2}(x-2)^{n+1}}{\frac{3^{2n}}{n+1}(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|3^2\cdot\frac{n+1}{n+2}\cdot(x-2)\right|$$

$$= 9|x-2|$$

$9|x-2|<1$

$|x-2|<\frac{1}{9}$

$x-2<\frac{1}{9} \vee x-2>-\frac{1}{9}$

$x<\frac{19}{9} \vee x>\frac{17}{9}$

$\frac{17}{9}<x<\frac{19}{9}$ this is the final interval of convergence.

when $x = 17/9$, $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}\left(\frac{17}{9}-2\right)^n = \sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}\left(-\frac{1}{9}\right)^n = \sum_{n=0}^{\infty} \frac{9^n}{n+1}\left(-\frac{1}{9}\right)^n$

$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1}$

when $x = 19/9$, $\sum_{n=0}^{\infty} \frac{3^{2n}}{n+1}\left(\frac{19}{9}-2\right)^n = \sum_{n=0}^{\infty} 1^n \frac{1}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{n+1}\right|$

①: $\lim_{n\to\infty}\left(\frac{1}{n+1}\right) = 0$ ②: $a_k > a_{k+1} > 0$ (diverging).

let $f(x) = (x+1)^{-1}$, $f'(x) = -(x+1)^{-2} = -\frac{1}{(x+1)^2}$

$(x+1)^2 > 0 \to \frac{1}{(x+1)^2} < 0 \to -\frac{1}{(x+1)^2} > 0$

* if alternating series already diverge, no way that the positive term series will converge :

to test : (ratio test) $\lim_{n\to\infty}\left|\frac{\frac{1}{k+2}}{\frac{1}{k+1}}\right| = 1$ ( inconclusive )

limit comparison test: $\sum_{n=0}^{\infty} \frac{1}{n+1} \to \sum_{n=0}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\right| = 1$ (can use limit comparison test)

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverge $\to \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverge.

# C6. Taylor's Series, Maclaurin's Series and Binomial Series.

## 6.1 Definition and Notation.

if $f(x)$, a function can be written as a power series:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

summation is to infinity (series)

then we can find $a_n$ by using the expression:

$$a_n = \frac{f^{(n)}(c)}{n!}$$

* extra cool stuff:
when can a function be written as power series?
1. The function is infinitely differentiable at anywhere. $f^{(n)}(x), \forall n \in \mathbb{Z}$
2. The function is complex differentiable.
These kind of function is also known as Analytic Function.

this is known as Taylor's Series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n}$$

Maclaurin's Series is just Taylor's Series but $c=0$:

$$a_n = \frac{f^{(n)}(0)}{n!} \rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is always $x^n$! even if $\sin(x^2)$, $\ln x^3$ etc.

* Remember that the series are exactly equal to $f(x)$, not estimate!

* when can you substitute $g(x)$ into a maclaurin's:
if known $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$
$x^2, x^3, 2x, \frac{1}{3}x$ can be sub into $x$
BUT ONLY IN EXPANDED SERIES
NOT $\sum a_n \underline{x}^n$!
$(x-2)$ cannot 2 term.

## 6.2 Approximation of $f(x)$

let $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k$,

if $n \to \infty$: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$; then this is exactly equal.

but it's hard to list out all term, so if $n$ is a finite value:

$$\underbrace{P_n(x)} = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k \underline{\approx} f(x)$$

just an approximation

$P_n(x)$ is "Polynomial of n degree"

$$\left( \lim_{n \to \infty} P_n(x) = f(x) \right)$$

## 6·3 Remainder of the approximation

since the polynomial, $P_n(x)$ is an approximation to $f(x)$. There must be a difference, $R_n(x)$ — known as remainder, is also dependant of $n$ (the higher the degree of polynomial, $n$, the smaller the remainder, $R_n(x)$.

$$f(x) = P_n(x) + R_n(x), \quad R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$, $z$ is between $x$ and $c$.

---

eg. Determine the number of correct decimal places if estimate $\sin(0.05)$ using a cubic polynomial.

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$f(x) = \sin(x)$, $f(0) = \sin(0) = 0$
$f'(x) = \cos(x)$, $f'(0) = \cos(0) = 1$
$f''(x) = -\sin(x)$, $f''(0) = -\sin(0) = 0$
$f^{(3)}(x) = -\cos(x)$, $f^{(3)}(0) = -\cos(0) = -1$

$$P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k$$
$$= \frac{f(0)}{0!}x + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$
$$= x - \frac{x^3}{3!}$$
$$= x - \frac{x^3}{6}$$

small till negligible.

$$P_3(0.05) = 0.05 - \frac{0.05^3}{6} = 0.05$$ ← This is our estimation of $\sin(0.05)$ using polynomial with degree 3 — cubic

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$, $n$ is the degree of polynomial.

$$R_3(x) = \frac{f^{(4)}(z)}{4!}(x)^4$$ $f^{(4)}(x) = \sin x$

$$R_3(0.05) = \frac{\sin(z)}{4!} \cdot (0.05)^4$$, where $0 \le z \le 0.05$

we know that: $|\sin z| \le 1$, $\forall z \in \mathbb{R}$

although we know
$\sin(0) \le \sin z \le \sin(0.05)$
$0 \le \sin z \le \sin(0.05)$
but 1. $\sin(0.05)$ need calculator
2. we want more room for error.

$|\sin z| \leq 1$

$$\left|\sin z \cdot \frac{0.05^4}{4!}\right| \leq 1 \cdot \frac{0.05^4}{4!}$$

$$|R_3(0.05)| \leq \frac{0.05^4}{4!} \approx 2.6 \times 10^{-7}$$ ← we now know that the difference between $P_3(0.05)$ and $f(0.05)$ is less than $2.6 \times 10^{-7}$

at $n^{th}$ decimel place, we will always look at $n+1^{th}$ decimel place,
- if it is greater than 5, we will round up (which mean the current $n^{th}$ decimel place is inaccurate)
- if it is less than 5, we will not do anything to the current $n^{th}$ decimal place.

that means for $n^{th}$ decimal place to be correct,

$$\varepsilon < 5 \times 10^{-(n+1)}$$
$$2.6 \times 10^{-7} < 5 \times 10^{-(n+1)}$$ for this statement to be true, $n=6$
$$2.6 \times 10^{-7} < 5 \times 10^{-7}$$ (smallest n)

at least 6 decimal places are correct.

---

eg2. Prove that $\sin x$ written as a polynomial is valid.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f^{(3)}(x) = -\cos x$
$f^{(4)}(x) = \sin x$

$$\therefore f^{(n+1)}(x) = \begin{cases} \pm \sin x \\ \pm \cos x \end{cases}$$

$$|f^{(n+1)}(z)| \leq 1, \forall z \in \mathbb{R}, \forall n \in \mathbb{Z}$$

$$\therefore |f^{(n+1)}(z)| \leq 1$$

$$\left|\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}\right| \leq \frac{(x-c)^{n+1}}{(n+1)!}$$

$$|R_n(x)| \leq \frac{(x-c)^{n+1}}{(n+1)!}$$

$$-\frac{(x-c)^{n+1}}{(n+1)!} \leq R_n(x) \leq \frac{(x-c)^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty}\left(\frac{(x-c)^{n+1}}{(n+1)!}\right) = 0 \qquad \lim_{n\to\infty}(R_n(x)) = 0$$ by squeeze theorem.

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4. Binomial series.

refer to "Pure Mathematic 3 Notes", very detailed explaination there.

85. Find the Maclaurin series for cos $(x)$ and prove that it represents cos $(x)$ for every real number $x$.

let $f(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f(x) = \cos(x)$ , $f(0) = 1$

$f'(x) = -\sin(x)$ , $f'(0) = 0$

$f''(x) = -\cos(x)$ , $f''(0) = -1$

$f^{(3)}(x) = \sin(x)$ , $f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos(x)$ , $f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin(x)$ , $f^{(5)}(0) = 0$

$$f(x) = \cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{1}{0!}x^0 + \frac{0}{1!}x^1 + \frac{(-1)}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \ldots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$ #

$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

$f'(z) = -\sin(z)$

$f''(z) = -\cos(z)$

$f'''(z) = \sin(z)$

$f^{(4)}(z) = \cos(z)$

we know $|f^{(n+1)}(z)| \leq 1, \quad \forall z \in \mathbb{R}, \forall n \in \mathbb{Z}$

$\left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| < \frac{x^{n+1}}{(n+1)!}$

$|R_n(z)| < \frac{x^{n+1}}{(n+1)!}$

factorial higher on hierarchy compare to exponent.

$\lim_{n\to\infty} \left( \frac{z^{n+1}}{(n+1)!} \right) = 0$

$|R_n(z)| < 0$

86. Based on a previously established Maclaurin series, obtain the Maclaurin series for the following functions:

(a) $f(x) = x \cdot \sin(3x)$ (b) $f(x) = \cos(-2x)$ (c) $f(x) = \cos^2(x)$

(a) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

$\frac{d}{dx}\left(\cos(x)\right) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}\right]$

$-\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot \cancel{2n}\, x^{2n-1}}{\cancel{2n} \cdot (2n-1)!}$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \times (-1) \cdot \frac{x^{2n-1}}{(2n-1)!}$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!}$

$\sin(3x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{(3x)^{2n-1}}{(2n-1)!}$

$x\sin(3x) = \sum_{n=0}^{\infty} (-1)^{n-1} \cdot \frac{(3x)^{2n}}{3(2n-1)!}$

$(3x)^{2n-1} = \frac{3^{2n}}{3} \cdot x^{2n-1}$

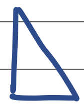
$x(3x)^{2n-1} = \frac{3^{2n}}{3} x^{2n}$

$= \frac{(3x)^{2n}}{3}$

87. Find the first three nonzero terms of the Taylor series for $f(x)$ at $c$:

(a) $f(x) = \sec(x), c = \frac{\pi}{3}$ (b) $f(x) = \arcsin(x), c = \frac{1}{2}$ (c) $f(x) = xe^x, c = -1$

(a) $f(x) = \sec(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$



$f(x) = \sec(x)$

$f'(x) = \sec(x)\tan(x)$

$f''(x) = \sec(x)\sec^2(x) + \sec(x)\tan(x)\tan(x)$

$= \sec^3(x) + \sec(x)\tan^2(x)$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f\left(\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = \frac{1}{\frac{1}{2}} = 2$

$f'\left(\frac{\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right) = 2\sqrt{3}$

$f''\left(\frac{\pi}{3}\right) = \sec^3\left(\frac{\pi}{3}\right) + \sec\left(\frac{\pi}{3}\right)\tan^2\left(\frac{\pi}{3}\right) = 2^3 + 2(\sqrt{3})^2 = 8 + 2(3) = 14$

$\sec(x) = \frac{2}{0!}\left(x - \frac{\pi}{3}\right)^0 + \frac{2\sqrt{3}}{1!}\left(x - \frac{\pi}{3}\right)^1 + \frac{14}{2!}\left(x - \frac{\pi}{3}\right)^2$

$= 2 + 2\sqrt{3}\left(x - \frac{\pi}{3}\right) + 7\left(x - \frac{\pi}{3}\right)^2$

90. Determine the Taylor's series for $f(x) = \sin(x)$ at $c = 0$ and at $c = \pi/3$. Subsequently, use the terms up to $n = 3$ to estimate $\sin(54°)$ and estimate the error in each approximation.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$c=0$:

$f(x) = \sin x \qquad f(0) = 0$
$f'(x) = \cos x \qquad f'(0) = 1$
$f''(x) = -\sin x \qquad f''(0) = 0$
$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1$
$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$$f^{(n)}(x) = \begin{cases} \pm 1, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$f^{(2k+1)}(x) = (-1)^k, \; n = 2k+1$

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}(x)^{2k+1}$$

$$P_3(x) = \frac{(-1)^0}{1!}x^1 + \frac{(-1)^3}{3!}x^3$$
$$= x - \frac{1}{6}x^3$$

$$P_3(0.3\pi) = 0.3\pi - \frac{1}{6}(0.3\pi)^3$$
$$= \frac{3\pi}{10} - \frac{9}{2000}\pi^3$$

$c = \frac{\pi}{3}$:

$f(\frac{\pi}{3}) = \sqrt{3}/2$
$f'(\frac{\pi}{3}) = 1/2$
$f''(\frac{\pi}{3}) = -\sqrt{3}/2$
$f^{(3)}(\frac{\pi}{3}) = -1/2$
$f^{(4)}(\frac{\pi}{3}) = \sqrt{3}/2$

$$f^{(n)}(x) = \begin{cases} \pm \frac{1}{2}, & n \text{ is odd} \\ \pm \frac{\sqrt{3}}{2}, & n \text{ is even} \end{cases}$$

$$\Rightarrow \begin{cases} f^{(2k+1)}(x) = (-1)^k\left(\frac{1}{2}\right), & n = 2k+1, \; k \in \mathbb{Z} \\ f^{(2k)}(x) = (-1)^k\left(\frac{\sqrt{3}}{2}\right), & n = 2k, \; k \in \mathbb{Z} \end{cases}$$

$$f(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{1}{2}\left(x - \frac{\pi}{3}\right)^{2k+1}, & n \text{ is odd} \\ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)^{2k}, & n \text{ is even.} \end{cases}$$

$$P_3(x) = \frac{\sqrt{3}}{2 \cdot 0!}\left(x - \frac{\pi}{3}\right)^0 + \frac{1}{2 \cdot 1!}\left(x - \frac{\pi}{3}\right)^1 - \frac{\sqrt{3}}{2 \cdot 2!}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{2 \cdot 3!}\left(x - \frac{\pi}{3}\right)^3$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3$$

$$\sin(0.3\pi) \approx P_3(0.3\pi)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{3\pi}{10} - \frac{1}{3}\pi\right) - \frac{\sqrt{3}}{4}\left(\frac{3\pi}{10} - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(\frac{3\pi}{10} - \frac{\pi}{3}\right)^3$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{\pi}{30}\right) - \frac{\sqrt{3}}{4}\left(-\frac{\pi}{30}\right)^2 - \frac{1}{12}\left(-\frac{\pi}{30}\right)^3$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{60}\pi - \frac{\sqrt{3}}{3600}\pi^2 + \frac{1}{324000}\pi^3$$

$$\approx 0.809012709$$

formula: $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

$$R_3(0.3\pi) = \frac{f^{(4)}(z)}{4!}(\pi)^4$$

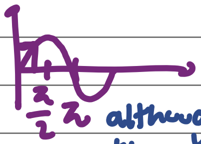
$$f^{(4)}(z) = \sin z$$

① $-1 \le \sin z \le 1, \forall z \in \mathbb{R}$ (although $0 \le z \le \frac{3\pi}{10}$, but since we don't know $\sin\left(\frac{3\pi}{10}\right)$ so why not take all? )

$$\left|\frac{\sin z}{4!} \cdot \frac{3\pi}{10}^4\right| \le \frac{\frac{3\pi}{10}^4}{4!}$$

$$|R_3(0.3\pi)| \le \frac{27}{80000}\pi^4$$

$$\approx 0.0329$$

② 

although the error should be less.

$$R_3(0.3\pi) = \frac{f^{(4)}(z)}{4!}\left(x - \frac{\pi}{3}\right)^4$$

$$f^{(4)}(z) = \sin z$$

$$|\sin z| \le 1, \forall z \in \mathbb{R}$$

$$\left|\frac{\sin z}{4!} \cdot \left(\frac{3\pi}{10} - \frac{\pi}{3}\right)^4\right| \le \frac{\left(\frac{3\pi}{10} - \frac{\pi}{3}\right)^4}{4!}$$

$$|R_3(0.3\pi)| \le \frac{\pi^4}{19\,440\,000}$$

$$\approx 5.01 \times 10^{-6}$$

92. Find a power series representation for the expression and state the radius of convergence of:

(a) $\sqrt{1-x^3}$ (b) $\sqrt[3]{8+x}$

$$(8+x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1 + \frac{1}{8}x\right)^{\frac{1}{3}}$$
$$= 2\left(1 + \frac{1}{8}x\right)^{1/3}$$

$${}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)\cancel{(n-r)!}}{\cancel{(n-r)!}\,r!}$$

$$= 2\left(1 + \frac{1}{3}\left(\frac{1}{8}x\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{1}{8}x\right)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{1}{8}x\right)^3 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!}\left(\frac{1}{8}x\right)^4 + \ldots\right)$$

$$= 2\left(1 + \frac{1}{24}x + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{3^n \cdot n!}\left(\frac{1}{8}x\right)^n (2)(5)(8)\ldots(3n-4)\right)$$

$$= 2 + \frac{1}{12}x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{24^n \cdot n!} x^n \cdot (2)(5)(8)\ldots(3n-4)$$

radius of convergence:

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{24^{n+1}\cdot(n+1)!}x^{n+1} \cdot \cancel{(2)(5)(8)}\ldots\cancel{(3n-4)}(3n-1)}{\frac{(-1)^{n+1}}{24^n \cdot n!}\cancel{x^n} \cdot \cancel{(2)(5)(8)}\ldots\cancel{(3n-4)}}\right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}\cdot(-1)}{\cancel{24^n}\, 24\cdot(n+1)\cancel{n!}} \cdot \frac{\cancel{24^n}\cdot\cancel{n!}}{\cancel{(-1)^{n+1}}} x(3n-1) \right|$$

$$= \lim_{n\to\infty} \left| -\frac{3n-1}{24(n+1)} x \right|$$

$$\begin{array}{r|l} & -\frac{1}{8} \\ \hline 24n+24 & -3n+1 \\ & -3n-3 \\ \hline & 4 \end{array} \qquad -\frac{1}{8} + \frac{4}{3n+3}$$

$$= \lim_{n\to\infty} \left| \left( -\frac{1}{8} + \underbrace{\frac{4}{3n+3}}_{0} \right) x \right|$$

$$= \lim_{n\to\infty} \left| -\frac{1}{8} x \right|$$

$$= \left| -\frac{1}{8} x \right|$$

$$L < 1$$

$$\left| -\frac{1}{8} x \right| < 1$$

$$-1 < -\frac{1}{8} x < 1$$

$$-1 < \frac{1}{8} x < 1$$

$$-8 < x < 8$$

$\therefore$ radius of convergence $= 8$

# C7. Integration

Integration topic will be divided into multiple part.

| | | |
|---|---|---|
| 7.1 | Integration Introduction | ( high-school level + pureMath 1 ) |
| 7.2 | Integration Intermediate | ( pureMath 3 and further pure 1 ) |
| 7.3 | Integration Advance | ( further pure 2 and Uni year 1 ) |

# 7.1 Integration Introduction

$\int f(x)\,dx = F(x) + c$ is known as indefinite integral (no limit)
$F(x)$ is antiderivative of $f(x)$
$c$ is known as arbitury constant.

## 7.1.1 Basic Rule for Integration.

$\int a\,dx = ax + c$ , where a is a constant

$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$ , $n \neq -1$

$\int (ax+b)^n = \dfrac{(ax+b)^{n+1}}{(n+1)\cdot a} + c$

$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int af(x)\,dx = a\int f(x)\,dx$

$\int e^x\,dx = e^x + c$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + c$

eg1. $\int x^2+4x\,dx = \frac{x^3}{3}+\frac{4x^2}{2} = \frac{1}{3}x^3+2x^2+c$

eg2. $\int \frac{(x^2-1)^2}{x^2}\,dx = \int \frac{x^4-2x^2+1}{x^2}\,dx$

$= \int x^2-2+x^{-2}\,dx$

$= \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1} + c$

$= \frac{1}{3}x^3 - 2x - \frac{1}{x} + c$

## 7.1.2 Definite Integral (Riemann's Sum etc. will be covered in 3)

$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = F(b)-F(a)$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

all indefinite integral rule works for definite.

eg. $\int_0^1 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}e^{2(1)} - \frac{1}{2}e^{2(0)}$

$= \frac{1}{2}e^2 - \frac{1}{2}$

## 7.2 Integration Intermediate.

### 7.2.1. Integration by parts.

let u and v be a function of $x \Rightarrow u=f(x)$, $v=g(x)$

$(uv)' = uv' + u'v$ (product rule)

$uv' = (uv)' - u'v$

$\int u\,dv = uv - \int v\,du$

Some thing to take note :

1. generally we follow the rule : LIATE to choose u.

| | |
|---|---|
| Logarithm | eg. $\ln x$, $\log x$ |
| Inverse function | eg. $\sin^{-1}x$, $\arctan(2x)$ |
| Algebraic | eg. $x^2$, $\sqrt{3x}$ |
| Trigonometry function | eg. $\sin(x)$, $\cos(x)$ |
| Exponential function | eg. $e^x$, $e^{-x}$ |

[3] (note that hyperbolic functions count as exponential)

$\sinh x = (e^x - e^{-x})/2$

2. However in some case, or when we have same type of function for u and dv, we always choose dv to be the one to easier integrate :

eg. $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$.

this is easier than $\int \sec x\,dx$ !

let $u = \sec x$ $dv = \sec^2 x\,dx$ ← $\int \sec^2 x\,dx = \tan x$

3. When doing integration by part multiple time. It's better to choose the same function as u continuously !

eg. $\int e^x \cos x\,dx = e^x \cos x\,dx + \int e^x \sin x\,dx$ $u = \cos x$ $dv = e^x\,dx$

... $du = -\sin x$ $v = e^x$

$u = \sin x$ $dv = e^x\,dx$

$du = \cos x$ $v = e^x$

4. Definite Integral by Parts.

$\int_a^b u\,dv = \underline{[uv]_a^b} - \int_a^b v\,du$

important !

eg1. $\int x\sin x\,dx$

$u = x$ $\qquad dv = \sin x\,dx$

$\frac{du}{dx} = 1 \qquad v = -\cos x$

$du = dx$

$= -x\cos x - \int -\cos x\,dx$

$= -x\cos x + \sin x + c$

eg2. $\int \ln x\,dx$

$u = \ln x \qquad dv = dx$

$\frac{du}{dx} = \frac{1}{x} \qquad v = x$

$du = \frac{1}{x}dx$

$= x\ln x - \int x\left(\frac{1}{x}\right)dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c$

eg3. $\int e^x\cos x\,dx$

$u = \cos x, \quad dv = e^x dx$

$du = -\sin x\,dx, \quad v = e^x$

$= e^x\cos x + \int e^x\sin x\,dx$

$u = \sin x, \quad dv = e^x dx$

$du = \cos x\,dx; \quad v = e^x$

$\int e^x\cos x\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$

: endless.

$2\int e^x\cos x\,dx = e^x\cos x + e^x\sin x$

$\int e^x\cos x\,dx = \frac{1}{2}e^x\cos x + \frac{1}{2}e^x\sin x + c$

eg4. $\int_0^1 xe^{2x}dx$

$u = x \qquad dv = e^{2x}dx$

$du = dx \qquad v = \frac{1}{2}e^{2x}$

$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \int_0^1 \frac{1}{2}e^{2x}dx$

$= \left[\frac{1}{2}(1)e^{2(1)} - \frac{1}{2}(0)e^{2(0)}\right] - \frac{1}{4}\left[e^{2x}\right]_0^1$

$= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - e^0)$

$= \frac{1}{4} - \frac{3}{4}e^2$ //

## 7·2·2 Integration using u-subsitution.

1. definition (hard—3)

$$\int f(g(x))\cdot g'(x)\,dx = F(g(x)) + c$$

* basically in normal term we say when we have an integral of a function, $g(x)$, we can sub it as $u$ to simplify our integration.

let $u = g(x)$
$du = g'(x)\,dx$

$$\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du = F(u) + c$$
$$= F(g(x)) + c$$

2. Indefinite Integral. (hard — 3)

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad , \text{ when } u = g(x)$$

* basically just remember to subsitute our limit also when doing u-sub!

eg1. $\int \frac{3}{\sqrt{5x-1}}dx = \int \frac{3}{\sqrt{u}}\left(\frac{du}{5}\right) = \frac{3}{5}\int u^{-\frac{1}{2}}du$

$= \frac{3}{5}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)+c$

let $u = 5x-1$

$du = 5dx$

$dx = \frac{du}{5}$

$= \frac{6}{5}\sqrt{u}+c$

$= \frac{6}{5}\sqrt{5x-1}+c$

eg2. $\int_{x=0}^{x=1} xe^{x^2}dx$

remember to sub your limit!

let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$x=0,\quad u=0^2=0$

$x=1,\quad u=1^2=1$

$= \int_{u=0}^{u=1} xe^{u}\frac{du}{2x}$

$= \frac{1}{2}\int_0^1 e^u du$

$= \frac{1}{2}\left[e^u\right]_0^1$

$= \frac{1}{2}(e-1)$

$= \frac{1}{2}e-\frac{1}{2}$ //

## 7.2.3 Method of recognition.

lets take $\int 2x \cos x^2 \, dx$ for an example.

know that : $\frac{d}{dx}(\sin x^2) = 2x \cos x^2$

$$\int \frac{d}{dx}(\sin x^2)\, dx = \int 2x \cos x^2 \, dx$$

$$\sin x^2 = \int 2x \cos x^2 \, dx.$$

notice that : $\int f'(x) \cos f(x) \, dx = \sin f(x) + c$ , hence :

so : If there is a derivate of a function next to what we want to integrate, integrate the "we want to integrate" function like how we used to do, and just throw away our $f'(x)$ .

list of some method of recognition formula :

• indicate somewhat important.
★ indicate extremely important.

- • $\int f'(x)\cos(f(x))\, dx = \sin(f(x)) + C$
- • $\int f'(x)\sin(f(x))\, dx = -\cos(f(x)) + C$
- • $\int f'(x)\sec(f(x))\tan(f(x))\, dx = \sec(f(x)) + C$
- • $\int f'(x)\sec^2(f(x))\, dx = \tan(f(x)) + C$
- • $\int f'(x)\csc(f(x))\cot(f(x))\, dx = -\csc(f(x)) + C$
- • $\int f'(x)\csc^2(f(x))\, dx = -\cot(f(x)) + C$
- $\int f'(x)\tan(f(x))\, dx = -\ln|\cos(f(x))| + C$
- $\int f'(x)\cot(f(x))\, dx = \ln|\sin(f(x))| + C$
- $\int f'(x)\sec(f(x))\, dx = \ln|\sec(f(x)) + \tan(f(x))| + C$
- $\int f'(x)\csc(f(x))\, dx = -\ln|\csc(f(x)) + \cot(f(x))| + C$
- ★ $\int f'(x)f(x)^n\, dx = \frac{f(x)^{n+1}}{n+1} + C \quad (n \neq -1)$ ★
- • $\int f'(x)e^{f(x)}\, dx = e^{f(x)} + C$
- ★ $\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C$ ★
- $\int f'(x)\sinh(f(x))\, dx = \cosh(f(x)) + C$
- $\int f'(x)\cosh(f(x))\, dx = \sinh(f(x)) + C$
- $\int \frac{f'(x)}{1 + f(x)^2}\, dx = \arctan(f(x)) + C$
- $\int \frac{f'(x)}{\sqrt{1 - f(x)^2}}\, dx = \arcsin(f(x)) + C$
- $\int \frac{f'(x)}{\sqrt{1 + f(x)^2}}\, dx = \operatorname{arsinh}(f(x)) + C$
- $\int f'(x)\arctan(f(x))\, dx = f(x)\arctan(f(x)) - \frac{1}{2}\ln(1 + f(x)^2) + C$
- $\int f'(x)\ln(f(x))\, dx = f(x)\ln(f(x)) - f(x) + C$

eg1. $\int (2x-1)\operatorname{cosec}^2(5x^2-5x)\,dx$

$= \int \frac{1}{5}(10x-5)\operatorname{cosec}^2(5x^2-5x)\,dx$

$= -\frac{1}{5}\cot(5x^2-5x)+c$

eg2. $\int \frac{5}{3}x^2\sec x^3\tan x^3\,dx$

$= \frac{5}{3}\cdot\frac{1}{3}\int 3x^2\sec x^3\tan x^3\cdot dx$

$= \frac{5}{9}\sec x^3+c$

eg3. $\int x^2\sec x^3\,dx$

$= \frac{1}{3}\int \underline{3x^2}\sec x^3\,\underline{dx}$   let $u=x^3$
$du=3x^2dx$

$= \frac{1}{3}\int \sec u\,du$

$= \frac{1}{3}\ln|\sec u+\tan u|+c$

$= \frac{1}{3}\ln|\sec x^3+\tan x^3|+c$

eg4. $\int \sec^2 2x\, e^{\tan 2x}\,dx$

$= \frac{1}{2}\int 2\sec^2 2x\, e^{\tan 2x}\,dx$

$= \frac{1}{2}e^{\tan 2x}+c$

eg5. $\int \cos x\sin^4 x\,dx$   (power rule: $\int f'(x)\cdot f(x)^n\,dx = \frac{f(x)^{n+1}}{n+1}+c$ )

$= \frac{1}{5}\sin^5 x+c$

eg6. $\int \frac{\ln x}{x} dx$

$= \int \frac{1}{x} (\ln x)' dx$

$= \frac{(\ln x)^2}{2} + c$

eg7. $\int \frac{e^x}{\sqrt{e^x - 2}} dx$

$= \int e^x (e^x - 2)^{-\frac{1}{2}} dx$

$= \frac{(e^x - 2)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 2\sqrt{e^x - 2} + c$

eg8. $\int \frac{3}{e^x + 1} dx$

$= \int \frac{3}{e^x + 1} \frac{\times e^{-x}}{\times e^{-x}} dx$

$= \int \frac{3e^{-x}}{1 + e^{-x}} dx$

$= -3 \int \frac{-e^{-x}}{1 + e^{-x}} dx$

$= -3 \ln |1 + e^{-x}| + c$

eg10. $\int \frac{2x^2 + 2x + 3}{(x-2)(x^2+1)} dx$

$\frac{2x^2+2x+3}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1}$

$= \int \frac{3}{x-2} + \frac{-x}{x^2+1} dx$

$2x^2 + 2x + 3 = A(x^2+1) + (Bx+C)(x-2)$

$= 3\ln|x-2| - \frac{1}{2}\ln|x^2+1| + c$

$A = 3, B = -1, C = 0$

this page contain (3 — integration advance)

eg11. $\int \frac{4}{3\cosh 4x - \sinh 4x}\,dx$

$= \int \frac{4}{\frac{3e^{4x}+3e^{-4x}}{2} - \frac{e^{4x}-e^{-4x}}{2}}\,dx$

$= 8\int \frac{1}{2e^{4x}+4e^{-4x}}\,dx$ (times with $e^{4x}$)

$= 4\int \frac{e^{4x}}{(e^{4x})^2+2}\,dx$

let $u = e^{4x}$

$du = 4e^{4x}\,dx$

$du = 4u\,dx$

$dx = \frac{du}{4u}$

$= \cancel{4}\int \frac{\cancel{u}}{u^2+2}\,\frac{du}{\cancel{4u}}$

$= \int \frac{1}{u^2+2}\,du$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}e^{4x}\right) + c$

$e^x, e^{2x}, e^{3x}$

eg12. $\int \frac{1}{e^x - e^{-x}}\,dx$

$\int \frac{e^x}{(e^x)^2 - 1}\,dx$

let $u = e^x$

$du = e^x\,dx$

$dx = \frac{du}{u}$

$\int \frac{\cancel{u}}{u^2-1}\,\frac{du}{\cancel{u}}$

$= \int \frac{1}{u^2-1}\,du$

$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c$

$= \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + c$

$= \frac{1}{2}\ln\left|\tanh\frac{x}{2}\right| + c$

$\frac{(e^x-1)e^{-\frac{x}{2}}}{(e^x+1)e^{-\frac{x}{2}}}$

$= \frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}$

$= \frac{\sinh\frac{x}{2}}{\cosh\frac{x}{2}}$

$= \tanh\frac{x}{2}$

## 7.2.4 More trigonometry Integration.

$$\int \sec x\,dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx.$$
$$= \ln\left|\sec x + \tan x\right| + c$$

$$\int \operatorname{cosec} x\,dx = \int \operatorname{cosec} x \cdot \frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x}\,dx$$
$$= \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x}\cdot dx$$
$$= -\int \frac{-\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x}\cdot dx$$
$$= -\ln\left|\operatorname{cosec} x + \cot x\right| + c$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$
$$= -\int \frac{-\sin x}{\cos x}\,dx$$
$$= -\ln\left|\cos x\right| + c$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$$
$$= \ln\left|\sin x\right| + c$$

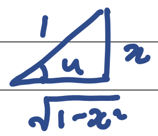
$\int \sin^{-1} x\,dx$

let $u = \sin^{-1} x$ $\quad dv = dx$

$\sin u = x$ $\quad v = x$

$\cos u\,du = dx$

$= uv - \int v\,du$




$= x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\cos u = \sqrt{1-x^2}$

$du = \frac{dx}{\sqrt{1-x^2}}$

$= x\sin^{-1} x - \int x(1-x^2)^{-\frac{1}{2}}\,dx$

$= x\sin^{-1} x + \frac{1}{2}\int -2x(1-x^2)^{-1/2}\,dx$

$= x\sin^{-1} x + \frac{\sqrt{1-x^2}}{2(\frac{1}{2})} + c$

$= x\sin^{-1} x + \sqrt{1-x^2} + c$

# 7.3. Integration Advance.

## 7.3.1 Some additional Integration Formula.

(a) Summary of integration rules involving inverse trigonometric functions:

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$

$$\int -\frac{1}{\sqrt{a^2-x^2}}dx = \arccos\left(\frac{x}{a}\right)+C \qquad \int \frac{1}{x\cdot\sqrt{x^2-a^2}}dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right)+C$$

(b) Summary of integration rules involving inverse hyperbolic functions:

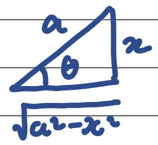
$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{a}\right)+C, \quad a>0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \operatorname{arcosh}\left(\frac{x}{a}\right)+C, \quad 0<a<x$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\operatorname{artanh}\left(\frac{x}{a}\right)+C, \quad |x|<a$$

$$\int \frac{1}{x\cdot\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{arsech}\left(\frac{|x|}{a}\right)+C, \quad 0<|x|<a$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\cancel{a}\sqrt{1-\sin^2\theta}}\cancel{a}\cos\theta\, d\theta = \int d\theta = \theta = \sin^{-1}\left(\frac{x}{a}\right)+C \;//$$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

or just (let $x = a\sin\theta$ by trig. sub

$dx = a\cos\theta\, d\theta$

$\frac{x}{a} = \sin\theta$, $\theta = \sin^{-1}\left(\frac{x}{a}\right)$ )

all example listed together later...

## 7.3.2 Trigonometry and Hyperbolic Subsitution.

1. $\int \sqrt{x^2+1}\,dx$ $x=\tan\theta$, $x=\sinh\theta$
2. $\int \sqrt{x^2-1}\,dx$ $x=\cosh\theta$, $x=\sec\theta$
3. $\int \sqrt{1-x^2}\,dx$ $x=\sin\theta$, $x=\cos\theta$, $x=\tanh\theta$

actually two formula is enough:

d1 : $\sin^2\theta+\cos^2\theta=1$

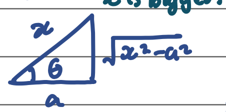
d2 : $-\sinh^2\theta+\cosh^2\theta=1$

☆ way to determine what to sub:

for eg1. we need to get rid of sqrt.

sqrt of what trigo or hyperbolic function give a single expression ?

method 2: $\int\sqrt{x^2-a^2}\,dx$ $x$ is biggest.

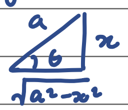


$\cos\theta = \frac{a}{x}$

$\sec\theta = \frac{x}{a}$

$x = a\sec\theta$

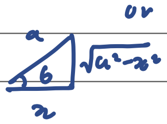
$\int\sqrt{x^2-a^2}$



$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

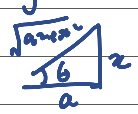
or



$\cos\theta = \frac{x}{a}$

$x = a\cos\theta$

$\int\sqrt{a^2+x^2}$



$\tan\theta = \frac{x}{a}$

$x = a\tan\theta$ .

## 7.3.3 Half Angle Subsitution (t-sub)

useful for integrating:

$$\int \frac{k}{a\sin f(x) + b\cos f(x) + c}\,dx$$ ; where $a, b, c, k$ are constants and $f(x)$ is a linear function

extra: for $\int \frac{k}{a\sinh f(x) + b\cosh f(x) + c}\,dx$, use $\sinh f(x) = \frac{e^{f(x)} - e^{-f(x)}}{2}$ ... to find...

*Trigonometry*

If $t = \tan\frac{1}{2}x$ then:

$$\sin x = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x = \frac{1-t^2}{1+t^2}$$

## 7.3.4 Reduction Formula.

### 7.3.4.1 Trigonometry Reduction (and hyperbolic).

$\sin^n x = \sin x \sin^{n-1} x$ $\longleftrightarrow$ same with $\cos x$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

same with $\sinh x$ $\qquad\qquad$ same with $\cosh x$.

$\tan^n x = \tan^2 x \tan^{n-2} x$ $\longleftrightarrow$ same with $\sec x$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

same with $\tanh x$. $\qquad\qquad$ same with $\operatorname{sech} x$.

$\cot^n x = \cot^2 x \cot^{n-2} x$ $\longleftrightarrow$ same with $\operatorname{cosec} x$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

same with $\coth x$. $\qquad\qquad$ same with $\operatorname{cosech} x$.

### 7.3.4.2 Reduction Formula — Integration By Parts.

### 7.3.4.3 Reduction Formula — Algebraic Manipulation

## 7.3.5 Definite Integral.

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b = F(b) - F(a)$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{let } u = g(x)$$

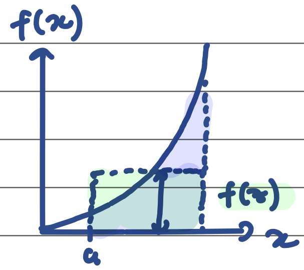
## 7·3·4·1 Mean value Theorem.

$$\int_a^b f(x)\,dx = f(z)\cdot(b-a)\ ,\ \text{for } z\in[a,b]$$

this can also be think as:

$$\frac{\int_a^b f(x)\,dx}{b-a} = f(z) \rightarrow \frac{\text{Area}}{\text{width}} = \text{average height.}$$

(average value of a function).




$$f(z)\cdot(b-a) = \int_a^b f(x)\,dx.$$

## 7.3.4.2 Derivative of Definite Integral. (fundamental theorem of Calculus)

$$g(x) = \int_a^x f(t)\,dt$$

$* F(x) = \int f(x)\,dx.$

$$g'(x) = \left[\int_a^x f(t)\,dt\right]'$$
$$= \left[F(t)\Big|_a^x\right]'$$
$$= [F(x) - F(a)]'$$
$$= f(x)$$

since $F(a)$ is a constant. substituting a constant, $a$ into $F(t)$ result in a constant.

this can also be written as:

prefer this

$$g(x) = \int_a^x f(t)\,dt$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[F(t)\Big|_{t=a}^{t=x}\right]$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}[F(x) - F(a)]$$
$$\frac{d}{dx}[g(x)] = f(x)$$

extras:

① what if: $g(x) = \int_a^b f(t)\,dt$?

$$\frac{d}{dx}[g(x)] = \frac{d}{dx}[F(b) - F(a)] = 0$$

② what if $\frac{d}{dt}\left[\int_a^b f(t)\,dt\right]$

$$= \frac{d}{dt}[F(b) - F(a)]$$
$$= 0$$

③ but, $\frac{d}{dt}\left[\int f(t)\,dt\right]$

$$= \frac{d}{dt}[F(t) + c]$$
$$= f(t) + c$$

$$\therefore \left(\int_a^x f(t)\,dt\right)' = f(x)$$

if $g(x) = \int_a^{h(x)} f(t)\,dt$

$$g'(x) = \left[\int_a^{h(x)} f(t)\,dt\right]'$$
$$g'(x) = \left[F(t)\Big|_{t=a}^{t=h(x)}\right]'$$
$$g'(x) = [F[h(x)] - F(a)]'$$
$$g'(x) = [F[h(x)]]' - [F(a)]'$$
$$g'(x) = h'(x) f[h(x)]$$

(chain rule)

$$\therefore \left(\int_a^{g(x)} f(t)\,dt\right)' = g'(x) f(g(x))$$

# 7.3.1 Additional Integration Formula.

$$\int \frac{1}{x^2+9}\,dx$$
$$= \int \frac{1}{x^2+3^2}\,dx$$
$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$$

$$\int \frac{5}{\sqrt{4-x^2}}\,dx$$
$$= 5\int \frac{1}{\sqrt{2^2-x^2}}\,dx$$
$$= 5\sin^{-1}\left(\frac{x}{2}\right)+c$$

$$\int \frac{2}{x^2+7}\,dx$$
$$= 2\int \frac{1}{x^2+\sqrt{7}^2}\,dx$$
$$= \frac{2}{\sqrt{7}}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+c$$

$$\int \frac{3}{\sqrt{x^2+5}}\,dx$$
$$= 3\int \frac{1}{\sqrt{x^2+\sqrt{5}^2}}\,dx$$
$$= 3\sinh^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

$$\int \frac{4}{\sqrt{x^2-16}}\,dx$$
$$= 4\int \frac{1}{\sqrt{x^2-4^2}}\,dx$$
$$= 4\cosh^{-1}\left(\frac{x}{4}\right)+c$$

$$\int \frac{3}{\sqrt{25-x^2}}\,dx$$
$$= 3\int \frac{1}{\sqrt{5^2-x^2}}\,dx$$
$$= 3\sin^{-1}\left(\frac{x}{5}\right)+c$$

$$\int \frac{5}{3x^2+1}\,dx$$
$$= \frac{5}{3}\int \frac{1}{x^2+\frac{1}{3}}\,dx$$
$$= \frac{1}{\sqrt{\frac{1}{3}}}\,\frac{5}{3}\tan^{-1}\left(\frac{x}{\sqrt{\frac{1}{3}}}\right)+c$$
$$= \frac{5}{3}\sqrt{3}\tan^{-1}\left(\sqrt{3}\,x\right)+c$$

$$\int \frac{4}{\sqrt{5x^2+2}}\,dx$$
$$= \frac{4}{\sqrt{5}}\int \frac{1}{\sqrt{x^2+\frac{2}{5}}}\,dx$$
$$= \frac{4}{\sqrt{5}}\sinh^{-1}\left(\frac{\sqrt{5}}{\sqrt{2}}x\right)+c$$

$$\sqrt{5\left(x^2+\frac{2}{5}\right)}$$
$$\sqrt{5}\left(\sqrt{x^2+\frac{2}{5}}\right)$$

eg 1. $\int \frac{1}{\sqrt{(x-3)^2+5}}\,dx$ let $u = x-3$

$du = dx$

$= \int \frac{1}{\sqrt{u^2+5}}\,du$

$= \sinh^{-1}\left(\frac{u}{\sqrt{5}}\right)+c$

$= \sinh^{-1}\left(\frac{x-3}{\sqrt{5}}\right)+c$

eg 2. $\int \frac{2}{\sqrt{4-(x+3)^2}}\,dx$ let $u = x+3$

$du = dx$

$= \int \frac{2}{\sqrt{4-u^2}}\,du$

$= 2\sin^{-1}\left(\frac{u}{2}\right)+c$

$= 2\sin^{-1}\left(\frac{x+3}{2}\right)+c$

eg 3. $\int \frac{4}{\sqrt{2(5x+1)^2-3}}\,dx$ let $5\sqrt{2}x+\sqrt{2} = u$

$du = 5\sqrt{2}\,dx$

$= \int \frac{4}{\sqrt{(5\sqrt{2}x+\sqrt{2})^2-3}}\,dx$ $dx = \frac{du}{5\sqrt{2}}$

$= \frac{4}{5\sqrt{2}} \int \frac{1}{\sqrt{u^2-3}}\,du$

$= \frac{4}{5\sqrt{2}} \cosh^{-1}\left(\frac{u}{\sqrt{3}}\right)+c$

eg 4. $\displaystyle\int \frac{4}{\sqrt{8(2x+1)^2+4}}\,dx$

$$= \frac{4}{\sqrt{8}}\int \frac{1}{\sqrt{(2x+1)^2+\frac{1}{2}}}\,dx$$

let $u = 2x+1$
$du = 2\,dx$

$$= \sqrt{2}\int \frac{1}{\sqrt{u^2+\frac{1}{2}}}\;\frac{du}{2}$$

$$= \frac{\sqrt{2}}{2}\left(\sinh^{-1}\left(\frac{u}{\sqrt{\frac{1}{2}}}\right)\right)+c$$

$$= \frac{\sqrt{2}}{2}\sinh^{-1}\left(2\sqrt{2}x+\sqrt{2}\right)+c$$

eg5. $\displaystyle\int \frac{1}{\sqrt{2x^2+5x+1}}\,dx$

$2x^2+5x+1$
$2\left(x^2+\frac{5}{2}x+\frac{1}{2}\right)$
$2\left(x^2+\frac{5}{2}x+\frac{5}{4}^2-\frac{5}{4}^2+\frac{1}{2}\right)$
$2\left(x+\frac{5}{4}\right)^2-\frac{17}{8}$

$$= \int \frac{1}{\sqrt{2\left(x^2+\frac{5}{2}x\right)+1}}\,dx$$

$$= \int \frac{1}{\sqrt{2}\left[\left(x+\frac{5}{4}\right)^2-\frac{17}{16}\right]}$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\left(x+\frac{5}{4}\right)^2-\frac{17}{4}}$$

$$= \frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{x+\frac{5}{4}}{\frac{\sqrt{17}}{4}}\right)$$

eg 6. $\displaystyle\int \frac{3}{\sqrt{4x^2+8x-9}}\,dx$

$4\left[x^2+2x-\frac{9}{4}\right]$

$=4\left[x^2+2x+1^2-1^2-\frac{9}{4}\right]$

$=4\left[(x+1)^2-\frac{13}{4}\right]$

$\displaystyle=\frac{3}{2}\int\frac{1}{\sqrt{u^2-\frac{13}{4}}}\,du$

$=\frac{3}{2}\cosh^{-1}\left(\frac{u}{\sqrt{\frac{13}{4}}}\right)+c$

let $u=x+1$

$du=dx$

$=\frac{3}{2}\cosh^{-1}\left(\frac{2x+2}{\sqrt{13}}\right)+c$

eg7. $\displaystyle\int\frac{4}{x^2+3x-10}\,dx$

$\displaystyle=\int\frac{4}{(x-2)(x+5)}\,dx$

$\frac{4}{(x-2)(x+5)}=\frac{A}{x-2}+\frac{B}{x+5}$

$4=A(x+5)+B(x-2)$

$-\frac{4}{7}=B \quad A=\frac{4}{7}$

$\displaystyle=\frac{4}{7}\int\frac{1}{x-2}-\frac{1}{x+5}\,dx$

$=\frac{4}{7}\left(\ln(x-2)-\ln(x+5)\right)+c$

$=\frac{4}{7}\ln\left(\frac{x-2}{x+5}\right)+c$

eg8. $\displaystyle\int\frac{3}{\sqrt{-x^2+4x+10}}\,dx$

$-(x^2-4x-10)$

$=-(x^2-4x+2^2-2^2-10)$

$=-\left[(x+2)^2-14\right]$

$\displaystyle=\int\frac{3}{\sqrt{14-(x+2)^2}}\,dx$

let $u=x+2$

$du=dx$

$\displaystyle=\int\frac{3}{\sqrt{14-u^2}}\,du$

$=3\sin^{-1}\left(\frac{u}{\sqrt{14}}\right)+c$

$=3\sin^{-1}\left(\frac{x+2}{\sqrt{14}}\right)+c$

# 7.3.2 Trigonometry and Hyperbolic Substitution Method.

1. $\int \sqrt{x^2+1}\,dx$ $\quad x = \tan\theta,\ \underline{x = \sinh\theta}$
2. $\int \sqrt{x^2-1}\,dx$ $\quad \underline{x = \cosh\theta},\ x = \sec\theta$
3. $\int \sqrt{1-x^2}\,dx$ $\quad \underline{x = \sin\theta,\ x = \cos\theta},\ x = \tanh\theta$

actually two formula is enough

d1 : $\sin^2\theta + \cos^2\theta = 1$

d2 : $-\sinh^2\theta + \cosh^2\theta = 1$

☆ way to determine what to sub:

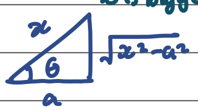
for eg1. we need to get rid of sqrt.

sqrt of what trigo or hyperbolic function give a single expression ?

from $\tan^2\theta + 1 = \sec^2\theta$ , $\sqrt{\tan^2\theta+1} = \sec\theta$ . $\therefore x = \tan\theta$

from $-\sinh^2\theta + \cosh^2\theta = 1$ , $\sqrt{1+\sinh^2\theta} = \cosh^2\theta$ $\quad \therefore x = \sinh\theta$

method 2: $\int \sqrt{x^2-a^2}\,dx$ $x$ is biggest.

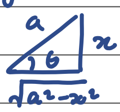



$\cos\theta = \frac{a}{x}$

$\sec\theta = \frac{x}{a}$

$x = a\sec\theta$

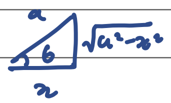
$\int \sqrt{x^2-a^2}$




$\sin\theta = \frac{x}{a}$

$x = a\sin\theta$

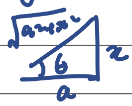
or




$\cos\theta = \frac{x}{a}$

$x = a\cos\theta$

$\int \sqrt{a^2+x^2}$




$\tan\theta = \frac{x}{a}$

$x = a\tan\theta$ .

eg1. $\int \sqrt{x^2+1}\,dx$ $\qquad$ let $x = \sinh\theta$

$dx = \cosh\theta\,d\theta$

$= \int \sqrt{\sinh^2\theta + 1}\,dx$

$= \int \sqrt{\cosh^2\theta}\ \cosh\theta\,d\theta$

$= \int \cosh^2\theta\,d\theta$

$= \int \frac{\cosh 2\theta + 1}{2}\,d\theta$

$= \frac{1}{4}\sinh 2\theta + \frac{1}{2}\theta + c$

$= \frac{1}{2}\sinh\theta\cosh\theta + \frac{1}{2}\theta + c$

$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right) + c$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cosh 2\theta = 2\cosh^2\theta - 1$

$\cosh^2\theta = \frac{\cosh 2\theta + 1}{2}$

$\theta = \sinh^{-1} x$

$= \ln\left(x + \sqrt{x^2+1}\right)$

$\sinh\theta = x$

$\cosh\theta = \sqrt{1+\sinh^2\theta}$

$= \sqrt{1+x^2}$

$$\int \sqrt{x^2+1}\,dx \qquad , \; x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \int \sec\theta \sec^2\theta\, d\theta$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$= \tan\theta\sec\theta - \int \tan^2\theta \sec\theta\, d\theta$$

let $u = \sec\theta \qquad dv = \sec^2\theta\, d\theta$

$du = \tan\theta\sec\theta\, d\theta \qquad v = \tan\theta$

$$= \tan\theta\sec\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec^3\theta - \sec\theta\, d\theta$$

$$= \tan\theta\sec\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \ln\left|\sec\theta + \tan\theta\right|$$

$x = \tan\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\sec\theta = \sqrt{1+x^2}$

$$\int \sec^3\theta\, d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|$$

$$= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left|\sqrt{1+x^2} + x\right|$$

eg2. $\int \sqrt{x^2-1}\,dx$

$x = \cosh\theta$

$dx = \sinh\theta\,d\theta$

$\sin^2\theta + \cos^2\theta = 1$

$-\sinh^2\theta + \cosh^2\theta = 1$

$\sinh^2\theta = \cosh^2\theta - 1$

$= \int \sqrt{\cosh^2\theta - 1}\,\sinh\theta\,d\theta$

$= \int \sinh\theta\sinh\theta\,d\theta$

$= \int \sinh^2\theta\,d\theta$

$\cos 2A = 1 - 2\sin^2 A$

$\cosh 2A = 1 + 2\sinh^2 A$

$\frac{\cosh 2A - 1}{2} = \sinh^2 A$

$= \int \frac{1}{2}\cosh 2\theta - \frac{1}{2}\,d\theta$

$= \frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta + c$

$x = \cosh\theta$

$\theta = \cosh^{-1}x$

$= \ln(x + \sqrt{x^2-1})$

$\sinh\theta = \sqrt{\cosh^2\theta - 1}$

$= \frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln(x + \sqrt{x^2-1}) + c$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= 2\sqrt{x^2-1}\,x$

eg3. $\int \sqrt{1-x^2}\,dx$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$= \int \sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$

$= \int \cos\theta\cos\theta\,d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \int \cos^2\theta\,d\theta$

$\cos^2\theta = \dfrac{\cos 2\theta + 1}{2}$

$\cos\theta = \sqrt{1-x^2}$

$= \int \frac{1}{2}\cos 2\theta + \frac{1}{2}\,d\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2x\sqrt{1-x^2}$

$= \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + c$

$= \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + c$

$\theta = \sin^{-1}x$

$\int \sqrt{a^2+x^2}\, dx$

so can factor out together with $a^2$

$= \int \sqrt{a^2+a^2\tan^2\theta}$ $\quad x = a\tan\theta$

eg4. $\int \sqrt{9+x^2}\, dx$ $\quad x = 3\sinh\theta$, $dx = 3\cosh\theta\, d\theta$

$-\sinh^2\theta + \cosh^2\theta = 1$

PICK +IVE ONLY! unless there's hint to -ive

$\cosh\theta = \sqrt{1+\sinh^2\theta}$

$= \int \sqrt{9+9\sinh^2\theta}\; 3\cosh\theta\, d\theta$

$\sinh\theta = \frac{1}{3}x \qquad \theta = \sinh^{-1}\left(\frac{1}{3}x\right)$

$= \ln\left(\frac{1}{3}x + \sqrt{\frac{1}{9}x^2+1}\right)$

$= \ln\left(\frac{1}{3}x + \frac{1}{3}\sqrt{x^2+9}\right)$

$= 3\int \sqrt{\cosh^2\theta} \cdot 3\cosh\theta\, d\theta$

$= 9\int \cosh^2\theta\, d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cosh 2\theta = 2\cosh^2\theta - 1$

$\cosh^2\theta = \frac{\cosh 2\theta + 1}{2}$

$= \frac{9}{2}\int \cosh 2\theta + 1\, d\theta$

$= \frac{9}{4}\sinh 2\theta + \frac{9}{2}\theta + C_1$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

$= 2\left(\frac{1}{3}x\right)\left(\sqrt{1+\frac{1}{9}x^2}\right)$

$= \frac{2}{9}x\sqrt{9+x^2}$

$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln\left(\frac{1}{3}x + \frac{1}{3}\sqrt{x^2+9}\right) + C_1$

$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln\left(x+\sqrt{x^2+9}\right) + \frac{9}{2}\ln\frac{1}{3} + C_1$

$= \frac{1}{2}x\sqrt{9+x^2} + \frac{9}{2}\ln\left(x+\sqrt{x^2+9}\right) + C_2$

fraction inside a surd is not neat, hence try to factorise it out.

eg5. $\int \sqrt{4-x^2}\, dx$ $\qquad x = 2\sin\theta$

$\qquad dx = 2\cos\theta\, d\theta$

$= \int \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$

$= 2\int \sqrt{1-\sin^2\theta}\; 2\cos\theta\, d\theta$

$= 4\int \cos^2\theta\, d\theta$ $\qquad \cos 2\theta = 2\cos^2\theta - 1$

$= 4\int \frac{1}{2}\cos 2\theta + \frac{1}{2}\, d\theta$ $\qquad \cos^2\theta = \frac{\cos 2\theta + 1}{2}$ $\qquad \sin\theta = \frac{1}{2}x$

$= \frac{2\sin 2\theta}{2} + 2\theta + c$ $\qquad \theta = \sin^{-1}\left(\frac{1}{2}x\right)$

$= \sin 2\theta + 2\theta + c$

$\qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{1}{4}x^2}$

$= 2\sin\theta\cos\theta + 2\theta + c$ $\qquad = \frac{1}{2}\sqrt{4-x^2}$

$= \frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{1}{2}x\right) + c$

eg6. $\int \sqrt{x^2-16}\,dx$

let $x = 4\cosh u$

$dx = 4\sinh u\,du$

$= \int \sqrt{16\cosh^2 u - 16}\ 4\sinh u\,du$

$-\sinh^2 u + \cosh^2 u = 1$

$\sinh^2 u = \frac{1}{16}x^2 - 1$

$\sinh u =$

$= \int 4\sinh u\,(4\sinh u)\,du$

$= \int 16\sinh^2 u\,du$

$\cos 2u = 1 - 2\sin^2 u$

$\cosh 2u = 1 + 2\sinh^2 u$

$\sinh^2 u = \frac{\cosh 2u - 1}{2}$

$= \int 8\cosh 2u - 8\,du$

$= 4\sinh 2u - 8u + c$

$\sinh 2u = 2\sinh u\cosh u$

$= 2\sqrt{\frac{1}{16}x^2-1} \times \frac{1}{4}x$

$= \frac{1}{2}x\frac{1}{4}\sqrt{x^2-16}$

$= \frac{1}{8}x\sqrt{x^2-16}$

$= \frac{1}{2}x\sqrt{x^2-16} - 8\left(\ln(x+\sqrt{x^2+16} - \ln 4\right)$

$u = \cosh^{-1}\left(\frac{1}{4}x\right)$

$= \ln\left(\frac{1}{4}x + \sqrt{\frac{1}{16}x^2+1}\right)$

$= \ln\left(\frac{1}{4}x + \frac{1}{4}\sqrt{x^2+16}\right)$

$= \ln\left(x + \sqrt{x^2+16}\right) + \ln\frac{1}{4}$

$= \ln\left(x + \sqrt{x^2+16}\right) - \ln 4$

eg8. $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

let $x = \sinh u$

$dx = \cosh u\,du$

$= \int \frac{\sqrt{1+\sinh^2 u}}{\sinh^2 u} \cosh u\,du$

$= \int \frac{\cosh^2 u}{\sinh^2 u}\,du$

$= \int \coth^2 u\,du$

$= \int (\operatorname{cosech}^2 u + 1)\,du$

$= -\coth u + u + c$

$= -\frac{1}{x}\sqrt{1+x^2} + \ln\left(x+\sqrt{x^2+1}\right) + c$

$\operatorname{cosech}^2\theta + 1 = \coth^2\theta$

$\frac{1}{x}^2 + 1 = \coth^2 u$

$\coth u = \sqrt{\frac{1}{x}^2+1}$

$= \frac{1}{x}\sqrt{1+x^2}$

$u = \sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$

eg9. $\int \frac{\sqrt{9-x^2}}{4x^2}\,dx$

let $x = 3\sin u$

$dx = 3\cos u\,du$

$\sin u = \frac{1}{3}x$

$= \int \frac{\sqrt{9-9\sin^2 u}}{4(3\sin u)^2} 3\cos u\,du$

$= \frac{1}{4}\int \frac{\cos u}{\sin^2 u} \times \cos u\,du$

$= \frac{1}{4}\int \frac{\cos^2 u}{\sin^2 u}\,du$

$= \frac{1}{4}\int \cot^2 u\,du$

$= \frac{1}{4}\int \operatorname{cosec}^2 u - 1\,du$

$= \frac{1}{4}(-\cot u - u) + c$

$= -\frac{1}{4}\cot u - \frac{1}{4}u + c$

$= -\frac{1}{4x}\sqrt{9-x^2} - \frac{1}{4}\sin^{-1}\left(\frac{x}{3}\right) + c$

$\sin^2 u + \cos^2 u = 1$

$\cot^2 u + 1 = \operatorname{cosec}^2 u$

$\cot^2 u + 1 = \frac{1}{\left(\frac{1}{3}x\right)^2}$

$\cot^2 u = \frac{1}{\frac{1}{9}x^2} - 1$

$\cot^2 u = \frac{9}{x^2} - 1 = \frac{9-x^2}{x^2}$

$\cot u = \frac{1}{x}\sqrt{9-x^2}$

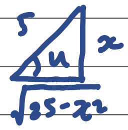
notice that:

no ± since we can only pick +ve.

eg10. $\displaystyle\int \frac{\sqrt{25-x^2}}{x^3}\,dx$

let $x = 5\sin u$
$dx = 5\cos u\,du$

$\sin u = \frac{1}{5}x$




$$= \int \frac{\sqrt{25-25\sin^2 u}}{125\sin^3 u}\,5\cos u\,du$$

$$= \int \frac{\cancel{25}\sqrt{1-\sin^2 u}\times\cos u\,du}{\cancel{125}\sin^3 u}$$

$$= \frac{1}{5}\int \frac{\cos^2 u}{\sin^3 u}\,du$$

$$= \frac{1}{5}\int \frac{1-\sin^2 u}{\sin^3 u}\,du$$

$$= \frac{1}{5}\int (\operatorname{cosec}^3 u - \operatorname{cosec} u)\,du$$

$$= \frac{1}{5}\left(-\frac{1}{2}\cot u\operatorname{cosec} u - \frac{1}{2}\ln|\operatorname{cosec} u + \cot u|\right)$$

$$+\frac{1}{5}\ln|\operatorname{cosec} x + \cot x| + c$$

$\int \operatorname{cosec}^3 u\,du$

$u = \operatorname{cosec} u \qquad dv = \operatorname{cosec}^2 u\,dv$
$du = -\cot u\operatorname{cosec} u \qquad v = -\cot u$

$$= -\cot u\operatorname{cosec} u - \int \cot^2 u\operatorname{cosec} u\,du$$

$$= -\cot u\operatorname{cosec} u - \int \operatorname{cosec} u(\operatorname{cosec}^2 u - 1)\,du$$

$$= -\cot u\operatorname{cosec} u - \int \operatorname{cosec}^3 u\,du + \int \operatorname{cosec} u\,du$$

$$2\int \operatorname{cosec}^3 u\,du = -\cot u\operatorname{cosec} u - \ln|\operatorname{cosec} u + \cot u|$$

$$\int \operatorname{cosec}^3 u\,du = -\frac{1}{2}\cot u\operatorname{cosec} u - \frac{1}{2}\ln|\operatorname{cosec} u + \cot u|$$

$$= -\frac{1}{10}\cot u\operatorname{cosec} u + \frac{1}{10}\ln|\operatorname{cosec} u + \cot u| + c$$

$$= -\frac{1}{10}\frac{\sqrt{25-x^2}}{x}\left(\frac{5}{x}\right) + \frac{1}{10}\ln\left|\frac{5}{x} + \frac{\sqrt{25-x^2}}{x}\right| + c$$

eg11.

$$\int \frac{\sqrt{x^2-1}}{x}\,dx \qquad \text{let } x = \cosh u$$
$$dx = \sinh u\, du$$

$$= \int \frac{\sinh u}{\cosh u} \sinh u\, du$$

$$= \int \frac{\sinh^2 u}{\cosh u}\, du$$

$$= \int \frac{\cosh^2 u - 1}{\cosh u}\, du$$

$$= \int \cosh u - \operatorname{sech} u\, du$$

$$\sinh^2 u = \cosh^2 u - 1 = \sqrt{x^2-1}$$

$$= \sinh u - 2\tan^{-1} e^u + c$$

$$u = \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$$

$$= \sqrt{x^2-1} - 2\tan^{-1}\left(x + \sqrt{x^2+1}\right) + c$$

$$e^u = e^{\ln(x+\sqrt{x^2-1})}$$
$$= x + \sqrt{x^2-1}$$

eg12. $\displaystyle\int \frac{5}{2x^2\sqrt{x^2+1}}\,dx$

$x = \sinh u$

$dx = \cosh u\, du$

$\displaystyle = \int \frac{5}{2\sinh^2 u\,\cancel{\sqrt{\cosh^2 u}}}\,\cancel{\cosh u}\, du$

$\sinh u = x$

$u = \sinh^{-1} x$

$u = \ln\left(x + \sqrt{x^2+1}\right)$

$\displaystyle = \frac{5}{2}\int \frac{1}{\sinh^2 u}\, du$

$\displaystyle = \frac{5}{2}\int \operatorname{cosech}^2 u\, du$

$\displaystyle = -\frac{5}{2}\coth u + c$

$\displaystyle = -\frac{5}{2}\left(\frac{e^u + e^{-u}}{e^u - e^{-u}}\right) + c$

let $x + \sqrt{x^2+1} = y$

$\displaystyle = -\frac{5}{2}\left(\frac{y + \frac{1}{y}}{y - \frac{1}{y}}\right) + c$

$y^2 = x^2 + 2x\sqrt{x^2+1} + x^2 + 1$

$y^2 + 1 = 2x^2 + 2x\sqrt{x^2+1} + 2$

$y^2 - 1 = 2x^2 + 2x\sqrt{x^2+1}$

$\displaystyle = -\frac{5}{2}\left(\frac{y^2+1}{y^2-1}\right) + c$

$\displaystyle \frac{y^2+1}{y^2-1} = \frac{x^2 + x\sqrt{x^2+1} + 1}{x^2 + x\sqrt{x^2+1}}$

$\displaystyle = 1 + \frac{1}{x^2 + x\sqrt{x^2+1}}$

$\displaystyle = -\frac{5}{2}\left(1 + \frac{1}{x^2 + x\sqrt{x^2+1}}\right) + c$

eg13. $\int \frac{5}{(4x^2-1)^{\frac{3}{2}}}\,dx$

$= \frac{5}{4^{\frac{3}{2}}}\int \frac{1}{(x^2-\frac{1}{4})^{\frac{3}{2}}}\,dx$

let $x = \frac{1}{2}\cosh u$

$dx = \frac{1}{2}\sinh u\,du$

$= \frac{5}{4^{\frac{3}{2}}}\int \frac{1}{(\frac{1}{4}\cosh^2 u - \frac{1}{4})^{\frac{3}{2}}}\,\frac{1}{2}\sinh u\,du$

$\cosh u = 2x$

$= \frac{5}{4^{\frac{3}{2}}}\int \frac{1}{(\frac{1}{4}\sinh^2 u)^{\frac{3}{2}}}\,\frac{1}{2}\sinh u\,du$

$= \frac{5}{2}\int \frac{1}{\sinh^3 u}\,\sinh u\,du$

$= \frac{5}{2}\int \operatorname{cosech}^2 u\,du$

$\sinh^2 u = \cosh^2 u - 1$

$= 4x^2 - 1$

$\sinh u = \sqrt{4x^2-1}$

$= -\frac{5}{2}\coth u + c$

$= -\frac{5}{2}\left(\frac{2x}{\sqrt{4x^2-1}}\right) + c$

$= -\frac{5x}{\sqrt{4x^2-1}} + c$

eg14. $\displaystyle\int \frac{3}{x(x^2+1)^{3/2}}\,dx$

let $x = \sinh u$

$dx = \cosh u \, du$

$$= \int \frac{3}{\sinh u\,(\sinh^2 u + 1)^{3/2}} \cosh u \, du$$

$$= \int \frac{3}{\sinh u \cosh^3 u} \cosh u \, du$$

$$= 3\int \frac{1}{\sinh u \cosh^2 u}\, du$$

$\operatorname{cosech} u \quad \operatorname{sech}^2 u$

$u = \operatorname{cosech} u$ $\qquad dv = \operatorname{sech}^2 u \, du$

$du = -\operatorname{cosech} u \coth u \, du$ $\qquad v = \tanh u$

$\tanh u \operatorname{cosech} u$

$+ \int \operatorname{cosech} u \, du$

$1 + \sinh^2 u = \cosh^2 u$

$\sqrt{1+x^2} = \cosh u$

$u = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$\tanh u \operatorname{cosech} u + \ln\left(\tanh \frac{u}{2}\right) + c$

$$\frac{\cancel{x}}{\sqrt{1+x^2}} \times \frac{1}{\cancel{x}} + \ln\left(\frac{e^u - 1}{e^u + 1}\right) + c$$

$$\frac{3}{\sqrt{1+x^2}} + 3\ln\left(\frac{x + \sqrt{x^2+1} - 1}{x + \sqrt{x^2+1} + 1}\right) + c$$

eg15. $\int \frac{x^3}{(x^2-4)^{3/2}}\,dx$

$x = 2\cosh u$

$dx = 2\sinh u\,du$

$= \int \frac{8\cosh^3 u}{(4\sinh^2 u)^{3/2}} \times 2\sinh u\,du$

$\sinh^2 u + 1 = \cosh^2 u$

$\sinh u = \sqrt{\left(\frac{1}{2}x\right)^2 - 1}$

$= \sqrt{\frac{1}{4}x^2 - 1}$

$= \int \frac{\cancel{8}\cosh^3 u}{\cancel{8}\sinh^3 u} \times 2\sinh u\,du$

$= 2\int \frac{\cosh^3 u}{\sinh^2 u}\,du$

$-\coth^2 u + 1 = -\operatorname{cosech}^2 u$

$= 2\int \coth^2 u \cosh u\,du$

$= 2\int (1 + \operatorname{cosech}^2 u)\cosh u\,du$

$= 2\int \cosh u + \coth u \operatorname{cosech} u\,du$

$\frac{\cosh u}{\sinh u}$ $\operatorname{cosech} u$

$\coth u \operatorname{cosech} u$

$= 2\sinh u - 2\operatorname{cosech} u + c$

$= \sqrt{x^2-4} - 2\left(\frac{1}{\frac{1}{2}\sqrt{x^2-4}}\right) + c$

$= \sqrt{x^2-4} - \frac{4}{\sqrt{x^2-4}} + c$

# 7.3.3 Half Angle Subsitution. (t-sub)

extra: for $\int \frac{k}{a\sinh f(x)+b\cosh f(x)+c}\,dx$, use $\sinh f(x)=\frac{e^{f(x)}-e^{-f(x)}}{2}$
... to find ...

useful for integrating:

$$\int \frac{k}{a\sin f(x)+b\cos f(x)+c}\,dx$$ ; where a, b, c, k are constants and $f(x)$ is a linear function

*Trigonometry*

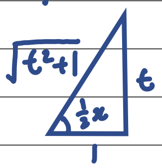
If $t=\tan\frac{1}{2}x$ then:

$$\sin x=\frac{2t}{1+t^2} \qquad \text{and} \qquad \cos x=\frac{1-t^2}{1+t^2}$$

if $f(x)\neq x$, eg: $f(x)=3x$ ; $t=\tan\frac{3}{2}x$

proving:

if $t=\tan\frac{1}{2}x$




$\sin\frac{1}{2}x=\frac{t}{\sqrt{t^2+1}}$

$\cos\frac{1}{2}x=\frac{1}{\sqrt{t^2+1}}$

$$\sin x = 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x = 2\left(\frac{t}{\sqrt{t^2+1}}\right)\left(\frac{1}{\sqrt{t^2+1}}\right) = \frac{2t}{1+t^2}$$

$$\cos x = \cos^2\tfrac{1}{2}x-\sin^2\tfrac{1}{2}x = \frac{1}{t^2+1}-\frac{t^2}{t^2+1} = \frac{1-t^2}{1+t^2}$$

when can $\tan(\tan^{-1}k)$ be reduced to k?

$\tan^{-1}k$ is an angle, so as long as $\tan^{-1}k$ exist, $\tan(\tan^{-1}k)=k$

only $\tan^{-1}(\tan k)$ need to be careful

if k is not a principle angle, $\tan^{-1}(\tan k)$ will not be equal to k eg1. $\displaystyle\int \frac{2}{3-2\sin x}\,dx$

$t = \tan\frac{1}{2}x$

$1+\tan^2\theta = \sec^2\theta$

$1+\tan^2\frac{1}{2}x = \sec^2\frac{1}{2}x$

$1+t^2 = \sec^2\frac{1}{2}x$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$

$dx = \frac{2dt}{\sec^2\frac{1}{2}x}$

$dx = \frac{2}{1+t^2}dt$

$$= \int \frac{2}{3-2\left(\frac{2t}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)$$

$$= \int \frac{4}{3+3t^2-4t}\,dt$$

$$= \frac{4}{3}\int \frac{1}{t^2-\frac{4}{3}t+1}\,dt$$

let $u = t-\frac{2}{3}$

$du = dt$

$$= \frac{4}{3}\int \frac{1}{\left(t-\frac{2}{3}\right)^2+\frac{5}{4}}\,dt$$

$$= \frac{4}{3}\int \frac{1}{u^2+\sqrt{\frac{5}{4}}^2}\,du$$

$$= \frac{4}{3}\left[\frac{3}{\sqrt{5}}\tan^{-1}\left(\frac{3t-2}{\sqrt{5}}\right)\right]+c$$

$$= \frac{4}{\sqrt{5}}\tan^{-1}\left(\frac{3t-2}{\sqrt{5}}\right)+c$$

$$= \frac{4}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan\frac{1}{2}x-2}{\sqrt{5}}\right)+c$$

eg2. $\int \frac{3}{5\cos x + 7}\,dx$

let $t = \tan\frac{1}{2}x$

$dt = \frac{1}{2}\sec^2\frac{1}{2}x\ dx$

$dx = \frac{2dt}{\sec^2\frac{1}{2}x}$

$= \frac{2}{t^2+1}\,dt$

$\tan^2\frac{1}{2}x + 1 = \sec^2\frac{1}{2}x$

$t^2+1 = \sec^2\frac{1}{2}x$

$= \int \frac{3}{5\left(\frac{1-t^2}{1+t^2}\right)+7}\left(\frac{2}{t^2+1}\,dt\right)$

$= \int \frac{6}{5(1-t^2)+7(t^2+1)}\,dt$

$= \int \frac{6}{5-5t^2+7t^2+7}\,dt$

$= \int \frac{6}{2t^2+12}\,dt$

$= \frac{6}{2}\int \frac{1}{t^2+6}\,dt$

$= \frac{3}{\sqrt{6}}\tan^{-1}\left(\frac{t}{\sqrt{6}}\right) + c$

$= \frac{3}{\sqrt{6}}\underbrace{\tan^{-1}\left[\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{6}}\right]}_{\text{cannot cancel out tan since there is } /\sqrt{6}} + c$

eg3. $\displaystyle\int \frac{2}{3\cos x - 2\sin x - 1}\,dx$

$t = \tan\frac{1}{2}x$

$dx = \frac{2}{1+t^2}\,dt$ ← don't memorise, cause this is only true if $\sin x$ or $\cos x$.

$= \displaystyle\int \frac{2}{3\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) - 1}\,\frac{2}{(1+t^2)}\,dt$

$= 4\displaystyle\int \frac{1}{3-3t^2-4t-1}\,dt$

$= \frac{4}{-3}\displaystyle\int \frac{1}{t^2+\frac{4}{3}t-\frac{2}{3}}\,dt$

$= -\frac{4}{3}\displaystyle\int \frac{1}{(t+\frac{2}{3})^2-\frac{10}{9}}\,dt$

$= -\frac{\cancel{4}^{\,2}}{3}\left[\frac{1}{\cancel{2}_{\,1}\sqrt{\frac{10}{9}}}\ln\left|\frac{t+\frac{2}{3}-\sqrt{\frac{10}{9}}}{t+\frac{2}{3}+\sqrt{\frac{10}{9}}}\right|\right]+c$

$= -\frac{\sqrt{10}}{5}\left(\ln\left(t+\frac{2-\sqrt{10}}{3}\right)-\ln\left(t+\frac{2-\sqrt{10}}{3}\right)\right)+c$

$= \frac{\sqrt{10}}{5}\ln\left(t+\frac{2-\sqrt{10}}{3}\right)-\frac{\sqrt{10}}{5}\ln\left(t+\frac{2+\sqrt{10}}{3}\right)+c$

$= \frac{\sqrt{10}}{5}\ln\left(\tan\frac{1}{2}x+\frac{2-\sqrt{10}}{3}\right)-\frac{\sqrt{10}}{5}\ln\left(\tan\frac{1}{2}x+\frac{2+\sqrt{10}}{3}\right)+c$

eg4 $\int \frac{4}{2\sin 3x+5}\,dx$

let $t=\tan\frac{3}{2}x$

$dt=\frac{3}{2}\sec^2\frac{3}{2}x\,dx$

$dx=\frac{2dt}{3\sec^2\frac{3}{2}x}$

$\tan^2\frac{3}{2}x+1=\sec^2\frac{3}{2}x$

$t^2+1=\sec^2\frac{3}{2}x$

$=\frac{8}{3}\int\frac{1}{2\left(\frac{2t}{1+t^2}\right)+5}\,\frac{dt}{t^2+1}$

$=\frac{8}{3}\int\frac{1}{4t+5t^2+5}\,dt$

$=\frac{8}{15}\int\frac{1}{t^2+\frac{4}{5}t+1}\,dt$

$=\frac{8}{15}\int\frac{1}{\left(t+\frac{2}{5}\right)^2+\frac{21}{25}}\,dt$

$=\frac{8}{15\sqrt{\frac{21}{25}}}\tan^{-1}\left(\frac{t+\frac{2}{5}}{\sqrt{\frac{21}{25}}}\right)+c$

$\frac{2\sqrt{21}}{21}$

$=\frac{8\sqrt{21}}{63}\tan^{-1}\left(\frac{5\sqrt{21}}{21}\tan\frac{3}{2}x+\frac{2\sqrt{21}}{21}\right)+c$

eg. $\int \frac{5}{\operatorname{cosec}x+1}\,dx$

let $t=\tan\frac{1}{2}x$

$= \int \frac{5}{\frac{1}{\sin x}+1}\,dx$

$dt=\frac{1}{2}\sec^2\frac{1}{2}x\,dx$

$dx=\frac{2\,dt}{\sec^2\frac{1}{2}x}$

$\tan^2\frac{1}{2}x+1=\sec^2\frac{1}{2}x$

$t^2+1=\sec^2\frac{1}{2}x$

$= \int \frac{5\sin x}{1+\sin x}\,dx$

$dx=\frac{2}{t^2+1}\,dt$

$$\begin{array}{r|l} & 5 \\ \hline 1+\sin x & 5\sin x \\ & \underline{5\sin x+5} \\ & -5 \end{array}$$

$= \int 5-\frac{5}{1+\sin x}\,dx$

$= 5x-5\int \frac{1}{1+\frac{2t}{1+t^2}}\,\frac{2}{t^2+1}\,dt$

$= 5x-10\int \frac{1}{t^2+2t+1}\,dt$

$= 5x-10\int \frac{1}{(t+1)^2}\,dt$

$= 5x-10\int (t+1)^{-2}\,dt$

$= 5x-10\left[\frac{(t+1)^{-1}}{-1}\right]+c$

$= 5x+\frac{10}{\tan\frac{1}{2}x+1}+c$

eg6. $$\int \frac{4}{1+\sec x}\,dx$$

let $t = \tan\frac{1}{2}x$

$dx = \frac{2}{t^2+1}\,dt$

$$= \int \frac{4}{1+\frac{1}{\cos x}}\,dx$$

$$= \int \frac{4}{\frac{\cos x+1}{\cos x}}\,dx$$

$\cos x + 1 \,\overline{)\, 4\cos x}$, quotient $4$: $4\cos x + 4$, remainder $-4$

$$= \int \frac{4\cos x}{\cos x+1}\,dx$$

$$= \int 4 - \frac{4}{\cos x+1}\,dx$$

$$= 4x - 4\int \frac{1}{\frac{1-t^2}{1+t^2}+1}\,\frac{2}{t^2+1}\,dt$$

$$= 4x - 8\int \frac{1}{1-t^2+t^2+1}\,dt$$

$$= 4x - 8\int \frac{1}{2}\,dt$$

$$= 4x - 8t + c$$

$$= 4x - 8\tan\tfrac{1}{2}x + c$$

eg7. $\int \frac{3}{2-\sin^2 2x}\,dx$

$\cos 4x = 1-2\sin^2 2x.$

$\sin^2 2x = 1-\cos 4x$

$= \int \frac{3}{2+\cos 4x-1}\,\frac{dt}{t^2+1}$

$= \int \frac{3}{1+\frac{1-t^2}{1+t^2}}\,\frac{dt}{t^2+1}$

$= 3\int \frac{1}{t^2+1+1-t^2}\,dt$

let $t = \tan 2x$

$dt = 2\sec^2 2x\,dx$

$\tan^2 2x+1 = \sec^2 2x$

$t^2+1 = \sec^2 2x$

$= 3\int \frac{1}{2}\,dt$

$dx = \frac{dt}{2\sec^2 2x}$

$= \frac{dt}{2(t^2+1)}$

$= \frac{3}{2}\tan 2x + c$

# 7.3.4 Reduction Formula.

1. Take a part of a function out.
2. Integration by Parts.
3. Substitution
4. Differentiation.

## 7.3.4.1 Function Manipulation.

$\sin^n x$ take $\sin x \sin^{n-1} x$
$\cos^n x$ $\cos x \cos^{n-1} x$
$\tan^n x$ $\tan^2 x \tan^{n-2} x$
$\cot^n x$ $\cot^2 x \cot^{n-2} x$

$\int \sin^2 x \, dx$ easy

$\int \sin^3 x \, dx$ still okay $\rightarrow \int \sin x \sin^2 x$

$\int \sin^{10} dx$ tedious.

so... $\int \sin^n x \, dx.$

$$\begin{pmatrix} \sin^n x \longleftrightarrow \cos^n x \\ \tan^n x \longleftrightarrow \cot^n x \\ \sec^n x \longleftrightarrow \csc^n x \end{pmatrix}$$

eg 1.a) $\int \sin^n x \, dx$

let $I_n = \int \sin^n x \, dx$

$= \int \sin x \sin^{n-1} x \, dx$

let $u = \sin^{n-1} x$
$du = (n-1)\sin^{n-2} x \cos x \, dx$

$dv = \sin x \, dx$
$v = -\cos x$

$\sin^2 x + \cos^2 x = 1$

$= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \, dx$

$= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x \, dx$

$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \, dx - (n-1)\int \sin^n x \, dx$

$= -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$I_n + (n-1) I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$n I_n = -\cos x \sin^{n-1} x + (n-1) I_{n-2}$

$I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$ , for $n \geq 2$

eg1.b) find $\int_{\pi/6}^{\pi/2} \sin^n x \, dx$

$$I_n = -\frac{1}{n}\left[\cos x \sin^{n-1} x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$$

$$= -\frac{1}{n}\left[0 - \frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^{n-1}\right] + \frac{n-1}{n} I_{n-2}$$

$$= -\frac{1}{n}\left(\frac{-\sqrt{3}}{2^n}\right) + \frac{n-1}{n} I_{n-2} \quad = \frac{\sqrt{3}}{n2^n} + \frac{n-1}{n} I_{n-2}$$

eg1.c) find $\int_{\pi/6}^{\pi/2} \sin^6 x \, dx$

$$I_6 = \frac{\sqrt{3}}{6(2^6)} + \frac{6-1}{6} I_4 = \frac{\sqrt{3}}{6(2^6)} + \frac{5}{6}\left(\frac{7\sqrt{3}}{64} + \frac{\pi}{8}\right) = \frac{3\sqrt{3}}{32} + \frac{5}{48}\pi$$

$$I_4 = \frac{\sqrt{3}}{4(2^4)} + \frac{4-1}{4} I_2 = \frac{\sqrt{3}}{96} + \frac{3}{4}\left(\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right) = \frac{7\sqrt{3}}{64} + \frac{\pi}{8}$$

$$I_2 = \frac{\sqrt{3}}{2(2^2)} + \frac{2-1}{2} I_0 = \frac{\sqrt{3}}{2(2^2)} + \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{8} + \frac{\pi}{6}$$

$$I_0 = \int_{\pi/6}^{\pi/2} \sin^0 x \, dx = \left[x\right]_{\pi/6}^{\pi/2} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$$\int \cos^n 2x \, dx$$

$$I_n = \int \cos 2x \cos^{n-1} 2x \, dx$$

let $u = \cos^{n-1} 2x$ $\quad dv = \cos 2x \, dx$

$v = \frac{1}{2} \sin 2x$

$du = (n-1)\cos^{n-2} 2x \, (-2\sin 2x) \, dx$

$$dx = \frac{du}{(2-n)\cos^{n-2} 2x \sin 2x}$$

$$= \frac{1}{2}\sin 2x \cos^{n-1} 2x + \int \sin 2x \, (n-1)\cos^{n-2} 2x \sin 2x \, dx$$

$$= \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1)\int \sin^2 2x \cos^{n-2} 2x \, dx$$

$$= \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1)\int \cos^{n-2} 2x \,(1 - \cos^2 2x)\, dx$$

$\sin^2 \theta + \cos^2 \theta = 1$

$\sin^2 2x = 1 - \cos^2 2x$

$$= \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1)\int \cos^{n-2} 2x \, dx - (n-1)\int \cos^n 2x \, dx$$

$$= \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1) I_{n-2}$$

$$n I_n = \frac{1}{2}\sin 2x \cos^{n-1} 2x + (n-1) I_{n-2}$$

$$I_n = \frac{1}{2n}\sin 2x \cos^{n-1} 2x + \frac{n-1}{n} I_{n-2}$$

3. $I_n = \int \tan^n x \, dx$

$= \int \tan^2 x \tan^{n-2} x \, dx$

$= \int (\sec^2 x - 1) \tan^{n-2} dx$

$= \int \left(\sec^2 x \tan^{n-2} x - \tan^{n-2} x\right) dx$

$I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$ #

4. $I_n = \int \cot^n x \, dx$

$1+\cot^2 x = \csc^2 x$

$\cot^2 x = \csc^2 x - 1$

$= \int \cot^2 x \cot^{n-2} x \, dx$

$= \int (\csc^2 x - 1) \cot^{n-2} x \, dx$

$= \int \csc^2 x \cot^{n-2} x - \int \cot^{n-2} x \, dx$

$I_n = \dfrac{-\cot^{n-1} x}{n-1} - I_{n-2}$ #

extra: if hyperbolic function, do it as how you treat trigonometry function

$I_n = \int_0^1 \sinh^n x \, dx$

let $u = \sinh^{n-1} x$

$du = (n-1)\sinh^{n-2} x \cosh x \, dx$

$= \int_0^1 \sinh^{n-1} x \sinh x \, dx$

$dv = \sinh x \, dx$

$v = \cosh x$

$= \left[\cosh x \sinh^{n-1} x\right]_0^1 - (n-1)\int_0^1 \sinh^{n-2} x \cosh^2 x \, dx$

$= \cosh 1 \sinh^{n-1} 1 - (n-1)\int_0^1 \sinh^{n-2} x(\sinh^2 x + 1) dx$ $\quad -\sinh^2 x + \cosh^2 x = 1$

$\cosh^2 x = \sinh^2 x + 1$

$= \left(\dfrac{e+e^{-1}}{2}\right)\left(\dfrac{e-e^{-1}}{2}\right)^{n-1} - (n-1)\int_0^1 \sinh^n x + \sinh^{n-2} x \, dx$

$= \frac{1}{2^n}\left(e+\frac{1}{e}\right)\left(e-\frac{1}{e}\right)^{n-1} - (n-1)I_n - (n-1)I_{n-2}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cosh 2\theta = 1 + 2\sinh^2\theta$

$\sinh^2\theta = \dfrac{\cosh 2\theta - 1}{2}$

$nI_n = \frac{1}{2^n}\left(e+\frac{1}{e}\right)\left(e-\frac{1}{e}\right)^{n-1} - (n-1)I_{n-2}$

$I_2 = \int_0^1 \sinh^2 x \, dx$

$6I_6 = \frac{1}{2^6}\left(e+\frac{1}{e}\right)\left(e-\frac{1}{e}\right)^5 - 5I_4 = 0.3089$

$= \frac{1}{2}\int_0^1 \cosh 2x - 1 \, dx$

$4I_4 = \frac{1}{2^4}\left(e+\frac{1}{e}\right)\left(e-\frac{1}{e}\right)^3 - 3I_2$

$= \frac{1}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^1$

$= \ldots$

5. $I_n = \int \sec^n x \, dx$ easier to integrate $\sec^2 x$ instead of $\sec x$

$= \int \sec^2 x \sec^{n-2} x \, dx$ let $u = \sec^{n-2} x$

$du = (n-2)\sec^{n-3} x (\sec x \tan x) \, dx$

$= \sec^{n-2} x \tan x$

$dv = \sec^2 x \, dx$

$- \int (n-2) \sec^{n-2} x \tan^2 x \, dx$ $v = \tan x$

$= \tan x \sec^{n-2} x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$

$= \tan x \sec^{n-2} x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

$I_n = \tan x \sec^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$

6. $I_n = \int \operatorname{cosec}^n 4x \, dx$

$= \int \operatorname{cosec}^2 4x \operatorname{cosec}^{n-2} 4x \, dx$

$u = \operatorname{cosec}^{n-2} 4x$

$du = (n-2)\operatorname{cosec}^{n-3} 4x(-\operatorname{cosec} 4x \cot 4x)$

$= -4(n-2)\operatorname{cosec}^{n-2} 4x \cot 4x \, dx$

$= -\frac{1}{4}\cot 4x \operatorname{cosec}^{n-2} 4x$

$- \int \cot^2 4x \,(n-2)\operatorname{cosec}^{n-2} 4x \, dx$

$dv = \operatorname{cosec}^2 4x \, dx$

$v = -\frac{1}{4}\cot 4x$

$= -\frac{1}{4}\cot 4x \operatorname{cosec}^{n-2} 4x - \overset{(n-2)}{\int} (\operatorname{cosec}^2 4x - 1) \operatorname{cosec}^{n-2} dx$

$1 + \cot^2 4x = \operatorname{cosec}^2 4x$

$I_n = -\frac{1}{4}\cot 4x \operatorname{cosec}^{n-2} 4x - (n-2) I_n + (n-2) I_{n-2}$

7. $I_n = \int x^n e^{ax}\,dx$

$u = x^n \qquad dv = e^{ax}\,dx$

$du = nx^{n-1}\,dx \qquad v = \frac{1}{a}e^{ax}$

$I_n = \frac{1}{a}x^n e^{ax} - \int \frac{1}{a}e^{ax} nx^{n-1}\,dx$

$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$

$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$

8. $I_n = \int x^n e^{ax^2} dx$

let $u = x^{n-1}$ $\quad$ $dv = 2axe^{ax^2} dx$

$du = (n-1)x^{n-2} dx$ $\quad$ $v = e^{ax^2}$

$= \frac{1}{2a}\int x^{n-1} 2axe^{ax^2} dx$

$= \frac{1}{2a}\left[x^{n-1}e^{ax^2} - (n-1)\int x^{n-2}e^{ax^2} dx\right]$

$= \frac{1}{2a}x^{n-1}e^{ax^2} - \frac{1}{2a}(n-1) I_{n-2}$

$$I_n = \int_0^{\pi/2} x^n \cos x \, dx \qquad \text{find } I_3$$

$u = x^n \qquad dv = \cos x \, dx$

$du = nx^{n-1} dx \qquad v = \sin x$

$$I_n = \left[x^n \sin x\right]_0^{\pi/2} - n\int_0^{\pi/2} \sin x \, x^{n-1} \, dx$$

let $u = x^{n-1} \qquad dv = \sin x \, dx$

$du = (n-1)x^{n-2} dx \qquad v = -\cos x$

$$= \left[\frac{\pi}{2}^n \sin\frac{\pi}{2} - 0^n \sin 0\right] - n\left\{\underbrace{\left[-\cos x \, x^{n-1}\right]_0^{\pi/2}}_{0} + (n-1)\int_0^{\pi/2} x^{n-2} \cos x \, dx\right\}$$

$$= \frac{\pi}{2}^n - n(n-1)I_{n-2}$$

$$I_3 = \frac{\pi}{2}^3 - 3(3-1)I_1$$

$$I_1 = \int_0^{\pi/2} x \cos x \, dx = \left[x \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx$$

let $u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$$= \frac{\pi}{2} \sin\frac{\pi}{2} - 0 \sin 0 - \left[-\cos x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - \left(-\cos\frac{\pi}{2} + \cos 0\right)$$

$$= \frac{\pi}{2} - 1$$

$$I_3 = \frac{\pi}{2}^3 - 3(2)\left(\frac{\pi}{2} - 1\right) = \frac{\pi^3}{8} - 3\pi + 6$$

8. $I_n = \int_0^1 x^n \sqrt{1-x^2}\, dx$

let $u = x^{n-1}$

$du = (n-1)x^{n-2}\, dx$

$dv = -2x(1-x^2)^{\frac{1}{2}}\, dx$

$v = \frac{2}{3}(1-x^2)^{\frac{3}{2}}$

$= -\frac{1}{2}\int_0^1 x^{n-1}(-2x)(1-x^2)^{\frac{1}{2}}\, dx$

$= -\frac{1}{2}\left\{\underbrace{\left[\frac{2}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1}_{0} - \int \frac{2}{3}(1-x^2)^{\frac{3}{2}}(n-1)x^{n-2}\, dx\right\}$

$= \cancel{-}\frac{1}{\cancel{2}}\left[\cancel{-}\frac{\cancel{2}}{3}(n-1)\int(1-x^2)(1-x^2)^{\frac{1}{2}}x^{n-2}\, dx\right]$

$= \frac{1}{3}(n-1)\left[\int_0^1(1-x^2)^{\frac{1}{2}}x^{n-2}\, dx - \int_0^1 x^n(1-x^2)^{\frac{1}{2}}\, dx\right]$

$I_n = \frac{1}{3}(n-1)\left[I_{n-2} - I_n\right]$

$I_n\left(1+\frac{1}{3}(n-1)\right) = \frac{1}{3}(n-1)I_{n-2}$

$I_n\left(\frac{2}{3}+\frac{1}{3}n\right) = \frac{1}{3}(n-1)I_{n-2}$

$I_n(n+2) = (n-1)I_{n-2}$

$I_5 = \frac{5-1}{5+2}I_3$

$I_3 = \frac{3-1}{3+2}I_1$

$= \frac{2}{5}\left(\frac{1}{3}\right)$

$= \frac{2}{15}$

$I_5 = \frac{4}{7}\left(\frac{2}{15}\right)$

$= \frac{8}{105}$

$I_1 = \int_0^1 x\sqrt{1-x^2}\, dx$

$= -\frac{1}{2}\int_0^1 -2x(1-x^2)^{\frac{1}{2}}\, dx$

$= -\frac{1}{2}\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= -\frac{1}{2}\left[-\frac{2}{3}\right]$

$= \frac{1}{3}$

11. $I_n = \int_0^{\pi/2} e^x \cos^n x \, dx$

let $u = \cos^n x$

$du = -n \cos^{n-1} x \sin x \, dx$

$= \left[ \cos^n x \, e^x \right]_0^{\pi/2} + n \int_0^{\pi/2} e^x \sin x \cos^{n-1} x \, dx \Big]$

$dv = e^x dx$

$v = e^x$

$= e^{\pi/2}(0) - 1 + n \int_0^{\pi/2} e^x \sin x \cos^{n-1} x \, dx$

$u = \sin x \cos^{n-1} x$

JUNE 2011(3)

Let

$$I_n = \int_0^{\frac{1}{4}\pi} \tan^n x \, dx,$$

where $n \geqslant 0$. Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that, for $n \geqslant 2$,

$$I_n = \frac{1}{n-1} - I_{n-2}.$$ [4]

Show that $I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{1}{4}\pi$. [4]

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int_0^{\pi/4} \sec^2 x \tan^{n-2} x - \tan^{n-2} x \, dx$$

$$= \left. \frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - I_{n-2}$$

$$= \frac{\tan^{n-1}(\pi/4)}{n-1} - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2}$$

$$I_0 = \int_0^{\pi/4} \tan^0 x \, dx = [x]_0^{\pi/4} = \frac{\pi}{4}$$

$$I_2 = \frac{1}{2-1} - I_0 = 1 - \frac{\pi}{4}$$

$$I_4 = \frac{1}{4-1} - I_2 = \frac{1}{3} - 1 + \frac{\pi}{4} = \frac{1}{3} - 1 + \frac{\pi}{4}$$

$$I_6 = \frac{1}{6-1} - I_4 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4}$$

$$I_8 = \frac{1}{8-1} - I_6 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4} \quad \text{(shown)}$$

JUNE 2012(1)

(a) Show that

$$\int_0^{\pi} e^x \sin x \,dx = \frac{1+e^{\pi}}{2}. \qquad [4]$$

(b) Given that

$$I_n = \int_0^{\pi} e^x \sin^n x \,dx,$$

ANS: $\frac{3}{13\pi}(1+e^{\pi})$

show that, for $n \geqslant 2$,

$$I_n = n(n-1)\int_0^{\pi} e^x \cos^2 x \sin^{n-2} x \,dx - nI_n,$$

(c) and deduce that

$$(n^2+1)I_n = n(n-1)I_{n-2}. \qquad [6]$$

A curve has equation $y = e^x \sin^5 x$. Find, in an exact form, the mean value of $y$ over the interval $0 \leqslant x \leqslant \pi$. [4]

(a)

$$\int_0^{\pi} e^x \sin x \,dx = e^x \sin x \Big]_0^{\pi} - \int_0^{\pi} e^x \cos x \,dx$$

let $u = \sin x$, $du = \cos x\,dx$, $dv = e^x dx$, $v = e^x$

$$= e^{\pi}\sin\pi - e^0 \sin 0 - \left[e^x \cos x\right]_0^{\pi} - \int_0^{\pi} e^x \sin x \,dx$$

let $u = \cos x$, $du = -\sin x\,dx$, $dv = e^x dx$, $v = e^x$

$$2\int_0^{\pi} e^x \sin x \,dx = -\left(e^{\pi}\cos\pi - e^0 \cos 0\right)$$

$$\int_0^{\pi} e^x \sin x \,dx = \frac{-e^{\pi}(-1)+1}{2}$$

$$= \frac{1+e^{\pi}}{2}$$

(b)

$$I_n = \int_0^{\pi} e^x \sin^n x \,dx$$

let $u = \sin^n x$, $du = n\sin^{n-1} x \cos x \,dx$, $dv = e^x dx$, $v = e^x$

$$= \underbrace{e^x \sin^n x \Big]_0^{\pi}}_{0} - n\int_0^{\pi} e^x \cos x \sin^{n-1} x \,dx$$

$\sin^2 x + \cos^2 x = 1$

$$= -n\left\{e^x \cos x \sin^{n-1} x\Big]_0^{\pi} - \int_0^{\pi} e^x \left[(n-1)\cos^2 x \sin^{n-2} x - \sin^n x\right] dx\right.$$

let $u = \cos x \sin^{n-1} x$

$du = (n-1)\cos^2 x \sin^{n-2} x - \sin x \sin^{n-1} x \,dx$

$= [(n-1)\cos^2 x \sin^{n-2} x - \sin^n x]\,dx$

$dv = e^x dx$, $v = e^x$

$$= n(n-1)\int_0^{\pi} e^x \cos^2 x \sin^{n-2} x \,dx - n\int_0^{\pi} e^x \sin^n x \,dx$$

$$= n(n-1)\int_0^{\pi} e^x \cos^2 x \sin^{n-2} x \,dx - nI_n$$

(shown)

(c) $I_n = n(n-1)\int_0^{\pi} e^x(1-\sin^2 x)\sin^{n-2} x \, dx - nI_n$

$$(n+1)I_n = n(n-1)\int_0^{\pi} e^x \sin^{n-2} x \, dx - n(n-1)\int_0^{\pi} e^x \sin^n x \, dx$$

$$= n(n-1)I_{n-2} - n(n-1)I_n$$

$$(n+1+n^2-n)I_n = n(n-1)I_{n-2}$$

$$(n^2+1)I_n = n(n-1)I_{n-2}$$

JUNE 2013(3)

Show that $\int_0^1 xe^{-x^2}\,dx = \frac{1}{2} - \frac{1}{2e}$. [2]

Let $I_n = \int_0^1 x^n e^{-x^2}\,dx$. Show that $I_{2n+1} = nI_{2n-1} - \frac{1}{2e}$ for $n \geqslant 1$. [3]

Find the exact value of $I_7$. [3]

$$\int_0^1 xe^{-x^2}dx = -\frac{1}{2}\int_0^1 -2xe^{-x^2}dx$$

$$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$$

$$= -\frac{1}{2}\left[e^{-1} - e^0\right]$$

$$= \frac{1}{2} - \frac{1}{2e} \quad \text{(shown)}$$

$$I_n = \int_0^1 x^n e^{-x^2}dx = \int_0^1 x^{n-1}xe^{-x^2}dx$$

let $u = x^{n-1}$

$du = (n-1)x^{n-2}dx$

$dv = xe^{-x^2}dx$

$v = -\frac{1}{2}e^{-x^2}$

$$= -\frac{1}{2}e^{-x^2}x^{n-1}\Big]_0^1 + \frac{1}{2}(n-1)\int_0^1 x^{n-2}e^{-x^2}dx$$

$$= -\frac{1}{2}e^{-1}1 + \frac{1}{2}e^{-0^2}0 + \frac{1}{2}(n-1)I_{n-2}$$

$$= -\frac{1}{2e} + \frac{1}{2}(n-1)I_{n-2}$$

let $n$ be $2n+1$

$$I_{2n+1} = -\frac{1}{2e} + \frac{1}{2}(2n+1-1)I_{2n+1-2}$$

$$I_{2n+1} = nI_{2n-1} - \frac{1}{2e}$$

$$I_7 = 3I_5 - \frac{1}{2e}$$

$$I_5 = 2I_3 - \frac{1}{2e}$$

$$I_3 = I_1 - \frac{1}{2e}$$

$$I_1 = \frac{1}{2} - \frac{1}{2e}$$

$$I_3 = \frac{1}{2} - \frac{1}{2e} - \frac{1}{2e} = \frac{1}{2} - \frac{1}{e}$$

$$I_5 = 2\left(\frac{1}{2} - \frac{1}{e}\right) - \frac{1}{2e}$$

$$= -\frac{2}{e} - \frac{1}{2e} + 1$$

$$= -\frac{5}{2e} + 1$$

$$I_7 = \frac{-15}{2e} + 3 - \frac{1}{2e} = 3 - \frac{8}{e}$$

JUNE 2015(1)

Let $I_n = \int_0^{\frac{1}{2}\pi} x^n \sin x \, dx$, where $n$ is a non-negative integer. Show that

$$I_n = n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2}, \quad \text{for } n \geqslant 2. \qquad [5]$$

Find the exact value of $I_4$. [4]

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$$

let $u = x^n$

$du = nx^{n-1}\,dx$

$dv = \sin x \, dx$

$v = -\cos x$

let $u = x^{n-1}$

$du = (n-1)x^{n-2}\,dx$

$dv = \cos x \, dx$

$v = \sin x$

$$= -x^n \cos x \Big]_0^{\pi/2} + n\int_0^{\pi/2} \cos x \, x^{n-1}\,dx$$

$$= -\frac{\pi}{2}^n \cos\frac{\pi}{2} + 0^n \cos 0 + n\left\{ \sin x \, x^{n-1} \Big]_0^{\pi/2} - (n-1)\int_0^{\pi/2} \sin x \, x^{n-2}\,dx \right\}$$

$$= n\left[\sin\frac{\pi}{2}\cdot\frac{\pi}{2}^{n-1} - \sin 0 \cdot 0^{n-1} - (n-1)I_{n-2}\right]$$

$$= n\left(\tfrac{1}{2}\pi\right)^{n-1} - n(n-1)I_{n-2} \quad , n \geqslant 2$$

$$I_4 = 4\left(\tfrac{1}{2}\pi\right)^3 - 4(3)I_2 = 4\left(\tfrac{1}{2}\pi\right)^3 - 12\left(\pi - 2\right) = \tfrac{1}{2}\pi^3 - 12\left(\tfrac{1}{2}\pi^{\prime} - 2\right)$$

$$I_2 = 2\left(\tfrac{1}{2}\pi\right)^1 - 2(1)I_0 = 2\left(\tfrac{1}{2}\pi\right) - 2$$

$$= \tfrac{1}{2}\pi^3 - 12\pi + 24$$

$$I_0 = \int_0^{\pi/2} \sin x \, dx = -\cos x \Big]_0^{\pi/2}$$

$$= -\cos \pi/2 + \cos 0$$

$$= 1$$

## 7.3.4.2 Substitution method under Reduction Formula (for reference only!)

**12.** Let

$$I_n = \int_0^1 (1+x^2)^{\frac{n}{2}-1}\,dx,$$

where $n \geq 0$. Use the fact that $1 + \tan^2\theta = \sec^2\theta$ to show that, for $n \geq 2$,

$$(n-1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\,. \qquad [6]$$

Hence, evaluate $I_3$ in the exact form. [3]

when $x = 1$, $\theta = \tan^{-1}(1)$

let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$I_n = \int_0^{\pi/4} \sec^2\theta^{\frac{n}{2}-1} \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta \sec^{n-2}\theta\, d\theta$$

$\sec^n\theta$

let $u = \sec^{n-2}\theta$

$du = (n-2)\sec^{n-3}\theta \sec\theta\tan\theta\, d\theta$

$$= \left[\tan\theta \sec^{n-2}\theta\right]_0^{\pi/4}$$

$dv = \sec^2\theta\, d\theta$

$v = \tan\theta$

**10.** If $I_n = \int \sec^2 y \sec^n(\tan y)\csc^n(\tan y)\,dy$, by using the substitution $x = \tan y$, show that $I_n = \int \sec^n x \csc^n x\,dx$ and find a reduction formula for $I_n$, in terms of $x$. [7]

Hence find $I_2$, $I_3$, and $I_4$. [5]

$$I_n = \int \sec^2 y \sec^n(\tan y)\,\text{cosec}^n(\tan y)\,dy$$

$$\tan^2 y + 1 = \sec^2 y$$
$$x^2 + 1 = \sec^2 y$$

$$= \int \cancel{(x^2+1)} \sec^n x\,\text{cosec}^n x\,\frac{dx}{\cancel{x^2+1}}$$

$$= \int \sec^n x\,\text{cosec}^n x\,dx$$

$$x = \tan y$$
$$dx = \sec^2 y\,dy$$
$$dy = \frac{dx}{x^2+1}$$

$$= \int \frac{1}{\cos^n x}\left(\frac{1}{\sin^n x}\right)dx$$

$$= \int \frac{1}{\frac{1}{2^n}\sin^n 2x}\,dx$$

$$1 + \cot^2 2x = \text{cosec}^2 2x$$

$$= \int 2^n \text{cosec}^n 2x\,dx$$

$$= 2^n \int \text{cosec}^2 2x\,\text{cosec}^{n-2} 2x\,dx$$

let $u = \text{cosec}^{n-2} 2x$

$du = (n-2)\,\text{cosec}^{n-1} 2x\left(-\frac{2\cot 2x}{\text{cosec}\,2x}\right)dx$

$dv = \text{cosec}^2 2x\,dx$

$v = -\frac{1}{2}\cot 2x$

$$= 2^n\left[-\frac{1}{2}\cot 2x\,\text{cosec}^{n-2} 2x - \cancel{\tfrac{1}{2}}\cancel{(2)}(n-2)\int \cot^2 2x\,\text{cosec}^{n-2} 2x\,dx\right]$$

$$= 2^n\left[-\frac{1}{2}\cot 2x\,\text{cosec}^{n-2} 2x - (n-2)\int(\text{cosec}^2 2x - 1)\,\text{cosec}^{n-2} 2x\,dx\right]$$

$$= -2^{n-1}\cot 2x\,\text{cosec}^{n-2} 2x - (n-2)I_n + 4(n-2)\underbrace{\int 2^n \times \frac{1}{4}\,\text{cose}^{n-2} 2x}_{I_{n-2}}$$

$$I_{n-2} = \int 2^{n-2}\,\text{cosec}^{n-2} 2x$$

$$= -2^{n-1}\cot 2x\,\text{cosec}^{n-2} 2x - (n-2)I_n + 4(n-2)I_{n-2}$$

$$I_n(1+n-2) = -2^{n-1}\cot 2x\,\text{cosec}^{n-2} 2x + 4(n-2)I_{n-2}$$

$$I_n = \frac{-2^{n-1}}{n-1}\cot 2x\,\text{cosec}^{n-2} 2x + \frac{4(n-2)}{n-1}I_{n-2}$$

JUNE 2014(3)

Using the substitution $u = \cos\theta$, or any other method, find $\int \sin\theta\cos^2\theta\,d\theta$. [1]

It is given that $I_n = \int_0^{\frac{1}{2}\pi} \sin^n\theta\cos^2\theta\,d\theta$, for $n \geqslant 0$. Show that, for $n \geqslant 2$,

$$I_n = \frac{n-1}{n+2} I_{n-2}.$$ [5]

let $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$d\theta = \frac{du}{-\sin\theta}$

$$\int \sin\theta\cos^2\theta\,d\theta$$

$$= \int \cancel{\sin\theta}\,u^2 \frac{du}{-\cancel{\sin\theta}}$$

$$= -\int u^2\,du$$

$$= -\frac{u^3}{3} = -\frac{\cos^3\theta}{3} + c$$

$$I_n = \int_0^{\pi/2} \sin^n\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^{n-1}\theta\sin\theta\cos^2\theta\,d\theta$$

let $u = \sin^{n-1}\theta$

$du = (n-1)\sin^{n-2}\theta\cos\theta\,d\theta$

$dv = \sin\theta\cos^2\theta\,d\theta$

$v = -\frac{\cos^3\theta}{3}$

$$= -\frac{1}{3}\cos^3\theta\sin^{n-1}\theta\Big]_0^{\pi/2} + \frac{1}{3}(n-1)\int_0^{\pi/2} \sin^{n-2}\theta\cos^4\theta\,d\theta$$

$$= 0 + \frac{1}{3}(n-1)\int_0^{\pi/2} \sin^{n-2}\theta\cos^2\theta(1-\sin^2\theta)\,d\theta$$

$$= \frac{1}{3}(n-1)\int_0^{\pi/2} \sin^{n-2}\theta\cos^2\theta\,d\theta - \frac{1}{3}(n-1)\int_0^{\pi/2} \sin^n\theta\cos^2\theta\,d\theta$$

$$I_n = \frac{1}{3}(n-1)I_{n-2} - \frac{1}{3}(n-1)I_n$$

$$I_n\left(1+\frac{1}{3}n-\frac{1}{3}\right) = \frac{1}{3}(n-1)I_{n-2}$$

$$I_n\left(\frac{2}{3}+\frac{1}{3}n\right) = \frac{1}{3}(n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n+2} I_{n-2}$$

☆ Algebraic manipulation is hard.

# 3.4.3 Differentiation with Algebraic Manipulation (Reduction Formula) (for reference only)

4. (i) Show that

$$\frac{d}{dx}\left[x\left(1-x^2\right)^n\right]=(2n+1)\left(1-x^2\right)^n-2n\left(1-x^2\right)^{n-1} \qquad [3]$$

$$\frac{d}{dx}\left[x(1-x^2)^n\right]=nx(1-x^2)^{n-1}(-2x)+(1-x^2)^n$$

$$=-2nx^2(1-x^2)^{n-1}+(1-x^2)^n$$

algebraic manipulation

☆ since I don't want $x^2$ since $-2n\int x^2(1-x^2)^{n-1}\,dx$ don't give us $I_n$, hence I'll try to combine it with $(1-x^2)^{n-1}$

$$=2n(-x^2)(1-x^2)^{n-1}+(1-x^2)^n$$

$$=2n(1-x^2-1)(1-x^2)^{n-1}+(1-x^2)^n$$

$$=2n\left[(1-x^2)^n-(1-x^2)^{n-1}\right]+(1-x^2)^n$$

$$=2n(1-x^2)^n-2n(1-x^2)^{n-1}+(1-x^2)^n$$

$$\frac{d}{dx}\left[x(1-x^2)^n\right]=(2n+1)(1-x^2)^n-2n(1-x^2)^{n-1}$$

(ii) It is given that

$$I_n=\int_0^1\left(1-x^2\right)^n\,dx \qquad (n\ge 0)$$

Use the result in part (i) to show that

$$I_n=\frac{2n}{2n+1}I_{n-1} \qquad (n\ge 1) \qquad [2]$$

Hence evaluate $I_5$, leaving your answer as a fraction in its lowest terms. [2]

$$\frac{d}{dx}\left[x(1-x^2)^n\right]=(2n+1)(1-x^2)^n-2n(1-x^2)^{n-1}$$

$$\int_0^1\frac{d}{dx}\left[x-(1-x^2)^n\right]dx=\int_0^1(2n+1)(1-x^2)^n-2n(1-x^2)^{n-1}$$

$$\left[x-(1-x^2)^n\right]_0^1=(2n+1)\int_0^1(1-x^2)^n-2n\int_0^1(1-x^2)^{n-1}dx$$

$$0-0=(2n+1)I_n-2nI_{n-1}$$

$$(2n+1)I_n=2nI_{n-1}$$

$$I_n=\frac{2n}{2n+1}I_{n-1}$$

JUNE 2004

Let

$$I_n = \int_e^{e^2} (\ln x)^n \, dx,$$

where $n \geqslant 0$. By considering $\frac{d}{dx}[x(\ln x)^{n+1}]$, or otherwise, show that

$$I_{n+1} = 2^{n+1}e^2 - e - (n+1)I_n. \qquad [4]$$

Find $I_3$ and deduce that the mean value of $(\ln x)^3$ over the interval $e \leqslant x \leqslant e^2$ is

$$2\left(\frac{e+1}{e-1}\right). \qquad [5]$$

$$\frac{d}{dx}\left[x(\ln x)^{n+1}\right] = x(n+1)(\ln x)^n\left(\frac{1}{x}\right) + (\ln x)^{n+1}(1)$$

$$= (n+1)(\ln x)^n + (\ln x)^{n+1}$$

$$\left[x(\ln x)^{n+1}\right]_e^{e^2} = \int_e^{e^2}(n+1)(\ln x)^n dx + \int_e^{e^2}(\ln x)^{n+1} dx$$

$$e^2(\ln e^2)^{n+1} - e(\ln e)^{n+1} = (n+1)I_n + I_{n+1}$$

$$e^2(2)^{n+1} - e = (n+1)I_n + I_{n+1}$$

$$I_{n+1} = 2^{n+1}e^2 - e - (n+1)I_n$$

JUNE 2017(3)

Let $I_n$ denote $\int_0^2 (4+x^2)^{-n}\,dx$.

**(i)** Find $\frac{d}{dx}\left(x(4+x^2)^{-n}\right)$ and hence show that

$$8nI_{n+1} = (2n-1)I_n + 2\times 8^{-n}. \qquad [5]$$

**(ii)** Use the result for integrating $\frac{1}{x^2+a^2}$ with respect to $x$, in the List of Formulae (MF10), to find the value of $I_1$ and deduce that

$$I_3 = \tfrac{3}{1024}\pi + \tfrac{1}{128}. \qquad [5]$$

(i) $\frac{d}{dx}\left[x(4+x^2)^{-n}\right] = x(-n)(4+x^2)^{-n-1}(2x) + (4+x^2)^{-n}$

$\left[x(4+x^2)^{-n}\right]_0^2 = \int_0^2 -2nx^2(4+x^2)^{-n-1} + (4+x^2)^{-n}\,dx$

since I don't want $x^2$ in my term, I algebraicly manipalated it so it combine together with $(4+x^2)^{-n}$

$2(4+2^2)^{-n} - 0(4+0^2)^{-n} = \int_0^2 -2n(4+x^2-4)(4+x^2)^{-n-1} + (4+x^2)^{-n}\,dx$

$2(8)^{-n} = -2n\int_0^2 (4+x^2)^{-n} - 4(4+x^2)^{-n-1}\,dx + \int_0^2 (4+x^2)^{-n}\,dx$

$2(8)^{-n} = -2nI_n + 8nI_{n+1} + I_n$

$= (-2n+1)I_n + 8nI_{n+1}$

$8nI_{n+1} = (2n-1)I_n + 2\times 8^{-n}$

JUNE 2005

ANS:
(iii) $B_n = (-1)^{n-1} n!$.

The integral $I_n$, where $n$ is a non-negative integer, is defined by

start from $I_{n+1}$

$$I_n = \int_0^1 e^{-x}(1-x)^n \, dx.$$

**(i)** Show that $I_{n+1} = 1 - (n+1)I_n$. [3]

**(ii)** Use induction to show that $I_n$ is of the form $A_n + B_n e^{-1}$, where $A_n$ and $B_n$ are integers. [4]

**(iii)** Express $B_n$ in terms of $n$. [2]

(i) $I_{n+1} = \int_0^1 e^{-x}(1-x)^{n+1} dx$ $\quad u = (1-x)^{n+1}$ $\quad dv = e^{-x} dx$

$du = (n+1)(1-x)^n(-1)\,dx$ $\quad v = -e^{-x}$

$= \left[-e^{-x}(1-x)^{n+1}\right]_0^1 - (n+1)\int_0^1 e^{-x}(1-x)^n dx$ $\quad = -(n+1)(1-x)^n dx$

$= e^{-0}(1-0)^{n+1} - (n+1)I_n$

$= 1-(n+1)I_n$

(ii) $I_{n+1} = 1-(n+1)I_n$ $\quad$ let $P_n$ be a statement that $I_n = A_n + B_n e^{-1}$

for base case where $n=0$, $I_0 = \int_0^1 e^{-x}(1-x)^0 dx = \left[-e^{-x}\right]_0^1 = -e^{-1}+e^{-0} = 1-e^{-1}$

where $A_0 = 1$, $B_0 = -1$

$P_0$

suppose $P_k$ is true, $I_k = A_k + B_k e^{-1}$ , tbpt : $I_{k+1} = A_{k+1} + B_{k+1} e^{-1}$

$P_{k+1}$: $I_{k+1} = 1-(k+1)\left[A_k + B_k e^{-1}\right]$

$= 1 - \left(kA_k + kB_k e^{-1} + A_k + B_k e^{-1}\right)$

$= 1 - A_k(k+1) - B_k(k+1)e^{-1}$

$A_{k+1} = 1 - A_k(k+1)$, $B_{k+1} = -B_k(k+1)$ $\quad$ (shown)

$P_{k+1}$ is true if $P_k$ is true, since $P_0$ is true. $P_k$ is true for all $n \geq 0$

JUNE 2014(1)

ANS: $3-\frac{8}{e}$

It is given that $I_n = \int_0^{\frac{1}{4}\pi} \frac{\sin^{2n} x}{\cos x}\,dx$, where $n \geqslant 0$. Show that

$$I_n - I_{n+1} = \frac{2^{-(n+\frac{1}{2})}}{2n+1}.$$ [5]

$$I_{n+1} = \int_0^{\pi/4} \frac{\sin^{2n+2}x}{\cos x}\,dx$$

should've start from $I_{n+1}$

Hence show that $\int_0^{\frac{1}{4}\pi} \frac{\sin^6 x}{\cos x}\,dx = \ln(1+\sqrt{2}) - \frac{73}{120}\sqrt{2}$. [5]

$$I_n = \int_0^{\pi/4} \frac{\sin^{2n}x}{\cos x}\,dx = \int_0^{\pi/4} \frac{\sin^2 x \sin^{2n-2}x}{\cos x}\,dx$$

$$= \int_0^{\pi/4} \frac{(1-\cos^2 x)\sin^{2n-2}x}{\cos x}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin^{2n-2}x\,dx}{\cos x} - \int_0^{\pi/4} \cos x \sin^{2n-2}x\,dx$$

$$= I_{n-1} - \left[\frac{\sin^{2n-1}}{2n-1}\right]_0^{\pi/4}$$

$$= I_{n-1} - \frac{2^{-(n-1/2)}}{2n-1}$$

$$I_{n-1} - I_n = \frac{2^{-(n-1/2)}}{2n-1}$$

let $n = n+1$

$$I_n - I_{n+1} = \frac{2-(n+1/2)}{2n+1}$$

OR.

$$I_n - I_{n+1} = \int_0^{\pi/4} \frac{\sin^{2n}x}{\cos x}\,dx - \int_0^{\pi/4} \frac{\sin^{2(n+1)}x}{\cos x}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin^{2n}x}{\cos x} - \frac{\sin^2 x \sin^{2n}x}{\cos x}\,dx$$

$$= \int_0^{\pi/4} \frac{\sin^{2n}x}{\cos x}\left(1-\sin^2 x\right)dx$$

$$= \int_0^{\pi/4} \sin^{2n}x \cos x\,dx$$

$$= \left[\frac{\sin^{2n+1}x}{2n+1}\right]_0^{\pi/4}$$

$$= \frac{\sin^{2n+1}\frac{\pi}{4}}{2n+1} = \frac{\left(\frac{\sqrt{2}}{2}\right)^{2n+1}}{2n+1} = \frac{\left(2^{-\frac{1}{2}}\right)^{2n+1}}{2n+1} = \frac{2^{-(n+\frac{1}{2})}}{2n+1}$$

JUNE 2002

Given that

$$C_n = \int_0^1 (1-x)^n \cos x \, dx \quad \text{and} \quad S_n = \int_0^1 (1-x)^n \sin x \, dx,$$

show that, for $n \geqslant 1$,

$$C_n = nS_{n-1} \quad \text{and} \quad S_n = 1 - nC_{n-1}. \qquad [3]$$

Hence find the value of $S_3$, correct to 6 decimal places. [3]

$u = (1-x)^n$ $\qquad dv = \cos x \, dx$

$du = n(1-x)^{n-1}(-1)\,dx$ $\qquad v = \sin x$

$du = -n(1-x)^{n-1}\,dx$

$$C_n = \left[(1-x)^n \sin x\right]_0^1 + n\int_0^1 (1-x)^{n-1} \sin x \, dx$$

$$= 0 - 0 + nS_{n-1}$$

$\therefore C_n = nS_{n-1}$

$u = (1-x)^n$ $\qquad dv = \sin x \, dx$

$du = -n(1-x)^{n-1}\,dx$ $\qquad v = -\cos x$

$$S_n = \left[\cos x (1-x)^n\right]_0^1 - n\int_0^1 (1-x)^{n-1} \cos x \, dx$$

$$= 1(1)^n - nC_{n-1}$$

$\therefore S_n = 1 - nC_{n-1}$

$S_3 = 1 - 3C_2 = 1 - 3\times 2(1-\sin 1) = 0.048826$

$C_2 = 2S_1 = 2(1-\sin 1)$

$S_1 = 1 - C_0 = 1 - \sin 1$

$$C_0 = \int_0^1 (1-x)^0 \cos x \, dx$$

$$= \int_0^1 \cos x \, dx$$

$$= \left[\sin x\right]_0^1 = \sin 1$$

JUNE 2008

**(i)** Given that

$$I_n = \int_0^{\frac{1}{2}\pi} t^n \sin t \, dt,$$

show that, for $n \geqslant 2$,

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}. \qquad [5]$$

(i) $I_n = \int_0^{\pi/2} t^n \sin t \, dt$ let $u = t^n$ $dv = \sin t \, dt$

$du = nt^{n-1} dt$ $v = -\cos t$

$= \left[-t^n \cos t\right]_0^{\pi/2} + \int_0^{\pi/2} nt^{n-1} \cos t \, dt$

let $u = t^{n-1}$ $dv = \cos t \, dt$

$= 0 - 0 + n\left\{\left[t^{n-1}\sin t\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} t^{n-2} \sin t \, dt\right\}$ $du = (n-1)t^{n-2} dt$ $v = \sin t$

$= n\frac{\pi}{2}^{n-1} \sin\frac{\pi}{2} - n(n-1)I_{n-2}$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$

JUNE 2009

Let

$$I_n = \int_0^1 t^n e^{-t}\, dt,$$

where $n \geqslant 0$. Show that, for all $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}. \qquad [3]$$

Hence prove by induction that, for all positive integers $n$,

$$I_n < n!. \qquad [5]$$

$I_n = \int_0^1 t^n e^{-t} dt$ let $u = t^n$, $du = nt^{n-1}dt$, $dv = e^{-t}dt$, $v = -e^{-t}$

$= \left[-t^n e^{-t}\right]_0^1 + n\int_0^1 t^{n-1}e^{-t}dt$

$= -1^n e^{-1} + 0^n e^{-0} + nI_{n-1}$

$I_n = nI_{n-1} - e^{-1}$ (shown)

let $P_n$ be the statement of $I_n < n!$

for $P_0$: $I_0 = \int_0^1 t^0 e^{-t} dt = \int_0^1 e^{-t} dt = [-e^{-t}]_0^1 = -e^{-1} + e^{-0} = 1 - \frac{1}{e}$

$n! = 0! = 1$

$-\frac{1}{e} < 0$

$1 - \frac{1}{e} < 1$

$I_0 < 0!$

LHS < RHS $P_0$ is true.

Suppose $P_k$ is true, $I_k < k!$

to be proven that $P_{k+1}$: $I_{k+1} < (k+1)!$

$I_{k+1} = (k+1)I_k - e^{-1}$

since $I_k < k!$

$(k+1)I_k < (k+1)k!$

$(k+1)I_k - e^{-1} < (k+1)k! - e^{-1} < (k+1)k!$

$I_{k+1} < (k+1)!$ (proven)

$P_{k+1}$ is true if $P_k$ is true, since $P_0$ is true, $P_n$ is true for $n \geqslant 0$

JUNE 2010(1)

Let

$$I_n = \int_1^e x(\ln x)^n \, dx,$$

where $n \geqslant 1$. Show that

start from $I_{n+1}$ $\longrightarrow$ $I_{n+1} = \frac{1}{2}e^2 - \frac{1}{2}(n+1)I_n$. [3]

Hence prove by induction that, for all positive integers $n$, $I_n$ is of the form $A_n e^2 + B_n$, where $A_n$ and $B_n$ are rational numbers. [6]

$$I_{n+1} = \int_1^e x(\ln x)^{n+1} dx$$

let $u = (\ln x)^{n+1}$

$du = (n+1)(\ln x)^n \left(\frac{1}{x}\right) dx$

$$= \left[\frac{x^2}{2}(\ln x)^{n+1}\right]_1^e - \frac{1}{2}(n+1)\int_1^e x(\ln x)^n dx$$

$dv = x\,dx$

$v = \frac{x^2}{2}$

$$= \frac{1}{2}e^2(\ln e)^{n+1} - \frac{1}{2}(1)^2(\ln 1)^{n+1} - \frac{1}{2}(n+1)I_n$$

$$= \frac{1}{2}e^2 - \frac{1}{2}(n+1)I_n$$

let $P_n$ be a statement of $I_n = A_n e^2 + B_n$

for base case, $P_1$ : $I_1 = \int_1^e x(\ln x)^1 dx = \int_1^e x \ln x \, dx$

let $u = \ln x$ $\quad$ $dv = x\,dx$

$du = \frac{1}{x} dx$ $\quad$ $v = \frac{x^2}{2}$

$$= \left[\frac{1}{2}x^2 \ln x\right]_1^e - \frac{1}{2}\int_1^e x\,dx$$

$$= \frac{1}{2}e^2 \ln e - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e$$

$$= \frac{1}{2}e^2 - \frac{1}{4}\left[e^2 - 1\right]$$

$= \frac{1}{4}e^2 + \frac{1}{4}$ , where $A_1 = \frac{1}{4}$, $B_1 = \frac{1}{4}$ ($P_1$ is true)

suppose $P_k$ is true, $I_k = A_k e^2 + B_k$ ; tbpt : $I_{k+1} = A_{k+1}e^2 + B_{k+1}$

$$I_{k+1} = \frac{1}{2}e^2 - \frac{1}{2}(k+1)I_k$$

$$= \frac{1}{2}e^2 - \frac{1}{2}(k+1)(A_k e^2 + B_k)$$

$$= \frac{1}{2}e^2 - \frac{1}{2}(kA_k e^2 + kB_k + A_k e^2 + B_k)$$

$= \left(\frac{1}{2} - \frac{1}{2}kA_k - \frac{1}{2}A_k\right)e^2 - \frac{1}{2}kB_k - \frac{1}{2}B_k$ ; where $A_{k+1} = \frac{1}{2} - \frac{1}{2}kA_k - \frac{1}{2}A_k$

$B_{k+1} = -\frac{1}{2}kB_k - \frac{1}{2}B_k$

JUNE 2013(1)

Let $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$. Prove that, for every positive integer $n$,

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n. \qquad [5]$$

Given that $I_1 = \frac{1}{4}\pi$, find the exact value of $I_3$. [3]

$$I_{n+1} = \int_0^1 \frac{1}{(1+x^2)^{n+1}}\,dx$$

$u = (1+x^2)^{-(n+1)}$

$du = -(n+1)(1+x^2)^{-(n+2)}(2x)\,dx$

$$= \int_0^1 (1+x^2)^{-n-1}\,dx$$

$dv = dx$

$v = x$

$$= \left[x(1+x^2)^{-(n+1)}\right]_0^1 + 2(n+1)\int_0^1 x^2(1+x^2)^{-(n+2)}\,dx$$

$$= 1(1+1^2)^{-(n+1)} + 2(n+1)\int_0^1 (1+x^2-1)(1+x^2)^{-(n+2)}\,dx$$

$$= 2^{-(n+1)} + 2(n+1)\left[\int_0^1 (1+x^2)^{-(n+1)}\,dx - \int_0^1 (1+x^2)^{-(n+2)}\,dx\right]$$

$$= 2^{-(n+1)} + 2(n+1)I_{n+1} - 2(n+1)I_{n+2}$$

$$I_{n+1}(1-2n-2) = 2^{-(n+1)} - 2(n+1)I_{n+2}$$

$$I_{n+1}(-1-2n) = 2^{-(n+1)} - 2(n+1)I_{n+2}$$

$$-I_{n+1}(2n+1) = 2^{-(n+1)} - 2(n+1)I_{n+2}$$

let $n = n-1$

$$2(n-1+1)I_{n-1+2} = 2^{-(n-1+1)} + (2n-2+1)I_n$$

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n$$

$$I_{n+1} = \frac{2^{-n}}{2n} + \frac{2n-1}{2n}I_n$$

$$I_3 = \frac{2^{-2}}{2^2} + \frac{2(2)-1}{2(2)}\left(\frac{1}{8}\pi + \frac{1}{4}\right) = \frac{3}{32}\pi + \frac{1}{4}$$

$$I_2 = \frac{2^{-1}}{2^1} + \frac{2(1)-1}{2(1)}\left(\frac{\pi}{4}\right) = \frac{1}{8}\pi + \frac{1}{4}$$

$$I_1 = \frac{\pi}{4}$$

JUNE 2019(3)

Let $I_n = \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \cot^n x \, dx$, where $n \geqslant 0$.

(i) By considering $\frac{d}{dx}(\cot^{n+1} x)$, or otherwise, show that

$$I_{n+2} = \frac{1}{n+1} - I_n.$$ [5]

$$\frac{d}{dx}(\cot^{n+1}x) = (n+1)\cot^n x(-\operatorname{cosec}^2 x)$$

$1 + \cot^2 x = \operatorname{cosec}^2 x$

$$= -(n+1)\cot^n x \operatorname{cosec}^2 x$$
$$= -(n+1)\cot^n x(1+\cot^2 x)$$
$$= -(n+1)\cot^n x - (n+1)\cot^{n+2} x$$

$$\left[\cot^{n+1} x\right]_{\pi/4}^{\pi/2} = -(n+1)\int_{\pi/4}^{\pi/2} \cot^n x \, dx - (n+1)\int_{\pi/4}^{\pi/2} \cot^{n+2} dx$$

$$-1 = -(n+1)I_n - (n+1)I_{n+2}$$

$$\frac{1}{n+1} = I_n + I_{n+2}$$

$$I_{n+2} = \frac{1}{n+1} - I_n$$

JUNE 2015(3)

Let $I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta}{\cos\theta}\,d\theta$, where $n$ is a non-negative integer.

(i) Use the identity $\sin P + \sin Q \equiv 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$ to show that

$$I_n + I_{n-1} = \frac{2}{2n-1}, \text{ for all positive integers } n. \quad [5]$$

(ii) Find the exact value of $\int_0^{\frac{1}{2}\pi} \frac{\sin 8\theta}{\cos\theta}\,d\theta$. [4]

$$I_n + I_{n-1} = \int_0^{\pi/2} \frac{\sin 2n\theta}{\cos\theta}\,d\theta + \int_0^{\pi/2} \frac{\sin 2(n-1)\theta}{\cos\theta}\,d\theta$$

SOMETIMES, we have to start from a combination of $I_n, I_n$...

$$= \int_0^{\pi/2} \frac{\sin 2n\theta + \sin 2(n-1)\theta}{\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{2\sin\frac{1}{2}(2n\theta + 2n\theta - 2\theta)\cos\frac{1}{2}(2n\theta - 2n\theta + 2\theta)}{\cos\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{2\sin\frac{1}{2}(4n\theta - 2\theta)\cancel{\cos\frac{1}{2}(2\theta)}}{\cancel{\cos\theta}}\,d\theta$$

$$= \int_0^{\pi/2} 2\sin(2n-1)\theta\,d\theta$$

$2n-1 \rightarrow$ odd number.

$\cos(2n-1)\frac{\pi}{2} = 0$

$$= 2\left[\frac{-\cos(2n-1)\theta}{2n-1}\right]_0^{\pi/2}$$

$$= \frac{2}{2n-1}\left[\underbrace{-\cos(2n-1)\times\frac{\pi}{2}}_{0} + \underbrace{\cos(2n-1)\times 0}_{1}\right]$$

$$= \frac{2}{2n-1} \quad \text{(proven)}$$

(ii) $I_n = \frac{2}{2n-1} - I_{n-1}$

$I_4 = \frac{2}{2(4)-1} - I_3 = \frac{\pi}{2} - \frac{152}{105}$

$I_3 = \frac{2}{2(3)-1} - I_2 = \frac{26}{15} - \frac{\pi}{2}$

$I_2 = \frac{2}{2(2)-1} - I_1 = \frac{\pi}{2} - \frac{4}{3}$

$I_1 = \frac{2}{2(1)-1} - I_0 = 2 - \frac{\pi}{2}$

$I_0 = \int_0^{\frac{\pi}{2}} d\theta = \theta\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

NOV 2015(1)

It is given that $I_n = \int_1^e (\ln x)^n \, dx$ for $n \geqslant 0$. Show that

$$I_n = (n-1)[I_{n-2} - I_{n-1}] \text{ for } n \geqslant 2.$$ [6]

$$I_n = \int_1^e (\ln x)^n dx$$

$u = (\ln x)^n$

$du = n(\ln x)^{n-1}\left(\frac{1}{x}\right) dx$

$$= \left[x(\ln x)^n\right]_1^e - n\int_1^e (\ln x)^{n-1} dx$$

$dv = dx$

$v = x$

$I_n = e - nI_{n-1} \leftarrow = e - n\int_1^e (\ln x)^{n-1} dx$

$e = I_n + nI_{n-1}$

$$= e - n\left\{\left[x(\ln x)^{n-1}\right]_1^e - (n-1)\int_1^e (\ln x)^{n-2} dx\right\}$$

$u = (\ln x)^{n-1}$

$du = (n-1)(\ln x)^{n-2}\left(\frac{1}{x}\right) dx$

$dv = dx$

$v = x$

$$= e - n\left[e - (n-1)I_{n-2}\right]$$

$$= e - ne + n(n-1)I_{n-2}$$

$$I_n = e - ne + n(n-1)I_{n-2}$$

$$= e(1-n) + n(n-1)I_{n-2}$$

$$= -e(n-1) + n(n-1)I_{n-2}$$

$$= (n-1)\left[-e + nI_{n-2}\right]$$

$$= (n-1)(-I_n - nI_{n-1} + nI_{n-2})$$

$$I_n = -(n-1)I_n + n(n-1)(I_{n-2} - I_{n-1})$$

$$I_n(\cancel{1+n-1}) = \cancel{n}(n-1)(I_{n-2} - I_{n-1})$$

$$I_n = (n-1)(I_{n-2} - I_{n-1})$$

prove that $\sinh^{-1}x = \ln(x \pm \sqrt{x^2+1})$

(i) $y = \sinh^{-1}x$

$\sinh y = x$

$\frac{e^y - e^{-y}}{2} = x$

$e^y - e^{-y} = 2x$

$e^y - \frac{1}{e^y} = 2x$

$(e^y)^2 - 2x(e^y) - 1 = 0$

$e^y = \frac{-(-2x) \pm \sqrt{4x^2 - 4(-1)}}{2(1)}$

$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$e^y = x \pm \sqrt{x^2+1}$

$y = \ln(x \pm \sqrt{x^2+1}) = \ln(x + \sqrt{x^2+1})$

$I_n = \int_0^{\alpha} \sinh^n\theta \, d\theta$

$= \int_0^{\alpha} \sinh\theta \sinh^{n-1}\theta \, d\theta$

$= [\cosh\theta \sinh^{n-1}\theta]_0^{\alpha} - (n-1)\int_0^{\alpha} \cosh^2\theta \sinh^{n-2}\theta \, d\theta$

$= \cosh(\ln(1+\sqrt{1^2+1}))\sinh^{n-1}(\ln(1+\sqrt{1^2+1}))$

$- \cosh 0 \sinh^{n-1} 0 - (n-1)\int_0^{\alpha} (1+\sinh^2\theta)\sinh^{n-2}\theta \, d\theta$

$= \sqrt{2}(1)^{n-1} - 1(0) - (n-1)I_{n-2} - (n-1)I_n$

let $u = \sinh^{n-1}\theta$

$du = (n-1)\sinh^{n-2}\theta \cosh\theta \, d\theta$

$dv = \sinh\theta \, d\theta$

$v = \cosh\theta$

$-\sinh^2\theta + \cosh^2\theta = 1$

$\sinh^{-1} 1 = \alpha$

$\sinh\alpha = 1$

$-1 + \cosh^2\alpha = 1$

$\cosh^2\alpha = 2$

$\cosh\alpha = \sqrt{2}$

$I_n(1+n-1) = \sqrt{2} - (n-1)I_{n-2}$

$nI_n = \sqrt{2} - (n-1)I_{n-2}$ $\qquad I_n = \frac{1}{n}\sqrt{2} - \frac{n-1}{n} I_{n-2}$

$I_4 = \frac{1}{4}\sqrt{2} - \frac{3}{4}I_2 = \frac{1}{4}\sqrt{2} - \frac{3}{4}\left(\frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(1+\sqrt{2})\right) = \frac{3}{8}\ln(1+\sqrt{2}) - \frac{1}{8}\sqrt{2}$

$I_2 = \frac{1}{2}\sqrt{2} - \frac{1}{2}I_0 = \frac{1}{2}\sqrt{2} - \frac{1}{2}\ln(1+\sqrt{2})$

$I_0 = \int_0^{\alpha} d\theta = [\theta]_0^{\alpha} = \alpha = \ln(1+\sqrt{1^2+1})$

example of hyperbolic reduction formula.

(i) $\sinh(\cosh^{-1}2)=\sqrt{3}$ (prove)

let $x=\cosh^{-1}2$

$\cosh x=2$

$-\sinh^2 x+\cosh^2 x=1$

$\sinh^2 x=2^2-1$

$\sinh x=\sqrt{3}$

(ii) $I_n=\int_0^{\beta}\cosh^n x\,dx$ , $\beta=\cosh^{-1}2$

DO IT THE SAME WAY AS TRIGO FUNCTION

$I_n=\int_0^{\beta}\cosh x\cosh^{n-1}x\,dx$

$u=\cosh^{n-1}x$

$du=(n-1)\cosh^{n-2}x(\sinh x)\,dx$

$dv=\cosh x\,dx$

$v=\sinh x$

$=\left[\sinh x\cosh^{n-1}x\right]_0^{\beta}-(n-1)\int_0^{\beta}\sinh^2 x\cosh^{n-2}x\,dx$

$=\sinh\beta\cosh^{n-1}\beta-0-(n-1)\int_0^{\beta}(\cosh^2-1)\cosh^{n-2}x\,dx$

$\cosh\beta=2$

$\sinh\beta=\sqrt{3}$

$=\sqrt{3}(2)^{n-1}-(n-1)\int_0^{\beta}\cosh^n x-\cosh^{n-2}x\,dx$

$=\sqrt{3}(2)^{n-1}-(n-1)I_n+(n-1)I_{n-2}$

$I_n(1+n-1)=\sqrt{3}(2)^{n-1}+(n-1)I_{n-2}$

$nI_n=2^{n-1}\sqrt{3}+(n-1)I_{n-2}$

$I_n=\frac{1}{n}2^{n-1}\sqrt{3}+\frac{n-1}{n}I_{n-2}$

$I_5=\frac{1}{5}2^4\sqrt{3}+\frac{4}{5}I_3=\frac{16}{5}\sqrt{3}+\frac{4}{5}(2\sqrt{3})=\frac{24}{5}\sqrt{3}$

$I_3=\frac{1}{3}2^2\sqrt{3}+\frac{2}{3}I_1=\frac{4}{3}\sqrt{3}+\frac{2}{3}\sqrt{3}=2\sqrt{3}$

$I_1=\int_0^{\beta}\cosh x\,dx=\left[\sinh x\right]_0^{\beta}=\sinh\beta-\sinh 0=\sqrt{3}$

(b) using $u = \tanh x$, find $\int \operatorname{sech}^2 x \tanh^2 x \, dx$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\operatorname{sech}^2 x = 1 - u$$

$$\int \operatorname{sech}^2 x \tanh^2 x \, dx = \int \operatorname{sech}^2 x \, u^2 \frac{du}{\operatorname{sech}^2 x}$$
$$= \int u^2 \, du$$
$$= \frac{1}{3} u^3 + c$$
$$= \frac{1}{3} \tanh^3 x + c$$

(c) $I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$

show that, $n \geq 2$:

$$(n+1) I_n = \frac{4}{5}^3 \left(\frac{3}{5}\right)^{n-2} + (n-2) I_{n-2}$$

$I_n = \int \operatorname{sech}^n x \tanh^2 x$

only take out $\tanh^2 x$ since we are refering back to

$\operatorname{sech}(\ln 3) = \frac{2}{e^{\ln 3} + e^{-\ln 3}}$
$= \frac{2}{3 + \frac{1}{3}} = \frac{3}{5}$

$$I_n = \int_0^{\ln 3} \operatorname{sech}^n x \tanh^2 x \, dx$$
$$= \int_0^{\ln 3} \operatorname{sech}^{n-2} x \operatorname{sech}^2 x \tanh^2 x \, dx$$
$$= \left[\frac{1}{3} \operatorname{sech}^{n-2} x \tanh^3 x\right]_0^{\ln 3} + \frac{n-2}{3} \int_0^{\ln 3} \operatorname{sech}^{n-3} x \tanh^4 x \, dx$$

let $u = \operatorname{sech}^{n-2} x$
$du = (n-2) \operatorname{sech}^{n-3} x \, (-\operatorname{sech} x \tanh x) \, dx$
$= -(n-2) \operatorname{sech}^{n-2} \tanh x \, dx$

$dv = \operatorname{sech}^2 x \tanh^2 x \, dx$
$v = \frac{1}{3} \tanh^3 x$

$$= \frac{3}{5}^{n-2} \left(\frac{4}{5}\right)^3 - 0 + \frac{n-2}{3} \int_0^{\ln 3} \operatorname{sech}^{n-2} x \tanh^2 x (1 - \operatorname{sech}^2 x) \, dx$$
$$\frac{n-2}{3} \int_0^{\ln 3} \operatorname{sech}^{n-2} x \tanh^2 x - \operatorname{sech}^n x \tanh^2 x \, dx$$
$$\frac{n-2}{3} I_{n-2} - \frac{n-2}{3} I_n$$

## 7.3.5 Definite Integral.

$$\int_a^b f(x)\,dx = \left[\int f(x)dx\right]_a^b = F(b)-F(a)$$

if u-sub,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \qquad \text{let } u=g(x)$$

remember to find what's u when your x change.

eg. $\int_2^{10} \frac{3}{\sqrt{5x-1}}dx = \int_9^{49} \frac{3}{\sqrt{u}}\frac{du}{5} = \frac{3}{5}\int_9^{49} u^{-\frac{1}{2}}du = \frac{3}{5}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_9^{49} = \frac{6}{5}\left(\sqrt{49}-\sqrt{9}\right)$

$= \frac{6}{5}(7-3)$

$= \frac{24}{5}$

let $u = 5x-1$

$du = 5\,dx$

when $x=2$, $u=5(2)-1=9$

" $x=10$, $u=5(10)-1=49$.

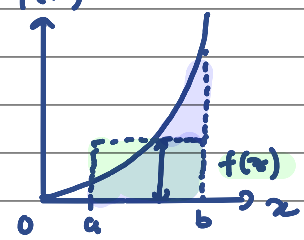
### 7.3.5.1 Mean value Theorem.

$$\int_a^b f(x)dx = f(z)\cdot(b-a), \text{ for } z \in [a,b]$$

this can also be think as:

$$\frac{\int_a^b f(x)dx}{b-a} = f(z) \rightarrow \frac{\text{Area}}{\text{width}} = \text{average height.}$$

(average value of a function).




$f(z)\cdot(b-a) = \int_a^b f(x)dx.$

eg. find $f_{average}$ of $f(x)=x^2$ for $x \in [1,3]$

$\int_1^3 x^2dx = f(z)\cdot(3-1)$

$\left.\frac{x^3}{3}\right|_1^3 = f(z)\cdot 2$

$\frac{27}{3} - \frac{1}{3} = f(z)\cdot 2$

$f_{average} = \frac{13}{3}$ //

find x at $f_{average}$.

$f(z) = \frac{13}{3}$

$z^2 = \frac{13}{3} \rightarrow z = \sqrt{\frac{13}{3}}$

## 7.3.5.2 Derivative of Definite Integral. (fundamental theorem of Calculus)

$$g(x) = \int_a^x f(t)\,dt$$

* $F(x) = \int f(x)\,dx$.

$$g'(x) = \left[\int_a^x f(t)\,dt\right]'$$
$$= \left[F(t)\Big|_a^x\right]'$$
$$= \left[F(x) - F(a)\right]'$$
$$= f(x)$$

since $F(a)$ is a constant. substituting a constant, a into $F(t)$ result in a constant.

this can also be written as:

$$g(x) = \int_a^x f(t)\,dt$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right]$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[F(t)\Big|_{t=a}^{t=x}\right]$$
$$\frac{d}{dx}[g(x)] = \frac{d}{dx}\left[F(x) - F(a)\right]$$
$$\frac{d}{dx}[g(x)] = f(x)$$

what if: $g(x) = \int_a^b f(t)\,dt$?

$$\frac{d}{dx}[g(x)] = \frac{d}{dx}[F(b) - F(a)]$$
$$= 0$$

what if $g(t) = \int_a^b f(t)\,dt$ ?

$$\frac{d}{dt}[g(t)] = \frac{d}{dt}[F(b) - F(a)]$$
$$g'(t) = 0$$

$$\therefore \left(\int_a^x f(t)\,dt\right)' = f(x)$$

MAIN PART!

if $g(x) = \int_a^{h(x)} f(t)\,dt$

$$g'(x) = \left[\int_a^{h(x)} f(t)\,dt\right]'$$
$$g'(x) = \left[F(t)\Big|_{t=a}^{t=h(x)}\right]'$$
$$g'(x) = \left[F[h(x)] - F(a)\right]'$$
$$g'(x) = \left[F[h(x)]\right]' - \left[F(a)\right]'$$
$$g'(x) = h'(x)\,f[h(x)] \quad \text{(chain rule)}$$

$$\therefore \left(\int_a^{g(x)} f(t)\,dt\right)' = g'(x)\,f(g(x))$$

eg1. $\frac{d}{dx}\left[\int_0^x \sqrt{t^2+4}\,dt\right] = \frac{d}{dx}\left[\int_0^x f(t)\,dt\right]$ let $f(t)=\sqrt{t^2+4}$

$= \frac{d}{dx}\left[F(t)\Big|_{t=0}^{t=x}\right]$

$= \frac{d}{dx}\left[F(x)-F(0)\right]$

$= f(x)$

$= \sqrt{x^2+4}$ //

eg2. $\frac{d}{dx}\left[\int_x^4 \sqrt{t^3+5}\,dt\right] = \frac{d}{dx}\left[\int_x^4 f(t)\,dt\right]$ let $f(t)=\sqrt{t^3+5}$

$= \frac{d}{dx}\left[F(t)\Big|_{t=x}^{t=4}\right]$

$= \frac{d}{dx}\left[F(4)-F(x)\right]$ $F'(4)=0$

$= -f(x)$

$= -\sqrt{x^3+5}$ //

eg3. $\frac{d}{dx}\left[\int_5^{x^3} \sqrt{t^3-4}\,dt\right] = \frac{d}{dx}\left[\int_5^{x^3} f(t)\,dt\right]$ let $f(t)=\sqrt{t^3-4}$

$= \frac{d}{dx}\left[F(t)\Big|_{t=5}^{t=x^3}\right]$

$= \frac{d}{dx}\left[F(x^3)-F(5)\right]$

$= 3x^2 f(x^3)$

$= 3x^2\sqrt{x^9-4}$

eg4. $\frac{d}{dx}\left[\int_{x^2}^{x^3} \sqrt{t^4-2}\,dt\right] = \frac{d}{dx}\left[\int_{x^2}^{x^3} f(t)\,dt\right]$ let $f(t)=\sqrt{t^4-2}$

$= \frac{d}{dx}\left[F(t)\Big|_{t=x^2}^{t=x^3}\right]$

my guess:

$3x^2\sqrt{x^{12}-2}-2x\sqrt{x^8-2}$

$= \frac{d}{dx}\left[F(x^3)-F(x^2)\right]$

$= 3x^2 f(x^3) - 2x f(x^2)$

$= 3x^2\sqrt{x^{12}-2} - 2x\sqrt{x^8-2}$

# 7·3·6 Improper Integral.

1. $\int_a^{\infty} f(x)\,dx$
2. $\int_{-\infty}^{b} f(x)\,dx$
3. $\int_{-\infty}^{\infty} f(x)\,dx$
4. $\int_a^{b} f(x)\,dx$, where $f(x)$ is undefined or discontinuous somewhere on $[a,b]$

eg. $\int_{-c}^{c} \frac{1}{x^2}dx$, where $c \in \mathbb{R}$

1. $$\int_a^{\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^{t} f(x)\,dx$$

2. $$\int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty} \int_t^{b} f(x)\,dx$$

3. $$\int_{-\infty}^{+\infty} f(x)dx = \lim_{t\to-\infty} \int_t^{c} f(x)\,dx + \lim_{t\to+\infty} \int_c^{t} f(x)\,dx$$

c be any finite number
usually 0 is a good choice

4. $\int_a^b f(x)dx$, if $f(x)$ is undefined on $x=c$, where $a<c<b$

$$\int_a^b f(x)dx = \lim_{t\to c^-} \int_a^{t} f(x)\,dx + \lim_{t\to c^+} \int_t^{b} f(x)\,dx$$

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx \text{ if } f(x) \text{ is discontinuous at } x=a.$$

(extra to 4)

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx \text{ if } f(x) \text{ is discontinuous at } x=b.$$

eg1. $$\int_2^{+\infty} \frac{1}{x}\,dx = \lim_{t\to+\infty} \int_2^t \frac{1}{x}\,dx$$
$$= \lim_{t\to+\infty} \left[\ln x\right]_2^t$$
$$= \lim_{t\to+\infty} \left[\ln t - \ln 2\right]$$
$$= \infty$$

eg2. $$\int_{-\infty}^{-3} \frac{1}{x^2}\,dx = \lim_{t\to-\infty} \int_t^{-3} x^{-2}\,dx$$
$$= \lim_{t\to-\infty} \left[\frac{x^{-1}}{-1}\right]_t^{-3}$$
$$= \lim_{t\to-\infty} \left[-\frac{1}{(-3)} + \frac{1}{t}\right]$$
$$= \frac{1}{3}$$

eg3. $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^0 \frac{1}{1+x^2}\,dx + \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx$$
$$= \lim_{t\to-\infty} \left[\tan^{-1}x\right]_t^0 + \lim_{t\to\infty} \left[\tan^{-1}x\right]_0^t$$
$$= \lim_{t\to-\infty} \left[0 - \tan^{-1}t\right] + \lim_{t\to\infty} \left[\tan^{-1}t - 0\right]$$
$$= \frac{\pi}{2} + \frac{\pi}{2}$$
$$= \pi$$

eg4. $$\int_0^4 \frac{1}{(x-3)^2}\,dx = \lim_{t\to3^-} \int_0^t \frac{1}{(x-3)^2}\,dx + \lim_{t\to3^+} \int_t^4 \frac{1}{(x-3)^2}\,dx$$

(discontinuing at $x=3$) $$= \lim_{t\to3^-} \left[\frac{(x-3)^{-1}}{-1}\right]_0^t + \lim_{t\to3^+} \left[\frac{(x-3)^{-1}}{-1}\right]_t^4$$
$$= \lim_{t\to3^-} \left[-\frac{1}{x-3}\right]_0^t + \lim_{t\to3^+} \left[-\frac{1}{x-3}\right]_t^4$$

$3^- - 3 \to -\infty$ $$= \lim_{t\to3^-} \left[\underbrace{-\frac{1}{t-3}}_{\text{approaches } +\infty} - \frac{1}{3}\right] + \lim_{t\to3^+} \left[-1 + \frac{1}{t-3}\right]$$

$\therefore$ the integral is divergent.

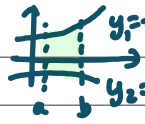
# 7.3.7 Integration Application.

## 1. Area.

$\int y\,dx$ , $\int x\,dy$

$\int y_1 - y_2\,dx$ , $\int x_1 - x_2\,dy$

another trick for Area, if:




can straight away: $A = \int_a^b y_1 - y_2\,dx$

cause we can move y up by:

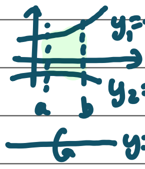
$A = \int_a^b (y_1 + \cancel{k}) - (y_2 + \cancel{k})\,dx$

## 2. Volume.

$\pi \int y^2\,dx$ , $\pi \int x^2\,dy$

$\pi \int y_1^2 - y_2^2\,dx$, $\pi \int x_1^2 - x_2^2\,dx$

for volume, if:

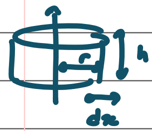



$V = \pi \int (y_1 - k)^2 - (y_2 - k)^2\,dx$

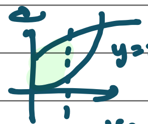
* just imagine you cut everything down the $y = k$ line

or shell method:

$dV = 2\pi r h \cdot dx$




eg.




$V = \int 2\pi x(\sqrt{x} + 1 - x^3)\,dx$

## 3. Arc Length.

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ , $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ , $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

(cartesian) (parametric) (polar)

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

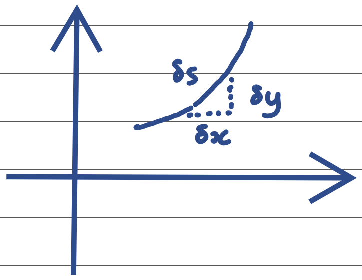
## 4. Surface Area.

| Rotate about | | Cartesian Equation ($y = f(x)$) | Parametric Equation ($y = f(t)$, $x = f(t)$) |
|---|---|---|---|
| x-axis | $\int 2\pi y\,ds$ | $\int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ | $\int 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ |
| y-axis | $\int 2\pi x\,ds$ | $\int 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$ | $\int 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ |

ds unchanged since $(ds)^2 = (dx)^2 + (dy)^2$

| rotate about: | Polar eqn. | |
|---|---|---|
| x-axis ($\theta = 0$) | $\int 2\pi\, r\sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ | $r = f(\theta)$ |
| y-axis ($\theta = \pi/2$) | $\int 2\pi\, r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ | $r = f(\theta)$ |

for (3) Arc Length — Derivation.




1. cartesian

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$\frac{(\delta s)^2}{(\delta x)^2} = \frac{(\delta x)^2}{(\delta x)^2} + \frac{(\delta y)^2}{(\delta x)^2}$$

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$$

as $\delta x \to 0$,

$$\frac{ds}{dx}^2 = 1 + \frac{dy}{dx}^2$$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\int \frac{ds}{dx}\,dx = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$s = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

2. parametric.

$$\frac{(\delta s)^2}{(\delta t)^2} = \frac{(\delta x)^2}{(\delta t)^2} + \frac{(\delta y)^2}{(\delta t)^2}$$

as $\delta t \to 0$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

for easier reference, formula for surface area continued below...

## 3. Polar coordinates.

$x = r\cos\theta$ , $y = r\sin\theta$

$$\frac{\delta s^2}{\delta\theta^2} = \frac{\delta x^2}{\delta\theta^2} + \frac{\delta y^2}{\delta\theta^2}$$

r is always a function of $\theta$, so both are variables !

as $\delta\theta \to 0$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$x = r\cos\theta$ where $r = f(\theta)$

$$\frac{dx}{d\theta} = -f(\theta)\sin\theta + f'(\theta)\cos\theta$$

$$\int \frac{ds}{d\theta}\, d\theta = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$y = r\cos\theta$

$$\frac{dy}{d\theta} = f(\theta)\cos\theta + f'(\theta)\sin\theta$$

$$s = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left(-f(\theta)\sin\theta + f'(\theta)\cos\theta\right)^2 + \left(f(\theta)\cos\theta + f'(\theta)\sin\theta\right)^2$$

$$= [f(\theta)]^2\sin^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + [f'(\theta)]^2\cos^2\theta$$

$$+ [f(\theta)]^2\cos^2\theta + 2f(\theta)f'(\theta)\sin\theta\cos\theta + [f'(\theta)]^2\sin^2\theta$$

$$= [f'(\theta)]^2(\sin^2\theta + \cos^2\theta) + [f(\theta)]^2(\sin^2\theta + \cos^2\theta)$$

$$= [f'(\theta)]^2 + [f(\theta)]^2$$

since $r = f(\theta)$, $\frac{dr}{d\theta} = f'(\theta)$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

## 5. Center of Mass.

for 3D: mass = volume × density
for 2D mass = Area × density.
(cartesian plane)

for $dx$, where $y=f(x)$

Mass $= \int \rho y \, dx$ or $\int \rho (y_1 - y_2) \, dx$

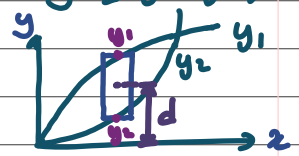
g can be ignored.

Moment = Force × distance.
(about y)

$M_y = \int \rho y \cdot x \, dx$ or $\int \rho (y_1 - y_2) x \, dx$.

$M_x = \int \rho y \cdot \frac{y}{2} \, dx$ or $\int \rho (y_1 - y_2) \cdot \frac{1}{2}(y_1 + y_2) \, dx$.

why $\frac{1}{2}(y_1+y_2)$?




distance for $M_x$, $d = y_2 + \frac{y_1 - y_2}{2}$
$= \frac{2y_2 - y_2 + y_1}{2}$
$= \frac{y_1 + y_2}{2}$

centre of mass: $\bar{x} = \frac{M_y}{m} = \frac{\int \rho (y_1 - y_2) x \, dx}{\int \rho (y_1 - y_2) \, dx}$

$\bar{y} = \frac{M_x}{m} = \frac{\int \rho (y_1 - y_2) \cdot \frac{1}{2}(y_1 + y_2) \, dx}{\int \rho (y_1 - y_2) \, dx}$.

alternatively, for $dy$: $x = f(y)$

Mass $= \int \rho (x_1 - x_2) \, dy$

$M_x = \int \rho (x_1 - x_2) y \, dy$

$M_y = \int \rho (x_1 - x_2) \cdot \frac{1}{2}(x_1 + x_2) \, dy$

centre of mass: $\bar{x} = \frac{M_y}{m} = \frac{\int \rho (x_1 + x_2) \cdot \frac{1}{2}(x_1 + x_2) \, dy}{\int \rho (x_1 + x_2) \, dy}$

$\bar{y} = \frac{M_x}{m} = \frac{\int \rho (x_1 + x_2) y \, dy}{\int \rho (x_1 + x_2) \, dy}$

continue next page ...

or can just recall the com formula in Further Mechanics:

$$\bar{x} = \frac{\sum m x_i}{\sum m} \quad , \quad \bar{y} = \frac{\sum m y_i}{\sum m}$$

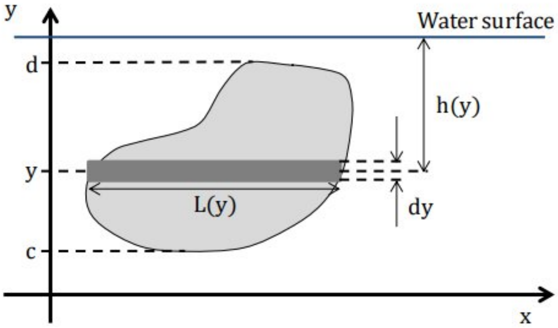
which can be derived into:

$$\bar{x} = \frac{\int \rho (y_1 - y_2) x \, dx}{\int \rho (y_1 - y_2) \, dx} \quad , \quad \bar{y} = \frac{\int \rho (x_1 - x_2) y \, dy}{\int \rho (x_1 - x_2) \, dy}$$

## 6. Forces. ( covered more on 'double integral application)

The force $F$ exerted by a fluid of constant unit weight $\gamma$ on one side of a submerged region of the type illustrated in the figure below is

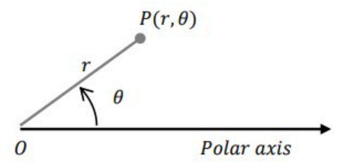
$$F = \int_c^d \gamma \cdot h(y) \cdot L(y) \, dy$$




Force on a general submerged shape.

(this is only applicable when force depend on a single dimension)

# c8 Polar Coordinates.




$x = r\cos\theta$, $y = r\sin\theta$ $\rightarrow$ $r = f(\theta)$

$r^2 = x^2 + y^2$, $\frac{y}{x} = \tan\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$x = f(\theta)\cos\theta$

$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$

$y = f(\theta)\sin\theta$

$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ $\rightarrow$ $dy/d\theta = 0$ (parallel to $\theta = 0$)

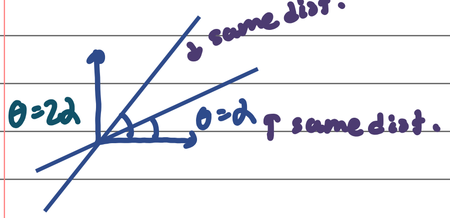
$dx/d\theta = 0$ (parallel to $\theta = \pi/2$)

$\frac{dr}{d\theta} = 0$ $\rightarrow$ furthest point away from center.

to find equation of symmetry: (MAIN WAY TO FIND SYMMETRY)

constant variable

formula : $f(2\alpha - \theta) = f(\theta)$ , where $\theta = \alpha$ is the equation of symmetry.



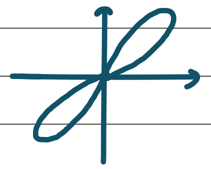
to test for symmetry :

(a) if $r = f(\theta) = f(-\theta)$ , then $\theta = 0$ is an eqn of symmetry .

(b) if $r = f(\pi - \theta) = f(\theta)$ , then $\theta = \pi/2$ is an eqn of symmetry .

(c) if $-r = r$ , for $\theta \in \mathbb{R}$ , then the curve is symmetric about the pole .
($r = 0$)

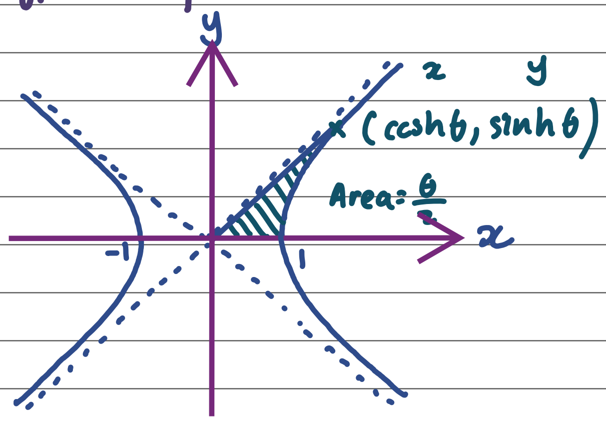
eg for (c)



rose / petal curve.

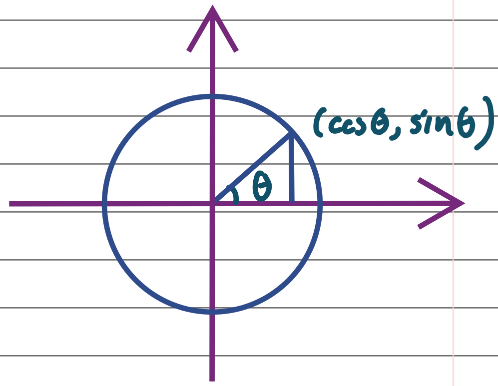
# c9 Hyperbolic Function

Hyperbolic function.




$x^2 - y^2 = 1$

Trigonometric function.




$x^2 + y^2 = 1$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}$$

$$\operatorname{sech}\theta = \frac{1}{\cosh\theta}$$

$$\operatorname{cosech}\theta = \frac{1}{\sinh\theta}$$

$$\coth\theta = \frac{1}{\tanh\theta}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

* to find $\cosh(\sinh^{-1} 5)$:

$\cosh(\sinh^{-1} 5) = \cosh x = \sqrt{26}$

let $\sinh^{-1} 5 = x$

$\sinh x = 5$

$-\sinh^2 x + \cosh^2 x = 1$

$-5^2 + \cosh^2 x = 1$

$\cosh^2 x = 26$

$\cosh x = \sqrt{26}$

# 9·2 Osborn's Rule

## OSBORN'S RULE

Conversion of formula / identity from trigonometry to hyperbolic function.

→ Product of $\underbrace{\sin A \text{ and } \sin B}_{\text{two sine function}}$ (multiply with a negative sign)

eg1. $\sin^2 x + \cos^2 x = 1$

→ $-\sinh^2 x + \cosh^2 x = 1$

eg2. $1 + \cot^2 x = \operatorname{cosec}^2 x$. → $\operatorname{cosec}^2 x = \frac{1}{\sin^2 x}$

→ $1 - \coth^2 x = -\operatorname{cosech}^2 x$

↳ $\cot^2 x = \frac{1}{\tan^2 x} = \frac{\cos^2 x}{\sin^2 x}$.

eg3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

→ $\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B$.

eg4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

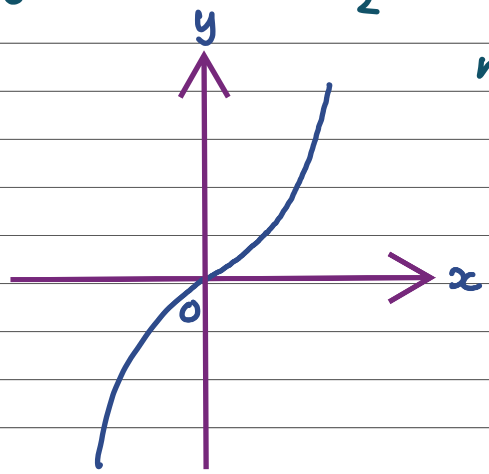
→ $\cosh(A-B) = \cosh A \cosh B - \sinh A \sinh B$

eg5. $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

→ $\tanh(A+B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$.

# 9·3 Sketching

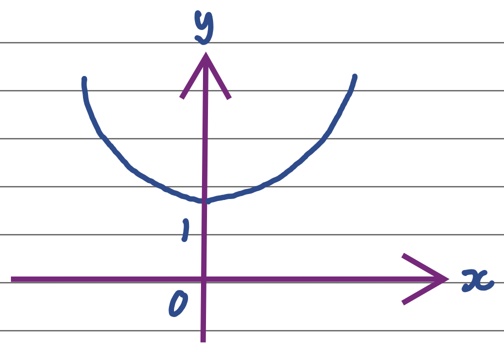
1. $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$




no asymptote.

$x \in \mathbb{R}$
$y \in \mathbb{R}$

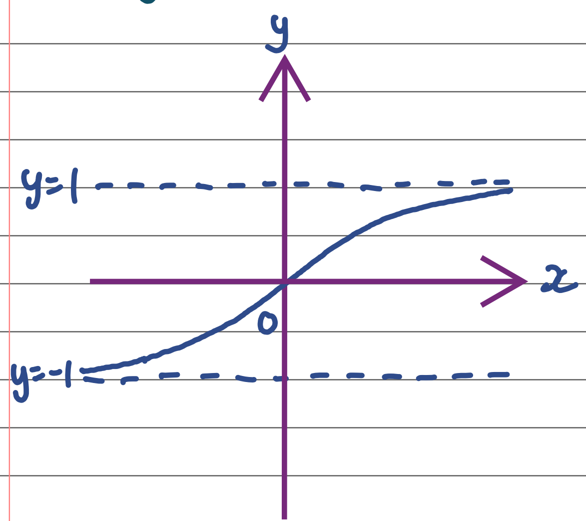
2. $y = \cosh x$




no asymptote.

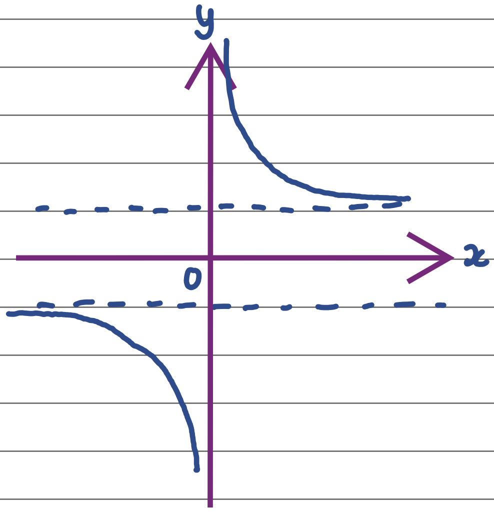
$x \in \mathbb{R}$
$\cosh x \geq 1$

3. $y = \tanh x$

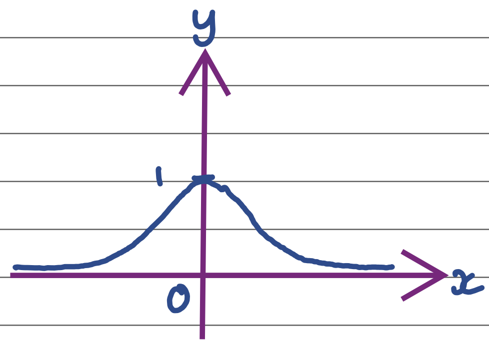



$x \in \mathbb{R}$
$-1 < \tanh x < 1$

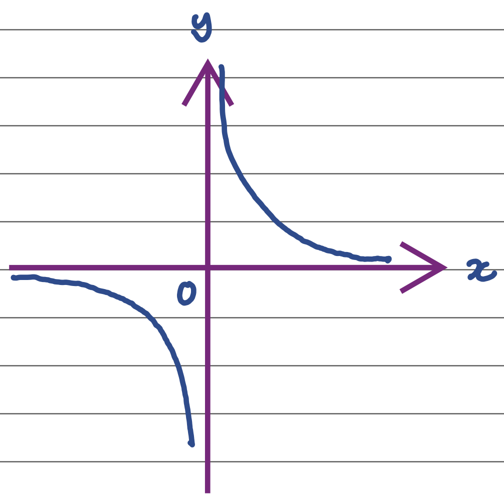
4. $y = \coth x$




5. $y = \operatorname{sech} x$




6. $y = \operatorname{cosech} x$



# 9.4 Solving Equation.

eg 1. $\sinh x = 4$ method 1 : calculator

$x = \sinh^{-1}(4)$

$= 2.0947$

or

$\sinh x = 4$ method 2: formula.

$x = \sinh^{-1}(4)$ $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

$= \ln\left(4 + \sqrt{4^2+1}\right)$

$= 2.0947$

or

$\dfrac{e^x - e^{-x}}{2} = 4$ method 3: definition

$e^x - e^{-x} = 8$

$(e^x)^2 - 1 = 8e^x$

$(e^x)^2 - 8e^x - 1 = 0$

$e^x = 4 \pm \sqrt{17}$

$x = \ln\left(4 \pm \sqrt{17}\right)$ $\ln(4-\sqrt{17})$ is rejected cause doesn't exist

$x = 2.0947$

★ let $y = \sinh^{-1}(x)$

$\sinh y = x$

$\dfrac{e^y - e^{-y}}{2} = x$

$e^y - \dfrac{1}{e^y} = 2x$

$(e^y)^2 - 1 = 2xe^y$ reject $y = \ln\left(x - \sqrt{x^2+1}\right)$

$(e^y)^2 - 2xe^y - 1 = 0$

$e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2(1)}$ $y = \ln\left(x \pm \sqrt{x^2+1}\right)$

eg2. $\cosh x = 5$

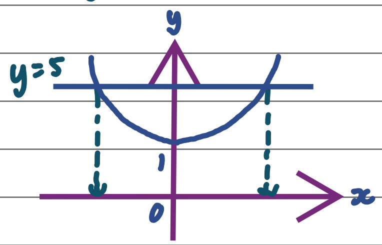
$\cosh x = 5$
$x = \cosh^{-1}(5)$
$x = 2.292$

$\cosh x = 5$
$x = \cosh^{-1}(5)$
$x = \ln(5+\sqrt{5^2-1})$
$x = 2.292$

$\cosh^{-1} x = \ln(x+\sqrt{x^2-1})$

HIGHLIGHT!

$y = \cosh x$




$x \in \mathbb{R}$
$\cosh x \geq 1$

Should have two solution instead of one!

but inverse function is one-to-one function so only get one.

NEED TO USE METHOD 3
(ONLY FOR COSH $x$)

$\cosh x = 5$

$\frac{e^x + e^{-x}}{2} = 5$

$e^x + e^{-x} = 10$

$e^x + \frac{1}{e^x} = 10$

$e^{2x} + 1 = 10e^x$

$(e^x)^2 - 10e^x + 1 = 0$

$e^x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)}$
$= 5 \pm 2\sqrt{6}$

$x = \ln(5 \pm 2\sqrt{6})$
$= \ln(5+2\sqrt{6}), \ln(5-2\sqrt{6})$
$= 2.292, -2.292$

eg3. $3\sinh x - \cosh x = 1$

$$3\left(\frac{e^x - e^{-x}}{2}\right) - \frac{e^x + e^{-x}}{2} = 1$$

$$\frac{3}{2}e^x - \frac{3}{2}e^{-x} - \frac{1}{2}e^x - \frac{1}{2}e^{-x} = 1$$

$$e^x - 2e^{-x} = 1$$
$$(e^x)^2 - 2 = e^x$$
$$(e^x)^2 - e^x - 2 = 0$$
$$(e^x - 2)(e^x + 1) = 0$$
$$e^x = 2 \;,\; e^x = -1$$
$$x = \ln 2 \;,\; x = \ln(-1) \text{ rejected}$$
$$= 0.6931$$

eg4. $4\tanh x = 1 + \operatorname{sech} x$

$$\frac{4\sinh x}{\cosh x} = 1 + \frac{1}{\cosh x}$$

$$4\sinh x = \cosh x + 1$$
$$2\cancel{4}\left(\frac{e^x - e^{-x}}{\cancel{2}_1}\right) = \frac{e^x + e^{-x}}{2} + 1$$
$$2e^x - 2e^{-x} = \frac{1}{2}e^x + \frac{1}{2}e^{-x} + 1$$

$$\frac{3}{2}e^x - \frac{5}{2}e^{-x} - 1 = 0$$

$$3(e^x)^2 - 5 - 2e^x = 0$$
$$3(e^x)^2 - 2e^x - 5 = 0$$
$$(e^x + 1)(3e^x - 5) = 0$$

$e^x = -1$

$x = \ln(-1)$ rejected

$3e^x - 5 = 0$

$e^x = \frac{5}{3}$

$x = \ln\frac{5}{3}$

# 9.5 Differentiation (and Inverse Differentiation)

$$[\sinh(f)]' = f' \cdot \cosh(f)$$
$$[\cosh(f)]' = f' \cdot \sinh(f)$$
$$[\tanh(f)]' = f' \cdot \text{sech}^2(f)$$
$$[\text{cotanh}(f)]' = -f' \cdot \text{cosech}^2(f)$$
$$[\text{sech}(f)]' = -f' \cdot \text{sech}(f) \cdot \tanh(f)$$
$$[\text{cosech}(f)]' = -f' \cdot \text{cosech}(f) \cdot \text{cotanh}(f)$$

$$[\text{arsinh}(f)]' = \frac{f'}{\sqrt{f^2+1}}$$
$$[\text{arcosh}(f)]' = \frac{f'}{\sqrt{f^2-1}}, \quad f > 1$$
$$[\text{artanh}(f)]' = \frac{f'}{1-f^2}, \quad f < 1$$
$$[\text{arsech}(f)]' = -\frac{f'}{f \cdot \sqrt{1-f^2}}, \quad 0 < f < 1$$

1. $\frac{d}{dx}(\sinh x)$
$= \frac{d}{dx}\left(\frac{e^x}{2} - \frac{e^{-x}}{2}\right)$
$= \frac{e^x}{2} + \frac{e^{-x}}{2}$
$= \frac{e^x + e^{-x}}{2}$
$= \cosh x$

2. $\frac{d}{dx}(\cosh x)$
$= \frac{d}{dx}\left(\frac{e^x}{2} + \frac{e^{-x}}{2}\right)$
$= \frac{e^x}{2} - \frac{e^{-x}}{2}$
$= \frac{e^x - e^{-x}}{2}$
$= \sinh x$

3. $\frac{d}{dx}(\tanh x)$
$= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$
$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$
$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$
$= \frac{1}{\cosh^2 x} = \text{sech}^2 x$

4. $\frac{d}{dx}(\text{sech} x)$
$= \frac{d}{dx}\left((\cosh x)^{-1}\right)$
$= -1(\cosh x)^{-2}(\sinh x)$
$= -\frac{\sinh x}{\cosh^2 x}$
$= -\frac{\sinh x}{\cosh x}\left(\frac{1}{\cosh x}\right)$
$= -\tanh x \,\text{sech} x$

5. $\frac{d}{dx}(\text{cosech} x)$
$= \frac{d}{dx}\left((\sinh x)^{-1}\right)$
$= -1(\sinh x)^{-2}(\cosh x)$
$= -\frac{\cosh x}{\sinh x}\left(\frac{1}{\sinh x}\right)$
$= -\coth x \,\text{cosech} x$

6. $\frac{d}{dx}(\coth x)$
$= \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right)$
$= \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x}$
$= \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x}$
$= 1 - \coth^2 x$
$= -\text{cosech}^2 x$

$\cot^2\theta + 1 = \text{cosec}^2\theta$
$-\coth^2\theta + 1 = -\text{cosech}^2\theta$
$1 - \coth^2\theta = -\text{cosech}^2\theta$

7. $\frac{d}{dx}(\sinh^{-1}x)$

let $y = \sinh^{-1}x$
$\sinh y = x$
$\cosh y \frac{dy}{dx} = 1$

$\cosh^2\theta - \sinh^2\theta = 1$
$\cosh^2 y = 1 + \sinh^2 y$
$\cosh y = \sqrt{1+x^2}$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$$

8. $\frac{d}{dx}(\cosh^{-1}x)$

let $y = \cosh^{-1}x$
$\cosh y = x$
$\sinh y \frac{dy}{dx} = 1$

$\sin^2\theta + \cos^2\theta = 1$
$-\sinh^2 y + \cosh^2 y = 1$
$\sinh^2 y = \cosh^2 y - 1$
$\sinh y = \sqrt{x^2-1}$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{x^2-1}}$$

9. $\frac{d}{dx}(\tanh^{-1}x)$

let $\tanh^{-1}x = y$
$\tanh y = x$
$-\text{sech}^2 y \cdot \frac{dy}{dx} = 1$

$\tan^2\theta + 1 = \sec^2\theta$
$-\tanh^2 y + 1 = \text{sech}^2 y$

$$\frac{dy}{dx} = -\frac{1}{\text{sech}^2 y} = \frac{1}{1-x^2}$$

eg1. $\frac{d}{dx}[\sinh(3x^2)] = 6x\cosh(3x^2)$

eg2. $\frac{d}{dx}[\operatorname{csech}(e^{2x})] = -2e^{2x}\operatorname{csech}(e^{2x})\coth(e^{2x})$

eg3. find $\frac{d}{dx}[\sinh^{-1}(f(x))]$

let $y = \sinh^{-1}(f(x))$

$\sinh(y) = f(x)$ $\longrightarrow$ $\sinh(y) = f(x)$

$\cosh(y) \cdot \frac{dy}{dx} = f'(x)$

$-\sinh^2(y) + \cosh^2(y) = 1$

$\cosh^2(y) = 1 + \sinh^2(y)$

$\cosh(y) = \pm\sqrt{1+\sinh^2 y}$

$\frac{dy}{dx} = \frac{f'(x)}{\cosh(y)}$

$\cosh(y) = \sqrt{1+\sinh^2 y}$

$\cosh(y) = \sqrt{1+\sinh^2 y}$

$\frac{d}{dx}[\sinh^{-1} f(x)] = \frac{f'(x)}{\sqrt{1+[f(x)]^2}}$

$\cosh(y) = \sqrt{1+[f(x)]^2}$

# 9.6 Integration

$$\int \sinh(u)\ du = \cosh(u) + C$$

$$\int \cosh(u)\ du = \sinh(u) + C$$

$$\int \operatorname{sech}^2(u)\ du = \tanh(u) + C$$

$$\int \operatorname{cosech}^2(u)\ du = -\operatorname{cotanh}(u) + C$$

$$\int \operatorname{sech}(u)\tanh(u)\ du = -\operatorname{sech}(u) + C$$

$$\int \operatorname{cosech}(u)\operatorname{cotanh}(u)\ du = -\operatorname{cosech}(u) + C$$

$$\int \frac{1}{\sqrt{a^2+u^2}}\ du = \operatorname{arsinh}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

$$\int \frac{1}{\sqrt{u^2-a^2}}\ du = \operatorname{arcosh}\left(\frac{u}{a}\right) + C, \qquad 0 < a < u$$

$$\int \frac{1}{a^2-u^2}\ du = \frac{1}{a}\operatorname{artanh}\left(\frac{u}{a}\right) + C, \qquad |u| < a$$

$$\int \frac{1}{u\sqrt{a^2-u^2}}\ du = -\frac{1}{a}\operatorname{arsech}\left(\frac{|u|}{a}\right) + C, \qquad 0 < |u| < a$$

1. $\int \sinh 4x\,dx = \frac{\cosh 4x}{4} + c$

2. $\int \operatorname{sech} 3x \tanh 3x\,dx = \frac{-\operatorname{sech} 3x}{3} + c$

3. $\int \cosh(4x-1)\,dx = \frac{\sinh(4x-1)}{4} + c$

4. $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \ln(\cosh x) + c$

5. $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx = \ln(\sinh x) + c$

6. $\int \sinh^2 x\,dx = \int\left(\frac{1}{2}\cosh 2x - \frac{1}{2}\right)dx = \frac{1}{4}\sinh 2x - \frac{1}{2}x + c$

$\cos 2x = 1 - 2\sin^2 x$
$\cosh 2x = 1 + 2\sinh^2 x$

7. $\int \cosh^2 x\,dx = \int\left(\frac{1}{2}\cosh 2x + \frac{1}{2}\right)dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + c$

$\cos 2x = 2\cos^2 x - 1$
$\cosh 2x = 2\cosh^2 x - 1$

8. $\int \tanh^2 x\,dx = \int 1 - \operatorname{sech}^2 x\,dx$

$= x - \tanh x + c$

$1 + \tan^2\theta = \sec^2\theta$
$1 - \tanh^2\theta = \operatorname{sech}^2\theta$
$\tanh^2\theta = 1 - \operatorname{sech}^2\theta$

# 9.7 Hyperbolic (relationship) with Trigonometric function.

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{i\theta}+e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)$

$e^{i\theta}+e^{-i\theta} = 2\cos\theta$

$\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \cosh(i\theta)$

$e^{i\theta}-e^{-i\theta} = \cos\theta + i\sin\theta - \cos(-\theta) - i\sin(-\theta)$

$e^{i\theta}-e^{-i\theta} = 2i\sin(\theta)$

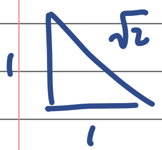
$i\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2} = \sinh(i\theta)$

given in exam!

$\cosh(i\theta) = \cos(\theta)$ and $\cos(i\theta) = \cosh(\theta)$

$\sinh(i\theta) = i\sin(\theta)$ and $\sin(i\theta) = i\sinh(\theta)$

eg $\sin\left(\frac{\pi}{4}(1+i)\right) = \sin\left(\frac{\pi}{4}+\frac{\pi}{4}i\right)$




$= \sin\frac{\pi}{4}\cos\left(\frac{\pi}{4}i\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}i\right)$

$= \frac{1}{\sqrt{2}}\left(\dfrac{e^{i\left(\frac{\pi}{4}i\right)}+e^{-i\left(\frac{\pi}{4}i\right)}}{2}\right) + \frac{1}{\sqrt{2}}\left(\dfrac{e^{i\left(\frac{\pi}{4}i\right)}-e^{-i\left(\frac{\pi}{4}i\right)}}{2i}\right)$

$= \frac{\sqrt{2}}{2}\left(\dfrac{e^{\frac{\pi}{4}}+e^{-\frac{\pi}{4}}}{2}\right) - \frac{\sqrt{2}\,i}{2}\left(\dfrac{e^{-\frac{\pi}{4}}-e^{\frac{\pi}{4}}}{2}\right)$

$= \frac{\sqrt{2}}{2}\cosh\left(\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}i\sinh\left(\frac{\pi}{4}\right)$

or

BETTER & FASTER

$\sin\left(\frac{\pi}{3}+i\right) = \sin\left(\frac{\pi}{3}\right)\cos(i) + \sin(i)\cos\left(\frac{\pi}{3}\right)$

$= \frac{\sqrt{3}}{2}\cos(i) + \frac{1}{2}\sin(i)$

$= \frac{\sqrt{3}}{2}\cosh(1) + \frac{1}{2}\left(\dfrac{-\sinh(1)}{i}\right) \times \frac{i}{i}$

$= \frac{\sqrt{3}}{2}\cosh(1) + \frac{1}{2}\sinh(1)\cdot i$

$i\sin\theta = \sinh(i\theta)$

$i\sin(\theta i) = -\sinh(\theta)$

$\cos\theta = \cos i\theta$

$\cos\theta i = \cosh\theta$

eg2.

$$\sinh\left(\frac{\pi}{3}(1+i)\right) = \sinh\left(\frac{\pi}{3}\right)\cosh\left(\frac{\pi}{3}i\right) + \cosh\left(\frac{\pi}{3}\right)\sinh\left(\frac{\pi}{3}i\right)$$

$$= \sinh\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right) + \cosh\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)i$$

$$= \frac{1}{2}\sinh\left(\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cosh\left(\frac{\pi}{3}\right)i$$

$\cos\theta = \cosh(i\theta)$

$i\sin\theta = \sinh(i\theta)$

# c10. Multivariable Function.

## 10.1 Introduction

given a function $y=f(x)$, $y$ is said to be DEPENDENT on $x$.

lets say a multivariable function $z=f(x,y)$, $z$ is said to be DEPENDENT on both $x$ and $y$, but $x$ and $y$ are independent of each other.

## 10.2 Domain of a multivariable function

recap: function of a single variable (domain)

$f(x)=\sqrt{x-3}$ $\quad D_f=\{x\in\mathbb{R}: x-3\geq 0\}=\{x\in\mathbb{R}: x\geq 3\}$ //

now:

eg1. $f(x,y)=\dfrac{xy-5}{2\sqrt{y-x^2}}$

$\mathbb{R}^2$ indicate domain is two dimensional.

$D_f=\{(x,y)\in\mathbb{R}^2: y-x^2>0\}=\{(x,y)\in\mathbb{R}^2: y>x^2\}$

eg2. $f(x,y)=\ln(xy)$

$D_f=\{(x,y)\in\mathbb{R}^2: xy>0\}=\{(x,y)\in\mathbb{R}^2: (x>0\wedge y>0)\cup(x<0\wedge y<0)\}$

QUITE HARD.

eg3. $f(x,y)=\log(x\log(y-x^2))$

$D_f=\{(x,y)\in\mathbb{R}^2: x\log(y-x^2)>0\}=\{(x,y)\in\mathbb{R}^2: x>0\wedge y>x^2+1\cup x<0\wedge y>x^2\wedge y<x^2+1\}$

$(x>0\wedge\log(y-x^2)>0\cup x<0\wedge\log(y-x^2)<0)\wedge(y-x^2>0)$

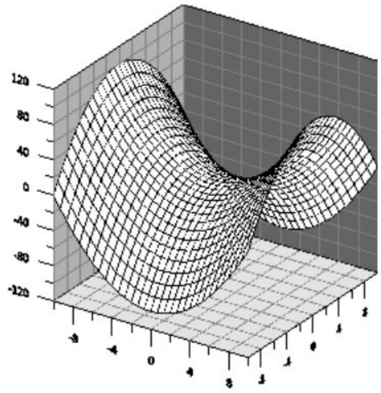
$(x>0\wedge y-x^2>1\cup x<0\wedge y-x^2<1)\wedge y-x^2>0$

$x>0\wedge\underbrace{y-x^2>1\wedge y-x^2>0}\cup x<0\wedge y-x^2>0\wedge y-x^2<1$

$x>0\wedge y>x^2+1\cup x<0\wedge y>x^2\wedge y<x^2+1$

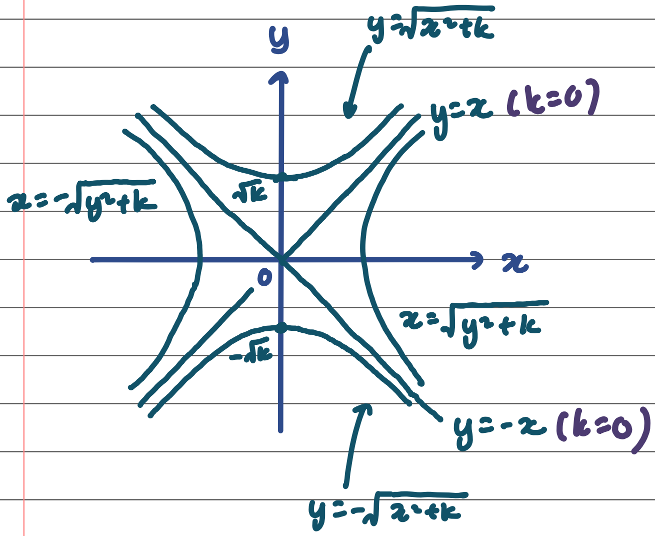
# 10·3 Representation of a multivariable function (Graphing)

$f(x,y) = y^2 - x^2$



3d graphing only achievable using computer software.

→ level curve:




$f(x,y) = y^2 - x^2$

let $k = y^2 - x^2$

$k=0$: $y^2 - x^2 = 0$

$y^2 = x^2$

$y = \pm\sqrt{x^2} = \pm x$

$y = x,\ y = -x$

$k>0$: $y^2 - x^2 = k$

$y^2 = x^2 + k$

$y = \pm\sqrt{x^2+k}$

$y = \sqrt{x^2+k},\ y = -\sqrt{x^2+k}$

$k<0$: $y^2 - x^2 = -|k|$

$|k| = x^2 - y^2$

$x^2 = y^2 + |k|$

$x = \pm\sqrt{y^2+|k|}$

$x = \sqrt{y^2+|k|},\ x = -\sqrt{y^2+|k|}$

# 10.4 Limits of a multivariable function

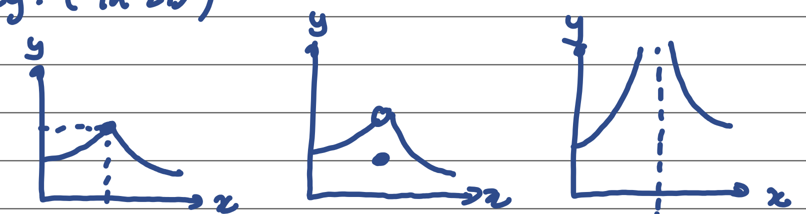
(note: for Y1 only covering 'prove limit does not exist')

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \ \ or \ \ f(x,y) \to L \ as \ (x,y) \to (a,b)$$

same concept as function of a single variable.
for a limit to exist :

as $(x,y)$ appraches $(a,b)$, if $f(x,y)$ approaches a FINITE value, let's say $L$, doesn't matter if $f(a,b) = L$, it will exist.

eg. (in 2D)



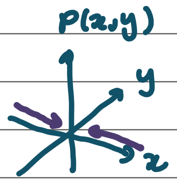

limit exist    limit exist    limit does not exist

HOWEVER, in a multivariable function, there is infinite way for a function to approch a point, let's say $(a,b)$. (for 2D case only two way, from left and from right)

to prove the limit doesn't exist, if found two way that $(x,y)$ appraches $(a,b)$, $f(x,y)$ doesn't approaches same value, limit doesn't exist .

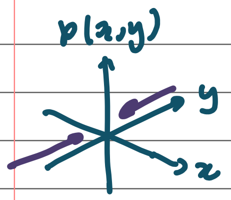
eg. show that the following limit doesn't exist.

$$\lim_{(x,y)\to(0,0)} \left( \frac{x^2-y^2}{x^2+y^2} \right) \qquad let \ P(x,y) = \frac{x^2-y^2}{x^2+y^2}$$




appreching $(0,0)$ along $x$-axis : $(y=0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2-0^2}{x^2+0^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1$$




appraching $(0,0)$ along $y$-axis : $(x=0)$

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1$$

doesn't approaches the same point, <u>limit doesn't exist</u>.

to prove that a limit exist: ( not in Year 1 )
we have to show all infinite way to approach $(x,y)$ all gives the same value.
Best method that work for all the case → Epsilon - Delta proof.
Other technique including : Polar coordinate trick.

$\forall \varepsilon > 0\ \exists \delta > 0\ (\forall x \in D_f): [0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$ — for single variable function.

$$\forall \varepsilon > 0\ \exists \delta > 0\ (\forall (x,y) \in \mathbb{R}^2): [0 < \|(x,y)-(a,b)\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon]$$

$$0 < \sqrt{(x-a)^2+(y-b)^2} < \delta \quad \text{(euclidean distance)}$$

eg. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2}$

1. given $\varepsilon > 0$
2. choose $\delta$...
3. suppose $0 < \sqrt{x^2+y^2} < \delta \quad (a=0, b=0)$
4. check $\left| \dfrac{3x^2y}{x^2+y^2} - 0 \right| < \varepsilon$ — try to start from here.

$$x^2 \leq x^2+y^2$$
$$\frac{x^2}{x^2+y^2} \leq 1$$
$$\frac{x^2|y|}{x^2+y^2} \leq |y|$$
$$\frac{3x^2|y|}{x^2+y^2} \leq 3|y|$$

since $0 \leq \sqrt{x^2+y^2} \leq \delta$

green is always +ive, can bring mod out

$$\frac{3x^2|y|}{x^2+y^2} \leq 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}$$
$$\left|\frac{3x^2y}{x^2+y^2}\right| \leq 3\delta$$

back to 2, choose $\delta = \dfrac{\varepsilon}{3}$

$\left|\dfrac{3x^2y}{x^2+y^2}\right| < \varepsilon$, hence by definition: $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y}{x^2+y^2} = 0$

## 10.5 Partial Derivatives

### 10.5.1 First Principle.

$$f'(x) = \frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$ (function of a single variable)

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$ (function of multivariable)

eg. $f(x,y) = x^2 - 2y^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \left( \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \right)$$
$$= \lim_{\Delta x \to 0} \left( \frac{(x+\Delta x)^2 - \cancel{2y^2} - x^2 + \cancel{2y^2}}{\Delta x} \right)$$
$$= \lim_{\Delta x \to 0} \left( \frac{\cancel{x^2} + 2x\cancel{\Delta x} + (\Delta x)^2 - \cancel{x^2}}{\Delta x} \right)$$
$$= \lim_{\Delta x \to 0} \left( 2x + \Delta x \right)$$
$$= 2x$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \left( \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right)$$
$$= \lim_{\Delta y \to 0} \left( \frac{\cancel{x^2} - 2(y+\Delta y)^2 - \cancel{x^2} + 2y^2}{\Delta y} \right)$$
$$= \lim_{\Delta y \to 0} \left( \frac{-2(\cancel{y^2} + 2y\Delta y + (\Delta y)^2) + \cancel{2y^2}}{\Delta y} \right)$$
$$= \lim_{\Delta y \to 0} \left( \frac{-4y\cancel{\Delta y} - 2(\Delta y)^{\cancel{2}}}{\cancel{\Delta y}} \right)$$
$$= \lim_{\Delta y \to 0} \left( -4y - 2\Delta y \right)$$
$$= -4y$$

## 10.5.2 Standard Notation

$$\frac{\partial}{\partial x} f_x = (f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y} f_x = (f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x} f_y = (f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y} f_y = (f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

* product rule, quotient rule, power rule, chain rule all apply to partial derivative too.

## 10.5.3 Second order Partial Derivative.

$$f_x(x,y) = \frac{\partial}{\partial x} f(x,y) = \frac{\partial w}{\partial x} = w_x$$

$$f_y(x,y) = \frac{\partial}{\partial y} f(x,y) = \frac{\partial w}{\partial y} = w_y$$

$$f_x = \frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x}\right)_y ; \; f_y = \frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial y}\right)_x$$

☆ $f_{xy} = f_{yx}$ IF $f$ is a continuous function.

eg. $f(x,y) = xy^2e^{xy}$, find $\frac{\partial^2 f}{\partial x \partial y}$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(2xy(e^{xy}) + xy^2(xe^{xy})\right)$$

$$= \frac{\partial}{\partial x}\left(2xye^{xy} + x^2y^2e^{xy}\right)$$

$$= 2y(e^{xy}) + 2xy(ye^{xy}) + 2y^2x(e^{xy}) + x^2y^2(ye^{xy})$$

$$= 2ye^{xy} + 2xy^2e^{xy} + 2xy^2e^{xy} + x^2y^3e^{xy}$$

$$= 2ye^{xy} + 4xy^2e^{xy} + x^2y^3e^{xy} \;//$$

# 10.6 Directional Derivative

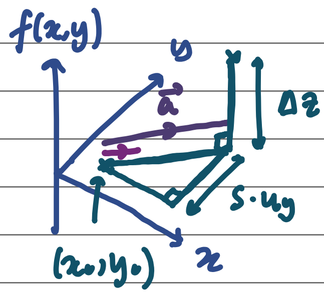
let $\vec{a}$ be the vector from original $(x,y)$ coords to the reference coordinate where the direction is where we want to differentiate along of.

$\vec{a} = S\cdot\vec{u}$, where $\vec{u}$ is a unit vector: $\vec{u} = u_x \underset{\sim}{i} + u_y \underset{\sim}{j}$ ☆

$\vec{a} = s\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} Su_x \\ Su_y \end{pmatrix}$

ALWAYS REMEMBER TO UNIT VECTOR YOUR $\vec{a}$!

Technically $S$ is $\|\vec{a}\|$




→ is $\vec{u}$, where $|\vec{u}| = 1$

directional derivative along $\vec{u}$

$$D_u f = \lim_{S\to 0}\left(\frac{f(x_0+Su_x, y+Su_y) - f(x_0,y_0)}{S}\right) \leftarrow \frac{\Delta z}{\|\vec{a}\|}$$

$$= \frac{\partial f}{\partial x}\cdot u_x + \frac{\partial f}{\partial y}\cdot u_y$$

$$\boxed{D_u f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix}}, \text{ where } \vec{u} = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

'unit vector of direction derivative'

also known as 'gradient vector of f': $\vec{\nabla f}$

☆ $D_u f = \vec{\nabla f}\cdot\vec{u}$ $\quad (a\cdot b = \|a\|\,\|b\|\cos\theta)$

$D_u f = \|\vec{\nabla f}\|\,\|\vec{u}\|\cos\theta$

since $\|\vec{u}\| = 1$,

$D_u f = \|\vec{\nabla f}\|\cos\theta$

$-1 \le \cos\theta \le 1$, for $\theta \in \mathbb{R}$.

$-\|\vec{\nabla f}\| \le \|\vec{\nabla f}\|\cos\theta \le \|\vec{\nabla f}\|$

$-\|\vec{\nabla f}\| \le D_u f \le \|\vec{\nabla f}\|$, $D_u f = 0$ when $\cos\theta = 0$, $\theta = \pi/2$

(level curve altitude doesn't change)

min ↑ $\quad$ ↑ max (when $\theta = 0$)

☆ since max of $D_u f$ is $\|\vec{\nabla f}\|$ when $\theta = 0$ (angle between $\vec{\nabla f}$ and $\underset{\sim}{\hat{u}}$) and $\vec{\nabla f}$ is fixed at a specific point, that means the max $D_u f$ is at:

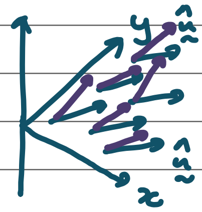
$\underset{\sim}{a} = \vec{\nabla f}$, $\underset{\sim}{\hat{u}} = \frac{\vec{\nabla f}}{\|\vec{\nabla f}\|}$

eg. $f(x,y) = x^3y^2$, find $D_{\underset{\sim}{u}}f$ at $P(-1,2)$ in the direction of $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$

$$\hat{\underset{\sim}{u}} = \frac{1}{\sqrt{3^2+4^2}}\begin{pmatrix}4\\-3\end{pmatrix} = \begin{pmatrix}4/5\\-3/5\end{pmatrix}$$

$f_x(x,y) = 3x^2y^2$ , $f_y(x,y) = 2x^3y$

$$D_uf(x,y) = \begin{pmatrix}3x^2y^2\\2x^3y\end{pmatrix}\cdot\begin{pmatrix}4/5\\-3/5\end{pmatrix}$$




$D_uf$ depend on $(x,y)$

$$D_uf(x,y) = \frac{4}{5}(3x^2y^2) - \frac{3}{5}(2x^3y)$$

$$D_uf(-1,2) = \frac{4}{5}\left(3(-1)^2(2)^2\right) - \frac{3}{5}\left(2(-1)^3(2)\right) = 12 \ \#$$

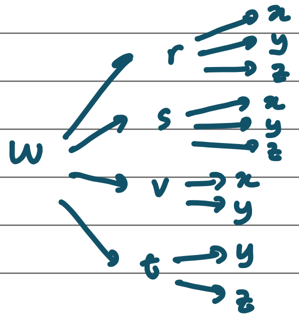
# 10.7 Chain Rule.

If $w = f(u, v)$ with $u = g(x, y)$ and $v = h(x, y)$, and if $f, g$ and $h$ are differentiable, then

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}$$

eg. find $\frac{\partial w}{\partial z}$ if $w = r^2 + sv + t^3$ with $r = x^2 + y^2 + z^2$, $s = xyz$, $v = xe^y$ and $t = yz^2$




$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial r} \times \frac{\partial r}{\partial z} + \frac{\partial w}{\partial s} \times \frac{\delta s}{\delta z} + \frac{\partial w}{\partial t} \times \frac{\partial t}{\partial z}$$

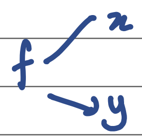
anything with z we include.

$$= 2r(2z) + v(xy) + 3t^2(2yz)$$

$$= 4z(x^2+y^2+z^2) + x^2ye^y + 6y^3z^5$$

# 10.8 Total Derivative and Implicit Differentiation

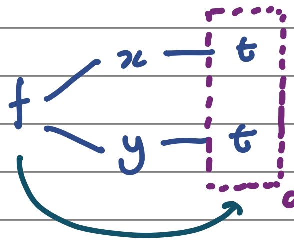
① If $f(x,y)$, where $x$ and $y$ are independent of each other:




$f_x = \frac{\partial f}{\partial x}$ → partial derivative because independency.

$f_y = \frac{\partial f}{\partial y}$

② If $f(x,y)$, with $x = g(t)$, $y = h(t)$



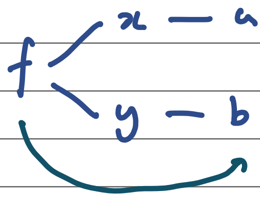

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \times \frac{dx}{dt} + \frac{\partial f}{\delta y} \times \frac{dy}{dt}$$

all same variable, any derivative respect this row will be total derivative. ☆

$\frac{df}{dt}$ cause last row all $t$

③ If $f(x,y)$, with $x=g(a)$ and $y=h(b)$

eg1



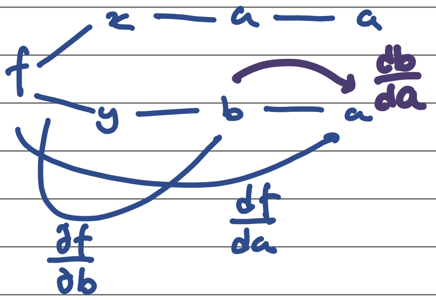

$\frac{\partial f}{\partial b}$.

$\frac{\partial f}{\partial b}$

although it look like b is indepent of a so it should be $\frac{df}{db}$ and $\frac{dy}{db}$ but NO!

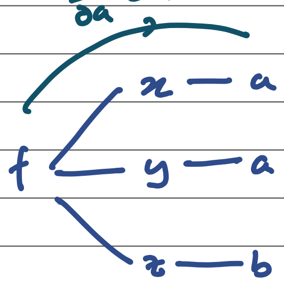
3rd row are not the same variable, there is still a chance that $b=f(a)$!

eg




$\frac{\partial f}{\partial a}$ cause there is b on 3rd row.

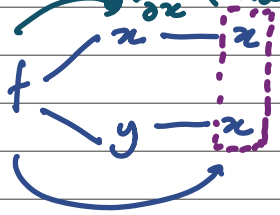
eg2.




④ If $f(x,y)=k$, where k is a constant.

one of $x,y$ is now dependent. → $y=g(x)$.

$\frac{\partial f}{\partial x}$ (notice the difference)



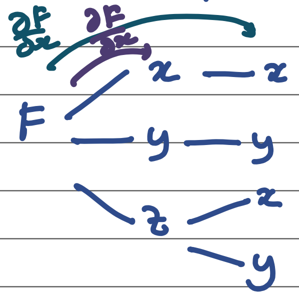

partial cause 2nd row still got y.

$$\frac{df}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\times\frac{dy}{dx}$$

$\frac{df}{dx}$ (total cause 3rd row same)

⑤ If $F(x,y,z) = 0$ $\quad \frac{\partial F}{\partial x} = 0$

→ $z$ is now dependent of $(x,y)$ → $z = f(x,y)$




This is not equal to zero! cause it's the $F(x,y,z)$ we define ourself in first place.

$$\underbrace{\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x}}_{\text{same symbol but represent different thing!}} + \frac{\partial F}{\partial z} \times \frac{\partial z}{\partial x}$$

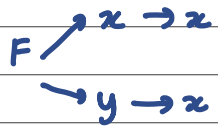
$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \times \frac{\partial z}{\partial x} = 0$$

☆ $$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}$$

185. Use partial derivatives to find $dy/dx$ if $y = f(x)$ is determined implicitly by the given equation.

(a) $2x^3 + x^2y + y^3 = 1$ (b) $6x + \sqrt{xy} = 3y - 4$

(a) $F(x,y) = 2x^3 + x^2y + y^3 - 1 = 0$




$$\frac{dF}{dx} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\left(\frac{dy}{dx}\right) = 0$$

or

$$\cancelto{=0}{\frac{dF}{dx}} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\left(\frac{dy}{dx}\right)$$

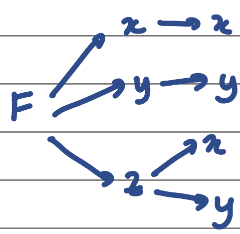
$$0 = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} = \frac{6x^2 + 2xy}{x^2 + 3y^2} \quad \#$$

187. Use partial derivatives to find $\partial z/\partial x$ and $\partial z/\partial y$ if $z = f(x,y)$ is determined implicitly by the given equation.

(a) $2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$ (b) $xe^{yz} - 2ye^{xz} + 3ze^{xy} = 1$

$$F(x,y,z) = 2xz^3 - 3yz^2 + x^2y^2 + 4z = 0$$



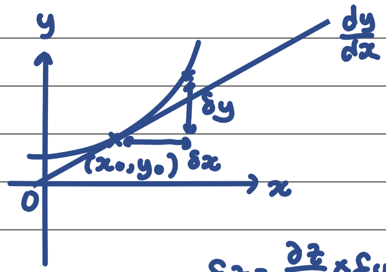

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial z}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{2z^3 + 2xy^2}{6xz^2 - 6yz + 4}$$

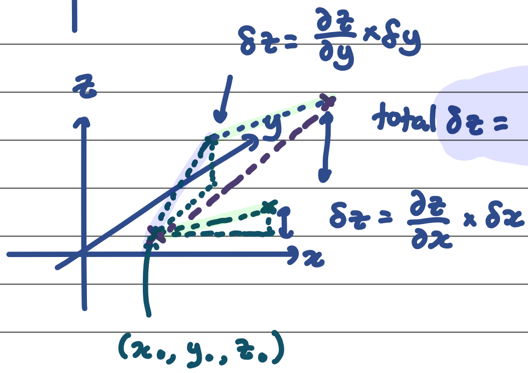
# 10.9 Applications

## 10.9.1 Increments.




$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

(in two dimension)




total $\delta z = \frac{\partial z}{\partial x} \times \delta x + \frac{\partial z}{\partial y} \times \delta y$ ☆

$\frac{\delta a}{a} = \pm 0.02$, $\frac{\delta m}{m} = \pm 0.02 \rightarrow \delta m = \pm 0.02m$

eg. $P(a, m) = a^2 \cos(m)$

measuring $a = 0.1$, $m = 30°$ with 2% error, find rel. error of P

$$\delta P = \frac{\partial P}{\partial a} \cdot \delta a + \frac{\partial P}{\partial m} \cdot \delta m \qquad \frac{\partial P}{\partial a} = 2a\cos(m)$$

$$\delta P = 2a\cos(m) \cdot \delta a - a^2 \sin(m) \cdot \delta m \qquad \frac{\partial P}{\partial m} = -a^2 \sin(m)$$

$$\frac{\delta P}{P} = \frac{2\cancel{a}\cancel{\cos(m)} \cdot \delta a}{a^{\cancel{2}}\cancel{\cos(m)}} - \frac{\cancel{a^2}\sin(m)\,\delta m}{\cancel{a^2}\cos(m)}$$

$$= 2(\pm 0.02) - \tan(m) \cdot (\pm 0.02m)$$

$$= 2(\pm 0.02) - \frac{\sqrt{3}}{3}\left(\pm 0.02 \cdot \frac{\pi}{6}\right)$$

$$= 2(+0.02) - \frac{\sqrt{3}}{3}\left(-0.02 \times \frac{\pi}{6}\right)$$

← one choose +ive and one choose −ive to produce biggest $\delta P / P$ !

$$= 0.04 + \frac{\sqrt{3}}{18}\pi(0.02) \; \#$$

188. Use differentials to approximate the change in $f$ if the independent variables change as indicated.

(a) $f(x,y) = x^2 - 3x^3y^2 + 4x - 2y^3 + 6,\qquad (-2,3)\ to\ (-2.02, 3.01)$

(b) $f(x,y,z) = x^2z^3 - 3yz^2 + x^{-3} + 2z\sqrt{y},\qquad (1,4,2)\ to\ (1.02, 3.97, 1.96)$

(a)

$$\delta f \approx \frac{\partial f}{\partial x} \times \delta x + \frac{\partial f}{\partial y} \times \delta y$$

$$= (2x - 9x^2y^2 + 4)(-0.02) + (-6x^3y - 6y^2)(0.01)$$

$$= (2(-2) - 9(-2)^2(3)^2 + 4)(-0.02) + (-6(-2)^3(3) - 6(3)^2)(0.01)$$

$$= 7.38$$

190. The dimensions of a closed rectangular box are measured as 30 $cm$, 40 $cm$ and 50 $cm$, with a possible error of $\pm 2$ $mm$ in each measurement. Use differentials to approximate the maximum error in the calculated value of

(a) the surface area

(b) the volume

(a) $S(x,y,z) = 2xy + 2xz + 2yz$

$$\delta S = \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z$$

$$= (2y+2z)(\pm 2) + (2x+2z)(\pm 2) + (2x+2y)(\pm 2)$$

$$= (2(400) + 2(500))(\pm 2) + (2(300) + 2(500))(\pm 2) + (2(300) + 2(400))(\pm 2)$$

$$= 9600\ mm^2$$

$$= 96\ cm^2$$

191. A given quantity, s, can be calculated using the formula:

$$s = \frac{a}{a-w}$$

where $a$ and $w$ are two physical quantities.

(a) Assuming that direct measurements indicate that $a = 12$ and $w = 5$, determine using partial derivatives the approximate change in $s$ when $a$ increases by 0.5 and $w$ decreases by 0.2.

(b) If the percentage errors in the measurements of $a$ and $w$ are, respectively, $\pm 2\%$ and $\pm 4\%$, express the maximum percentage error in the calculated value of $s$ as a function of $a$ and $w$.

(a) $\delta s \approx \frac{\partial s}{\partial a} \times \delta a + \frac{\partial s}{\partial w} \times \delta w$

$s = a(a-w)^{-1}$

$\frac{ds}{da} = -a(a-w)^{-2} + (a-w)^{-1}$

$\frac{ds}{dw} = -a(a-w)^{-2}(-1)$

$\delta s \approx \left(-\frac{a}{(a-w)^2} + \frac{1}{a-w}\right)(0.5) + \left(\frac{a}{(a-w)^2}\right)(-0.2)$

$\approx \left(-\frac{12}{(12-5)^2} + \frac{1}{12-5}\right)\left(\frac{1}{2}\right) + \left(\frac{12}{(12-5)^2}\right)\left(-\frac{1}{5}\right)$

$\approx -0.1$

$\frac{\delta a}{a} = \pm 2\%$

$\frac{\delta w}{w} = \pm 4\%$

(b) $\delta s \approx \left(-\frac{a}{(a-w)^2} + \frac{1}{a-w}\right) \times \delta a + \frac{a}{(a-w)^2} \times \delta w$

$\frac{\delta s}{s} \approx \frac{\cancel{a-w}}{a}\left(-\frac{a}{(a-w)^{\cancel{2}}} + \frac{1}{\cancel{a-w}}\right) \times \delta a + \frac{\cancel{a-w}}{\cancel{a}}\left(\frac{\cancel{a}}{(a-w)^{\cancel{2}}}\right) \times \delta w$

$\approx \frac{\delta a}{a}\left(1 - \frac{a}{a-w}\right) + \frac{1}{a-w} \times (\pm 4\% \cdot w)$

$\approx \pm 2\% \left(1 - \frac{a}{a-w}\right) + \frac{w}{a-w}(\pm 4\%)$

$\approx \pm 2\% \left(\frac{a-w-a}{a-w}\right) + \frac{w}{a-w}(\pm 4\%)$

$\approx \pm 2\% \left(-\frac{w}{a-w}\right) + \frac{w}{a-w}(\pm 4\%)$

choose correctly + or − make $\delta s$ biggest

$\approx -2\% \left(-\frac{w}{a-w}\right) + \frac{w}{a-w}(4\%)$

$\approx 6\% \cdot \frac{w}{a-w}$

192. Show that if

$$u = x^a y^b$$

and the percentage errors in the measurements of $x$ and $y$ are $e$ and $f$, then the approximate percentage error in $u$ (i.e. $du/u$) is

$$a\,e + b\,f$$

The value of the acceleration of gravity, $g$, may be calculated from the period, $T$, of a simple pendulum of length, $L$:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

If the percentage errors in $L$ and $T$ are 3 and -2, find the approximate percentage error in $g$.

$$\frac{\delta x}{x} = e \qquad \frac{\delta y}{y} = f$$

$$\delta u \approx \frac{\partial u}{\partial x} \times \delta x + \frac{\partial u}{\partial y} \times \delta y$$

$$\delta u \approx a x^{a-1} y^b \cdot \delta x + b x^a y^{b-1} \delta y$$

$$\frac{\delta u}{u} \approx \frac{a x^{a-1} y^b \delta x}{x^a y^b} + \frac{b x^a y^{b-1} \delta y}{x^a y^b}$$

$$\approx \frac{a \delta x}{x} + \frac{b \delta y}{y}$$

$$\approx ae + bf \qquad \text{(shown)}$$

$$T = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}}$$

$$T^2 = 4\pi^2\left(\frac{L}{g}\right)$$

$$\frac{L}{4\pi^2 T^2} = g$$

$$g = \frac{1}{4\pi^2} L T^{-2} \qquad \frac{\delta g}{g} \approx ae + bf = 1(3) - 2(-2) = 7\%$$

## 10.9.2 Extrema of multivariable function

let $z = f(x,y)$

to find maximum or minimum point:

$\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ (solve simultaneous eqn.)

obtained point $(a,b)$

to check whether it is a maximum or minimum point (or saddle point :)

let's say : $D(x,y) = \frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ or $f_{xx} \cdot f_{yy} - f_{xy}^2$

Test $D(a,b)$

- $D(a,b) > 0$ ($f_{xx}$ and $f_{yy}$ have same sign)
  - $f_{xx}(a,b) > 0$ or check with $f_{yy}(a,b) > 0$ $\Rightarrow$ local minimum
  - $f_{xx}(a,b) < 0$ or check with $f_{yy}(a,b) < 0$ $\Rightarrow$ local maximum
- $D(a,b) < 0$ ($f_{xx}$ and $f_{yy}$ may have opposite sign) $\longrightarrow$ saddle point. (don't need to check whether $f_{xx}$ or $f_{yy}$ have opposite sign or not : as long as $D(a,b) < 0$, it is a saddle point ! )

**NOTE**: This method cannot be used when:

(i) $D(a,b) = 0$

(ii) $f_x(a,b)$ or $f_y(a,b)$ does not exist.

eg. find all the critical point (inc. extrema and saddle point, if any) of:

$$f(x,y) = x^2 - 4xy + y^3 + 4y$$

$f_x(x,y) = 2x - 4y$ $\quad$ $f_y(x,y) = -4x + 3y^2 + 4$

$f_x(x,y) = 0$

$2x - 4y = 0$

$x = 2y$

$f_y(x,y) = 0$

$-4x + 3y^2 + 4 = 0$

$-4(2y) + 3y^2 + 4 = 0$

$-8y + 3y^2 + 4 = 0$

$3y^2 - 8y + 4 = 0$

$y = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(4)}}{2(3)}$

$y = 2 \cup y = 2/3$

$(x,y) = (4,2) \vee (x,y) = (4/3, 2/3)$

$f_x(x,y) = 2x - 4y \quad f_y(x,y) = -4x + 3y^2 + 4$

$f_{xx}(x,y) = 2 \quad f_{yy}(x,y) = 6y \quad f_{xy} = \frac{\partial}{\partial y}(2x - 4y) = -4$

$D(x,y) = 2(6y) - (-4)^2 = 12y - 16$

when $(x,y) = (4,2)$:

$D(4,2) = 12(2) - 16 = 8 > 0$, since $f_{xx}(4,2) = 2 > 0$, therefore $(4, 2, f(4,2))$ is a local minimum.

when $(x,y) = (4/3, 2/3)$

$D(4/3, 2/3) = 12(2/3) - 16 = -8 < 0$, therefore $(4/3, 2/3, f(4/3, 2/3))$ is a saddle point.

## 10.9.3 Optimisation (not inc. in 'Summary')

eg. Room volume = $12m^3$. Given cost, $C(x,y,z) = \frac{2}{5}xy + \frac{3}{5}xz + \frac{2}{5}zy$, find minimum cost. ($V = xyz = 12$)

$$C(x,y,z) = \frac{2}{5}xy + \frac{3}{5}xz + \frac{2}{5}zy$$

$$\downarrow \quad z = \frac{12}{xy}$$

$$C(x,y) = \frac{2}{5}xy + \frac{36}{5y} + \frac{24}{5x}$$

$\frac{\partial C}{\partial x} = 0$ $\qquad$ $\frac{\partial C}{\partial y} = 0$

$\frac{2}{5}y - \frac{24}{5x^2} = 0$ $\qquad$ $\frac{2}{5}x - \frac{36}{5y^2} = 0$

$y = \frac{12}{x^2}$ $\qquad$ $\frac{2}{5}x - \frac{36}{5}\left(\frac{x^4}{144}\right) = 0$

$x = 0 \cup x^3 = 8$

$x = 0 \cup x = 2$

(rejected)

$y = \frac{12}{2^2} = 3$

basically just $\frac{\partial C}{\partial x} = 0$ and $\frac{\partial C}{\partial y} = 0$ solve for $x$ and $y$

$(x,y) = (2,3)$ * need to prove its minimum!

$$\frac{\partial C}{\partial x} = \frac{2}{5}y - \frac{24}{5x^2} \qquad \frac{\partial C}{\partial y} = \frac{2}{5}x - \frac{36}{5y^2}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{48}{5x^3} \qquad \frac{\partial^2 C}{\partial y^2} = \frac{72}{5y^2}$$

$$\frac{\partial^2 C}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial C}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2}{5}x - \frac{36}{5y^2}\right) = \frac{2}{5}$$

$$D(2,3) = \frac{48}{5(2)^3} \cdot \left(-\frac{36}{5(3)^2}\right) - \frac{2}{5} = \frac{12}{25} > 0 \quad \text{(local minimum)}$$

195. Find and classify the stationary points of f.

(a) $f(x,y) = (x-y)(x^2+y^2-1)$ (b) $f(x,y) = xy^2 - x - x^2$

(c) $f(x,y) = (x^2+y^2)^2 - 2(x^2-y^2)$ (d) $f(x,y) = \frac{1}{2}x^4 - 2x^3 + 4xy + y^2$

(a) $f_x = 0$

$(x-y)(2x) + (x^2+y^2-1) = 0$

$f_y = 0$

$(x-y)(2y) - (x^2+y^2-1) = 0$

$(x-y)(2x) + (x-y)(2y) = 0$

$x(x-y) + y(x-y) = 0$

$(x+y)(x-y) = 0$

$x = -y, \; x = y$

$x = y$:

$y^2 + y^2 - 1 = 0$

$2y^2 = 1$

$y^2 = 1/2$

$y = \pm\sqrt{2}/2, \; x = \pm\sqrt{2}/2$

$x = -y$:

$-2y(-2y) + (2y^2 - 1) = 0$

$4y^2 + 2y^2 - 1 = 0$

$6y^2 - 1 = 0$

$y^2 = \frac{1}{6} \quad y = \pm\sqrt{\frac{1}{6}} = \pm\frac{\sqrt{6}}{6}$

$x = \mp\frac{\sqrt{6}}{6}$

$f_{xx} = (x-y)\cdot 2 + 2x + 2x$

$= 2x - 2y + 4x$

$= 6x - 2y$

$f_{yy} = (x-y)\cdot 2 - 2y - 2y$

$= 2x - 2y - 2y - 2y$

$= 2x - 6y$

$f_{xy} = -2x + 2y$

... continue next page.

| | $f_{xx}$ | $f_{yy}$ | $f_{xy}$ | $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$ |
|---|---|---|---|---|
| $(\sqrt{2}/2, \sqrt{2}/2)$ | $3\sqrt{2} - \sqrt{2} = 2\sqrt{2}$ | $\sqrt{2} - 3\sqrt{2} = -2\sqrt{2}$ | $0$ | $-8 < 0$ saddle pt. |
| $(-\sqrt{2}/2, -\sqrt{2}/2)$ | $-3\sqrt{2} + \sqrt{2} = -2\sqrt{2}$ | $-\sqrt{2} + 3\sqrt{2} = 2\sqrt{2}$ | $0$ | $-8 < 0$ saddle pt. |
| $(\sqrt{6}/6, -\sqrt{6}/6)$ | $\sqrt{6} + \frac{1}{3}\sqrt{6} = \frac{4}{3}\sqrt{6}$ +ve | $\frac{1}{3}\sqrt{6} + \sqrt{6} = \frac{4}{3}\sqrt{6}$ | $-\frac{4}{\sqrt{6}}$ | $48/6$ minimum pt. |
| $(-\sqrt{6}/6, \sqrt{6}/6)$ | $-\frac{1}{3}\sqrt{6} - \sqrt{6} = -\frac{4}{3}\sqrt{6}$ -ve. | $-\frac{1}{3}\sqrt{6} - \sqrt{6} = -\frac{4}{3}\sqrt{6}$ | $\frac{4}{\sqrt{6}}$ | $48/6$ maximum pt. |

☆ 2 method: 1- find $V(x,y)$ using $z = f(x,y)$ 2- chain rule.

196. Find the dimensions of a rectangular box of given surface area A which has maximum volume.

☆ $V(x,y,z) = xyz$

$\downarrow z = f(x,y)$

$$2xy + 2xz + 2yz = A$$
$$xy + xz + yz = \frac{A}{2}$$
$$xz + yz = A/2 - xy$$
$$z(x+y) = A/2 - xy = \frac{A - 2xy}{2}$$
$$z = \frac{A - 2xy}{2(x+y)}$$

$$V(x,y) = \frac{Axy - 2x^2y^2}{2x+2y}$$

$$\frac{\partial V}{\partial x} = 0$$

$$\frac{(2x+2y)\cdot(Ay - 4xy^2) - (Axy - 2x^2y^2)(2)}{(2x+2y)^2} = 0$$

$$(2x+2y)(Ay - 4xy^2) - 2(Axy - 2x^2y^2) = 0 \quad \text{——— } ①$$

$$\frac{\partial V}{\partial y} = 0$$

$$\frac{(2x+2y)(Ax - 4x^2y) - (Axy - 2x^2y^2)(2)}{(2x+2y)^2} = 0$$

$$(2x+2y)(Ax - 4x^2y) - (Axy - 2x^2y^2)(2) = 0 \quad \text{——— } ②$$

$$(2x+2y)(Ay - 4xy^2) - (2x+2y)(Ay - 4x^2y) = 0 \quad \text{——— } ① - ②$$

$$(2x+2y)(4x^2y - 4xy^2) = 0$$

$x = -y \qquad 4xy(x-y) = 0$

$\underbrace{x = 0, y \in \mathbb{R};\ x \in \mathbb{R}, y = 0}_{\text{rejected cause } x \neq 0,\, y \neq 0};\ x = y;\ \underbrace{x = -y}_{\text{rejected cause } x > 0,\, y > 0}$

sub $x=y$ into ①:

$$(2x+2x)(Ax-4x^3)-2(Ax^2-2x^4)=0$$

$$4Ax^2-16x^4-2Ax^2+4x^4=0$$
$$2Ax^2-12x^4=0$$
$$x^2(2A-12x^2)=0$$

$x^2=0$ (rejected)

$$2A-12x^2=0$$
$$A=6x^2$$
$$x^2=A/6$$
$$x=\pm\sqrt{\frac{A}{6}}=\pm\frac{\sqrt{6}}{6}A=\frac{\sqrt{6}}{6}A$$

rejected cause $x>0$.

$$z=\frac{A-2xy}{2(x+y)}=\frac{A-2\left(\frac{\sqrt{6}}{6}A\right)\left(\frac{\sqrt{6}}{6}A\right)}{2\left(\frac{\sqrt{6}}{6}A+\frac{\sqrt{6}}{6}A\right)}=\frac{A-2\left(\frac{1}{6}A\right)}{\frac{2}{3}\sqrt{6}A}=\frac{\cancel{\frac{2}{3}}\cancel{A}}{\cancel{\frac{2}{3}}\sqrt{6}\cancel{A}}=\frac{\sqrt{6}}{6}A$$

to prove it is maximum:

$$\frac{\partial V}{\partial x}=\frac{(2x+2y)\cdot(Ay-4xy^2)-(Axy-2x^2y^2)(2)}{(2x+2y)^2}$$

$$\frac{\partial^2 V}{\partial x^2}=\frac{y^2(A+2y^2)}{(x+y)^3}\qquad\frac{\partial^2 V}{\partial y^2}=\frac{x^2(A+2x^2)}{(x+y)^3}\qquad\frac{\partial^2 V}{\partial y\partial x}=\frac{xy}{(x+y)^3}\left(A-2(x^2+3xy+y^2)\right)$$

when $x=y=\sqrt{6}/6\,A$:

$$D=\frac{\partial^2 V}{\partial x^2}\cdot\frac{\partial^2 V}{\partial y^2}-\left(\frac{\partial^2 V}{\partial y\partial x}\right)^2=\frac{A}{8}>0\rightarrow\underline{\text{extremum}}$$

since $\frac{\partial^2 V}{\partial x^2}<0$, $x=y=\sqrt{6}/6\,A$ is a maximum.

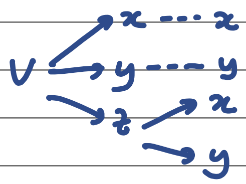
alternative method ↓

$$-\frac{(2x+2y)\cdot 2y - 2(A-2xy)}{(2x+2y)^2}$$

method 2 : chain rule.

$2xy + 2xz + 2yz = A \qquad V(x,y,z) = xyz$

$$z = \frac{A-2xy}{2(x+y)}$$



$$\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial z} \times \frac{\partial z}{\partial x} = 0$$

$$yz + xy\left(\frac{2(x+y)(-2y) - 2(A-2xy)}{4(x+y)^2}\right) = 0 \quad \text{———} \quad ①$$

$$\frac{\partial V}{\partial y} = 0$$

$$\frac{\partial V}{\partial y} + \frac{\partial V}{\partial z} \times \frac{\partial z}{\partial y} = 0$$

$$xz + xy\left(\frac{2(x+y)(-2x) - 2(A-2xy)}{4(x+y)^2}\right) = 0 \quad \text{———} \quad ②$$

① - ② :

$$yz - xz + xy\left(\frac{-\cancel{4}y\cancel{(x+y)} + \cancel{4}x\cancel{(x+y)}}{\cancel{4}(x+y)^{\cancel{2}}}\right) = 0$$

$$yz - xz + xy\left(\frac{x-y}{x+y}\right) = 0$$

$$-z(x-y) + \frac{xy}{x+y}(x-y) = 0$$

$$(x-y)\left(\frac{xy}{x+y} - z\right) = 0$$

$$x = y \quad \cup \quad \frac{A-2xy}{2\cancel{(x+y)}} = \frac{xy}{\cancel{x+y}}$$

$$4xy = A$$

$$xy = A/4$$

## 10.9.4 Multivariable Taylor Expansions. (not inc. in 'Summary')

recall that for single variable function,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

let $x - c = \delta x$,

$$f(c+\delta x) = \sum \frac{f^{(n)}(c)}{n!}(\delta x)^n$$

this can then be converted into multivariable function, (let c be coordinate we evaluating at → $(x,y)$)

$$f(x+\delta x, y+\delta y) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left[ \begin{pmatrix} f^{(n)}(x) \\ f^{(n)}(y) \end{pmatrix} \cdot \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \right]$$

$$f(x+\delta x, y+\delta y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(f_x(x,y)\cdot\delta x + f_y(x,y)\cdot\delta y\right)^n$$

hence,

$$f(a+h, b+k) = f(a,b) + \frac{1}{1!}\left(h f_x(a,b) + k f_y(a,b)\right) + \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)$$
$$+ \dots$$

If $f(x,y)$ is defined in a region $R$ of the $xy$-plane and all its partial derivatives of orders up to and including the $(n+1)$th are continuous in $R$, then for any point $(a,b)$ in this region

$$f(a+h,b+k) = f(a,b) + {}^{*}Df(a,b) + \frac{1}{2!}\,{}^{*}D^2 f(a,b) + \cdots + \frac{1}{n!}\,{}^{*}D^n f(a,b) + E_n$$

where ${}^{*}D$ is the differential operator defined as:

$${}^{*}D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

Therefore,

$${}^{*}Df(a,b) = h\frac{\partial f}{\partial x}\bigg|_{a,b} + k\frac{\partial f}{\partial y}\bigg|_{a,b} = \left[hf_x + kf_y\right]_{a,b}$$

Moreover,

$${}^{*}D^r = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r$$

meaning that, for example, if $r = 2$:

$${}^{*}D^2 f(a,b) = \left(h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial}{\partial x}\frac{\partial}{\partial y} + k^2\frac{\partial^2}{\partial y^2}\right)f(a,b) = \left[h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right]_{a,b}$$

The error term $E_n$ is given by

$$E_n = \frac{1}{(n+1)!}\,{}^{*}D^{n+1} f(a+\theta h, b+\theta k)$$

where $0 < \theta < 1$. Note that this restriction to the value of $\theta$ means that the point at which ${}^{*}D^{n+1}f(x,y)$ is evaluated (i.e. $(a+\theta h, b+\theta k)$) is located between the point about which the expansion is carried out (i.e. $(a,b)$) and the point at which an approximation is being obtained (i.e. $(a+h,b+k)$). This is the generalisation of the procedure adopted for single-variable functions, where the evaluation of the *(n+1)*th derivative was carried out at a point $z$, located between $c$ and $x$.

Furthermore, note that if only terms up to order one are used, then

$$f(a+h,b+k) \approx f(a,b) + {}^{*}Df(a,b) \Rightarrow f(a+h,b+k) - f(a,b) \approx {}^{*}Df(a,b) \Rightarrow \Delta f \approx {}^{*}Df(a,b)$$

Expanding the right-hand side and considering that $\Delta x = a + h - a = h$ and $\Delta y = b + k - b = k$:

$$\Delta f \approx {}^{*}Df(a,b) = \left[hf_x + kf_y\right]_{a,b} = \Delta x\, f(a,b) + \Delta y\, f(a,b)$$

which coincides with the expression previously employed to estimate changes in the value of functions of two variables due to increments of $x$ and $y$.

**[Example]** Expand the function $f(x,y) = e^{xy}$ about the point $(2,3)$ ignoring terms of order three or higher.

*For terms of order one or two:*

$f_x = ye^{xy}, \quad f_x(2,3) = 3e^6$

$f_y = xe^{xy}, \quad f_y(2,3) = 2e^6$

$f_{xx} = y^2e^{xy}, \quad f_{xx}(2,3) = 9e^6$

$f_{xy} = e^{xy} + xye^{xy}, \quad f_{xy}(2,3) = e^6(1+6) = 7e^6$

$f_{yy} = x^2e^{xy}, \quad f_{yy}(2,3) = 4e^6$

*Therefore, with* $h = x - 2$ *and* $k = y - 3$:

$$e^{xy} \approx e^6 + 3(x-2)e^6 + 2(y-3)e^6 + \frac{1}{2}\left(9e^6(x-2)^2 + 2\cdot 7e^6(x-2)(y-3) + 4e^6(y-3)^2\right)$$

$$\Rightarrow e^{xy} \approx e^6\left(1 + 3(x-2) + 2(y-3) + \frac{9}{2}(x-2)^2 + 7(x-2)(y-3) + 2(y-3)^2\right)$$

# C11. Ordinary Differential Equation

$\dfrac{dy}{dx} = \dfrac{f(x,y)}{g(x,y)}$

(linear, homogeneous)

if both $f(x,y)$ and $g(x,y)$ have same degree homogenity, ie $f(kx,ky) = k^n f(x,y)$; $g(kx,ky) = k^n g(x,y)$; both $n$ are equal :

- substitute $y = ux$, $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{ax+by+c}{fx+gy+h}$

(linear, non-homogeneous)

if $ag - bf \neq 0$

- solve $ax_0 + by_0 + c = 0$ ; $fx_0 + gy_0 + h = 0$
- substitute $x = x_0 + X$, $y = y_0 + Y$, obtain $\dfrac{dY}{dX} = \dfrac{aX+bY}{fX+gY}$
- continue with homogeneous method, ie. substitute $Y = uX$

if $ag - bf = 0$

- $\dfrac{dy}{dx} = \dfrac{k(fx+gy)+c}{fx+gy+h}$ or $\dfrac{dy}{dx} = \dfrac{ax+by+c}{k(ax+by)+h}$
- substitute $u = fx+gy$ or $u = ax+by$

$\dfrac{dy}{dx} + p(x)y = g(x)$

- find integrating factor, $I(x) = e^{\int p(x)\,dx}$
- multiply the whole thing by $I(x)$
- $\dfrac{d}{dx}\left(I(x)y\right) = I(x)g(x)$

$P(x,y)dx + Q(x,y)dy = 0$

(exact ODE)

- $P = \dfrac{\partial F}{\partial x}$, $Q = \dfrac{\partial F}{\partial y}$

if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ ie $\dfrac{\partial}{\partial y}\left(\dfrac{\partial F}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y}\right)$ (exact ODE)

- $\partial F = P\partial x$

$F = \int P\,dx + g(y)$

$\dfrac{\partial F}{\partial y} = \dfrac{d}{dy}\int P\,dx + g'(y)$

$Q = \dfrac{d}{dy}\int P\,dx + g'(y)$

solve for $g'(y) \to g(y)$

or $\partial F = Q\partial y$

$F = \int Q\,dy + h(x)$

$\dfrac{\partial F}{\partial x} = \dfrac{d}{dx}\int Q\,dy + h'(x)$

$P = \dfrac{d}{dx}\int Q\,dy + h'(x)$

solve for $h'(x) \to h(x)$

$F = \int P\,dx + g(y) = C$

$F = \int Q\,dy + h(x) = C$

$P(x,y)dx + Q(x,y)dy = 0$ – $P = \frac{\partial F}{\partial x}$ , $Q = \frac{\partial F}{\partial y}$

(non exact ODE) if $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ ie. $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \neq \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$

- find Integrating factor.

$$I(x) = e^{\int \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q} dx}$$

- if can't solve for $I(x)$

$$I(y) = e^{\int \frac{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}{P} dy}$$

- multiply $IP(x,y)dx + IQ(x,y)dy = 0$
  and continue with exact ODE method.

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$

(second order, linear ODE with constant coeff.)

if $f(x) = 0$ (homogeneous)

- solve auxiliary eqn: $a\lambda^2 + b\lambda + c = 0$
  $y_c = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$ (two different roots)
  $y_c = (A_1 + A_2x)e^{\lambda x}$ (two equal roots)
  $y_c = e^{Px}(A_1\cos Qx + A_2\sin Qx)$ (complex roots)

if $f(x) \neq 0$ (non homogeneous)

- find $y_p$ , and final answer:
  $y = y_c + y_p$

## 11.1 Convert function to ODE. (not important)

only this important

- n order LINEAR differential equation have n number of arbitrary constant.
- hence if we have n arbitrary constant, we need to differentiate n times to get rid of it.

eg1. $y = (x+A)\sin x$

$y = x\sin x + A\sin x$

$\frac{dy}{dx} = x\cos x + \sin x + A\cos x$

$A\cos x = \frac{dy}{dx} - x\cos x - \sin x$

$A = \frac{dy}{dx}\left(\frac{1}{\cos x}\right) - x - \frac{\sin x}{\cos x}$

$y = x\sin x + \sin x\left(\frac{dy}{dx}\left(\frac{1}{\cos x}\right) - x - \frac{\sin x}{\cos x}\right)$

$y = x\sin x - x\sin x + \frac{dy}{dx}\left(\frac{\sin x}{\cos x}\right) - \frac{\sin^2 x}{\cos x}$

$y = \frac{dy}{dx}\tan x - \sin x \tan x.$

eg2. $y = A\sin x + B\cos x$

$\frac{dy}{dx} = A\cos x - B\sin x$

$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$

$\frac{d^2y}{dx^2} = -(A\sin x + B\cos x)$

$\frac{d^2y}{dx^2} = -y$

# 11·2 Linear ODE, non seperable, homogeneus

a function is said to be homogeneus when:

$f(kx, ky) = k^n f(x,y)$ , where n is the degree of homogenity

an ODE is said to be homogeneous when:

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

f(x,y) and g(x,y) have the same degree of homogenity.

↪ this means f(x,y) and g(x,y) are both homogeneous

↪ and $dy/dx$ get back to itself after factoring k out.

**METHOD:** For Linear ODE, nonseperable but homogeneous:

substitute $y = ux$

eg1.

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$$

check for homogenity:

$$\frac{ky+\sqrt{k^2x^2+k^2y^2}}{kx} = \frac{\cancel{k}(y+\sqrt{x^2+y^2})}{\cancel{k}x} \checkmark$$

let $y = ux$

$$\frac{dy}{dx} = u + \frac{du}{dx}x$$

$$u + \frac{du}{dx}x = \frac{ux+\sqrt{x^2+u^2x^2}}{x}$$

$$u + \frac{du}{dx}x = \frac{u\cancel{x}+\cancel{x}\sqrt{1+u^2}}{\cancel{x}}$$

$$\cancel{u} + \frac{du}{dx}x = \cancel{u} + \sqrt{1+u^2}$$

$$\frac{du}{dx}x = \sqrt{1+u^2}$$

$$\int \frac{1}{\sqrt{1+u^2}}\,du = \int \frac{1}{x}\,dx$$

$$\sinh^{-1}(u) = \ln|x| + c$$

$$\sinh^{-1}\left(\frac{y}{x}\right) = \ln|x| + c \quad //$$

eg2. $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$ $\qquad \frac{2(kx)(ky)}{k^2x^2-k^2y^2}$ ✓

let $y = ux$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{2x(ux)}{x^2-u^2x^2}$

$u + x\frac{du}{dx} = \frac{2u\cancel{x^2}}{\cancel{x^2}(1-u^2)}$

$x\frac{du}{dx} = \frac{2u}{1-u^2} - u$

$x\frac{du}{dx} = \frac{2u-u(1-u^2)}{1-u^2}$

$x\frac{du}{dx} = \frac{u+u^3}{1-u^2}$

$\frac{1-u^2}{u+u^3}du = \frac{1}{x}dx$

$\frac{1-u^2}{u(1+u^2)} = \frac{A}{u} + \frac{Bu+C}{1+u^2}$

$1-u^2 = A(1+u^2) + (Bu+C)u$

let $u=0$ $\qquad$ let $u=1$ $\qquad$ let $u=-1$

$1=A$ $\qquad$ $0 = 2 + B + C$ $\qquad$ $0 = 2-(-B+C)$

$0 = 2+B-C$

$0 = 2+B+B+2$ $\qquad$ $C = B+2$

$2B = -4$ $\qquad$ $C = -2+2 = 0$

$B = -2$

$\frac{1-u^2}{u(1+u^2)} = \frac{1}{u} + \frac{-2u}{1+u^2}$

$\int \frac{1}{u} - \frac{2u}{1+u^2}du = \int \frac{1}{x}dx$

$\ln|u| - \ln|1+u^2| = \ln|x| + c$

$e^{\ln\left|\frac{u}{1+u^2}\right|} = e^{\ln|x|+c}$

$e^{\ln\left|\frac{u}{1+u^2}\right|} = c_1 e^{\ln|x|}$

$\frac{u}{1+u^2} = c_1 x$

$\frac{\frac{y}{x}}{1+\frac{y}{x}^2} = c_1 x$

$\frac{y}{x+\frac{y^2}{x}} = c_1 x$

$y = c_1(x^2+y^2)$

$x^2+y^2 = \frac{1}{c_1}y$

$x^2+y^2 = c_2 y$

## 11·3 Linear ODE, non seperable, non homogeneous.

if the ODE is in the form of:

$$\frac{dy}{dx} = \frac{ax+by+c}{fx+gy+h}$$

$ag-bf \neq 0$ (form A) ; $ag-bf = 0$ (form B)

### 11·3·1 Form A: $ag-bf \neq 0$

step 1. apply transformation $x = x_0 + X$, $y = y_0 + Y$ ☆

where $x_0$ and $y_0$ are from simultaneous equation:

$$ax_0 + by_0 + c = 0$$
$$fx_0 + gy_0 + h = 0$$

step 2. obtain $\frac{dY}{dX} = \frac{aX+bY}{fX+gY}$

do usub as it is now homogeneous, ie $Y = uX$

step 3. redo all the substitution, from $u = \frac{X}{Y}$ and $X = x - x_0$, $Y = y - y_0$

eg1. $\frac{dy}{dx} = \frac{2x+2y-2}{3x+y-5}$

$2(1)-2(3) = -4 \neq 0$ → need to apply transformation

apply transformation

$x = x_0 + X$
$y = y_0 + Y$

$2x_0 + 2y_0 - 2 = 0$ → $x_0 + y_0 - 1 = 0$ → $x_0 = 1 - y_0$
$3x_0 + y_0 - 5 = 0$

$3(1-y_0) + y_0 - 5 = 0$ $\quad$ $x_0 = 1-(-1) = 2$
$3 - 2y_0 - 5 = 0$
$-2 - 2y_0 = 0$
$y_0 = -1$

$$\frac{dy}{dx} = \frac{d(y_0+Y)}{d(x_0+X)} = \frac{dY}{dX}!$$

$$\frac{dY}{dX} = \frac{2(2+X)+2(-1+Y)-2}{3(2+X)+(-1+Y)-5} = \frac{2X+2Y}{3X+Y}$$ (now it's homogeneous)

...

u sub
$Y=uX$

let $Y=uX$

$$\frac{dY}{dX}=u+X\frac{du}{dX}$$

$$u+X\frac{du}{dX}=\frac{2X+2uX}{3X+uX}$$

$$u+X\frac{du}{dX}=\frac{\cancel{X}(2+2u)}{\cancel{X}(3+u)}$$

$$X\frac{du}{dX}=\frac{2+2u}{3+u}-u$$

$$X\frac{du}{dX}=\frac{2+2u-u(3+u)}{3+u}$$

$$X\frac{du}{dX}=\frac{-u^2-u+2}{3+u}$$

$$\frac{3+u}{-u^2-u+2}du=\frac{1}{X}dX$$

$$\int\frac{u+3}{(u-1)(u+2)}du=\int-\frac{1}{X}dX$$

$$\int\frac{4}{3u-3}-\frac{1}{3u+6}du=\int-\frac{1}{X}dX$$

$$\frac{4}{3}\ln|u-1|-\frac{1}{3}\ln|u+2|=-\ln|X|+c_1$$

$$\frac{u+3}{(u-1)(u+2)}=\frac{A}{u-1}+\frac{B}{u+2}$$

$$u+3=A(u+2)+B(u-1)$$

let $u=-2$ | let $u=1$
$1=B(-3)$ | $4=A(3)$
$B=-1/3$ | $A=4/3$

sub u back
$u=Y/X$

$$\frac{1}{3}\ln\left|\frac{(u-1)^4}{u+2}\right|=\ln|X^{-1}|+c_1$$

$$e^{\ln\left|\frac{(Y/X-1)^4}{Y/X+2}\right|^{\frac{1}{3}}}=e^{\ln\left|\frac{1}{X}\right|+c_1}$$

$$\frac{(Y/X-1)^{4/3}}{(Y/X+2)^{1/3}}=\frac{1}{X}\cdot e^{c_1}$$

$x=2+X$ $y=-1+Y$
$X=x-2$ $Y=y+1$

sub $X=x-x_0$
$Y=y-y_0$

$$\frac{\left(\frac{y+1}{x-2}-1\right)^{4/3}}{\left(\frac{y+1}{x-2}+2\right)^{1/3}}=\frac{c}{x-2}$$

//

## 11.3.2 Form B : $ag - bf = 0$

step1: substitute $f = ak$ ; $g = bk$ (or just simply factorise either top or bottom)

step2: substitute $u = ax + by$

eg. $\dfrac{dy}{dx} = \dfrac{2x - 4y + 3}{-x + 2y}$ $\qquad 2(2) - (-4)(-1) = 0$ Form B.

$$\frac{dy}{dx} = \frac{-2(-x+2y)+3}{-x+2y}$$

let $u = -x + 2y$

$u + x = 2y$

$\frac{1}{2}u + \frac{1}{2}x = y$

$\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{du}{dx} + \dfrac{1}{2}$

$$\frac{1}{2}\frac{du}{dx} + \frac{1}{2} = \frac{-2u+3}{u}$$

$$\frac{du}{dx} = \frac{-4u+6}{u} - 1$$

$$\frac{du}{dx} = \frac{-5u+6}{u}$$

$$-5u+6 \overline{)\, u} = -\tfrac{1}{5}, \quad u - \tfrac{6}{5}$$

$$\frac{u}{-5u+6}\,du = dx$$

$$\int -\frac{1}{5} + \frac{6/5}{-5u+6}\,du = \int dx$$

$$-\frac{1}{5}u + \frac{6}{5}\left(-\frac{1}{5}\right)\ln\left|-5u+6\right| = x + c$$

$$-\frac{1}{5}(-x+2y) - \frac{6}{25}\ln\left|-5(-x+2y)+6\right| = x + c$$

$$\frac{1}{5}x - \frac{2}{5}y - \frac{6}{25}\ln\left|5x - 10y + 6\right| = x + c$$

need to combine with c !

$$\frac{1}{5}x - \frac{2}{5}y - \frac{6}{25}\ln\left|5x - 10y + 24\right| = x + c_1$$

OR $\dfrac{dy}{dx} = \dfrac{2x - 4y + 3}{-\frac{1}{2}(2x - 4y)}$

let $u = 2x - 4y$

$\dfrac{du}{dx} = 2 - 4\dfrac{dy}{dx}$

$4\dfrac{dy}{dx} = 2 - \dfrac{du}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{4}\dfrac{du}{dx}$

$$\frac{1}{2} - \frac{1}{4}\frac{du}{dx} = \frac{u+3}{-\frac{1}{2}u}$$

$$\frac{1}{4}\frac{du}{dx} = \frac{1}{2} + \frac{2u+6}{u} = \frac{\frac{5}{2}u + 6}{u}$$

$$10u+24 \overline{)\, u} = \tfrac{1}{10}, \quad u + \tfrac{24}{10}, \quad -\tfrac{12}{5}$$

$$u\,du = 10u + 24\,dx$$

$$\int \frac{u}{10u+24}\,du = \int dx$$

$$\int \frac{1}{10} - \frac{12/5}{10u+24}\,du = \int dx$$

$$\frac{1}{10}u - \frac{12}{5}\left(\frac{1}{10}\right)\ln\left|10u+24\right| = x + c$$

$$\frac{1}{10}(2x-4y) - \frac{6}{25}\ln\left|10(2x-4y)+24\right| = x + c$$

$$\frac{1}{5}x - \frac{2}{5}y - \frac{6}{25}\ln\left|20x - 40y + 24\right| = x + c$$

$$\frac{1}{5}x - \frac{2}{5}y - \frac{6}{25}\ln\left|4(5x - 10y + 24)\right| = x + c$$

# 11·4 Integrating Factor

when linear ODE, non seperable is in the form of:

$$\frac{dy}{dx} + p(x)y = q(x)$$

integrating factor, $I(x) = e^{\int p(x)\,dx}$

☆ always multiply I on the eqn that coeff. of $dy/dx = 1$ !

$$\rightarrow I(x)\frac{dy}{dx} + I(x)p(x)y = I(x)q(x)$$

$$\frac{d}{dx}(yI(x)) = I(x)q(x)$$

$$yI(x) = \int I(x)q(x)\,dx \quad \text{(+c after integrating)}$$

note that if:

1. there is anything in front of $\frac{dy}{dx}$, divide the whole equation by it.
2. Integrating factor need to take whole $p(x)$.

eg. $\frac{dy}{dx} - p(x)y = q(x)$

$\rightarrow I(x) = e^{\int -p(x)\,dx}$

eg. $x\frac{dy}{dx} + 2y = x^2 - x + 1$

$$\frac{dy}{dx} + \frac{2}{x}y = x - 1 + \frac{1}{x}$$

$$x^2\frac{dy}{dx} + 2xy = x^3 - x^2 + x$$

$$\frac{d}{dx}(yx^2) = x^3 - x^2 + x$$

$$yx^2 = \int x^3 - x^2 + x\,dx$$

$$yx^2 = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + c$$

$$y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + cx^{-2}$$

$$I(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x^2|} = x^2$$

# 11·5 First order, linear, exact ODEs

If we have ODE in the form of:

$$P\,dx + Q\,dy = 0 \quad , \text{where } P(x,y) = \frac{\partial F}{\partial x} , \; Q(x,y) = \frac{\partial F}{\partial y}$$

we first have to check:

1. $P_y \overset{?}{=} Q_x$ ; if yes → exact
   if no → non exact.

   $P_y$ means $\frac{\partial P}{\partial y}$
   $Q_x$ means $\frac{\partial Q}{\partial x}$

   $$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \overset{?}{=} \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

   when this is equal, the ODE is basically a derivatives of a multivariable function, $F(x,y)$, and it is continuous and differentiable.

2. $P = \frac{\partial F}{\partial x}$    $+g(y)$ when $dx$
   $\partial F = P\,\partial x$
   $F(x,y) = \int P\,dx + g(y)$

   OR

   $Q = \frac{\partial F}{\partial y}$    $+h(x)$ when $dy$
   $\partial F = Q\,\partial y$
   $F(x,y) = \int Q\,dy + h(x)$

   why $+g(y)$? eg. $F(x,y) = 2xy + \cos(x^2) + y^2$ ← some of the y term disappears!
   $\partial F/\partial x = 2y - 2x\sin(x^2)$

3. this is Q!   * don't need +c here

   $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int P\,dx + g'(y)$$

   $$Q = \frac{\partial}{\partial y}\int P\,dx + g'(y)$$ would not have any $x$!

   solve for $g'(y)$ and find $g(y)$ by $g(y) = \int g'(y)\,dy$

   OR

   this is P!   don't need +c here.

   $$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int Q\,dy + h'(x)$$

   $$P = \frac{\partial}{\partial x}\int Q\,dy + h'(x)$$ would not have any $y$!

   solve for $h'(x)$ and find $h(x)$ by $h(x) = \int h'(x)\,dx$

4. $F(x,y) = \int P\,dx + g(y) = C$   OR   $F(x,y) = \int Q\,dy + h(x) = C$

$F(x,y) = \underline{C}$ derived all these, don't forget $= C$!

## 11·6 First Order, Linear, Non-exact ODEs.

from 11·5, if $P_y \neq Q_x$ ie $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \neq \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$

find integrating factor with formula:

$$I(x) = e^{\int \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q} dx}$$

, and if it doesn't work, try:

$$I(y) = e^{\int \frac{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}{P} dy}$$

multiplying $I$ with $Pdx + Qdy = 0 \rightarrow IPdx + IQdy = 0$
and continue with checking $P_y = Q_x$ and with step 2, 3, ...

eg1. $y\cos x + 2xe^y + (\sin x + x^2 e^y - 1)\dfrac{dy}{dx} = 0$ (exact ODE)

$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2e^y - 1)\,dy = 0$$

$$P = \frac{\partial F}{\partial x} = y\cos x + 2xe^y \qquad \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \cos x + 2xye^y$$

$$Q = \frac{\partial F}{\partial y} = \sin x + x^2e^y - 1 \qquad \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \cos x + 2xe^y$$

$$\Big\} \; P_y = Q_x$$

$$\partial F = P\,\partial x$$

$$\begin{aligned} F(x,y) &= \int P\,dx + g(y) \\ &= \int y\cos x + 2xe^y\,dx + g(y) \\ &= \underbrace{y\sin x + x^2e^y + g(y)}_{\text{don't need } +c!} \end{aligned}$$

don't need $+c$! (will combined with $c_1$ eventually)

$$\frac{\partial F}{\partial y} = \sin x + x^2ye^y + g'(y)$$

$$\cancel{\sin x} + \cancel{x^2e^y} - 1 = \cancel{\sin x} + \cancel{x^2ye^y} + g'(y)$$

$$g'(y) = -1$$

$$g(y) = \int g'(y)\,dy = \int -1\,dy = -y + c_1$$

$$F(x,y) = y\sin x + x^2e^y - y + c_1 = C_2$$

due to level curve! (deduction)

$$\boxed{y\sin x + x^2e^y - y = C_3}$$

eg2. $(3x^2 - 2xy)\frac{dy}{dx} + 5x^2 + 12xy - 3y^2 = 0$ (non-exact), I(x) will work.

$$(3x^2 - 2xy)dy + (5x^2 + 12xy - 3y^2)dx = 0$$

check exact or not

$P = \frac{\partial F}{\partial x} = 5x^2 + 12xy - 3y^2$, $P_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = 12x - 6y$

$Q = \frac{\partial F}{\partial y} = 3x^2 - 2xy$, $Q_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = 6x - 2y$

$P_y \neq Q_x$ not exact

find I(x) if doesn't work then I(y)

$$I(x) = e^{\int \frac{\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)}{Q} dx} = e^{\int \frac{(12x - 6y) - (6x - 2y)}{3x^2 - 2xy} dx}$$

$$= e^{\int \frac{6x - 4y}{3x^2 - 2xy} dx}$$

$$= e^{\int \frac{2\cancel{(3x - 2y)}}{x\cancel{(3x - 2y)}} dx}$$

$$= e^{\int \frac{2}{x} dx}$$

$$= e^{2\ln|x|}$$

$$= x^2$$

I(x) × Pdx+Qdy=0

$$(3x^4 - 2x^3y)dy + (5x^4 + 12x^3y - 3x^2y^2)dx = 0$$

check exact or not

$P = \frac{\partial F}{\partial x} = 5x^4 + 12x^3y - 3x^2y^2$ $P_y = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = 12x^3 - 6x^2y$

$Q = \frac{\partial F}{\partial y} = 3x^4 - 2x^3y$ $Q_x = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = 12x^3 - 6x^2y$

$P_y = Q_x$ exact

find its potential function.

$\partial F = Q\partial y$

$F(x,y) = \int Q dy + h(x)$

$= \int (3x^4 - 2x^3y)dy + h(x)$

$= 3x^4y - x^3y^2 + h(x)$

$\frac{\partial F}{\partial x} = 12x^3y - 3x^2y^2 + h'(x)$

$5x^4 + \cancel{12x^3y} - \cancel{3x^2y^2} = \cancel{12x^3y} - \cancel{3x^2y^2} + h'(x)$

$h'(x) = 5x^4$

$h(x) = \int 5x^4 dx = x^5 + C_1$

$F(x,y) = 3x^4y - x^3y^2 + x^5 + C_1 = C_2$

$3x^4y - x^3y^2 + x^5 = C_3$ //

eg3. $2xy^3+y^4+(xy^3-2y)y'=0$ (non exact), $I(x)$ will not work!

$$(2xy^3+y^4)\,dx+(xy^3-2y)\,dy=0$$

$$P=\frac{\partial F}{\partial x}=2xy^3+y^4, \quad \frac{\partial P}{\partial y}=6xy^2+4y^3$$

$$Q=\frac{\partial F}{\partial y}=xy^3-2y, \quad \frac{\partial Q}{\partial x}=y^3$$

$P_y \neq Q_x$ <u>non exact</u>

$$I(x)=e^{\int \frac{\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)}{Q}dx}=e^{\int \frac{(6xy^2+4y^3)-y^3}{xy^3-2y}dx}$$

$$=e^{\int \frac{6xy^2+3y^3}{xy^3-2y}dx}$$

$$=e^{\int \frac{3y^2(2x+y)}{y(xy^2-2)}dx}$$ (can't continue, try $I(y)$)

$$I(y)=e^{\int \frac{\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)}{P}dy}=e^{\int \frac{y^3-(6xy^2+4y^3)}{2xy^3+y^4}dy}$$

$$=e^{\int \frac{-3y^3-6xy^2}{2xy^3+y^4}dy}$$

$$=e^{\int \frac{-3\cancel{y^2}\cancel{(y+2x)}}{y^{\cancel{3}}\cancel{(2x+y)}}dy}$$

$$=e^{\int -\frac{3}{y}dy}$$

$$=e^{-3\ln|y|}$$

$$=e^{\ln|y^{-3}|}$$

$$=\frac{1}{y^3}$$

$$\left[(2xy^3+y^4)\,dx+(xy^3-2y)\,dy=0\right]\times\frac{1}{y^3}$$

$$(2x+y)\,dx+(x-2y^{-2})\,dy=0$$

$P = \frac{\partial F}{\partial x} = 2x + y$ , $\frac{\partial P}{\partial y} = 1$
$Q = \frac{\partial F}{\partial y} = x + \frac{2}{y^2}$ , $\frac{\partial Q}{\partial x} = 1$ $\Bigg\}$ $P_y = Q_x$ exact

$P\partial x = \partial F$
$F(x,y) = \int P\,dx + g(y)$
$= \int (2x+y)\,dx + g(y)$
$= x^2 + yx + g(y)$

$\frac{\partial F}{\partial y} = x + g'(y)$
$Q = x + g'(y)$
$\cancel{x} + \frac{2}{y^2} = \cancel{x} + g'(y)$
$g'(y) = \frac{2}{y^2}$
$g(y) = \int g'(y)\,dy$
$= \int 2y^{-2}\,dy$
$= \frac{2y^{-1}}{-1} + C_1$
$= -2/y + C_1$

$F(x,y) = \underbrace{x^2 + yx - \frac{2}{y} + C_1 = C_2}$
$x^2 + yx - \frac{2}{y} = C_3$ //

answer checking.
$2x + y + \frac{dy}{dx}x - \frac{2}{y^2}\frac{dy}{dx} = 0$
$2xy^2 + y^3 + xy^2\frac{dy}{dx} - 2\frac{dy}{dx} = 0$
$\frac{dy}{dx}(xy^2 - 2) + (2xy^2 + y^3) = 0$
$(2xy^2 + y^3)\,dx + (xy^2 - 2)\,dy = 0$

## 11.7 Second Order Linear Differential Equation, with constant coefficient.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

eg of non linear 2nd order ODE:

$$\frac{d^2y}{dx^2} + \tan\left(\frac{dy}{dx}\right) + \sqrt{y} = 0$$

easiest way to approximate solution is use Taylor's series!

if a, b, c are constant!
if $f(x)=0$, the ODE is homogeneous
if $f(x)\neq 0$, the ODE is non homogeneous.

### 11.7.1 Homogeneous case.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

For second order differential equations of the form $a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$:

*From the auxiliary equation* $a\lambda^2 + b\lambda + c = 0$:

- If $\lambda_1 = \alpha, \lambda_2 = \beta$ then the complementary function is $y = Ae^{\alpha x} + Be^{\beta x}$.
- If $\lambda_1 = \lambda_2 = \alpha$ then the complementary function is $y = (Ax + B)e^{\alpha x}$.
- If $\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$ then the complementary function is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

proof of auxiliary equation:

let $y = Ae^{\lambda x}$
$y' = A\lambda e^{\lambda x}$
$y'' = A\lambda^2 e^{\lambda x}$

$ay'' + by' + cy = 0$
$a(A\lambda^2 e^{\lambda x}) + b(A\lambda e^{\lambda x}) + c(Ae^{\lambda x}) = 0$
$Ae^{\lambda x}(a\lambda^2 + b\lambda + c) = 0$

since $y = Ae^{\lambda x} \neq 0$ (non trivial case)
$a\lambda^2 + b\lambda + c = 0$ (auxiliary eqn).

two different roots
two same roots
complex roots
– depends on $b^2 - 4ac$

eg. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ (two different roots)

$\lambda^2 - \lambda - 6 = 0$

$(\lambda+2)(\lambda-3) = 0$

$\lambda = -2, \lambda = 3$

$y = A_1 e^{-2x} + A_2 e^{3x}$

eg. $y'' + 4y' + 4y = 0$ , $y = f(x)$ (two same roots)

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)(\lambda+2) = 0$

$\lambda = -2$

$y = A_1 e^{-2x} + A_2 x e^{-2x}$

eg. $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 41y = 0$ (two complex roots)

$\lambda^2 + 8\lambda + 41 = 0$

$\lambda = \frac{-8 \pm \sqrt{8^2 - 4(1)(41)}}{2}$

$= \frac{-8 \pm \sqrt{-100}}{2}$

$= \frac{-8 \pm 10i}{2}$

$= -4 \pm 5i$

$y = e^{-4x}(A_1 \cos 5x + A_2 \sin 5x)$

## 11·7·2 Non-homogeneous case.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

1. solve for homogeneous case first, to get $y_c$ (complimentary function)

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y_c = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$$

2. find $y_p$ (particular Integral) (note that there is no arbitrary constant in $y_p$!)

NOT THE WHOLE $y_p$!

☆ IF a part of $y_p$ is equal to a part of $y_c$, multiply that specific part of $y_p$ by $x$, if $xp(x)$ is still a part of $y_c$, multiply it again by $x$

eg. $y_c = Ae^{2x} + Be^{4x}$

① if $y_p = \alpha e^{2x}$
change it to $y_p = \alpha x e^{2x}$

② if $y_p = \alpha e^{2x} + x^2 - 2x - 3$
change it to $y_p = \alpha x e^{2x} + \underbrace{x^2 - 2x - 3}_{\text{no change.}}$

③ if $y_p = e^{2x}(A\sin 3x + B\cos 3x)$
no change since $p(x) = e^{2x}\sin 3x$ not $e^{2x}$ (only change when exactly the same!)

3. differentiate twice $y_p$ to find $y_p'$ and $y_p''$ to get $y_p$ without unknown.

$$ay_p'' + by_p' + cy_p = f(x) \quad \text{solve for } y_p$$

4. $y = y_c + y_p$

if $f(x) = kx$
ignore $k$,
$y_p = \alpha x + \beta$
since we going to solve for it later.

| $f(x)$ | particular integral |
|---|---|
| $e^{mx}$ | $\alpha e^{mx}$ |
| constant : $k$ | $\alpha$ |
| linear : $mx + c$, $mx$ | $\alpha x + \beta$ |
| quadratic : $mx^2 + nx + p$, $mx^2 + nx$, $mx^2$, $mx^2 + p$ | $\alpha x^2 + \beta x + \gamma$ |
| cubic | $\alpha x^3 + \beta x^2 + \gamma x + \delta$ |
| trigonometry: $k \sin bx$, $k \cos bx$, $k_1 \sin bx + k_2 \cos bx$ | $\alpha \sin bx + \beta \cos bx$ |
| $k_1 \sinh bx + k_2 \cosh bx$ | $Ae^{x} + Be^{-x}$ |
| add / multiply | add / multiply. |

if $f(x) = \sin 2x + \cos 3x$
$y_p = \alpha \sin 2x + \beta \cos 2x + \gamma \sin 3x + \delta \cos 3x$

eg1. $\frac{d^2y}{dx^2}+3\frac{dy}{dx}-4y=7e^{3x}$ (simplest case)

1. find $y_c$, auxiliary eqn: $\lambda^2+3\lambda-4=0$

$\lambda^2+3\lambda-4=0$
$(\lambda-1)(\lambda+4)=0$
$\lambda=1, \lambda=-4$
$y_c=A_1e^{x}+A_2e^{-4x}$

2. find $y_p$, first define general form $y_p=\alpha e^{3x}$ (double check no repetition with $y_c$) (non-linear dependent)

$y_p'=3\alpha e^{3x}$
$y_p''=9\alpha e^{3x}$

$9\alpha e^{3x}+3(3\alpha e^{3x})-4(\alpha e^{3x})=7e^{3x}$
$14\alpha e^{3x}=7e^{3x}$
$14\alpha=7$
$\alpha=1/2$
$y_p=\frac{1}{2}e^{3x}$

3. $y=y_c+y_p=A_1e^{x}+A_2e^{-4x}+\frac{1}{2}e^{3x}$

eg2. $\frac{d^2y}{dx^2}+4y=3\sin 2x$

$\lambda^2+4=0$
$\lambda=\pm 2i$
$y_c=e^{0x}(A_1\cos 2x+A_2\sin 2x)$
$y_c=A_1\cos 2x+A_2\sin 2x$

$y_p=\alpha\sin 2x+\beta\cos 2x$ (linear dependent with $y_c$!)
{ both $\sin 2x$ and $\cos 2x$ linear dependent so both $\times\, x$!

$y_p=\alpha x\sin 2x+\beta x\cos 2x$
$y_p'=2\alpha x\cos 2x+\alpha\sin 2x-2\beta x\sin 2x+\beta\cos 2x$
$y_p''=-4\alpha x\sin 2x+2\alpha\cos 2x+2\alpha\cos 2x-4\beta x\cos 2x-2\beta\sin 2x-2\beta\sin 2x$

$\therefore\ y=A_1\cos 2x+A_2\sin 2x-\frac{3}{4}x\cos 2x$

$-4\alpha x\sin 2x+4\alpha\cos 2x-4\beta\sin 2x-4\beta x\cos 2x-2\beta\sin 2x+4(\alpha x\sin 2x+\beta x\cos 2x)=3\sin 2x$

$4\alpha\cos 2x-\cancel{4\beta x\cos 2x}+\cancel{4\beta x\cos 2x}=0$
$4\alpha=0$
$\alpha=0$

$-\cancel{4\alpha x\sin 2x}-4\beta\sin 2x+\cancel{4\alpha x\sin 2x}=3\sin 2x$
$-4\beta=3$
$\beta=-3/4$

eg3. $\frac{d^2y}{dx^2}+\frac{dy}{dx}-x=0 \rightarrow \frac{d^2y}{dx^2}+\frac{dy}{dx}=x$ ← $f(x)$ is linear

$\lambda^2+\lambda=0$

$\lambda(\lambda+1)=0$

$\lambda=0,\ \lambda=-1$

$y_c=A_1e^{0x}+A_2e^{-x}$

$y_c=A_1+A_2e^{-x}$

linear write it full. $\alpha x$ only is wrong!

$y_p=\alpha x+\beta$ ← realise $\beta$ is linearly dependant of $y_c$

$y_p=\alpha x^2+\beta x$ ← multiply both by $x$, although we always say multiply the "repeated" part by $x$

BUT in this case:

$y_p=\alpha x+\beta x$

they will become linearly dependent again.

$y_p'=2\alpha x+\beta$

$y_p''=2\alpha$

$2\alpha+2\alpha x+\beta=x$

$2\alpha x=x \qquad 2\alpha+\beta=0$

$2\alpha=1 \qquad 1+\beta=0$

$\alpha=1/2 \qquad \beta=-1$

$y_p=\frac{1}{2}x^2-x$

$\therefore y=A_1+A_2e^{-x}+\frac{1}{2}x^2-x$

eg4. $y''+8y'+8y=12\cosh 2x$ ← $y_p$ is $\alpha e^{2x}+\beta e^{-2x}$ not $\alpha\sinh(2x)+\beta\cosh(2x)$ cause of linear dependancy!

$\lambda^2+8\lambda+8=0$

$\lambda=\frac{-8\pm\sqrt{8^2-4(8)}}{2}$

$=\frac{-8\pm\sqrt{32}}{2}$

$=\frac{-8\pm4\sqrt{2}}{2}$

$=-4\pm2\sqrt{2}$

$y_c=A_1e^{(-4+2\sqrt{2})x}+A_2e^{(-4-2\sqrt{2})x}$

$=e^{-4x}(A_1e^{2\sqrt{2}x}+A_2e^{-2\sqrt{2}x})$

$y_p=\alpha e^{2x}+\beta e^{-2x}$

$y_p'=2\alpha e^{2x}-2\beta e^{-2x}$

$y_p''=4\alpha e^{2x}+4\beta e^{-2x}$

$4\alpha e^{2x}+4\beta e^{-2x}+8(2\alpha e^{2x}-2\beta e^{-2x})+8(\alpha e^{2x}+\beta e^{-2x})$

$=12\left(\frac{e^{2x}+e^{-2x}}{2}\right)$

$\cancel{e^{2x}}(4\alpha+16\alpha+8\alpha)=6\cancel{e^{2x}}$

$28\alpha=6$

$\alpha=3/14$

$\cancel{e^{-2x}}(4\beta-16\beta+8\beta)=6\cancel{e^{-2x}}$

$-4\beta=6$

$\beta=-3/2$

$y=e^{-4x}(A_1e^{2\sqrt{2}x}+A_2e^{-2\sqrt{2}x})+\frac{3}{14}e^{2x}-\frac{3}{2}e^{-2x}$

# c12 Matrices

## 12.1 Rank.

Rank is the dimension span by column vector in a matrix. (ie column space)

rank is: 1. number of independent pieces of information to solve $Ax=b$
2. always less than or equal to number of row and column. ($r \leq m$, $r \leq n$)

$Ax=b$ : $\text{rank}(A) + \dim(\text{Ker}(A)) = n$

(A is a square matrix) $n \times n$

where $\dim(\text{Ker}(A))$ is the dimension of the solution, $x$

$n$ is the number of row or column.

point → 0D
line → 1D
plane → 2D
⋮

eg. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\text{rank}(A)=2$, $n=3$, hence solution will be a line! ☆

ie $\dim(\text{Ker}(A)) = 3-2 = 1$

☆ if A is invertible ($\det A \neq 0$) $\text{rank}(A) = n$ → full rank

## 12.2 Determinant.

Determinant defined for square matrix only.

given $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- choose a row or a column to be reference axis.
- multiply each element with cofactor $(-1)^{i+j}$ with minor $M_{ij}$

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \leftarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \leftarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- determinant remain the same when multiple of a row or column is added to a given row or column. ☆☆

  eg. $R_2' = \underline{3R_1} + \underline{R_2}$, $R_3' = \frac{1}{3}R_1 + \underline{R_3}$ the current row need to be the same factor!

  other row can be not!

- determinant is multiplied by k if whole row is multiplied by k

  eg. $R_2' = 3R_2$, $R_3' = 2R_1 + \underline{3}R_3$ if not! have to multiply det by k! ☆☆

  $R_3' = 3R_3$ (det ×3)

  $R_3'' = 2R_1 + R_3'$ (det =)

  (better do it seperately ie don't do $\alpha R_1 + \beta R_2$)

- determinant is multiplied with −1 if we swap two rows.
- det = 0 if there is a whole row or column of zeros.

☆ shortcut: determinant of a 3×3 matrix can be found by:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \times \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

triple scalar product (volume of a parallelpiped)

special things to take note:

$\det(A^T) = \det(A)$

$\det(AB) = \det(A) \times \det(B)$

$\det(I) = 1$

$\det\begin{pmatrix} a & b \\ ka & kb \end{pmatrix} = 0$

$\det(A^{-1}) = \dfrac{1}{\det(A)}$

☆ $\det(cA) = c^n \det(A)$ if $A$ is a $n \times n$ matrix

→ if I have matrix $P$ such that $\det(P) = k$ and matrix $Q = cP$

if I want matrix $Q$ such that $\det(Q) = 1$

$Q = \dfrac{1}{\sqrt[n]{\det(P)}} P$ cause $\det\left((\det(P))^{-\frac{1}{n}} \cdot P\right) = \left(\det(P)^{-\frac{1}{n}}\right)^n \cdot \det(P)$

$= \det(P)^{-1} \cdot \det(P)$

$= 1$

generally if want $\det(cA) = k$:

$c^n \det(A) = k$

$c = \left(\dfrac{k}{\det(A)}\right)^{\frac{1}{n}}$

eg. find $\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix}$

$\det\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$ $R_1' = R_2 + R_1$

$= \det\begin{pmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$ $C_2' = -C_1 + C_2$

☆ can do both row and column manipulation.

$= \det\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -2 & 1 & 1 \\ -1 & 2 & 1 & 1 \end{pmatrix}$

$= 2\begin{vmatrix} 0 & -1 & 1 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

$= 2\left(-1 \times -\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix}\right) = 2(-4 + (-4)) = 16$ //

## 12.2 System of Linear Equation

### Method 1 : Gaussian's Elimination

$$\begin{aligned} ax+by+cd &= e \\ fx+gy+hd &= i \\ jx+ky+ld &= m \end{aligned}$$ can be rewritten as $$\left(\begin{array}{ccc|c} a & b & c & e \\ f & g & h & i \\ j & k & l & m \end{array}\right)$$

- can exchange any row without needing to do anything else
- can add or subtract multiple of any row.

  eg. $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right)$ $R_2' = R_2 - 3R_1$ or $R_2' = 3R_1 - R_2$ both can.

- can multiply or divide a row by any scalar without needing to do anything else.

$$\left(\begin{array}{ccc|c} a & b & c & g \\ 0 & d & e & h \\ 0 & 0 & f & i \end{array}\right)$$

- if $f=0$, $i=0$ infinite solution (solution is 1D or higher)
  if $f=0$, $i\neq0$ no solution.
  if $f\neq0$ unique solution (solution is 0D ie a point)

note that: $\left(\begin{array}{ccc|c} a & b & c & g \\ 0 & d & e & h \\ 0 & 0 & f & 0 \end{array}\right)$ is not no solution! $fz=0 \rightarrow z=0$.

### Method 2: Cramer's Rule

explanation with example : $Ax=b$ where $A=\begin{pmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{pmatrix}$, $b=\begin{pmatrix} 20 \\ 13 \\ 0 \end{pmatrix}$

$$\det A = 1\begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} - 2\begin{vmatrix} 7 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 7 & 3 \\ 1 & 6 \end{vmatrix} = 91$$

$$x_1 = \frac{\begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix}}{91} = \frac{182}{91} = 2 \qquad x_2 = \frac{\begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix}}{91} = \frac{-273}{91} = -3$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix}}{91} = \frac{728}{91} = 8$$

12·3 Inverse of a matrix

## Method 1: Definition.

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Adj}(A)$$

☆☆☆ DO NOT SWAP ROW, ADD MULTIPLE ROW OR MULTIPLY ROW WHEN FINDING ADJ!

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\operatorname{Adj}(A) = \begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}^T$$

☆ don't forget to transpose.

☆ not same as det! we do not multiply by element!

$\times\ a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

* Adj(A) shortcut for 3×3 matrix:

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\operatorname{Adj}(A) = \left[ \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \times \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \quad \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \times \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \quad \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \times \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \right]^T$$

## Method 2: Gauss-Jordan Elimination.

$$\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{gauss jordan elimination}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & a' & b' & c' \\ 0 & 1 & 0 & d' & e' & f' \\ 0 & 0 & 1 & g' & h' & i' \end{array}\right)$$

(the right block $a' \ldots i'$ is labelled: inverse)

Special things to note:

$(AB)^{-1} = B^{-1}A^{-1}$

$(A^{-1})^T = (A^T)^{-1}$

☆ a matrix is only invertible if $\lambda \neq 0$, $\det \neq 0$

# 12.4 Eigenvalue and Eigenvector.

$Ae = \lambda e$

$(A-\lambda I)\underset{\sim}{e} = 0$

$\underset{\sim}{e} = 0$ (trivia solution)

$\det(A-\lambda I) = 0$ (non-trivia solution)

trivia means we all know $\underset{\sim}{e} = 0$ is a solution that do not need to find

why can we do $\det(A-\lambda I)=0$
- we all know $\underset{\sim}{e}$ have infinite solution by definition. even if geometric multiplicity = 1
- rank $(A-\lambda I) < n$ A is n×n matrix.

$\det(A-\lambda I) = 0$ is also known as characteristic eqn.

$(\lambda-a)^{M_1}(\lambda-b)^{M_2}\dots = 0$

$\lambda = a$, $\lambda = b$ are eigenvalues. where $M_1, M_2, \dots$ are its corresponding algebraic multiplicity.

$\sum_{\forall i} M_i = M_1 + M_2 + \dots = n$ (n is number of row or column of A)

algebraic multiplicity of an eigenvalue tell you what is the biggest dimension of the eigenspace. eg. if $M=2$, eigenspace can most be a plane, might be a line only.

the dimension of eigenspace is known as m, geometric multiplicity.
m is always less than or equal to M, algebraic multiplicity. : $m_i \le M_i$

eg. if all $M=1$ and $m=1$, all eigenspace will be a line.
if $M_1=2$, $m_1=2$, eigenspace for that eigenvalue, $\lambda_1$ will be a plane.
if $M_2=2$, $m_1=1$, eigenspace for that eigenvalue, $\lambda_2$ will be a line.

eg1 (where all $m=1$)

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 4 \end{pmatrix}$$

$\det(A-\lambda I) = 0$

$$\det\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{pmatrix} = 0$$

$(2-\lambda)(3-\lambda)(4-\lambda) = 0$

$(\lambda-2)^1(\lambda-3)^1(\lambda-4)^1 = 0$ (all algebraic multiplicity is 1 → geometric multiplicity must be 1)

$\lambda=2$, $\lambda=3$, $\lambda=4$

$(A-\lambda I)\underset{\sim}{e} = \underset{\sim}{0}$

$\lambda = 2$:

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$ swap $R_1, R_3$ then $R_3' = R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y = 0$ $\quad x + 2z = 0$

let $z = \lambda \in \mathbb{R}$

$x = -2\lambda$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = 3$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad R_2' = R_2 + R_1$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y + z = 0$ $\quad -x + y = 0$

let $y = \mu \in \mathbb{R}$ $\quad -x + \mu = 0$

$z = -\mu$ $\quad x = \mu$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$\lambda = 4$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \quad R_3' = 2R_3 + R_1$$

$$\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-y = 0$ $\quad -2x + y = 0$ $\quad z = \mu$

$y = 0$ $\quad x = 0$

$$\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

eigenvalue: $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 4$

eigenvector: $\underset{\sim}{e_1} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\underset{\sim}{e_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\underset{\sim}{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

each eigenvalue have one eigenvector,

$M_i = m_i$

## eg2. (where $M > 1$, $m > 1$)

$$\begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$$\det \begin{pmatrix} 2-\lambda & -2 & 3 \\ -2 & -1-\lambda & 6 \\ 1 & 2 & -\lambda \end{pmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} -1-\lambda & 6 \\ 2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 6 \\ 1 & -\lambda \end{vmatrix} + 3\begin{vmatrix} -2 & -1-\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$-(\lambda-2)[-\lambda(-1-\lambda)-12] + 2[2\lambda-6] + 3[-4+\lambda+1] = 0$$

$$-(\lambda-2)(\lambda^2+\lambda-12) + 4\lambda - 12 + 3\lambda - 9 = 0$$

$$\lambda^3 - \lambda^2 - 21\lambda + 45 = 0$$

$$(\lambda+5)(\lambda-3)^2 = 0$$

$\lambda = -5$ $(M=1)$, $\lambda = 3$ $(M=2)$ $\quad M > 1$

when $\lambda = 3$:

$(A - \lambda I)\underset{\sim}{e} = \underset{\sim}{0}$

$$\left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ -2 & -4 & 6 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

rank $= 1$, $n = 3 \rightarrow \dim(\ker) = 2$ (plane)

$-x - 2y + 3z = 0$

let $y = \alpha \in \mathbb{R}$, $z = \beta \in \mathbb{R}$:

$-x - 2\alpha + 3\beta = 0$

$x = -2\alpha + 3\beta$

$$\underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}$$

eigenspace is a plane span by basis vector

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \& \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$m = 2$ $(m > 1)$

when $\lambda = -5$

$$\left(\begin{array}{ccc|c} 7 & -2 & 3 & 0 \\ -2 & 4 & 6 & 0 \\ 1 & 2 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ -2 & 4 & 6 & 0 \\ 7 & -2 & 3 & 0 \end{array}\right) \quad \begin{array}{l} R_2 + 2R_1 \\ R_3 - 7R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 8 & 16 & 0 \\ 0 & -16 & -32 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$y + 2z = 0$ $\quad$ let $z = \mu \in \mathbb{R}$:

$y = -2z$ $\quad y = -2\mu$

$z = -\frac{1}{2}(y)$

$x + 2y + 5(-\frac{y}{2})$

$x + -\frac{1}{2}y = 0$

$x = \frac{1}{2}(-2\mu), = -\mu$

$$\underset{\sim}{e} = \begin{pmatrix} -1 \\ -2 \\ 1. \end{pmatrix}$$

extra things to note:

1. for any triangular matrices (upper or lower)

$$A=\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}, \quad \lambda = a, d, f$$

2. sum of eigenvalues and multiple of eigenvalue.

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \quad \sum \lambda = a+e+i \quad \prod \lambda = \det(A)$$

, but note that a, e, i probably are not the eigenvalue.

THIS MEAN IF $\lambda=0$, $\det=0$ !!!

## 12.5 Diagonalisation.

Similarity — A square matrix B is said to be similar to matrix A iff there is an invertible matrix P such that:

$$B = P^{-1}AP$$

→ P('his') then apply 'my' A then $P^{-1}$('my')

change of basis
$B = P^{-1}AP$

(change of basis don't change the inherent properties of A such as rank, det and $\lambda$)

if P is a matrix that write 'his' basis vector in 'my' language: This matrix can translate any of 'his' vectors into 'my' language.

eg. $P=\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ 'his' basis vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

if he say: $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$, he means $P\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 3 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Diagonalisation — A square matrix A is said to be diagonalisable iff it is similar to a diagonal matrix D, such that:

$$D = P^{-1}AP$$

important things to take note!

A matrix, A is only diagonalisable iff $m_i = M_i$ for $\forall i$!
(even if $\lambda=0$, it still can be diagonalised if $m_i = M_i$ for $\forall i$! )
(even if all $\lambda=0$, sometimes it can still be diagonalised! eg. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ )

(iii) Consider the following system of differential equations in which the unknowns are three functions $x(t)$, $y(t)$ and $z(t)$:

$$\frac{dx}{dt} = y(t) + z(t)$$
$$\frac{dy}{dt} = -\frac{1}{2}x(t) + \frac{3}{2}y(t) - \frac{1}{2}z(t)$$
$$\frac{dz}{dt} = \frac{3}{2}x(t) - \frac{3}{2}y(t) + \frac{1}{2}z(t)$$

Use the result of (ii) to rewrite this system in the form of a system of differential equations for functions $u(t)$, $v(t)$ and $w(t)$, in which each function appears once and

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & -1/2 \\ 3/2 & -3/2 & 1/2 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} -\lambda & 1 & 1 \\ -1/2 & 3/2-\lambda & -1/2 \\ 3/2 & -3/2 & 1/2-\lambda \end{pmatrix} = 0$$

$$-\lambda\begin{vmatrix} 3/2-\lambda & -1/2 \\ -3/2 & 1/2-\lambda \end{vmatrix} - 1\begin{vmatrix} -1/2 & -1/2 \\ 3/2 & 1/2-\lambda \end{vmatrix} + 1\begin{vmatrix} -1/2 & 3/2-\lambda \\ 3/2 & -3/2 \end{vmatrix} = 0$$

$$-\lambda\left[(3/2-\lambda)(1/2-\lambda) - 1/2(3/2)\right] - \left[-1/2(1/2-\lambda) + 1/2(3/2)\right] + \left[1/2(3/2) - 3/2(3/2-\lambda)\right] = 0$$

$$-\lambda\left[\cancel{\tfrac{3}{4}} - \tfrac{3}{2}\lambda - \tfrac{1}{2}\lambda + \lambda^2 - \cancel{\tfrac{3}{4}}\right] - \left[-\tfrac{1}{4} + \tfrac{1}{2}\lambda + \tfrac{3}{4}\right] + \left[\tfrac{3}{4} - \tfrac{9}{4} + \tfrac{3}{2}\lambda\right] = 0$$

$$-\lambda(\lambda^2 - 2\lambda) - \left(\tfrac{1}{2}\lambda + \tfrac{1}{2}\right) + \left(-\tfrac{3}{2} + \tfrac{3}{2}\lambda\right) = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$
$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$
$$(\lambda+1)(\lambda-2)(\lambda-1) = 0$$
$$\lambda = -1, \lambda = 1, \lambda = 2$$

to find basis of eigenspace:

when $\lambda = 1$:

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ -1/2 & 1/2 & -1/2 & 0 \\ 3/2 & -3/2 & -1/2 & 0 \end{array}\right) \begin{array}{l} \\ 2R_2 - R_1 \\ 2R_3 + 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-2z = 0$
$z = 0$

$-x + y + z = 0$
$-x + y = 0$
$x = y$
let $y = \mu \in \mathbb{R}$. $\rightarrow x = \mu$

$\underset{\sim}{e_1} = \mu\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

when $\lambda = -1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -1/2 & 5/2 & -1/2 & 0 \\ 3/2 & -3/2 & 3/2 & 0 \end{array}\right) \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - 3R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & -6 & 0 & 0 \end{array}\right) \begin{array}{l} \\ \\ R_3 + R_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$6y = 0$
$y = 0$

$x + y + z = 0$
$x + z = 0$
$x = -z$
let $z = \mu \in \mathbb{R}$:
$x = -\mu$

$\underset{\sim}{e_2} = \mu\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\underset{\sim}{e}: \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

when $\lambda = 2$:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ -1/2 & -1/2 & -1/2 & 0 \\ 3/2 & -3/2 & -3/2 & 0 \end{array}\right)$$

$\underset{\sim}{e} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$$\boxed{P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

found P and D

idea is: I want to change this into <u>Diagonal matrix</u>

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & -1/2 \\ 3/2 & -3/2 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$PDP^{-1} = A$

$D = P^{-1}AP$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\downarrow$

$P^{-1}AP$

multiply matrix in front or at back of matrix is easy:

$B = C$ $\quad$ $B = C$

$AB = AC$ $\quad$ $BA = CA$

but in between

$BC = DE$

$BAC = DAE$ $\times$

need to:

let $AK = E$

$BC = DE$

$BC = DAK$

let $P\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & -1/2 \\ 3/2 & -3/2 & 1/2 \end{pmatrix}$

$\downarrow$

$$P\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$\therefore P\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = AP\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = P^{-1}AP\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = D\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$u' = u \qquad v' = -v \qquad w' = 2w$

$\frac{du}{dt} = u \qquad \frac{dv}{dt} = -v \qquad \frac{dw}{dt} = 2w$

$\frac{1}{u}du = dt \qquad \frac{1}{v}dv = -dt \qquad \frac{1}{w}dw = 2dt$

$\ln|u| = t + c \qquad \ln|v| = -t \qquad \ln|w| = 2t$

$u = e^{t+c} \qquad v = e^{-t} \qquad w = e^{2t}$

$u = C_1e^{t} \qquad v = C_2e^{-t} \qquad w = C_3e^{2t}$

$\rightarrow$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = P\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} C_1e^{t} \\ C_2e^{-t} \\ C_3e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} C_1e^{t} + C_2e^{-t} \\ C_1e^{t} - C_3e^{2t} \\ -C_2e^{-t} + C_3e^{2t} \end{pmatrix}$$

# C13 Vectors

# VECTORS

## Lines

- Vector Equation

$$\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

- Parametric Equation

$$x = a_1 + \lambda b_1$$
$$y = a_2 + \lambda b_2$$
$$z = a_3 + \lambda b_3$$

express $\lambda$ in terms of others.

- Cartesian Equation.

$$\frac{x-a_1}{b_1} = \frac{y-a_2}{b_2} = \frac{z-a_3}{b_3}$$

let each one $= \lambda$

$$\frac{x-a_1}{b_1} = \lambda, \dots$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \to x = 1, \frac{y-2}{2} = \frac{z-3}{3}$

$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to x = 1, y = 2, z \in \mathbb{R}$

extra equation:

$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$ is eqn of plane.

$(\underset{\sim}{r} - \underset{\sim}{a}) \times \underset{\sim}{n} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is eqn of line.

## Line & Plane

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

## Planes

- Vector Equation

$$\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} + \mu \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

- Parametric Equation

$$x = a_1 + \lambda b_1 + \mu c_1$$
$$y = a_2 + \lambda b_2 + \mu c_2$$
$$z = a_3 + \lambda b_3 + \mu c_3$$

- Normal Equation

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}, \quad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

- Cartesian Equation.

$$ax + by + cz = \underset{\sim}{a} \cdot \underset{\sim}{n} = d$$

This two a isn't the same! $a$ is a constant, $\underset{\sim}{a}$ is a position vector on the plane.

* V.E. to N.E. → C.E.

$$\underset{\sim}{n} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \times \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \to ax + by + cz = d$$

$$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$$

* C.E. to V.E.

eg. $ax + by + cz = d$

let $x = \lambda \in \mathbb{R}, y = \mu \in \mathbb{R}$

$a\lambda + b\mu + cz = d$

$$z = \frac{d - a\lambda - b\mu}{c}$$

$$z = \frac{d}{c} - \frac{a}{c}\lambda - \frac{b}{c}\mu$$

$$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{d}{c} \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -\frac{a}{c} \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ -\frac{b}{c} \end{pmatrix}$$

$ax + by = c$ can be a plane in 3d!

$ax + by = c$

let $y = \lambda \in \mathbb{R}$ $z = \mu \in \mathbb{R}$

$ax + b\lambda = c$

$x = \frac{c}{a} - \frac{b}{a}\lambda$

$$\underset{\sim}{r} = \begin{pmatrix} c/a \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -b/a \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

## Lines

### point of intersection

$$\underset{\sim}{r_1} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\underset{\sim}{r_2} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

solve: $a_1 + \lambda b_1 = c_1 + \mu d_1$
$a_2 + \lambda b_2 = c_2 + \mu d_2$
$a_3 + \lambda b_3 = c_3 + \mu d_3$

if 3 all consistent, two lines are the same.

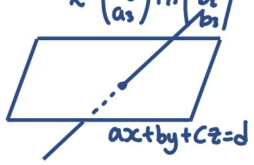
if 2 got consistent but one not → parallel

if 3 eqn all got solution that are consistent when check. → POI → when solve 2 eqn got sol. ins sol. into $3^{rd}$ eqn still consistent

if solve 2 got solution but one not → skewed, no POI. → when solve 2 eqn got sol. but ins sol. into $3^{rd}$ eqn not consistent.

## Line and Plane

$$\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

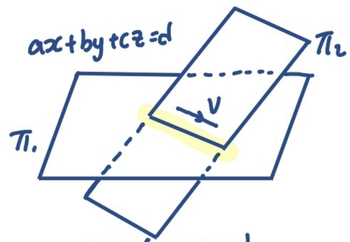



$a(a_1+\lambda b_1) + b(a_2+\lambda b_2) + c(a_3+\lambda b_3) = d$

solve for $\lambda$ put in $\underset{\sim}{r}$ find position vector.

## Planes

### line of intersection




$ex+fy+gz=h$

$$\underset{\sim}{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} e \\ f \\ g \end{pmatrix}$$

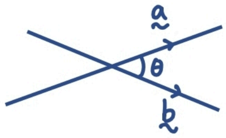
$1^{st}$ way: find any point on the line by substituting $x=0$, solve for $y, z$ → $by+cz=d$, $fy+gz=h$ simultaneous eqn.
$(0, y, z)$

ans: $\underset{\sim}{r} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} + \lambda \underset{\sim}{v}$

$2^{nd}$ way: eliminate one variable and let one of the other two to be a parameter.

# Lines

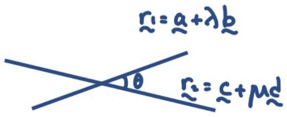
## Angle between vectors.




$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

* direction is important!

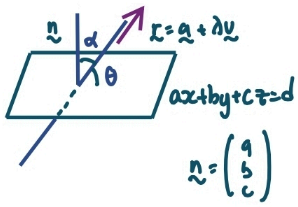
## Angle between lines (skew or intersecting)




$$\cos\theta = \frac{\underset{\sim}{b}\cdot\underset{\sim}{d}}{|\underset{\sim}{b}||\underset{\sim}{d}|}$$

* direction isn't important...

# Line and Plane



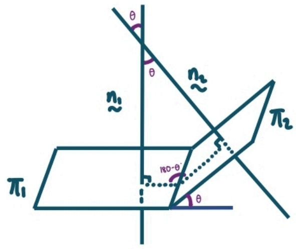

$$\sin\theta = \frac{\underset{\sim}{n}\cdot\underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|}$$

proof: $\cos\alpha = \frac{\underset{\sim}{n}\cdot\underset{\sim}{v}}{|\underset{\sim}{n}||\underset{\sim}{v}|}$

$\cos(90-\theta) = \dots$

$\sin\theta = \dots$

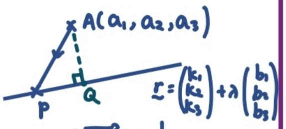
# Planes




$$\cos\theta = \frac{\underset{\sim}{n_1}\cdot\underset{\sim}{n_2}}{|\underset{\sim}{n_1}||\underset{\sim}{n_2}|}$$

# Lines

## shortest distance between a point and a line




formula: $\dfrac{|\overrightarrow{AP} \times \underset{\sim}{b}|}{|\underset{\sim}{b}|}$

manual: $Q = \begin{pmatrix} k_1 + \lambda b_1 \\ k_2 + \lambda b_2 \\ k_3 + \lambda b_3 \end{pmatrix}$

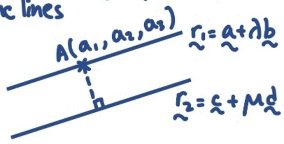
$\overrightarrow{AQ} = \begin{pmatrix} k_1 + \lambda b_1 - a_1 \\ k_2 + \lambda b_2 - a_2 \\ k_3 + \lambda b_3 - a_3 \end{pmatrix}$

$\overrightarrow{AQ} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$, solve for $\lambda$

insert $\lambda$ into $\overrightarrow{AQ}$ and find $|\overrightarrow{AQ}|$

## shortest distance between two parallel line.

use the exact method as point to line but use a fix point on one of the lines




find A to $r_2$ using point to line method.

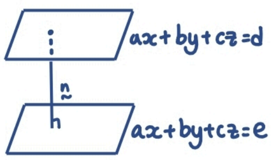
# Line and Plane

## shortest distance between line and plane that is parallel

same method as point to plane, just choose any point on the line as reference.

**DO NOT**

factorise or simplify direction vector when finding any kind of shortest distance!




New Theory: if $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a unit vector

i.e. $\sqrt{a^2 + b^2 + c^2} = 1$

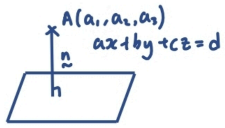
$\Rightarrow$ distance between two plane is equal to $|d - e| = |e - d|$ ← modulus

hence: distance between two parallel plane

$$= \left| \frac{d}{\sqrt{a^2+b^2+c^2}} - \frac{e}{\sqrt{a^2+b^2+c^2}} \right| = \frac{|d-e|}{|\underset{\sim}{n}|}$$

# Planes

## shortest distance between a point and a plane.




formula:

distance $= \dfrac{|d - A \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|}$ ← modulus, ← magnitude.

manual: $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

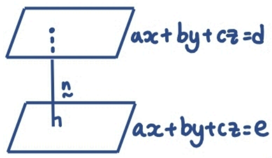
$x = a_1 + \lambda a$
$y = a_2 + \lambda b$
$z = a_3 + \lambda c$

$a(a_1 + \lambda a) + b(a_2 + \lambda b) + c(a_3 + \lambda c) = d$

solve for $\lambda$, obtain foot point.

$\left| \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|$ = shortest distance.

## shortest distance between two parallel plane.

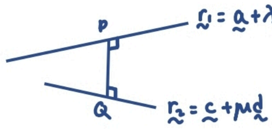



1st way: when $x = 0$, $y = 0$, $z = \frac{d}{c}$

$\dfrac{|d - A \cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|} = \dfrac{\left| e - \begin{pmatrix} 0 \\ 0 \\ d/c \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right|}{\sqrt{a^2 + b^2 + c^2}}$

# shortest distance between two skew line.

1st way:




coords of $P = (a_1+\lambda b_1,\ a_2+\lambda b_2,\ a_3+\lambda b_3)$

coords of $Q = (c_1+\mu d_1,\ c_2+\mu d_2,\ c_3+\mu d_3)$

$$\overrightarrow{PQ} = \begin{pmatrix} c_1+\mu d_1 - a_1 - \lambda b_1 \\ c_2+\mu d_2 - a_2 - \lambda b_2 \\ c_3+\mu d_3 - a_3 - \lambda b_3 \end{pmatrix}$$

$\overrightarrow{PQ} \perp \mathbf{r}_1$: $\overrightarrow{PQ} \cdot \mathbf{b} = 0$

obtain 1st eqn of $\mu$ and $\lambda$

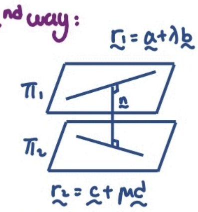
$\overrightarrow{PQ} \perp \mathbf{r}_2$: $\overrightarrow{PQ} \cdot \mathbf{d} = 0$

obtain 2nd eqn of $\mu$ and $\lambda$

solve $\mu$ and $\lambda$, insert into $\overrightarrow{PQ}$ to find $|\overrightarrow{PQ}|$

→ shortest distance $= |\overrightarrow{PQ}|$

2nd way:




suppose two skew lines on two parallel plane (sharing same normal)

① $\mathbf{n} \perp \mathbf{r}_1$, $\mathbf{n} \perp \mathbf{r}_2$

$\mathbf{n} = \mathbf{b} \times \mathbf{d}$

② $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{c} \cdot \mathbf{n}$

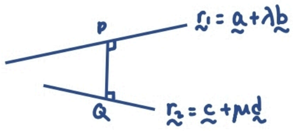
↑

find distance between two planes

* can use $\dfrac{|d - A\mathbf{n}|}{|\mathbf{n}|}$

$$\frac{|\mathbf{a}\cdot\mathbf{n} - \mathbf{c}\cdot\mathbf{n}|}{|\mathbf{n}|}$$

formula way:

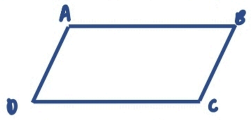



shortest distance $= \dfrac{|\mathbf{b}\times\mathbf{d} \cdot (\mathbf{a}-\mathbf{c})|}{|\mathbf{b}\times\mathbf{d}|}$ ← modulus, ← magnitude.

} triple scalar product.

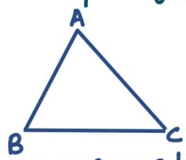
## Area and Volume.

### 1. Area of Parallelogram



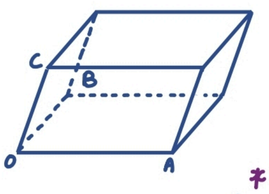

$A = |\overrightarrow{AB} \times \overrightarrow{AD}|$

### 2. Area of Triangle.




$A = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}|$

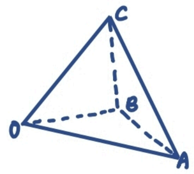
### 3. Volume of Parallelepiped.




$V = \underbrace{|\overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC}|}_{\text{Modulus}}$

≠ triple scalar product!

### 4. Volume of Tetrahedron.




$V = \frac{1}{6} \underbrace{|\overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC}|}_{\text{Modulus.}}$

triple scalar product!

## Extra vector properties:

$|\underset{\sim}{u} + \underset{\sim}{v}| \leq |\underset{\sim}{u}| + |\underset{\sim}{v}|$

$|\underset{\sim}{u} \cdot \underset{\sim}{v}| \leq |\underset{\sim}{u}||\underset{\sim}{v}|$

$|\underset{\sim}{u} \times \underset{\sim}{v}| \leq |\underset{\sim}{u}||\underset{\sim}{v}|$

*end*