IMPERIAL COLLEGE LONDON

MEng Examination 2024

PART II

This paper is also taken for the relevant examination for the Associateship.

CIVE 50010: STRUCTURAL MECHANICS 2

Wednesday 22 May 2024, 09:30 to 12:30 BST

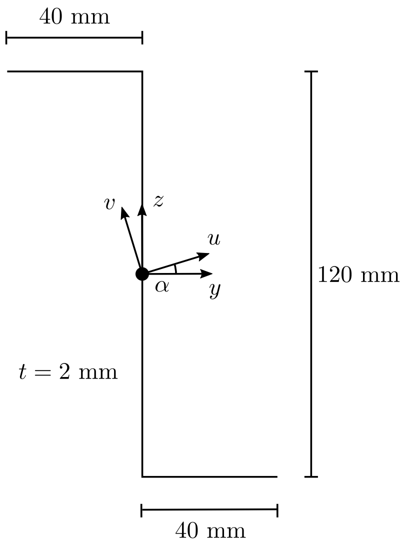
*This paper contains **FIVE** questions.*

*Answer **ALL FIVE** questions.*

Questions 1, 3 and 4 carry 20 marks.
Questions 2 and 5 carry 10 marks.

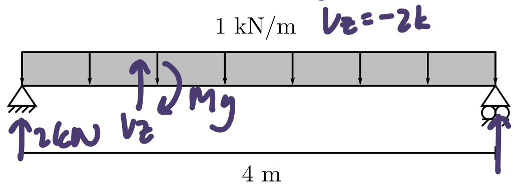
Formula Sheets are provided.

1. A Z-section with the centreline dimensions shown in Figure Q1 is subject to the loading also shown in the figure.



at $x=0$,
$\Sigma F_z = 0$
$V_z + 2k = 0$
$V_z = -2k$

at $x = 4m$,
$\Sigma F_z = 0$
$V_z + 2k - 4k = 0$
$V_z = 2k$



at $x = 2m$,
$\Sigma M_y = 0$
$2k(2) - 1k\left(\frac{2^2}{2}\right) + M_y = 0$
$M_y = -2$kNm

(b)

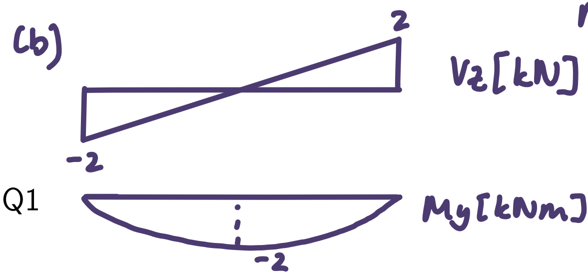


Figure Q1

a) Calculate the second moments of area I_{yy}, I_{zz}, I_{yz}, $I_{\max} = I_{uu}$ and $I_{\min} = I_{vv}$, and the angle α. Your solution should include a sketch of Mohr's circle showing the position of the pole of normals and indicating orientation of the major and minor bending axes in comparison to the y and z axes.

(12 marks)

b) Sketch the shear force and bending moment diagrams.

(2 marks)

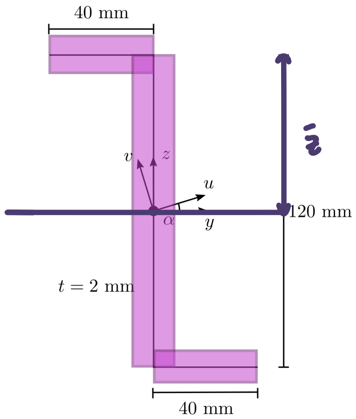
c) If the Z-section is used in the orientation that maximises the flexural rigidity calculate the normal stress distribution due to bending at the location along the beam where the maximum stresses due to bending will occur and sketch this indicating any significant values. Include a clear sketch to show the orientation of the cross-section indicating the major bending axis.

(4 marks)

d) Demonstrate that integrating the normal stress over each cross-sectional area that it acts, multiplied by the associated lever arm for each area, returns the expected bending moment.

$$M_y = \int \sigma_{xx}\ z\ \mathrm{d}A$$

(2 marks)



$$I_{yy} = \int z^2 \, dA$$

$$I_{yy}' = \sum I_{yy} + \sum A\bar{z}^2$$

$$I_{yy}' = 2\left(\frac{40\times 2^3}{12}\right) + \frac{2\times 120^3}{12} + (40\times 2)\times \frac{120}{2}^2 + (40\times 2)\times\left(-\frac{120}{2}\right)^2$$

$$= 864\,053 \text{ mm}^4$$

$$I_{zz} = \int y^2 \, dA, \quad I_{zz}' = \sum I_{zz} + \sum A\bar{y}^2$$

$$I_{zz}' = 2\left(\frac{2\times 40^3}{12}\right) + \frac{120\times 2^3}{12} + 2\left[(40\times 2)\times \frac{40}{2}^2\right]$$

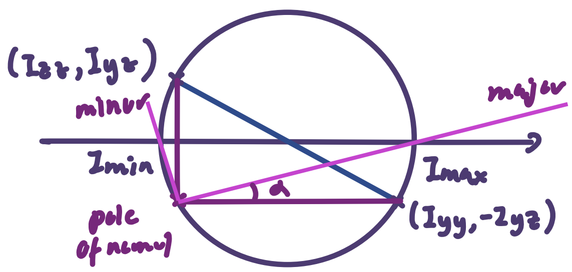
$$= 85\,413 \text{ mm}^4$$

0 (all 3 rectangle has at least one symmetry)

$$I_{yz}' = \cancel{\sum I_{yz}} - \sum A\bar{y}\bar{z}$$

$$= -\left\{(40\times 2)\left(-\frac{40}{2}\right)\left(\frac{120}{2}\right) + (40\times 2)\left(\frac{40}{2}\right)\left(-\frac{120}{2}\right)\right\}$$

$$= 192\,000 \text{ mm}^4$$



$$c = \frac{I_{yy} + I_{zz}}{2} = 474\,733$$

$$r = \frac{1}{2}\sqrt{(I_{yy} - I_{zz})^2 + (2I_{yz})^2}$$

$$= 434\,090$$

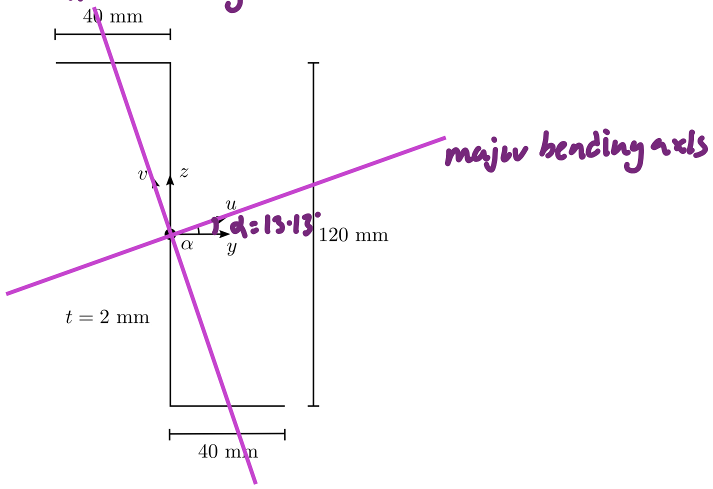
$$I_{max} = c + r = 908\,823 \text{ mm}^4$$

$$I_{min} = c - r = 40\,643 \text{ mm}^4$$

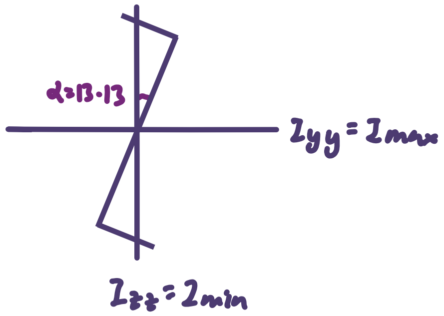
$$\tan\alpha = \frac{I_{yz}}{I_{max} - I_{zz}}$$

$$\alpha = 13.13^\circ$$

minor bending axis.



(c)

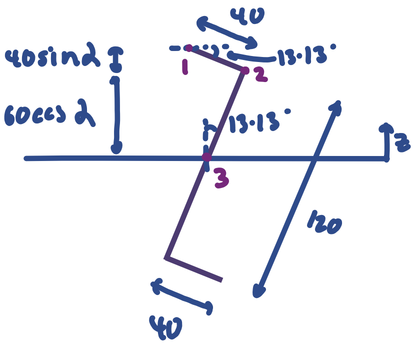


Since the moment applied, M_y is in the same orientation as the max and min I, it is biaxial bending:

$$\sigma_{xx} = \frac{M_y}{I_{yy}} z + \cancel{\frac{M_z}{I_{zz}} y}^{\,0} \quad (\text{only } M_y,\ M_z = 0)$$

Maximum σ_{xx} occurs at max M_y (mid span)

$M_y = -2\text{kNm}$



at node 1: $z = 40\sin\alpha + 60\cos\alpha$,

$$\sigma_{xx} = \frac{-2k}{908823}(40\sin 13.13 + 60\cos 13.13)$$

$= -0.1486 \text{ kN/mm}^2$

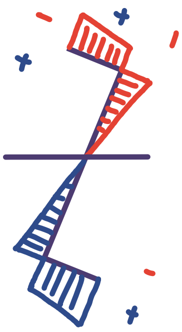
$= -148.6 \text{ N/mm}^2$

at node 2: $z = 60\cos\alpha$,

$\sigma_{xx} = -128.6 \text{ N/mm}^2$

at node 3: $z = 0$

$\sigma_{xx} = 0$



/// (-) compression

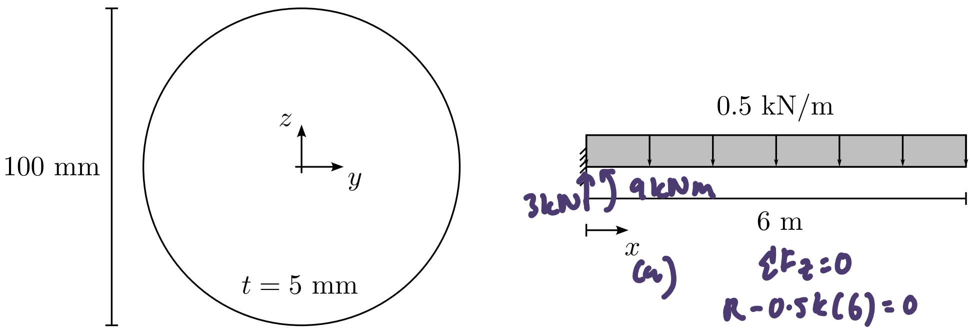
/// (+) tension.

(d) $M_y = \int \sigma_{xx} z \, dA$

$$= 2\left\{ -128.6 \times 40 \times 2 \times -\left(60\cos\alpha + \frac{40\sin\alpha}{2}\right) + \frac{1}{2}(-148.6 + 128.6) \times 40 \times 2 \times -\left(60\cos\alpha + \frac{2}{3} \times 40\sin\alpha\right)\right.$$

$$\left. + \frac{1}{2} \times -128.6 \times 60 \times 2 \times -\frac{2}{3}(60\cos\alpha)\right\} = -2 \times 10^6 \text{ Nmm} = -2 \text{ kNm}$$

2. A hollow circular section with the centreline dimensions shown in Figure Q2 is subject to the loading also shown in the figure.



(a)

$\Sigma F_z = 0$
$R - 0.5k(6) = 0$
$R = 3kN$

$\Sigma M_y = 0$
$M + 0.5\left(\frac{6^2}{2}\right) = 0$
$M = -9kNm.$

Figure Q2

V_z

a) Find the vertical and moment reactions at the fixed end.

$\frac{dM}{dx} = V$

V_z [kN]:

at $x = 6m$,
$V_z + 3k - 0.5k(6) = 0$
$V_z = 0$

(1 marks)

V is less and less negative.
hence gradient of M,
is less and less negative too!

b) Sketch the shear force and bending moment diagrams.

M_y [kNm]:

at $x = 3m$,
$M_y - 9k + 3k(3) - 0.5k\left(\frac{3^2}{2}\right) = 0$
$M_y = 2.25kNm.$

at $x = 6m$,
$M_y - 9k + 3k(6) - 0.5k\left(\frac{6^2}{2}\right) = 0$
$M_y = 0$

(2 marks)

c) Calculate the normal stress distribution due to bending at the location along the beam where the maximum bending moment occurs and sketch this indicating any significant values. You may calculate the value of the second moment of area of a hollow circular section as:

$M_y = 9kNm$ (+ive)

It's a biaxial loading case:

$\sigma_{xx} = \frac{M_y}{I_{yy}} z + \cancel{\frac{M_z}{I_{zz}} y}^{\,0}$ $\underline{\underline{M_z = 0!}}$

$= \frac{9 \times 10^6}{1.963 \times 10^6} z$

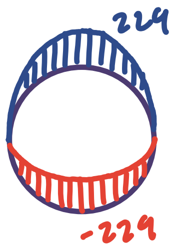
$I_{yy} \approx \pi r^3 t$
$= \pi\left(\frac{100}{2}\right)^3(5)$
$= 1963495 \text{ mm}^4$

$$I_{yy} = I_{zz} \approx \pi r^3 t$$

(3 marks)

at $z = +r$
$\sigma_{xx} = 229 \text{ N/mm}^2$
at $z = -r$
$\sigma_{xx} = -229 \text{ N/mm}^2$

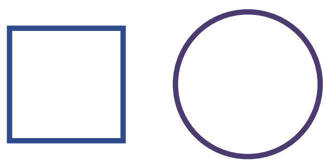
d) Calculate the depth of the sides of a square hollow box section that has the same cross-sectional area as the hollow circular section, assuming the thickness remains the same. If the square hollow box section is used instead of the hollow circular section, subject to the same loading, go on to calculate the normal stress distribution due to bending at the location along the beam where the maximum bending moment occurs and sketch this indicating any significant values.



/// tension.
/// compression.

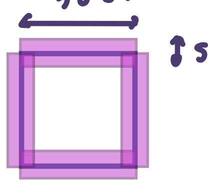
(d)

(4 marks)



$P = 4d$ $P = 2\pi r$

$2\pi(50) = 4d$
$d = 78.54mm.$



$I_{yy} = \frac{78.54(5)^3}{12} \times 2 + \frac{5(78.54)^3}{12} \times 2$
$+ (5 \times 78.54) \times \frac{78.54}{2}^2 \times 2$
$= 1.617 \times 10^6 \text{ mm}^4$

$\sigma_{xx} = \frac{M_y}{I_{yy}} z$ ...

I am lazy to continue sure tho.

3. The two-span beam ABCDME is shown in Figure Q3. It is pinned at A and supported through rollers at D, and E. Point loads of 30 kN are applied at B (upwards) and C (downwards). The flexural rigidity of all the members is $EI=10000\ kNm^2$

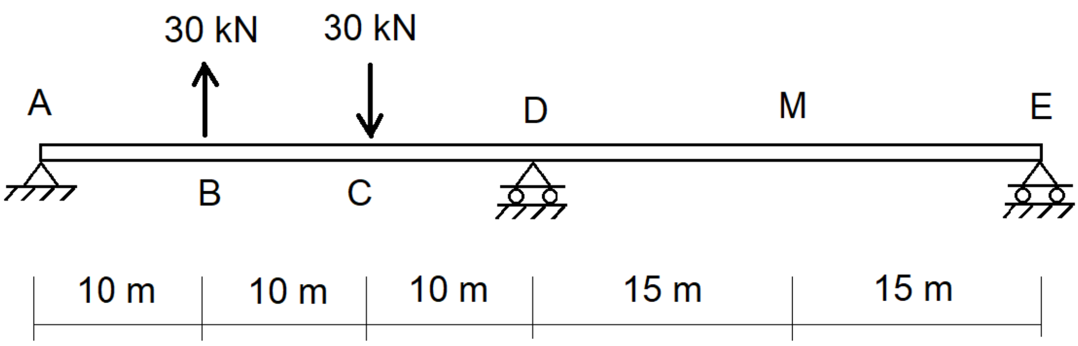


Figure Q3

(a) Calculate the redundancy of the structure.

static indeterminancy (redundancy) = 4-3=1

[1 mark]

(b) Describe how you would obtain the vertical reaction at E using the flexibility method. State very clearly which statically determinate structure, load cases, and compatibility equation need to be considered, and how to obtain the corresponding flexibility coefficients. State the magnitude that each of the flexibility coefficients is representing.

- remove support at E and replace with a reaction force, X_1
- split into two case, one without the reaction force, X_1 and one without any external force but only that reaction force X_1
- $f_{10} + X_1 f_{11} = 0$

[6 marks]

(c) Obtain the bending moment diagram in each of the load cases stated in your response to point b).

[4 marks]

(d) Calculate the flexibility coefficients stated in your response to point b).

[4 marks]

(e) Calculate the vertical reaction at E.

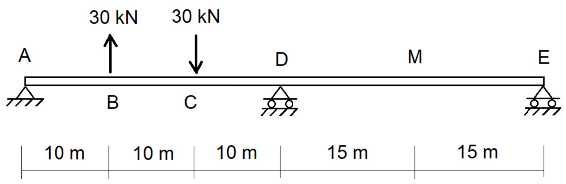
[1 mark]

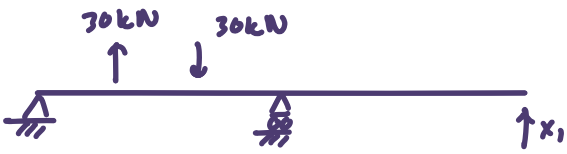
(f) Calculate and sketch the bending moment diagram and annotate the values of the bending moments at A, B, C, D, M, and E.

[2 marks]

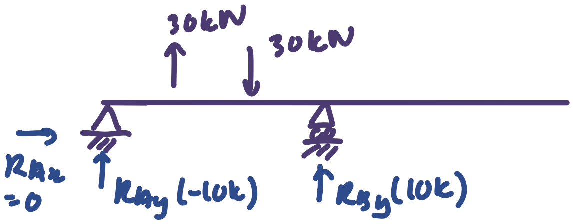
(g) Calculate the vertical displacement of the beam at M.

[2 marks]





Case 0:



Case 1:

$\Sigma F_y = 0$
$R_{Ay} + R_{By} = 0$
$R_{Ay} = -10kN$

$\Sigma M_A = 0$
$-30k(10) + 30k(20) - R_{By}(30) = 0$
$R_{By} = 10kN$

$\Sigma F_y = 0$
$R_{Ay} + R_{By} + 1k = 0$
$R_{Ay} = 1kN$

$\Sigma M_A = 0$
$-R_{By}(30) - 1k(60) = 0$
$R_{By} = -2kNm$

to find BMD:

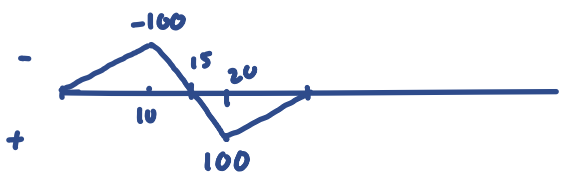
$\Sigma M_{\uparrow 30kN} = 0$
$-10k(10) - M = 0$
$M = -100k$

$\Sigma M_{\downarrow 30kN} = 0$
$-10k(20) + 30k(10) - M = 0$
$M = 100k$

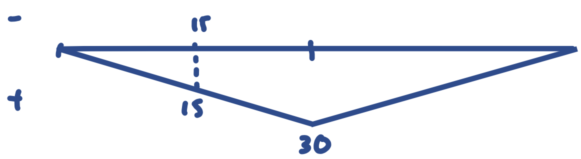
$\Sigma M_B = 0$
$1k(30) - M = 0$
$M = 30kNm$

$\Sigma M_{\uparrow 1kN} = 0$
$1k(60) - 2k(30) - M = 0$
$M = 0$

$\Sigma M_B = 0$
$-10k(30) + 30k(20) - 30k(10) - M = 0$
$M = 0$



$M(x)$ / kNm

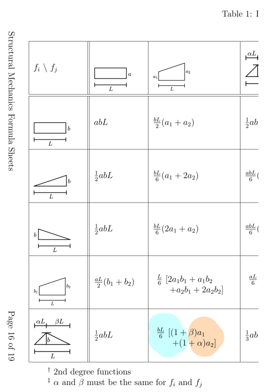


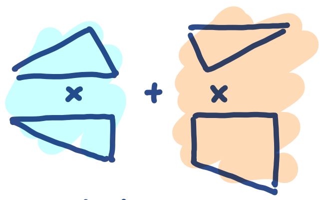
$m(x)$ / kNm

$$f_{10} = \int m \frac{M}{EI} dx$$

f_{10} and f_{11} is flexibility coeff.

$$f_{11} = \int m \frac{m}{EI} dx$$



$=$ 

$= \frac{-100(15)}{6}\left[\left(1+\frac{10}{15}\right)\times 15\right]$

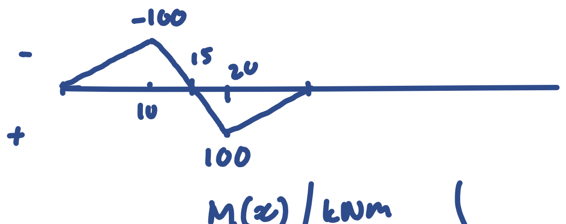
$+ \frac{100(15)}{6}\left[\left(1+\frac{10}{15}\right)\times 15 + \left(1+\frac{5}{15}\right)\times 30\right]$

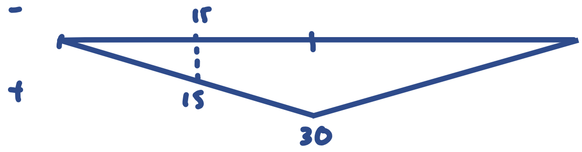
$= 10000/EI = 1k$

$= \frac{30}{6}\left(0 + 4\times 15^2 + 30^2\right)\times 2$

$= 1.8k$

$f_{10} + X_1 f_{11} = 0$
$X_1 = -0.5556$
$R_E = 0.5556$ (downwards)

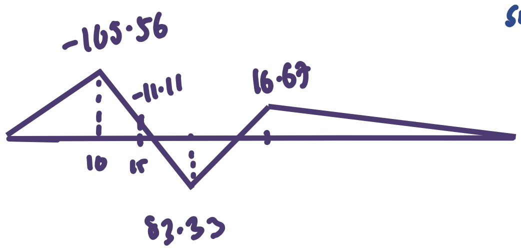


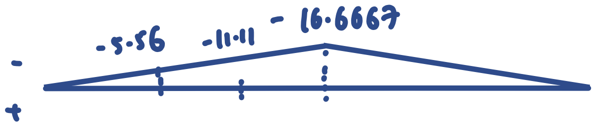


multiply by -0.5556

sum

=

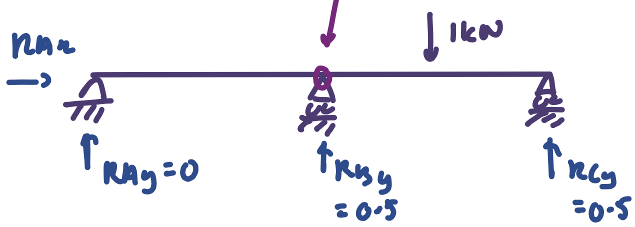


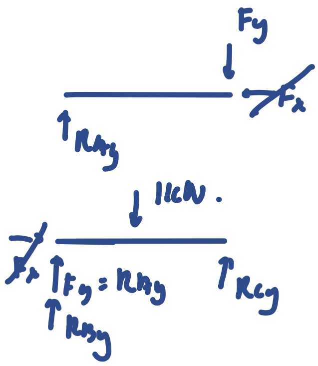


to find displacement at M.

- we need BMD of the real case (which we just got) ✓
- we need BMD of a virtual case where there is a 1kN unit load at M ✗

add a hinge here to simplify.





$\Sigma F_y = 0$ (global)

$R_{Ay} + R_{By} + R_{Cy} - 1k = 0$

$R_{Ay} = 0$

$\Sigma M_A = 0$

$-R_{By}(30) + 1(45) - 0.5(60) = 0$

$R_{By} = 0.5$

$\Sigma M_B = 0$

$-R_{Cy}(30) + 1k(15) = 0$

$R_{Cy} = 0.5\text{kN}$

$\Sigma M_{\downarrow 1kN} = 0$

$0.5(15) - M = 0 \quad M = 7.5\text{ kNm}$

$$f = \int m \frac{M}{EI}\,dx$$

$$= \frac{1}{EI}\left\{ \frac{15}{6}\left[4\left(\frac{7.5}{2} \times \frac{3}{4}(-16.67)\right) + \left(7.5 \times \frac{-16.67}{2}\right)\right] + \frac{15}{6}\left[7.5 \times \frac{-16.67}{2} + 4\left(\frac{7.5}{2} \times \frac{1}{4}(-16.67)\right)\right]\right\}$$

$= -0.0938\text{m}$

$= 93.8$ mm (upwards.

FULL WORKING CHECK RECORDING

4. Figure Q4 shows a plane frame with fixed supports at A and D and a roller support at B that prevents the horizontal displacement of beam BC. The frame, which is to be modelled using three inextensible elements AB, BC and CD, is subjected to a nodal force M_1 at the support B. Element lengths, flexural rigidities EI_1, EI_2 and EI_3 and load are displayed in the figure.

(i) $\beta = 2$

   i. Identify the kinematic indeterminacy in the proposed modelling for the frame, determine the system stiffness matrix and the vector of the equivalent nodal forces.

   [6 marks]

   ii. Solve the system of stiffness equations for the frame and determine the values of the independent nodal displacements.

   [3 marks]

   (9)

   iii. Sketch the bending moment distribution, indicating all significant values, for all the members.

   [6 marks]

   iv. Determine the system stiffness matrix of the plane structure which is obtained from the frame in Figure Q4 by removing the support at B, thus allowing horizontal displacements of the beam BC.

   [5 marks]

$u_1 = 1, u_2 = 0$ $u_2 = 1, u_1 = 0$

(i) $K = \begin{bmatrix} \overbrace{K_{1,1}}^{\text{mode 1}} & \overbrace{K_{1,2}}^{\text{mode 2}} \\ K_{2,1} & K_{2,2} \end{bmatrix}$

$k = \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix}$ (mode 3, mode 6)

u_3 u_6

$u_3 = 1, u_6 = 0$ $u_6 = 1, u_3 = 0$

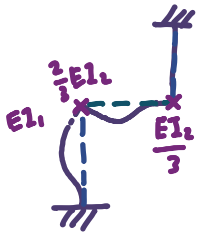
$\frac{4EI_3}{3}$ $\cancel{\frac{2EI_3}{3}}$ beam 3

mode 6

$2EI_2/6$ $4EI_2/6$

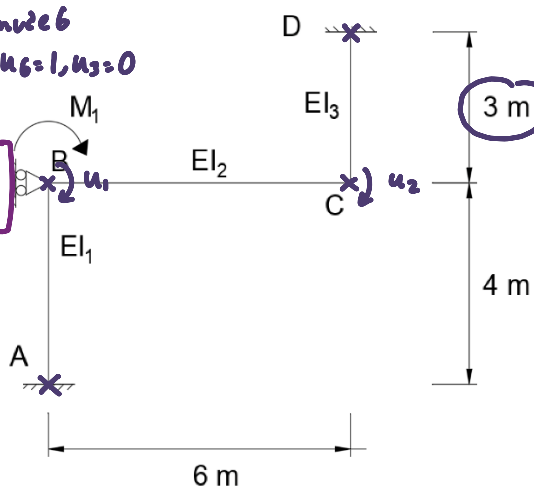
$\frac{EI_2}{3}$ $\frac{2EI_2}{3}$

$\frac{2}{3}EI_2$ EI_1 $\frac{EI_2}{3}$



$$K = \begin{bmatrix} EI_1 + \frac{2}{3}EI_2 & EI_2/3 \\ \frac{EI_2}{3} & \frac{2}{3}EI_2 + \frac{4}{3}EI_3 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 10 \\ 10 & 30 \end{bmatrix} \text{MNm}$$



$EI_1 = 15$ MNm^2; $EI_2 = 2 \cdot EI_1$ MNm^2; $EI_3 = 0.5 \cdot EI_1$ MNm^2

$M_1 = 18$ kNm

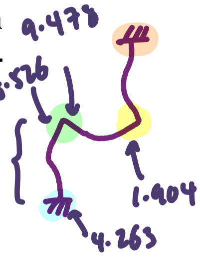
Figure Q4

$P = KU$

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 35 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 35 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 35 & 10 \\ 10 & 30 \end{bmatrix}^{-1} \begin{bmatrix} 18 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5684 \\ -0.1840 \end{bmatrix}$$



$$\begin{bmatrix} P_3 \\ P_6 \end{bmatrix} = \frac{15}{4} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5684 \end{bmatrix} = \begin{bmatrix} 4.263 \\ 8.526 \end{bmatrix}$$

$u = 0.5684$

$u_3 = 0.5684$ $u_6 = 0.1840$

$$\begin{bmatrix} P_3 \\ P_6 \end{bmatrix} = \frac{30}{6} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0.5684 \\ -0.1840 \end{bmatrix} = \begin{bmatrix} 9.478 \\ 1.404 \end{bmatrix}$$

5. Figure Q5 shows a continuous beam ABCD with three spans of flexural rigidity EI_1 = 12 MNm^2, EI_2 = 12 MNm^2 and EI_3 = 8 MNm^2. The beam has fixed supports at A and D and is continuous over simple supports at B and C. Spans AB and BC are subjected to midspan point loads of 40 kN. Considering the moment distribution method:

i. Calculate the distribution factors.

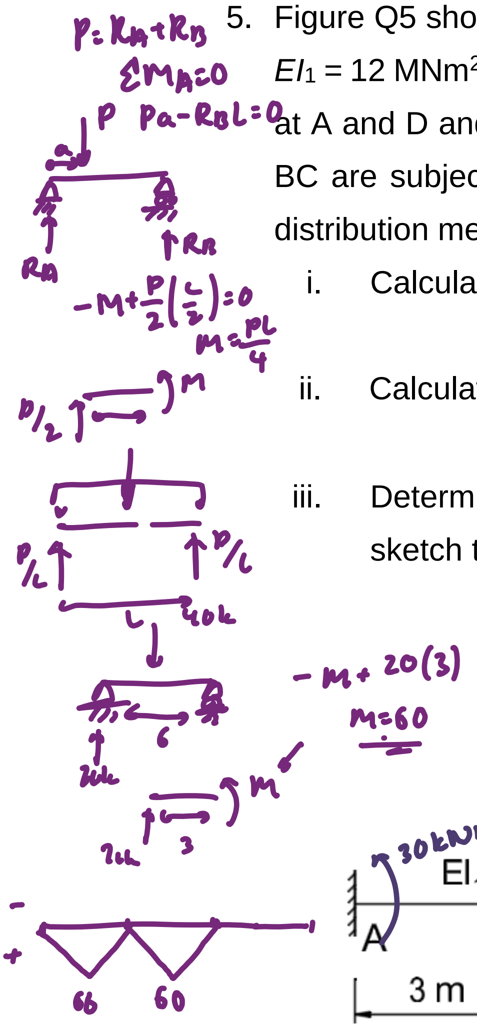
[2 marks]

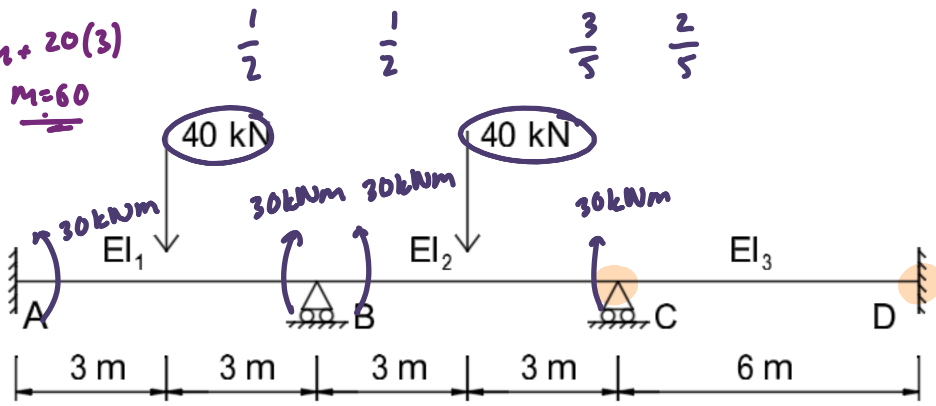
ii. Calculate the fixed end moments.

[2 marks]

iii. Determine the bending moments at the supports A, B, C and D, and sketch the bending moment distribution, indicating all significant values.

[6 marks]





$$EI_1 = EI_2 = 12\ \text{MNm}^2;\quad EI_3 = 8\ \text{MNm}^2$$

Figure Q5

	A	B	B	C	C	D
DF		1/2	1/2	3/5	2/5	
FEM	-30	30	-30	30		
		0	0	-18	-12	
	0		-9	0		-6
		4.5	4.5	0	0	
	2.25		0	2.25		0
		0	0	-1.35	-0.9	
	0		-0.675	0		-0.45
		0.3375	0.3375	0	0	
	-27.75	34.84	-35.18	12.9	-12.9	-6.45