CIVE50005 FLUID MECHANICS

Relevant tutorial derivations are labelled TxQy for Tutorial x Question y.

1 FUNDAMENTALS

Langrangian: Considers individual fluid particle trajectories

Eulerian: Considers fluid motion at a fixed point in space

2D Mass Continuity

- Cartesian: $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$
- Polar: $\frac{\partial u_\theta}{\partial \theta} + u_r + r\frac{\partial u_r}{\partial r} = 0$ (T5Q1)

Vorticity

$$\Omega = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

- **Irrotationality:** $\Omega = 0 \Rightarrow \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$

Velocity Potential

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- Assumes irrotationality & mass continuity
- Integrating irrotationality:
  - Cartesian: $u = \frac{\partial \phi}{\partial x}$, $w = \frac{\partial \phi}{\partial z}$
  - Polar: $u_r = \frac{\partial \phi}{\partial r}$, $u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$
- Represents fluid potential field

Stream Function

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

- Assumes irrotationality & mass continuity
- Integrating mass continuity:

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}$$

- Represents movement of fluid particles

Unsteady Bernoulli Equation

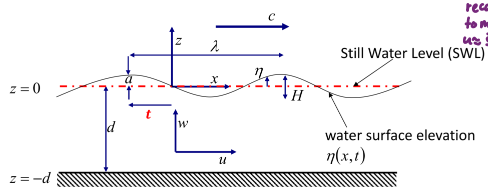
$$\rho\frac{\partial \phi}{\partial t} + p + \rho g z + \rho\left[\frac{u^2 + w^2}{2}\right] = \text{constant}$$

- Derived from 2D Euler equations, where flow is irrotational and therefore ϕ exists (T1Q5)
- Flow is also inviscid

2 SMALL AMPLITUDE WAVE THEORY

NEED TO CHECK IF STEEPNESS, AK < 0.1 (SAWT IS ONLY ACCURATE IF AK < 0.1)
NEED TO CHECK IF A/D < 0.1 (SAWT IS ONLY ACCURATE IF A/D < 0.1)

DEFINITIONS



Wave Amplitude: $a = \frac{H}{2}$

Wave Frequency: $\omega = \frac{2\pi}{T}$

Wavenumber: $k = \frac{2\pi}{\lambda}$

Phase Velocity: $c = \frac{\lambda}{T} = \frac{\omega}{k}$

memorise these.

(speed individual wave crest propagates)

Wave Phase/Phase Angle: $\omega t - kx$

Surface Elevation: $\boxed{\eta = a\sin(\omega t - kx)}$

important (but given in data sheets)

ASSUMPTIONS (LINEAR REGULAR WAVE THEORY)

- Mass Continuity: Incompressib id
- Irrotationality: Negligible/zero viscosity and not near any boundaries
- Small Wave Amplitude: $a \ll \lambda$ and $a \ll d$
- Unsteady Bernoulli equation is applicable
- Waves are periodic in x and t

EQUATIONS

FREE SURFACE BOUNDARY CONDITION:

kinematic

1. KFSBC (to derive u and w) water surface is streamline : $\frac{\partial \eta}{\partial t} = w - u\frac{\partial \eta}{\partial x} \approx w$
2. DFSBC pressure at surface = P_{atm} : $p(z=\eta) = 0$

dynamic (to derive c)

MORE BOUNDARY CONDITION

3. SEABED BC $w = 0$ at $z = -d$

(simplify expression for w → use KFSBC next)

Velocity Components

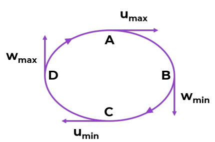
given in data sheets

$$\begin{cases} u = \dfrac{a\omega\cosh k(z+d)}{\sinh(kd)}\sin(\omega t - kx) \\ w = \dfrac{a\omega\sinh k(z+d)}{\sinh(kd)}\cos(\omega t - kx) \end{cases}$$

- Solution to Laplace's equation with boundary conditions at the bed and the kinematic free-surface
- u decays exponentially to zero at solid boundaries due to viscous forces
- Deep Water: $(d \to \infty)$

$$u = a\omega e^{kz}\sin(\omega t - kx)$$
$$w = a\omega e^{kz}\cos(\omega t - kx)$$

  - Deep water orbits are circular (T3Q3)



TO CHECK WHEN DEEP WATER ASSUMPTION IS ALLOWED, FIND TANH(KD). IF IT IS CLOSE TO 1, WE CAN USE DEEP WATER ASSUMPTIONS (or $kd > \pi$)

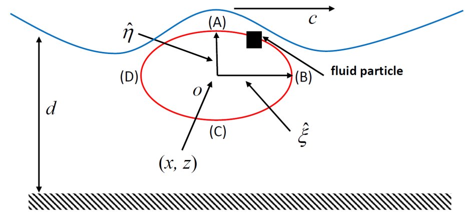
Particle Orbits

not given recommended to memorise $u \approx \frac{\partial \xi}{\partial t}$ and $w = \frac{\partial \eta}{\partial t}$ and derive!

$$u \approx \frac{\partial \widehat{\xi}}{\partial t} \Rightarrow \widehat{\xi} = \frac{-a\cosh k(z+d)\cos(\omega t - kx)}{\sinh(kd)}$$

$$w \approx \frac{\partial \widehat{\eta}}{\partial t} \Rightarrow \widehat{\eta} = \frac{a\sinh k(z+d)\sin(\omega t - kx)}{\sinh(kd)}$$

- Elliptical, since $\frac{\widehat{\xi}^2}{\cosh^2 k(z+d)} + \frac{\widehat{\eta}^2}{\sinh^2 k(z+d)} = c$ (T3Q1)
- Both major and minor axes of the ellipses decay exponentially with depth d



Dispersion Equation

$c = \frac{\omega}{k} = \left(\frac{g}{k}\tanh(kd)\right)^{1/2}$

$$\omega^2 = gk\tanh(kd)$$ this is valid for all depth.

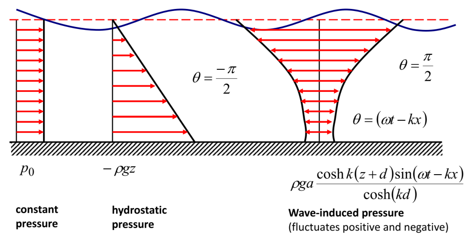
☆ when waves advance into shallow water : ω is kept constant! $gk_o = gk\tanh(kd)$ deep → shallow.

- Deep Water: $\tanh(kd) \to 1 \Rightarrow \omega_o^2 = gk_o$

Total Pressure Distribution

$$p = p_0 - \rho g z + \rho g a\frac{\cosh k(z+d)}{\cosh(kd)}\sin(\omega t - kx)$$

- p_0 = atmospheric pressure at $z = 0$
- $\rho g z$ = hydrostatic pressure
- Last term = wave-induced pressure (T3Q5)



3 WAVES ADVANCING INTO SHALLOW WATER

ASSUMPTIONS

- **Sufficiently slow variation in depth:** Velocities and pressures defined at all points
- **No energy losses:** No friction and wave breaking
- Thus, wave period stays constant
- Wave is linear (neglect velocity head term for energy conservation)

EQUATIONS

k advancement into shallow water: $gk_o = gk\tanh(kd)$

Total Work Per Wave Cycle

$$W_{\text{cycle}} = \frac{T\rho g a^2\omega}{4k}\left[1 + \frac{2kd}{\sinh(2kd)}\right]$$

Energy Conservation ☆

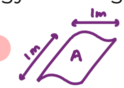
simplification for deep water.

$$\left\{\frac{a_o^2}{k_o}\right\} = \frac{a^2}{k}\left[1 + \frac{2kd}{\sinh(2kd)}\right] = \text{constant}$$

deep → shallow.

- Transfer of energy must be equal along any vertical section to prevent energy building up, causing the wave to break

Wave Energy Per Unit Plan Area



$$PE_{\text{waves}} = \frac{\rho g a^2}{4} = KE_{\text{waves}}$$

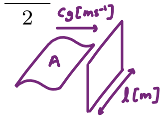
$$E_{\text{waves}} = PE_{\text{waves}} + KE_{\text{waves}} = \frac{\rho g a^2}{2}$$

[Jm^{-2}]

Group Velocity

(speed overall wave energy transports)

power = $\frac{\rho g a^2}{2} \times l \times c_g$
[W] [Jm^{-2}][m][ms^{-1}]



$$c_g = \frac{c}{2}\left[1 + \frac{2kd}{\sinh(2kd)}\right]$$

- Mean velocity of an isolated group of waves or of wave energy transportation
- Deep water ($kd \to \infty$): $c_g = \frac{c}{2}$
- Shallow water ($kd \to 0$): $c_g = c$

☆ when advancing into shallow water, wave might break. This occurs when $kd \to 0$ ($k = 0$) → $\frac{2kd}{\sinh(2kd)} \to 1$ not 0!

4 FLUID LOADING

☆ WAVES BREAK WHEN: 1. $kd \to 0$, 2. $\frac{u_{max}}{c} \to 1$

Dynamic Viscosity: μ

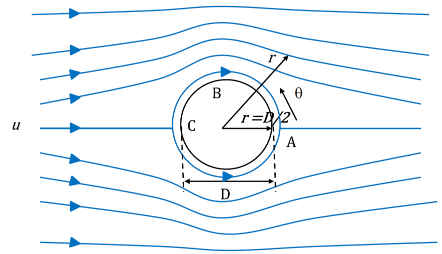
Kinematic Viscosity: $\nu = \frac{\mu}{\rho}$

Reynolds Number: $Re = \frac{\rho u D}{\mu} = \frac{uD}{\nu}$

POTENTIAL FLOW AROUND A CYLINDER

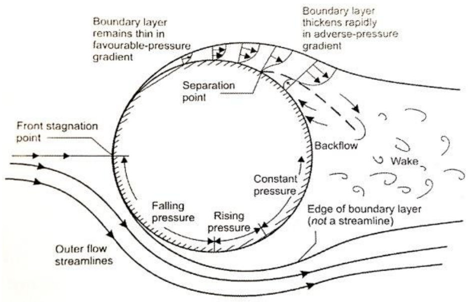
$$\phi = u\left(r + \frac{D^2}{4r}\right)\cos\theta, \quad r \geq \frac{D}{2}$$

- Assuming inviscid and irrotational flow, inviscid pressure: $P = \frac{1}{2}\rho u^2(1 - 4\sin^2\theta)$
- D'Alembert's Paradox: No net force in flow direction (T5Q2)



FLOW SEPARATION

- Occurs at significant flow velocity with viscosity causing strong shear forces in the boundary layer ($Re \sim 10$)



1. Upstream stagnation point: High pressure, low velocity
2. Flow speeds up around cylinder: Low pressure & maximum velocity
3. Flow encounters higher pressures from the downstream stagnation point and separates to avoid pressure gradient
4. Steady downstream wake forms with 2 recirculating eddies
   - As Re increases:
     - Flow separation occurs earlier along the cylinder surface
     - Wake size and length increases linearly with Re
     - Separation of eddy cores increases with $\sqrt{Re}$
     - Drag force magnitude increases
   - Approximate steady state limit at about $Re = 41.0$ for a cylinder
5. **Von Karman Vortex Street:**
   - Eddy vortices grow to cut into the fluid flow due to increasing shear
     - Flow pulls away part of the bigger vortex downstream, allowing the smaller one to expand and fill its space
     - Process alternates sides repeatedly
     - Loss of horizontal symmetry creates a net lift force on the cylinder, perpendicular to the vortex street
   - **Strouhal Number:** $St = \frac{f_{vs}D}{u}$, where f_{vs} is the vortex shedding frequency
     - Used to ensure f_{vs} is much lower than natural frequency of structure to prevent resonance
     - For $Re < 5 \times 10^5$, $St = 0.2 \Rightarrow f_{vs} = \frac{0.2u}{D}$
6. Wake becomes unsteady and turbulent ($Re \geq 10^5$)

STEADY FLOW

fd : due to pressure drag (i.e. pressure difference upstream and downstream) and friction drag. → From flow seperation. (wake forming)

fl : due to eddie shedding.

Drag & Lift Coefficients

$$C_d = \frac{f_d}{0.5\rho u^2 A}, \quad \boxed{C_l = \frac{f_l}{0.5\rho u^2 A}}$$

note that this f_l has alternates perpendicular to the flow direction. (f_l represent max magnitude)

- f_d, f_l = total drag/lift forces $f_l = C_l \frac{1}{2}\rho u|u|D$ always u! never w!
  - Difficult to calculate, even numerically
  - When boundary layer becomes turbulent ($Re \geq 10^5$), wake becomes narrower and $f_d \downarrow$
- $A = DL$ = diameter × length
  - Area of cylinder's silhouette

UNSTEADY FLOW

Inertia Force

f_m: a potential flow force. arise in unsteady flow caused by fluid acceleration due to pressure gradient.

$$f_m = C_m \frac{\pi D^2}{4}\rho\frac{\partial u}{\partial t}$$

- f_m = inertia force per unit length
- C_m = intertia coefficient
  - $C_m = 2$ for ideal cylinder flow
  - $C_m < 2$ usually if a wake is created, due to viscous effects

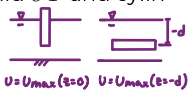
MORISON'S EQUATION

note that u and $\frac{\partial u}{\partial t}$ are 90° out of phase. → fd and fm are 90° out of phase.

$$f = C_d\frac{1}{2}\rho u|u|D + C_m\rho\frac{\pi D^2}{4}\frac{\partial u}{\partial t}$$

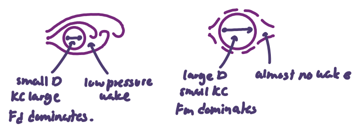
- f = total fluid load per unit length $F = \int f\,dz$
  - First term = drag force per unit length
    - Modulus to ensure drag force is always in the direction of the flow
  - Second term = inertia force per unit length
- **Slender Body Regime:** Centreline values of u and $\frac{\partial u}{\partial t}$ are used
  - Assumes $D \ll \lambda$, so does not account for structure's disturbance to the flow
- **Keulegan-Carpenter Number:** Ratio between displacement of fluid UT and cylinder diameter D

$$KC = \frac{UT}{D}$$



  - U = flow velocity, T = oscillation period, D = cylinder diameter
  - $KC < 5$: Inertia dominates, omit drag
    - Fluid only moves a bit, so wake formation is limited
    - When KC is very small, $C_d \rightarrow 0$ and $C_m \rightarrow 2$ (T6Q2)
  - $KC > 20$: Drag dominates, omit inertia
    - Fluid moves far enough to separate and form a wake before flow oscillates back

- **Using Morison's Equation:** $F_m = 0$!
  1. If steady current ($\frac{\partial u}{\partial t} = 0$), only drag forces are important
  2. If unsteady, check KC value to determine dominant forces
  3. If in deep water, use approximations   vertical cylinder
  4. Calculate $\boxed{F_x = F_d \sin\theta|\sin\theta| + F_m\cos\theta,}$ where $\theta = \omega t - kx$ very important, memorise!
     - Use the velocity component that is perpendicular to the cylinder!
     - $u^2 = \frac{a^2\omega^2\sin^2\theta}{\sinh^2(kd)}\cosh^2 k(z+d)$
     - $\frac{\partial u}{\partial t} = \frac{a\omega^2\cos\theta}{\sinh(kd)}\cosh k(z+d)$
     - $\int \cosh k(z+d)dz = \frac{1}{k}\sinh k(z+d)$
     - $\int \cosh^2 k(z+d)dz = \frac{1}{4k}\sinh 2k(z+d) + \frac{z}{2}$
  5. Set $\frac{\partial F_x}{\partial\theta} = 0$ to get maximum F_x
  6. Repeat for F_z if applicable
     - Remember to multiply by cylinder length if needed!



Morisson Equation (to find fx, Fx) ↪ evaluate at z=-d+h (fx or Fx both!)

horizontal column.

if fx or fz get rid of l

$F_x = C_d \cdot \rho\frac{1}{2}u_{z=-d+h}|u_{z=-d+h}|D \cdot l + \underbrace{C_m \cdot \rho\frac{\pi D^2}{4}\left(\frac{\partial u}{\partial t}\right)_{z=-d+h}}_{F_m = 0 \text{ if steady } (\frac{\partial u}{\partial t}=0)} \cdot l$

- u and $\frac{\partial u}{\partial t}$ always evaluated at z=-d+h no matter uniform or non-uniform flow (i.e. doesn't matter if u=U₀ or u=u(z))
- simple, no integration needed

$F_z = C_d \cdot \rho\frac{1}{2}w_{z=-d+h}|w_{z=-d+h}|D \cdot l + C_m \cdot \rho\frac{\pi D^2}{4}\left(\frac{\partial w}{\partial t}\right)_{z=-d+h} \cdot l$

- only exist if non-steady (there's waves)
- if waves doesn't approach perpendicular to the horizontal column → need integration dl

note that both Fx and Fz are still both function of θ=ωt-kx to find max, $\frac{\partial F}{\partial \theta} = 0$! (If $\underbrace{\text{find seperately}}_{\text{or when can ignore either Fm or Fd}}$, sinθ or cosθ = 1 is enough)

vertical column

$f_x = C_d\rho\frac{1}{2}u|u|D + C_m\rho\frac{\pi D^2}{4}\left(\frac{\partial u}{\partial t}\right)$
u=umax(z=0) if evaluated together.
If fd only u=umax(z=a)
If fm only u=umax(z=0)

$F_x = C_d \cdot \rho\frac{1}{2}D\int_{z=-d}^{z=0} u|u|dz + \underbrace{C_m \cdot \rho\frac{\pi D^2}{4}\int_{z=-d}^{z=0}\frac{\partial u}{\partial t}dz}_{F_m = 0 \text{ if steady } (\frac{\partial u}{\partial t}=0)}$

and solution to this (should be memorised):

$F_x = F_d\sin\theta|\sin\theta| + F_m\cos\theta$, θ=ωt-kx   Fd and Fm are Fd,max and Fm,max shown below.

note that Fx is still function of θ=ωt-kx to find max, $\frac{\partial F_x}{\partial\theta} = 0$!

- note that all integration we did should be from z=-d to z=0!
- even if told to find Fd and Fm seperately.
- UNLESS told to find Fd,max and Fm,max SEPERATELY:

$F_{d,max} = C_d \cdot \rho\frac{1}{2}D\int_{z=-d}^{z=a} u|u|dz$ (when evaluating u, sinθ = 1)

$F_{m,max} = C_m \cdot \rho\frac{\pi D^2}{4}\int_{z=-d}^{z=0}\frac{\partial u}{\partial t}dz$ (when evaluating u, cosθ = 1)

when find Fmax or fmax, remember to evaluate u at max! (might need to figure out θ=ωt-kx)

→ basically when told to find Fmax seperately, ← i.e. use umax, $\frac{\partial u}{\partial t}|_{max}$, wmax, $\frac{\partial w}{\partial t}|_{max}$.
1. evaluate θ=ωt-kx manually. (sinθ=1, cosθ=1)
2. if integration is involved, fix upper limit accordingly (if needed)

→ If told to find fmax (force per unit length)
1. have to sub z=0 (fm,max) and z=a (fd,max); z=0 for fmax (total)
DON'T NEED INTEGRATION.

just Fmax need integrate and fmax is z=0!

If find together fm and fd, or Fm and Fd remember have to find θ=ωt-kx through $\frac{df}{d\theta} = 0$ and not just "sin(~~ωt-kx~~) cos(~~ωt-kx~~)"
1   1