Dr Li Ma (Spring Term)

Velocity Potential ϕ and Stream Function ψ

In Cartesian co-ordinates

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial z} \quad , \quad w = \frac{\partial \phi}{\partial z} = -\frac{\partial \psi}{\partial x}$$

In Polar Co-ordinates

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad , \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Mass Continuity for 2-D flow

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Vorticity in a 2-D flow

$$\Omega = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$$

Equations of Motion for a 2-D flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad ; \quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g$$

Unsteady Bernoulli's Equation

$$\rho\frac{\partial \phi}{\partial t} + P + \rho g z + \rho\left(\frac{u^2 + w^2}{2}\right) = \text{constant}$$

Small Amplitude Wave Theory

$$\eta = a\sin(\omega t - kx)$$

$u = a\omega e^{kz}\sin(\omega t - kx)$; $w = a\omega e^{kz}\cos(\omega t - kx)$ If deep water.

$$\boxed{u = \frac{a\omega\cosh k(z+d)}{\sinh(kd)}\sin(\omega t - kx) \;\; ; \;\; w = \frac{a\omega\sinh k(z+d)}{\sinh(kd)}\cos(\omega t - kx) \; ;}$$

$$p = \frac{\rho g a\cosh k(z+d)}{\cosh(kd)}\sin(\omega t - kx) - \rho g z$$

$$c = \frac{\omega}{k} = \left[\frac{g}{k}\tanh(kd)\right]^{1/2}$$

$$\frac{c_g}{c} = \frac{1}{2}\left[1 + \frac{2kd}{\sinh(2kd)}\right]$$

Variation of Wave Height in Shallow Water

$$\frac{a^2}{k}\left[1+\frac{2kd}{\sinh(2kd)}\right]=\text{constant}$$

Energy in one wavelength

$$\boxed{PE_{WAVES}=KE_{WAVES}=\rho g\frac{a^2\lambda}{4}}$$

Instead of this I remember:
$E=\frac{\rho g a^2}{2}$ $[Jm^{-2}]$ ← more flexible: $[Jm^{-2}][m][ms^{-1}]$
can find Power $= E \times \lambda \times c_g$
and etc.

Potential flow about a vertical cylinder

$$\phi = u\left(r+\frac{D^2}{4r}\right)cos\theta \quad \text{for} \quad r \geq \frac{D}{2}$$

Morison's Equation

$$F = C_D\frac{1}{2}\rho u|u|D + C_M\rho\frac{\pi D^2}{4}\frac{du}{dt} \qquad \text{(N/m)}$$

Useful identities

$$\cosh^2(\theta)=\frac{1}{2}[\cosh(2\theta)+1]$$

$$\sinh(\theta)=\frac{\left(e^{\theta}-e^{-\theta}\right)}{2},\ \cosh(\theta)=\frac{\left(e^{\theta}+e^{-\theta}\right)}{2}$$

Dr H Burridge (Autumn Term)

Real Fluids

Shear Stress $\tau = \mu \dfrac{\partial u}{\partial y}$

Viscosity $\upsilon = \dfrac{\mu}{\rho}$

	ρ [kg/m^3]	ν [m^2/s]
Air	1.2	$1.5 \cdot 10^{-5}$
Water	1000	$1.0 \cdot 10^{-6}$
Oil	910	$4 \cdot 10^{-4}$

Navier Stokes

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Notation: $\bar{u} = \bar{u}(\boldsymbol{x}) = \frac{1}{T}\int_0^T u(\boldsymbol{x})\,\mathrm{d}t$, and $U = \frac{1}{A}\int_0^A \bar{u}(\boldsymbol{x})\,\mathrm{d}A$, i.e. $Q = U\,A$.

$dA = 2\pi r\,dr$

Laminar (steady) flow between two very large parallel plates

$$u_{\max} = \frac{-B^2}{2\mu}\frac{d(p+\rho g\xi)}{dx}$$

$$u(y) = u_{\max}\left[1-\left(\frac{y}{B}\right)^2\right]$$

The drag force on a sphere in laminar (steady) flow is:

$$F_S = 3\pi\mu u D$$

Reynolds number:

$$\mathrm{Re} = \frac{UL}{\nu}$$

Steady Pipe Flow

$$\frac{p_1}{\rho g} + z_1 + \frac{U_1^2}{2g} + H_{Pump} = \frac{p_2}{\rho g} + z_2 + \frac{U_2^2}{2g} + h_f + h_L$$

$$h_f = \frac{\tau_0 PL}{\rho g A}$$ (τ_0 and h_f interchangeable)

(τ_0 and f or U interchangeable) $$\tau_0 = \frac{f}{8}\rho U^2$$

$$h_f = f\frac{L}{D}\frac{U^2}{2g}$$

$$f\left(\mathrm{Re}, \frac{k_s}{D}\right)$$

first three rows are general eqn. that can be used for any pipe flow!

$$\boxed{f = \frac{64}{\mathrm{Re}}}$$ Re < 2300

$$\boxed{\frac{1}{f^{1/2}} = -2\log_{10}\left[\underbrace{\frac{k_s/D}{3.71}}_{k_s/D=0 \text{ if smooth}} + \underbrace{\frac{2.51}{\mathrm{Re}\, f^{1/2}}}_{\text{can get rid of this term if Re very large. } (10^7)}\right]}$$ generally for any Re.

$$h_L = \xi\frac{U^2}{2g}$$

$R_H = 4D$ $$R_H = \frac{A}{P}$$

$$\boxed{u(r) = -\frac{1}{4\mu}\frac{dp}{dx}\left(R^2 - r^2\right) \quad \text{laminar } u(r) \text{ and } \tau(r) \quad \tau(r) = \mu\frac{du}{dr} = \frac{1}{2}\frac{dp}{dx}r}$$

$$\boxed{\frac{u_{\max} - \bar{u}(y)}{u_*} = -\frac{1}{\kappa}\ln\frac{y}{R}}$$ velocity defect law 3

$$\kappa = 0.4$$

$$\boxed{\frac{\bar{u}(y)}{u_*} = \frac{y}{\nu/u_*}}$$ viscous sub layer 1

then to find τ_0

$$u_* = \sqrt{\tau_0/\rho}$$

$$\boxed{\frac{\bar{u}(y)}{u_*} = \frac{1}{\kappa}\ln\frac{y}{k_s} + 8.5}$$ law of the wall 2

$$-\frac{\rho\overline{u'v'}}{\tau_0} = -\frac{\rho\overline{u'v'}}{\rho u_*^2} \approx 1 - \frac{y}{R}$$

$$\tau_0 = \rho u_*^2 = \frac{f}{8}\rho U^2$$

$$u_* = \sqrt{\frac{f}{8}}U$$

$$\boxed{U = u_{\max} - \frac{3}{2}\frac{u_*}{\kappa}}$$ velocity defect law

used to find u_*

Pipe Systems

$$h_f + h_L = \left(f\frac{L}{D} + \sum_j \xi_j\right)\frac{Q^2}{2gA^2} = KQ^2$$

$$K = \frac{8}{g\pi^2 D^4}\left(f\frac{L}{D} + \sum_j \xi_j\right) \approx \frac{8fL}{g\pi^2 D^5}$$

$$f \approx \left[\log_{10}\left(\frac{k_s/D}{3.71}\right)^{-2}\right]^{-2}$$

Gradually-Varied Open Channel Flow

Bed Slope

$$S_0 = -\frac{dZ}{dx} \qquad \tau_0 = \frac{f}{8}\rho U^2$$

$$\tau_0 = \rho g R S$$

Froude number:

$$Fr = \frac{U}{(gh)^{1/2}}$$

this h is not h_n, h_c or current h! It's h_m (mean depth) where $h_m = \frac{A}{w}$ ← cross sectional area of flow, ← width at surface.

Friction Factors

$$U = \frac{Q}{A} = \left(\frac{8g}{f}\right)^{1/2} R^{1/2} S^{1/2}$$

S=So if uniform
S=Sf if used in GVF

$$U = \frac{1}{n} R^{2/3} S^{1/2}$$

$$U = C R^{1/2} S^{1/2}$$

$$\frac{1}{f^{1/2}} = -2\log_{10}\left(\frac{2.51}{\mathrm{Re}\, f^{1/2}} + \frac{k_s/(4R)}{3.71}\right) \approx -2\log_{10}\left(\frac{k_s/R}{14.84}\right)$$

$$f = \frac{8g}{C^2}$$

$$f = 8gn^2 / R^{1/3}$$

Specific Energy / Momentum

with respect to channel bed!

$$\boxed{E(h) = \underbrace{h}_{\text{no elevation head!}} + \frac{q^2}{2gh^2} \qquad J(h) = \frac{1}{2}\rho g h^2 + \underbrace{\frac{\rho q^2}{h}}_{\text{no weight component!}}} \qquad Fr = \frac{Q/A}{\sqrt{gA/b}}$$

Normal and Critical Depth

If not rectangular channel: Fr=1 (to find hc)

$\frac{U}{\sqrt{g h_m}} = 1$ where $h_m = \frac{A}{w}$ ← top width w

$$\boxed{h_c = \left(\frac{q^2}{g}\right)^{1/3}}$$

note that q is Q/w
- valid for ANY rectangular channel.

$$\boxed{F(h_n) = 0 = \frac{Q}{A} - \left(\frac{8g}{f}\right)^{1/2}\left(\frac{A}{P}\right)^{1/2} S^{1/2}}$$

Used to find hn given Q
→ any channel (rectangular or not)

S=So (cause uniform flow!)

$$\boxed{h_n = \left(\frac{fq^2}{8gS_0}\right)^{1/3}}$$

used to find hn given Q
→ valid for ONLY wide rectangular channel.

Gradually Varied Flow

$$\frac{d\eta}{dx} = \frac{Q^2}{gA^3}\frac{dA}{dx} - \frac{fQ^2}{8gA^2R} \qquad \boxed{\frac{dh}{dx} = S_0 \frac{1-(h_n/h)^3}{1-(h_c/h)^3}}$$

only valid for wide rectangular channel

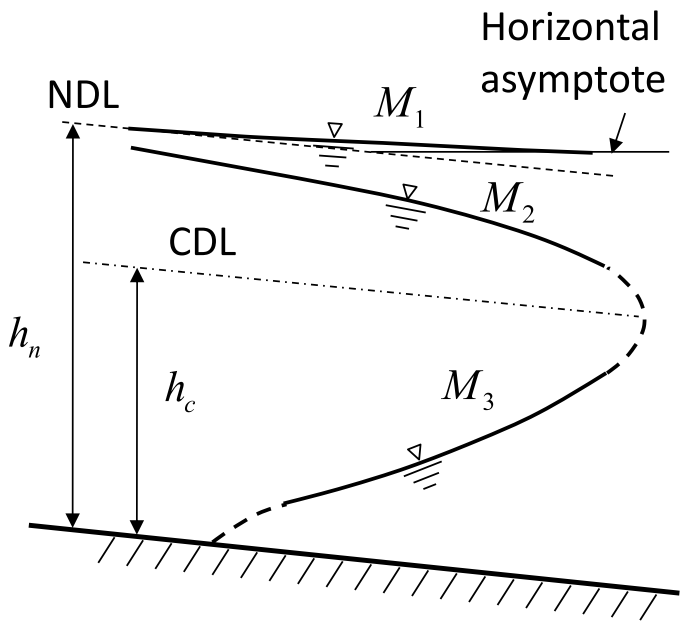
$$\frac{S_0}{h_n}(x_2 - x_1) = (\gamma_2 - \gamma_1) - \left[1 - \left(\frac{h_c}{h_n}\right)^3\right](\Phi(\gamma_2) - \Phi(\gamma_1))$$

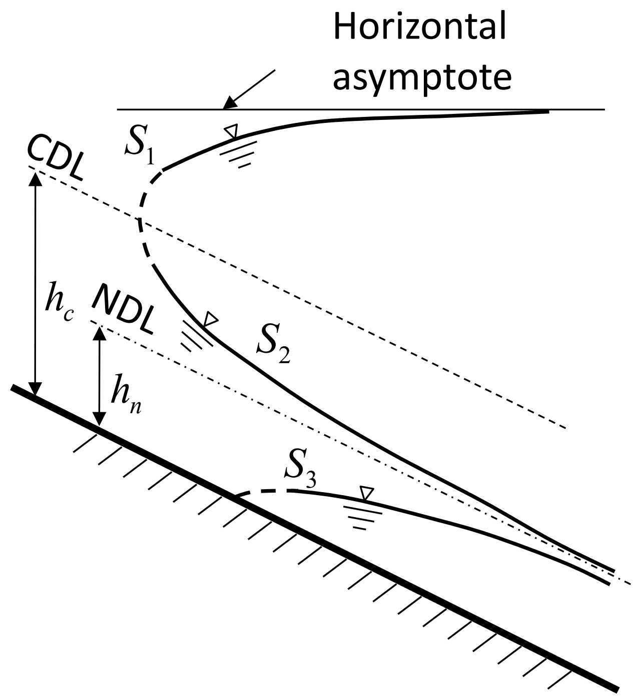
aim: to find x_2

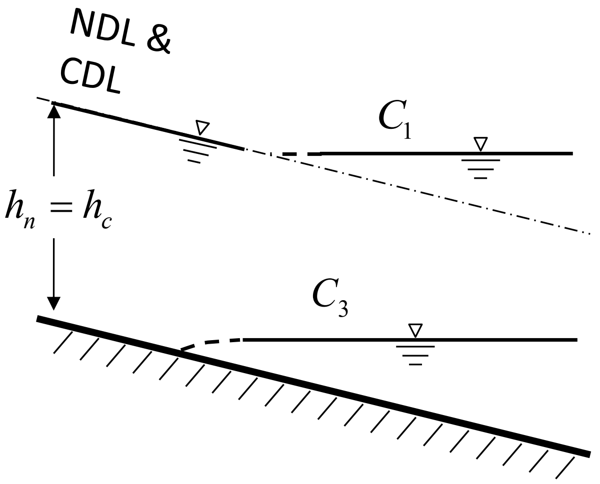
$$x_2 = x_1 + \frac{h_n}{S_0}\left\{(\gamma_2 - \gamma_1) - \left[1 - \left(\frac{h_c}{h_n}\right)^3\right](\Phi(\gamma_2) - \Phi(\gamma_1))\right\}$$

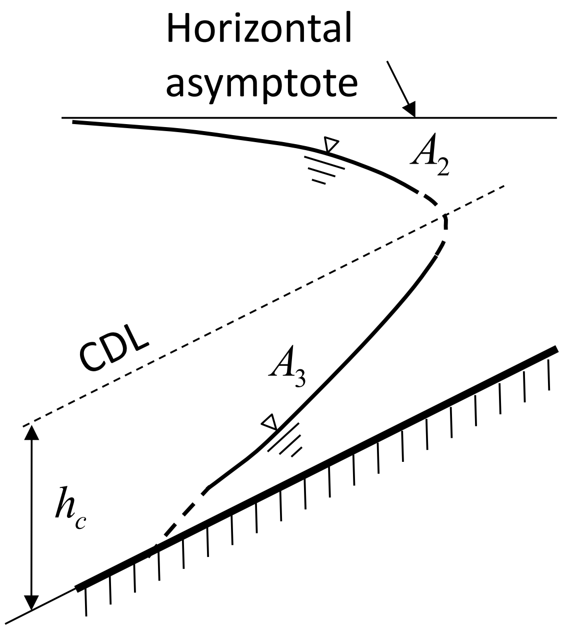
$$\Phi(\gamma) = \frac{1}{6}\ln\frac{\gamma^2 + \gamma + 1}{(\gamma - 1)^2} + \frac{1}{\sqrt{3}}\arctan\frac{2\gamma + 1}{\sqrt{3}}$$

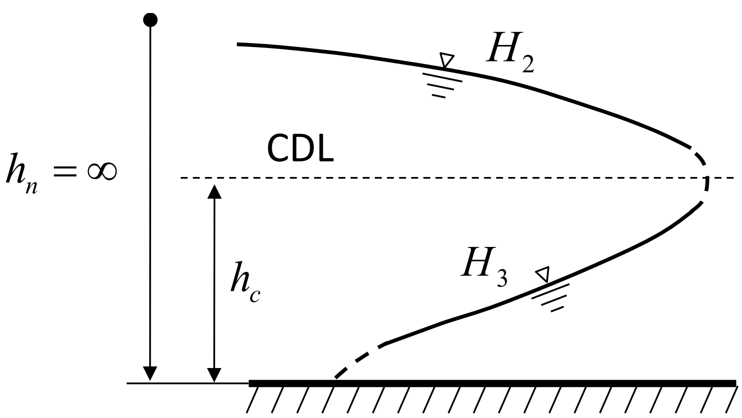
$$\gamma \equiv \frac{h}{h_n} \qquad \Phi(0.2) = 0.5027 \qquad \Phi(1.2) = 1.3867$$

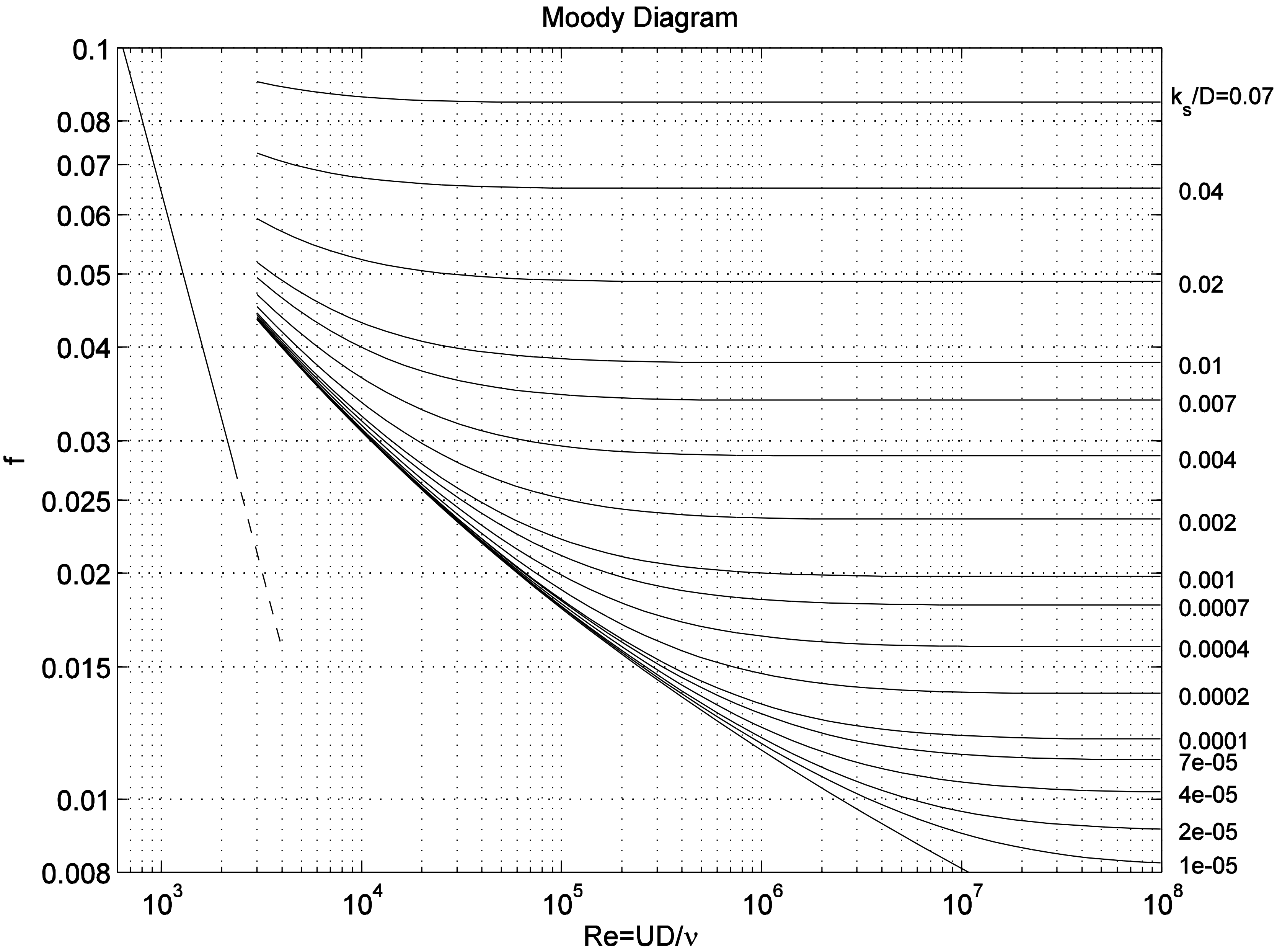

Horizontal asymptote
NDL
M_1
M_2
CDL
h_n
h_c
M_3


Horizontal asymptote
CDL
S_1
h_c
NDL
S_2
h_n
S_3


NDL & CDL
C_1
$h_n = h_c$
C_3


Horizontal asymptote
A_2
CDL
A_3
h_c


H_2
CDL
$h_n = \infty$
h_c
H_3


Moody Diagram
f
Re=UD/ν
0.1
0.08
0.07
0.06
0.05
0.04
0.03
0.025
0.02
0.015
0.01
0.008
10^3
10^4
10^5
10^6
10^7
10^8
k_s/D=0.07
0.04
0.02
0.01
0.007
0.004
0.002
0.001
0.0007
0.0004
0.0002
0.0001
7e-05
4e-05
2e-05
1e-05