Tutorial 1: Governing Equations

Question 1

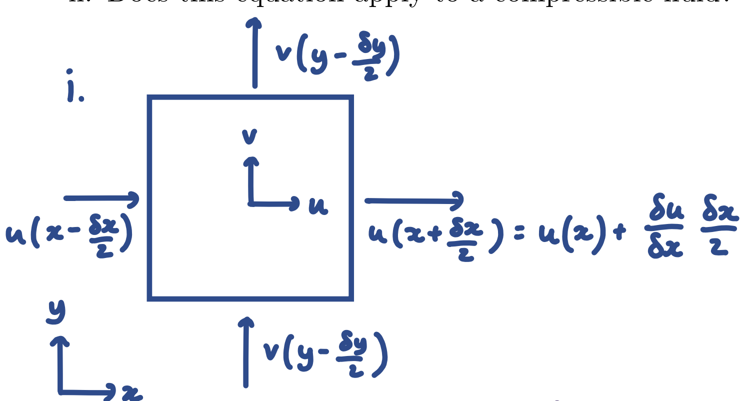
In answering this question consider a **control surface** for a 2D incompressible flow.

i. Show that the 2D equation of mass continuity is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

ii. Does this equation apply to a compressible fluid?

i.



flux, Q = UA !

flux,out,x − flux,in,x + flux,out,y − flux,in,y = 0

$$\left(\cancel{u} + \frac{\delta u}{\delta x}\frac{\delta x}{2}\right)\delta y - \left(\cancel{u} - \frac{\delta u}{\delta x}\frac{\delta x}{2}\right)\delta y + \left(\cancel{v} + \frac{\delta v}{\delta y}\frac{\delta y}{2}\right)\delta x - \left(\cancel{v} - \frac{\delta v}{\delta y}\frac{\delta y}{2}\right)\delta x = 0$$

$$\delta x\,\delta y\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y}\right) = 0$$

since $\delta x \neq 0$, $\delta y \neq 0$

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = 0$$

ii. It doesn't apply to a compressible fluid.
The reason being the equation treat that there will be no ACCUMULATION of fluid in the CV

Question 2

In answering this question make use of the pressure distribution on a rectangular **fluid particle**.

i. For an incompressible and inviscid fluid, show that the 2D equations of motion (2D Euler's equation) are given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g$$

ii. Are these equations appropriate to the description of a viscous fluid?

iii. Write down the 3D Euler's equation, think about whether you'd like to use vector notation!

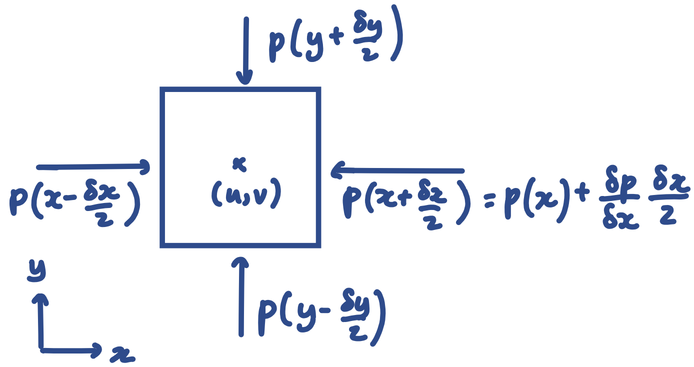
i. $F = ma$

when in 2D,

$F_x = ma_x$

$F = PA$ or Pl (2D)

$m = V\rho$ or $A\rho$ (2D)



$$\underbrace{\left[\cancel{p} - \frac{\partial p}{\partial x}\frac{\delta x}{2} - \left(\cancel{p} + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\right]}_{P} \times \underbrace{\delta y}_{l} = \underbrace{\delta x \delta y}_{A} \times \underbrace{\rho}_{\rho} \times \underbrace{a_x}_{a} \quad \text{(in } x\text{-direction)}$$

$u(x,y,t)$

$$-\frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)$$

$$a_x = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

x-direction eqn.

$F_y = ma_y$

$$-\frac{\partial p}{\partial y} - \rho g = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)$$

$$\boxed{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g}$$

y-direction eqn.

ii. no, doesn't apply.
There is no viscous stress term.

iii. $$\frac{D\underset{\sim}{U}}{Dt} = -\frac{1}{\rho}\nabla p - g\underset{\sim}{k}$$

Question 3

The x-component of the Navier-Stokes equation is given by

$$\rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

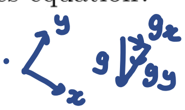
i. Provide meaningful names for each of the terms involved. The terms in parentheses may be dealt with as a single contribution.

ii. In deriving this result, which assumption was adopted for the stress-strain relationship?

iii. Is this equation appropriate to the description of turbulent flows? Explain your arguments.

iv. How is the Euler equation related to the Navier-Stokes equation?

v. State the x-component of the Euler equation.

eg. 

i. $$\underbrace{\overbrace{\rho\frac{\partial u}{\partial t}}^{\text{unsteady}} + \overbrace{\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}^{\text{convective}}}_{\text{acceleration in } x \text{ direction.}} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{pressure gradient in } x} + \overbrace{\rho g_x}^{x \text{ component of: gravity}} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\text{viscous stress term (due to shear stress)}}$$

ii. $\tau = \mu \frac{du}{dy}$

shear stress is linearly dependent on rate of strain (or strain rate)

(not strain!)

iii. In this form, the equation can describe turbulent flow but the interactions between the turbulence within the flow and the mean flow are not obvious. To clarify these interactions, the velocity components (u, v, w) and pressure (p) could be separated into a mean and a fluctuating contribution. Subsequently, a process called Reynolds averaging needs to be employed.

iv. Euler → inviscid flows (no viscous stress term)
$\underbrace{\text{Navier Stokes eqn}}_{\text{both eqn are identical if } \mu = 0 \text{ (viscosity)}}$ → real fluid

v. $$\rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \cancelto{0}{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}$$

$$\rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x$$

Question 4

Consider the steady, laminar and fully developed flow between two infinitely large parallel plates as discussed in §2.3 of the lecture notes.

i. Simplify the three dimensional Navier-Stokes equations provided on your Datasheet by eliminating all terms that are zero in this particular problem.

ii. Does the simplified set of equations lead to a problem that can be solved analytically? Explain your arguments.

i. flow is steady → $\frac{\partial}{\partial t} = 0$

flow is fully developed → $\frac{\partial u}{\partial x} = 0$

very large plate, effect boundaries ignored → $\frac{\partial}{\partial z} = 0$

very large plate, assume no flow in z-direction → $w = 0$

from continuity equation, $\underbrace{\cancel{\frac{\partial u}{\partial x}}}_{0} + \frac{\partial v}{\partial y} + \underbrace{\cancel{\frac{\partial w}{\partial z}}}_{0} = 0 \rightarrow \frac{\partial v}{\partial y} = 0$

wall is impermeable → $\left.\begin{array}{l} v(y=b) = 0 \\ \frac{\partial v}{\partial y} = 0 \end{array}\right\} v = 0$

$$\left.\right\} \quad \begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \rho g_x + \mu \frac{\partial^2 u}{\partial y^2} \\ 0 &= -\frac{\partial p}{\partial y} + \rho g_y \\ 0 &= -\frac{\partial p}{\partial z} + \rho g_z \end{aligned}$$

ii. $u(x, y, z)$ has now been simplified into $u(y)$
it can be solved analytically.