C1. Complex Variables

1.1 Differentiation and Analycity

If $f(z)$ is continuous at $z=z_0$, then: $\lim\limits_{z\to z_0} f(z) = f(z_0)$

If $f(z)$ is differentiable at $z=z_0$, then: $f'(z_0) = \lim\limits_{z\to z_0} \left.\dfrac{f(z)-f(z_0)}{z-z_0}\right|_{z=z_0+\Delta z}$

or $f'(z_0) = \lim\limits_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

eg1. is $f(z)=z^3$ differentiable?

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$
$$= \lim_{z\to z_0} \frac{z^3-z_0^3}{z-z_0}$$
$$= \lim_{z\to z_0} \frac{\cancel{(z-z_0)}(z^2+z_0z+z_0^2)}{\cancel{z-z_0}}$$
$$= \lim_{z\to z_0} (z^2+z_0z+z_0^2)$$
$$= \underline{\underline{3z_0^2}}$$

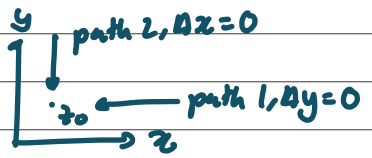
function can be differentiable.

eg2. is $f(z)=z^*$ differentiable?

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$
$$= \lim_{z\to z_0} \frac{z^*-z_0^*}{z-z_0}$$
$$= \lim_{z\to z_0} \frac{x-yi-(x_0-y_0i)}{x+yi-(x_0+iy_0)}$$

let $\Delta x = x-x_0$, $\Delta y = y-y_0$ $\rightarrow$ $x=\Delta x+x_0$, $y=\Delta y+y_0$

$$f'(z_0) = \lim_{\Delta x\to 0,\Delta y\to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$



path 1: substitute $\Delta y=0$

$$f'(z_0) = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1$$

path 2: substitute $\Delta x=0$

$$f'(z_0) = \lim_{\Delta y\to 0} -\frac{i\Delta y}{i\Delta y} = -1$$

answer different $\rightarrow$ $f(z)=z^*$ <u>not</u> differentiable

If $f(z) = u(x,y) + iv(x,y)$

then two important expressions are:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad , \quad f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

only if $f(z)$ is differentiable or known as ANALYTIC!

when told to find $f'(z)$, can use limit, or use either of these eqn.

IMPORTANT!

If $f(z)$ is ANALYTIC:

$\longleftrightarrow$ It can be differentiated

$\longleftrightarrow$ u and v satisfy Cauchy-Riemann eqn.

$\longleftrightarrow$ u and v are harmonic and satisfy Laplace eqn.

$\longrightarrow$ $u = c$ and $v = c'$ form orthogonal trajectories.

1.2 Cauchy-Riemann Equations.

$$f(z) = u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Cauchy-Riemann Equations

2D Laplace Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

function that satisfy this equation are called HARMONIC FUNCTION!

eg1. is $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ analytic?

can use C-R eqn (easier)

can use Laplace, test both u and v

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$$u(x,y) = x^3 - 3xy^2 \quad ; \quad v(x,y) = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy \quad \text{-ive} \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \checkmark \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \checkmark$$

Cauchy-Riemann eqn satisfied $\rightarrow$ analytic.

eg2. Find $f(z)$ given that $u = xy + x$ (find u given v or vice versa) ☆

$$\frac{\partial u}{\partial x} = y+1 \qquad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \qquad \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad , \; u = xy + x \text{ is harmonic}$$

check whether $u = xy + x$ is harmonic

☆ must check or else can't use C-R eqn.
→ If not... CAN'T FIND

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$y+1 = \frac{\partial v}{\partial y}$$

$$v = \int y+1 \, dy + h(x)$$

$$v = \frac{1}{2}y^2 + y + h(x)$$

$$\frac{\partial v}{\partial x} = h'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$h'(x) = -x$$

$$h(x) = \int -x \, dx$$

$$h(x) = -\frac{1}{2}x^2 + C$$

$$u = xy + x \; ; \; v = \frac{1}{2}y^2 + y - \frac{1}{2}x^2 + C$$

$$f(z) = xy + x + i\left(\frac{1}{2}y^2 + y - \frac{1}{2}x^2 + c\right) = z$$

1.3 Elementary functions of Complex Variables.

1. Exponential, e^z

$$e^z = e^x(\cos y + i\sin y)$$

$$\frac{d}{dz}(e^z) = e^z$$

$$e^{i\pi} = -1$$

$e^z \neq 0$ for all z

2. Trigonometric function, and Hyperbolic Function.

$\cos z = \frac{1}{2}(e^{iz}+e^{-iz})$ $\cosh z = \frac{1}{2}(e^{z}+e^{-z})$

$\sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$ $\sinh z = \frac{1}{2}(e^{z}-e^{-z})$

$\frac{d}{dz}(\cos z) = -\sin z$ $\frac{d}{dz}(\cosh z) = \sinh z$

$\frac{d}{dz}(\sin z) = \cos z$ $\frac{d}{dz}(\sinh z) = \cosh z$

3. Natural Logarithm, $\ln(z)$

$z = x+iy = re^{i\theta}$, where $r=\sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$.

$\ln(z) = \ln(re^{i\theta}) = \ln(r) + i(\theta+2n\pi)$, $n \in \mathbb{Z}$

if $Ln(z)$, $Ln(z) = \ln(r) + i\underline{\underline{\theta}}$ principal sol: $-\pi < Arg(z) \leq \pi$

4. Cauchy-Riemann for polar coordinates

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$

☆ useful for $f(z)$ in terms of r and θ, (like $Ln(z)$)

☆ $Ln(z)$ is analytic at all point except: $z=0$; $R(z)<0$ and $I(z)=0$ (negative real axis) because $Ln(z)$ is discontinuous.

5. To the power of complex number.

trick: $z^{\alpha} = e^{Ln(z^{\alpha})} = e^{\alpha Ln(z)}$

everytime you see $Ln(z)$, solve it! $Ln(z) = \ln r + i\theta$! (even if it is $\ln 2$, solve it!) ln(real)

and every time you see e^{z}, do $e^{x+iy} = e^{x}(\cos y + i\sin y)$!

eg1 $\tan^{-1}(2i)$ ← Trigonometric question.

let $z = \tan^{-1}(2i)$

$\tan z = 2i$

$\frac{\sin z}{\cos z} = 2i$

$\frac{(e^{iz}-e^{-iz})/\cancel{2}i}{(e^{iz}+e^{-iz})/\cancel{2}} = 2i$

$\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = -2$

let $w = e^{iz}$

$\frac{w-\frac{1}{w}}{w+\frac{1}{w}} = -2$

$\frac{w^2-1}{w^2+1} = -2$

$w^2-1 = -2w^2-2$

$3w^2 = -1$

$w^2 = -1/3$

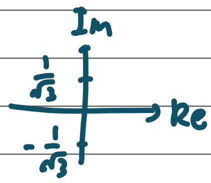
$w = \pm\sqrt{-\frac{1}{3}}$

$= \pm\sqrt{-1}\sqrt{\frac{1}{3}}$

$= \pm i\frac{1}{\sqrt{3}}$

$\ln re^{i\theta}$

$\ln r + i\theta$



$e^{iz} = \pm i\frac{1}{\sqrt{3}}$

$iz = \ln\left(\pm i\frac{1}{\sqrt{3}}\right)$ $\quad \left|\pm i\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}}$

$iz = \ln\left(\frac{1}{\sqrt{3}}\right) + i\left(\pm\frac{\pi}{2}+2\pi n\right)$ $\quad \arg\left(\pm i\frac{1}{\sqrt{3}}\right) = \underbrace{\pm\frac{\pi}{2}+2\pi n}_{\text{can be written as } \frac{\pi}{2}+\pi n,\ \forall n\in\mathbb{Z}}, \forall n\in\mathbb{Z}$

$iz = -\frac{1}{2}\ln 3 \pm i\left(\frac{\pi}{2}+2\pi n\right), \forall n\in\mathbb{Z}$

$z = \frac{i}{2}\ln 3 + \left(\frac{\pi}{2}+\pi n\right), \forall n\in\mathbb{Z}$

eg2. $\ln(1-i)$ ← Natural logarithm.

$|1-i| = \sqrt{1^2+(-1)^2} = \sqrt{2}$

$\arg(1-i) = \tan^{-1}(-1/1) = \tan^{-1}(-1) = -\pi/4 + 2\pi k, \forall k\in\mathbb{Z}$

$\ln(1-i) = \ln\left(\sqrt{2}e^{i(-\pi/4+2\pi k)}\right), \forall k\in\mathbb{Z}$

$\ln(1-i) = \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4}+2\pi k\right), \forall k\in\mathbb{Z}$

eg3. $(1+i)^{2i} = e^{\ln(1+i)^{2i}}$ $\longleftarrow z_1^{z_2}$

$= e^{2i\ln(1+i)}$

$\ln(1+i) = \ln\left(\sqrt{2}e^{i(\pi/4+2\pi k)}\right), \forall k \in \mathbb{Z}$

$= \frac{1}{2}\ln 2 + i(\pi/4 + 2\pi k), \forall k \in \mathbb{Z}$

$(1+i)^{2i} = e^{2i\left(\frac{1}{2}\ln 2 + i(\pi/4+2\pi k)\right)}, \forall k \in \mathbb{Z}$

$= e^{i\ln 2 - \pi/2 - 4\pi k}, \forall k \in \mathbb{Z}$

$= e^{-(\pi/2+4\pi k)} \cdot (\cos\ln 2 + i\sin\ln 2), \forall k \in \mathbb{Z}$

eg4. $2^{3+2i} = 2^3 \cdot 2^{2i}$ $\longleftarrow \varphi^z$

$= 8 \cdot 2^{2i}$

$= 8e^{2i\ln 2}$ when got ln, solve ln! ☆

$\ln 2 = \ln 2e^{i(2n\pi)}, \forall n \in \mathbb{Z}$

$= \ln 2 + i(2n\pi)$

$2^{3+2i} = 8e^{2i(\ln 2 + 2n\pi i)}$

$= 8e^{2\ln 2 i - 4n\pi}$ when got e^z, solve it! ☆

$= 8e^{-4n\pi} \cdot e^{2\ln 2 i}$ $\longrightarrow e^{i\theta} = \cos\theta + i\sin\theta$

$= 8e^{-4n\pi}(\cos 2\ln 2 + i\sin 2\ln 2)$

1.4 Orthogonal Trajectories

If $f(z)$ is analytic $\longrightarrow$ $u(x,y) = c_1$ and $v(x,y) = c_2$ forms orthogonal trajectories.

$\not\leftarrow$

the other way around is NOT always true. (can be true, just not always)

$$m_1 = \left(\frac{dy}{dx}\right)_1 = \frac{-\partial u/\partial x}{\partial u/\partial y}$$ derived from $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$, $du = 0$

$$m_2 = \left(\frac{dy}{dx}\right)_2 = \frac{-\partial v/\partial x}{\partial v/\partial y}$$ derived from $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$, $dv = 0$

eg1. Does z^2 forms sets of orthogonal trajectories?

$f(z) = z^2$ (is analytic → will form, let us test:)

$z = x + iy$

$f(z) = (x+iy)^2 = x^2 - y^2 + 2xyi$

$u(x,y) = x^2 - y^2$; $v(x,y) = 2xy$

$$m_1 = \frac{-\partial u/\partial x}{\partial u/\partial y} = \frac{x}{y} \quad ; \quad m_2 = \frac{-\partial v/\partial x}{\partial v/\partial y} = -\frac{y}{x}$$

$$m_1 m_2 = \frac{x}{y}\left(-\frac{y}{x}\right) = -1$$

$\therefore f(z) = z^2$ will form O.T.

☆ case where u is harmonic → $f(z)$ is analytic.

eg2. Find Orthogonal Trajectories of $u = e^{3x}\cos(3y)$

1. Is $u(x,y)$ harmonic? → is $f(z) = u + iv$ analytic?

$$\frac{\partial u}{\partial x} = 3e^{3x}\cos(3y) \qquad \frac{\partial u}{\partial y} = -3e^{3x}\sin(3y)$$

$$\frac{\partial^2 u}{\partial x^2} = 9e^{3x}\cos(3y) \qquad \frac{\partial^2 u}{\partial y^2} = -9e^{3x}\cos(3y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 9e^{3x}\cos(3y) - 9e^{3x}\cos(3y) = 0$$

$\therefore u = e^{3x}\cos(3y)$ is harmonic

2. find $v(x,y)$ ☆ can use Cauchy-Riemann cause Analytic!

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$3e^{3x}\cos(3y) = \frac{\partial v}{\partial y}$$

$$v = \int 3e^{3x}\cos(3y)\,dy + h(x)$$

$$v = e^{3x}\sin(3y) + h(x)$$

$$\frac{\partial v}{\partial x} = 3e^{3x}\sin(3y) + h'(x)$$

$$\cancel{3e^{3x}\sin(3y)} + h'(x) = \cancel{3e^{3x}\sin(3y)}$$

$$h'(x) = 0$$

$$h(x) = \int h'(x)\,dx$$

$$h(x) = \int 0\,dx$$

$$h(x) = C_1$$

$$\therefore v = e^{3x}\sin(3y) + C_1$$

$\therefore$ orthogonal trajectories: $e^{3x}\cos(3y) = c$

$e^{3x}\sin(3y) = c'$

☆ case where u is harmonic → $f(z)$ is not analytic!

eg3. Find Orthogonal Trajectories of $u = e^{3x}\cos(3y)$

1. Is $u(x,y)$ harmonic? → is $f(z) = u + iv$ analytic?

$$\frac{\partial u}{\partial x} = 2e^{2x}\cos(3y) \qquad \frac{\partial u}{\partial y} = -3e^{2x}\sin(3y)$$

$$\frac{\partial^2 u}{\partial x^2} = 4e^{2x}\cos(3y) \qquad \frac{\partial^2 u}{\partial y^2} = -9e^{2x}\cos(3y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4e^{2x}\cos(3y) - 9e^{2x}\cos(3y) = -5e^{2x}\cos(3y) \neq 0$$

$\therefore u = e^{2x}\cos(3y)$ is not harmonic → $f(z) = u + iv$ is not harmonic!

2. find $v(x,y)$

$$m_1 = \frac{-\partial u/\partial x}{\partial u/\partial y} = \frac{\cancel{-}2e^{2x}\cos(3y)}{\cancel{-}3e^{2x}\sin(3y)} = \frac{2\cos(3y)}{3\sin(3y)}$$

$$m_2 = -1/m_1$$

$$\frac{dy}{dx} = -\frac{3\sin(3y)}{2\cos(3y)}$$

in this case why we don't use $\frac{-\partial v/\partial x}{\partial v/\partial y}$ cause we will introduce v which complicates stuff.

$$\int \frac{\cos(3y)}{\sin(3y)}\,dy = \int -\frac{3}{2}\,dx$$

$$\frac{1}{3}\ln(\sin(3y)) = -\frac{3}{2}x + C$$

$$\sin(3y) = e^{-\frac{9}{2}x + C_1}$$

$$\sin(3y) = c' e^{-9/2x}$$

$$c' = \underbrace{\sin(3y)e^{9/2x}}_{v} \quad \text{and} \quad c = \underbrace{e^{2x}\cos(3y)}_{u}$$

Application to 2D Fluid Flow

$f(z) = u + iv$, if $f(z)$ is analytic, u and v forms orthogonal trajectories.
since in fluid, potential ϕ and stream function ψ are also orthogonal trajectories,

$u \to \phi$, $v \to \psi$

$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$, $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

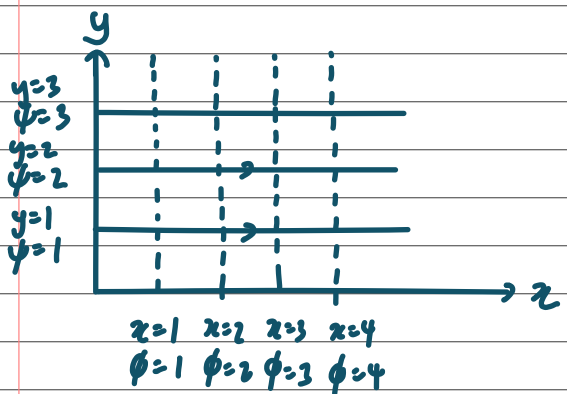
eg1. Plot ϕ and ψ for $h(z) = z$ (left to right is positive)

$h(z) = z = x + iy$

for ϕ: $\mathbb{R}\{h(z)\} = c$ $\quad$ for ψ: $\mathbb{I}\{h(z)\} = c'$

$x = c$ $\quad$ $y = c'$

$\phi = c$ and $\psi = c'$ form orthogonal trajectories.



eg2. Plot ϕ and ψ for $h(z) = i\operatorname{Ln}(z)$

$h(z) = i\operatorname{Ln}(z) = i(\ln(r) + i\theta) = i\ln r - \theta$

for ϕ: $\mathbb{R}\{h(z)\} = c$

$-\theta = c$

$\theta = c_1$

(represent ϕ)

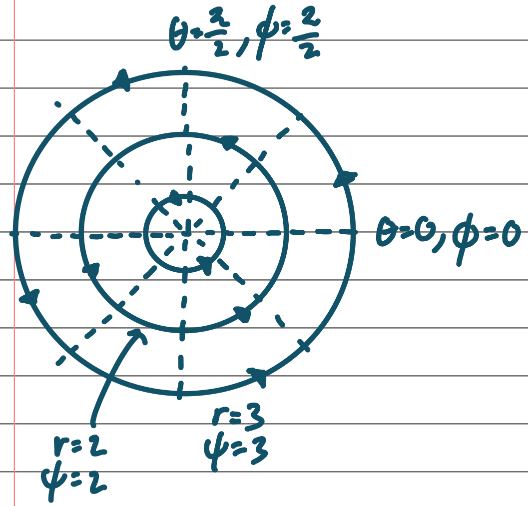
for ψ: $\mathbb{I}\{h(z)\} = c$

$\ln r = c$

$r = e^c$

$r = c_2$

(represent ψ)



eg 3. Draw the streamline and level surface of. $f(z) = e^{z}$

$$f(z) = e^{z} = e^{x+iy} = e^{x}(\cos y + i \sin y) = e^{x}\cos y + e^{x}\sin y \, i$$

$$\underbrace{e^{x}\cos y = c \quad \text{and} \quad e^{x}\sin y = c'}_{\text{two sets of orthogonal trajectories.}}$$

c2 Vector Calculus.

2.1 Scalar function of a scalar.

$f(x)$, $f'(x) = \frac{df}{dx}$.

2.2 Vector function of a scalar.

$\underset{\sim}{r}(t)$, $\frac{d\underset{\sim}{r}}{dt} = \frac{d\underset{\sim}{r_1}}{dt}\underset{\sim}{i} + \frac{d\underset{\sim}{r_2}}{dt}\underset{\sim}{j} + \frac{d\underset{\sim}{r_3}}{dt}\underset{\sim}{k}$

general rule of differentiation:

$\frac{d}{dt}(\underset{\sim}{A}\cdot\underset{\sim}{B}) = \frac{d\underset{\sim}{A}}{dt}\cdot\underset{\sim}{B} + \underset{\sim}{A}\cdot\frac{d\underset{\sim}{B}}{dt}$ sequence not important.

$\frac{d}{dt}(\underset{\sim}{A}\times\underset{\sim}{B}) = \frac{d\underset{\sim}{A}}{dt}\times\underset{\sim}{B} + \underset{\sim}{A}\times\frac{d\underset{\sim}{B}}{dt}$ sequence important, $\underset{\sim}{A}\times\underset{\sim}{B} = -\underset{\sim}{B}\times\underset{\sim}{A}$

2.3. Scalar function of a vector (known as SCALAR FIELD) → like multivariable function.

$\varphi(\underset{\sim}{r})$ input vector

output scalar.

$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$

let $\underset{\sim}{r}$ be a 3-D vector, depends on $x, y,$ and z:

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\underset{\sim}{i} + \frac{\partial\varphi}{\partial y}\underset{\sim}{j} + \frac{\partial\varphi}{\partial z}\underset{\sim}{k}$$

note that grad (scalar function) = vector.

$\nabla\varphi$:

1. $\nabla\varphi$ is the direction in which the rate of change is the largest.
2. if want to find the direction vector remember to $\nabla\varphi / |\nabla\varphi|$ to get unit vector.
3. $|\nabla\varphi|$ indicates the magnitude of the rate of change. $|\nabla\varphi| = \sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2}$
4. Always towards the higher value of $\varphi(x, y, z)$

2.4 Vector function of a vector (known as VECTOR FIELD)

in scalar field, ∇ can only $\nabla\varphi$ (one type of operation)

in vector field: 1. $\nabla \cdot \underset{\sim}{A}$ (div $\underset{\sim}{A}$)

2. $\nabla \times \underset{\sim}{A}$ (curl $\underset{\sim}{A}$)

so if we write ∇k without noting whether there is $\cdot$ between '∇' and 'k'
if k is scalar then $\nabla k = \text{grad}(k)$
if k is vector then $\nabla k = \text{div}(k)$

$$\text{div}(\underset{\sim}{A}) = \nabla \cdot \underset{\sim}{A}$$

$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$= \underbrace{\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}}_{\text{rate of change in direction of } \underset{\sim}{A}}$$

(notice that $\nabla \cdot \underset{\sim}{A}$, dot product answer must be <u>SCALAR</u>)

$$\text{curl}(\underset{\sim}{A}) = \nabla \times \underset{\sim}{A}$$

$$= \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

$$= \underbrace{\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\underset{\sim}{k}}_{\text{rate of change of vector perpendicular to direction of } \underset{\sim}{A}}$$

(notice that $\nabla \times \underset{\sim}{A}$, cross product answer must be <u>VECTOR</u>)

small notes in fluid mechanics:

- fluid flow diverges if $\text{div}(\underset{\sim}{v}) > 0$
- fluid flow converges if $\text{div}(\underset{\sim}{v}) < 0$
- " incompressible if $\text{div}(\underset{\sim}{v}) = 0$

- fluid flow is irrotational if $\text{curl}(\underset{\sim}{v}) = 0$
- " is rotational if $\text{curl}(\underset{\sim}{v}) \neq 0$

☆ Laplacian term (del squared) ∇^2:

if applied onto scalar field,

$$\nabla^2\varphi = \nabla\cdot(\nabla\varphi) = div(\nabla\varphi) = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$$

so div

remember grad φ becomes vector

if applied onto vector field,

$$\nabla^2\underset{\sim}{A} = (\nabla\cdot\nabla)\underset{\sim}{A} = \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\cdot\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\underset{\sim}{A}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\underset{\sim}{A}$$

scalar multiply with a vector.

$$k\underset{\sim}{A} = k\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix} = \begin{pmatrix}kA_1\\kA_2\\kA_3\end{pmatrix}$$

$$= \left.\begin{pmatrix}\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\\ \frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_2}{\partial z^2}\\ \frac{\partial^2 A_3}{\partial x^2} + \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_3}{\partial z^2}\end{pmatrix}\right\}$$

Answer is a vector

(it's weird cause applying ∇^2 onto a vector is like applying ∇ onto a scalar)

☆ Other important result (φ and ϕ are scalar field, A and B are vector field)

- $curl(\nabla\varphi) = 0$
- $div(curl(\underset{\sim}{A})) = 0$
- $curl(curl(\underset{\sim}{A})) = \nabla(div(\underset{\sim}{A})) - \nabla^2\underset{\sim}{A}$

* NO $div(div(\underset{\sim}{A}))$ or $curl(div(\underset{\sim}{A}))$ cause $div(\underset{\sim}{A})$ is a scalar !

- $\nabla(\varphi\phi) = \varphi\nabla\phi + \phi\nabla\varphi$
- $div(\varphi\underset{\sim}{A}) = \varphi\, div(\underset{\sim}{A}) + \underset{\sim}{A}\cdot\nabla\varphi$
- $curl(\varphi\underset{\sim}{A}) = \varphi\, curl(\underset{\sim}{A}) + \nabla\varphi\times\underset{\sim}{A}$
- $div(\underset{\sim}{A}\times\underset{\sim}{B}) = \underset{\sim}{B}\, curl(\underset{\sim}{A}) - \underset{\sim}{A}\cdot curl(\underset{\sim}{B})$

Some extra stuff:

- conservative vector field:

  $curl(\underset{\sim}{v}) = 0 \rightarrow v = \nabla\varphi$

- solenoidal vector field:

  $div(\underset{\sim}{v}) = 0 \rightarrow v = \nabla\times\underset{\sim}{A}$

C3. Integration

3.1 Line Integral (with respect to dx or dy only)

this dy is not the same as: $I = \int f(x,y)\frac{dx}{dy}dy$

$I = \int_c f(x,y)\,dx$ (with respect to x-direction) $I = \int_c f(x,y)\,dy$ (respect y-direction)

☆ EXTREMELY IMPORTANT : IN LINE INTEGRAL MUST INTEGRATE ALL VARIABLE SO NEED TO SUB ALL VARIABLE INTO ONE OF YOUR CHOICE

<u>CANNOT SKIP</u>

1. use $y=g(x)$ and integrate with respect to $\underline{\underline{x}}$

$$\int_c f(x,y)\,dx = \int_{x=x_0}^{x=x_1} f(x,g(x))\,dx$$

2. use $x=h(y)$ and integrate with respect to $\underline{\underline{y}}$

$$\int_c f(x,y)\,dx = \int_{y=y_0}^{y=y_1} f(h(y),y)\underbrace{\frac{dx}{dy}}_{\text{just chain rule.}}dy$$

3. use $x=g(t)$ and $y=h(t)$ and integrate with respect to t.

$$\int_c f(x,y)\,dx = \int_{t=t_0}^{t=t_1} f(g(t),h(t))\frac{dx}{dt}dt$$

along 1-dimension integration.

if ds is along curve.

★ if $\int P(x,y)dx + Q(x,y)dy$

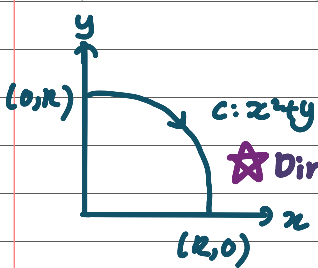
$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ (exact differential)

→ path independent.

→ can simplify path.

eg1. Calculate $\int_c x/y\,dx$ along curve: $x^2+y^2=R^2$ between $(0,R)$ and $(R,0)$

step 1: sketch the path.



☆ Direction is important!

step 2: Integrate, by using the best method.

for this question, circular/elliptical use dt,

$x = R\cos\theta$, $y = R\sin\theta$

at θ_0 : $\theta_0 = \pi/2$, $x=0$, $y=R$

θ_1 : $\theta_1 = 0$, $x=R$, $y=0$

$$\int_c \frac{x}{y}\,dx = \int_{\theta_0=\pi/2}^{\theta_1=0} \frac{R\cos\theta}{R\sin\theta}\,\frac{dx}{d\theta}\,d\theta$$

$$= \int_{\pi/2}^{0} \frac{\cos\theta}{\sin\theta}\,(-R\sin\theta\,d\theta)$$

$$= R\int_0^{\pi/2} \cos\theta\,d\theta$$

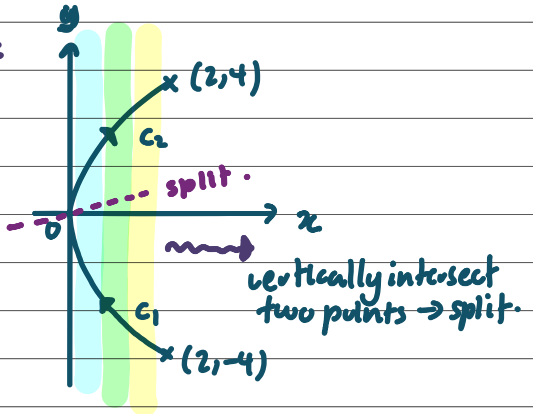
$$= R\left[\sin\theta\right]_0^{\pi/2}$$

$$= R(1-0)$$

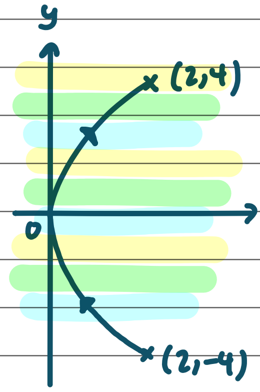
$$= R \,//$$

eg2. $I = \int_c (x^2 - 2xy)\,dx$ along $c: y^2 = 8x$ from $(2,-4)$ to $(2,4)$

dx:



dy:



doesn't intersect two ore more points.

☆ IF DO 'dx'

swipe a vertical line along x-axis, if it touches two or more point at any given time, have to split so it doesn't! (MUST!)

split at (0,0)

$y^2 = 8x$, if dx need to find $y = g(x)$

$y = \pm\sqrt{8x}$

$\underbrace{y = \sqrt{8x}}_{c_2}$ $\quad \underbrace{y = -\sqrt{8x}}_{c_1}$

$$I = \int_{c_1} (x^2 - 2xy)\,dx + \int_{c_2} (x^2 - 2xy)\,dx$$

$y^2 = 8x$, if dy need to find $x = h(y)$

$x = y^2/8$

$\frac{dx}{dy} = \frac{1}{4}y$

$$I = \int_c (x^2 - 2xy)\,dx$$

$$= \int_{-4}^{4} \left(\frac{1}{64}y^4 - \frac{1}{4}y^3\right) \frac{dx}{dy}\cdot dy$$

$$= \int_{-4}^{4} \left(\frac{1}{64}y^4 - \frac{1}{4}y^3\right)\left(\frac{1}{4}y\right) dy$$

$$= \frac{-128}{5} \,//$$

$$I = \int_2^0 \left(x^2 - 2x(-\sqrt{8x})\right) dx + \int_0^2 \left(x^2 - 2x(\sqrt{8x}\right) dx$$

$$= -128/5 \;//$$

3.2 Line Integral (with respect to the arc of path)

$$I = \int_C f(x,y)\, ds$$

$$\left(\begin{array}{ll} ds = \sqrt{1+\frac{dy}{dx}^2}\, dx & ds = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt \\ ds = \sqrt{1+\frac{dx}{dy}^2}\, dy & ds = \sqrt{r^2 + \frac{dr}{d\theta}^2}\, d\theta \end{array} \right)$$

from previous math notes Integration

1. use $y = g(x)$ and integrate with respect to $\underline{\underline{x}}$

$$\int_C f(x,y)\, ds = \int_{x=x_0}^{x=x_1} f(x, g(x)) \sqrt{1+\frac{dy}{dx}^2}\, dx$$

2. use $x = h(y)$ and integrate with respect to $\underline{\underline{y}}$

$$\int_C f(x,y) ds = \int_{y=y_0}^{y=y_1} f(h(y), y) \sqrt{1+\frac{dx}{dy}^2}\, dy$$

3. use $x = g(t)$ and $y = h(t)$ and integrate with respect to t.

$$\int_C f(x,y)\, ds = \int_{t=t_0}^{t=t_1} f(g(t), h(t)) \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

EXTREMELY IMPORTANT!

FOR LINE INTEGRAL WITH RESPECT TO ARC LENGTH (ie 'ds')

MAKE SURE LIMIT 0 < LIMIT 1 WHEN FROM START TO END.

(lower limit) (upper limit)

IF IT DOESN'T,

MULTIPLY ds by -1!

only for ds! for normal case limit 0 can be > limit 1

eg. $I = \int_2^1 f(x, g(x))\, ds$

$$= \int_2^1 f(x, g(x)) \left[-\sqrt{1+\frac{dy}{dx}^2}\, dx \right]$$

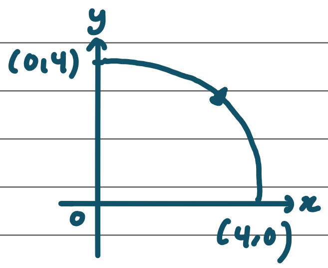
Add a negative.

$$= \underbrace{\int_1^2}_{} f(x, g(x)) \sqrt{1+\frac{dy}{dx}^2}\, dx$$

negative used to invert the sign.

eg1. Evaluate $\int_c (x^2-xy)ds$ along c: $x^2+y^2=16$ from (0,4) to (4,0)

Step 1: Sketch the path:



Step 2: Choose which method:

for this question, circular/elliptical use parametric (polar coordinate)

let $x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2=16 \rightarrow r^2=16 \rightarrow r=\pm 4$ (choose +4)

$x = 4\cos\theta$ $\quad\quad$ $y = 4\sin\theta$

$dx/d\theta = -4\sin\theta$ $\quad$ $dy/d\theta = 4\cos\theta$

Step 3: Find ds.

$$ds = \pm\sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2}\, d\theta$$

$$= \pm\sqrt{16\sin^2\theta + 16\cos^2\theta}\ d\theta$$

$$= \pm 4 d\theta$$

Step 4: Integrate.

$$I = \int_c (x^2 - xy)\, ds$$

$$= \int_{\theta_0=\pi/2}^{\theta_1=0} [16\cos^2\theta - 4\cos\theta(4\sin\theta)](-4d\theta)$$

☆ reason on choosing (+4dθ) instead of (−4dθ) need to keep $\theta_0 \le \theta \le \theta_1$

$$= \int_{\theta_0=0}^{\theta_1=\pi/2} 64\cos^2\theta - 64\sin\theta\cos\theta\ d\theta$$

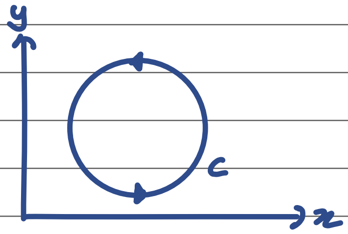
$$= \int_{\theta_0=0}^{\theta_1=\pi/2} 32(\cos 2\theta + 1) - 32\sin 2\theta\ d\theta$$

$$= \left[16\sin 2\theta + 32\theta + 16\cos 2\theta\right]_0^{\pi/2}$$

$$= \cancelto{0}{16\sin\pi} + 32\left(\frac{\pi}{2}\right) + 16\cos\pi - \cancelto{0}{16\sin 0} - 16\cos 0$$

$$= 16\pi - 32 \quad //$$

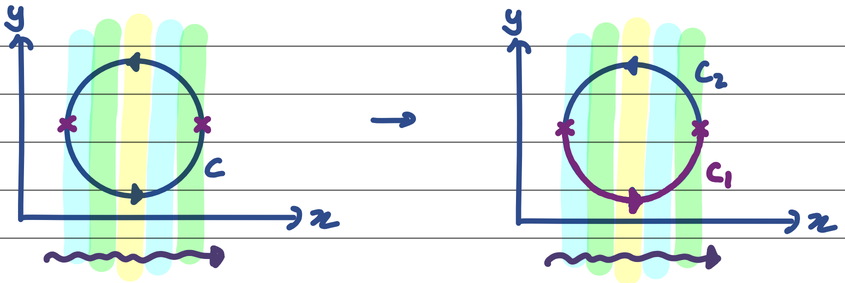
3.3 Closed Line Integral



(sometime use $d\theta$ might not need split)

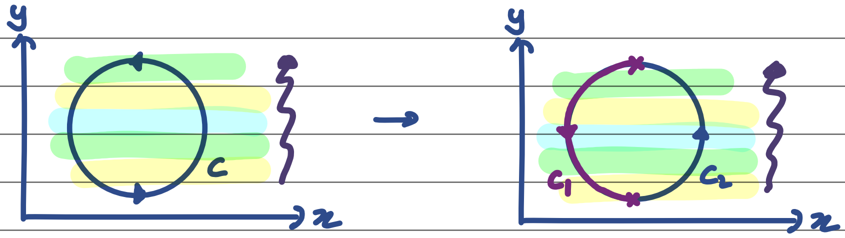
to solve a closed curve integral, have to split into two (or more) region.

if $I = \oint_C f(x,y)\,dx$, have to split it such that when swiping vertical strip along the x-axis, each of the splitted path cannot intersect the strip twice.



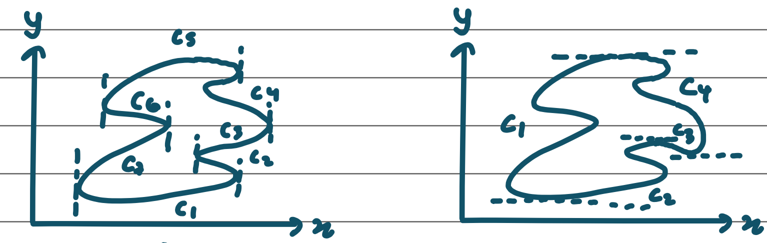
$$I = \oint_C f(x,y)\,dx = \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx$$

if $I = \oint_C f(x,y)\,dy$, have to split it such that when swiping horizontal strip along the y-axis, each of the splitted path cannot intersect the strip twice.



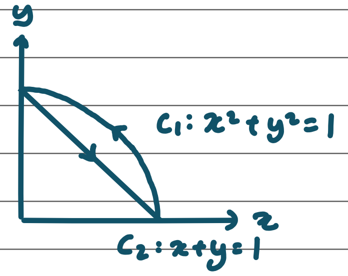
$$I = \oint_C f(x,y)\,dy = \int_{C_1} f(x,y)\,dy + \int_{C_2} f(x,y)\,dy$$

for complicated path:



$$\oint_C f(x,y)\,\underline{dx} = \sum_{i=1}^{N} \int_{C_i} f(x, g_i(x))\,dx \qquad \oint f(x,y)\,\underline{dy} = \sum_{i=1}^{N} \int_{C_i} f(h_i(y), y)\,dy$$

eg. Evaluate $\oint_c \left(\frac{x\,dy - y\,dx}{x^2+y^2+1}\right)$ where c is closed curve formed from intersection of circle $x^2+y^2=1$ and straight line $x+y=1$. (Follow anti-clockwise direction)



C_1: $x^2+y^2=1$, $r=1$; $x=r\cos\theta$; $y=r\sin\theta$; $x=\cos\theta$; $y=\sin\theta$; $dx/d\theta=-\sin\theta$; $dy/d\theta=\cos\theta$

C_2: $x+y=1$; $y=1-x$; $dy/dx=-1$; $dy=-dx$ or $dy \to \frac{dy}{dx}dx$

$$\oint_c \frac{x\,dy-y\,dx}{x^2+y^2+1}$$

choose $d\theta$ method. easier.

choose either dx or dy method. CAN do both but harder.

$$= \int_{C_1} \frac{x\,dy-y\,dx}{x^2+y^2+1} + \int_{C_2} \frac{x\,dy-y\,dx}{x^2+y^2+1}$$

$$= \int_{\theta_0=0}^{\theta_1=\pi/2} \frac{\cos\theta(\cos\theta\,d\theta)-\sin\theta(-\sin\theta\,d\theta)}{1^2+1} + \int_{x_0=0}^{x_1=1} \frac{x(-dx)-(1-x)dx}{x^2+(1-x)^2+1}$$

☆ for each integral, have to choose which method (dx, dy or $d\theta$) doesn't mean each curve can only one method, eg $\int_{C_2} f(x,y)dy - g(x,y)dx$ can be split: $\int_{C_2} f(h(y),y)dy - \int_{C_2} g(x,m(x))dx$ but will be harder

$$= \frac{1}{2}\int_{\theta_0=0}^{\theta_1=\pi/2} d\theta + \int_{x_0=0}^{x_1=1} -\frac{1}{x^2+1-2x+x^2+1}\,dx$$

$$= \frac{1}{2}[\theta]_0^{\pi/2} - \int_{x_0=0}^{x_1=1} \frac{1}{2x^2-2x+2}\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\int_{x_0=0}^{x_1=1} \frac{1}{x^2-x+1}\,dx$$

$x^2-x+1 = x^2-x+\left(-\frac{1}{2}\right)^2-\left(-\frac{1}{2}\right)^2+1$
$= \left(x-\frac{1}{2}\right)^2+\frac{3}{4}$

$$= \frac{\pi}{4} - \frac{1}{2}\int_{x_0=0}^{x_1=1} \frac{1}{(x+\frac{1}{2})^2+3/4}\,dx$$

let $u = x-\frac{1}{2}$ $\quad du = dx$
when $x_0=0$, $u=-1/2$; when $x_1=1$, $u=1/2$

$$= \frac{\pi}{4} - \frac{1}{2}\int_{u_0=-1/2}^{u_1=1/2} \frac{1}{u^2+\sqrt{\frac{3}{4}}^2}\,du$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right)\right]_{u_0=-1/2}^{u_1=1/2}$$

$$= \frac{\pi}{4} - \frac{1}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)$$

$$= \frac{\pi}{4} - \frac{\pi}{3\sqrt{3}} \quad //$$

☆ Relation of line integral to vector calculus.

- conservative field, ie $\mathrm{curl}(\underset{\sim}{F}) = 0$

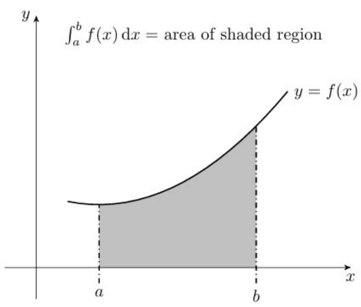
$$\downarrow$$

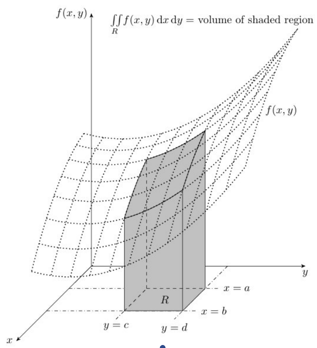
$$\int_C \underset{\sim}{F}\, d\underset{\sim}{r} = \int_C f(x)dx + f(y)dy + f(z)dz \text{ is path independent}$$

$$\downarrow$$

$$\oint_C \underset{\sim}{F}\, d\underset{\sim}{r} = 0$$

3.5 Double Integral





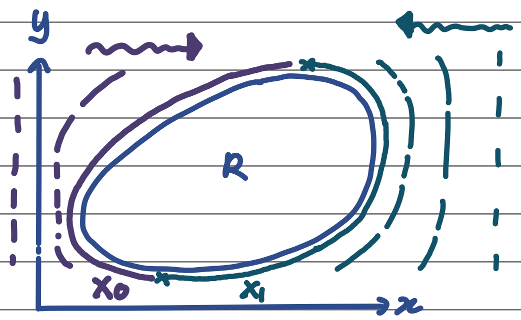
$$A = \int_a^b f(x)dx \qquad\qquad V = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$I = \iint_R f(x,y)\,\underbrace{dx\,dy}$$

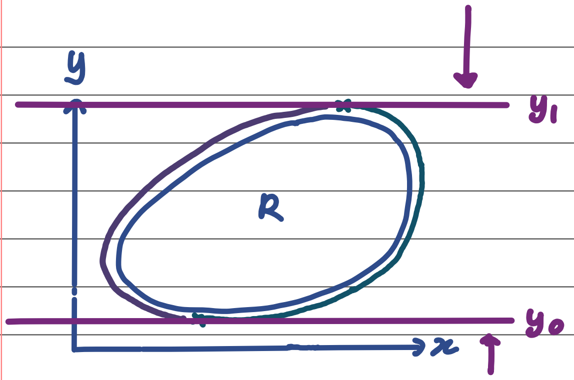
doing dx or dy first will result in the same exact solution!

How to determine x_0, x_1, y_0, y_1?

CASE 1: if $I = \iint_R f(x,y)\,dx\,dy$ (dx first then dy)



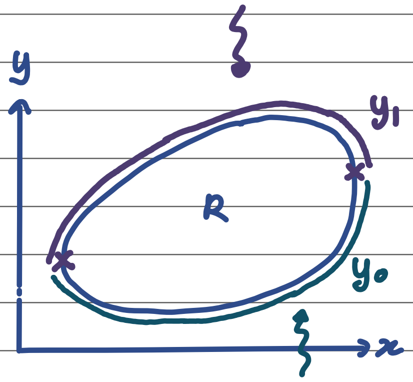
cover from LEFT and RIGHT and that is your x_0 and x_1 equation.
(imagine they are flexible and mold onto the region R)



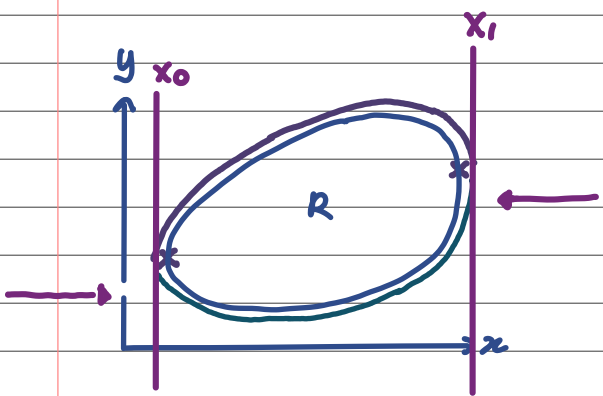
cover from **UP** and **DOWN** that is your y_0 and y_1
(imagine they are NOT flexible and cannot bend, can only touch)

☆ x_0 and x_1 can be a function of y, but y_0 and y_1 are constant only !
(for dx first then dy)

CASE 2 : if $I = \iint_R f(x,y)\, dydx$ (dy first then dx)



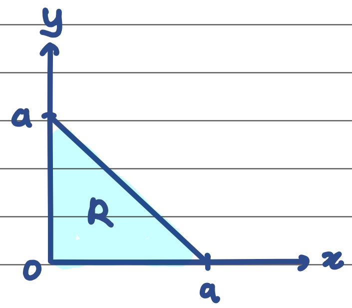
cover from UP and DOWN that is your y_0 and y_1
(imagine they are flexible and mold onto the region R)

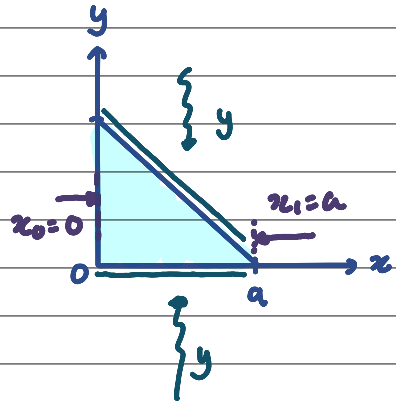


cover from **LEFT** and **RIGHT** and that is your x_0 and x_1 equation .
(imagine they are NOT flexible and cannot bend, can only touch)

☆ y_0 and y_1 can be a function of x, but x_0 and x_1 are constant only !
(for dy first then dx)

eg1. Find the area of region R, defined by the figure below





outer integral limit must be constant!

$$I=\int_{x_0=0}^{x_1=a}\left(\int_{y_0=0}^{y_1=a-x} 1\,dy\right)dx$$

$$=\int_0^a [y]_0^{a-x}\,dx$$

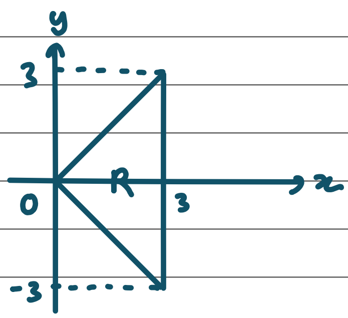
$$=\int_0^a (a-x)\,dx$$

$$=\left[ax-\tfrac{1}{2}x^2\right]_0^a$$

$$=a^2-\tfrac{1}{2}a^2$$

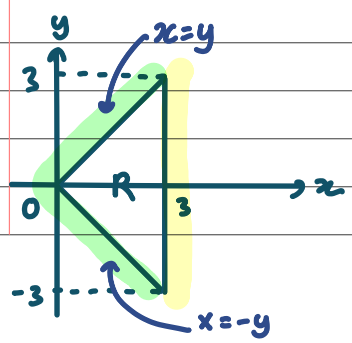
$$=\tfrac{1}{2}a^2 \text{ unit}^2$$

eg2. Find the area of region R, defined by the figure below



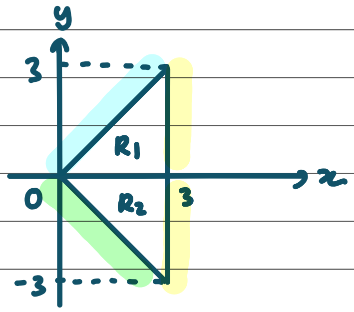
step1. choose dx or dy first.

→ IF 'dx' FIRST: ☆ Realise that lower boundary, x_0 cannot be expressed in single eqn, but only with two equation ie.

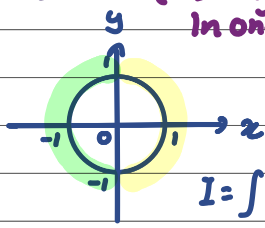


$$x_0=\begin{cases} y, & \text{when } y\geq 0 \\ -y, & \text{when } y<0 \end{cases}$$

this is equivalent to splitting into two region:



✫ Note that for double integral, it is not like line integral, swiping strips and check number of intersection: sometimes, let's say dx, if vertical strip intersect two points, still don't have to split region: (as long as it can be express in one function!)

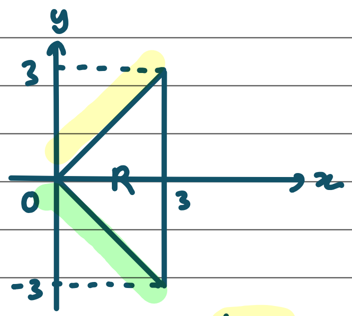


$$I = \int \left(\int_{x_0 = -\sqrt{1-y^2}}^{x_1 = +\sqrt{1-y^2}} dx \right) dy$$

$$I = \iint_{R_1} dx\,dy + \iint_{R_2} dx\,dy$$

$$= \int_{y_0=0}^{y_1=3} \left(\int_{x_0=y}^{x_1=3} dx \right) dy + \int_{y_0=-3}^{y_1=0} \left(\int_{x_0=-y}^{x_1=3} dx \right) dy$$

→ IF 'dy' FIRST:



✫ upper boundary y_1 and lower boundary y_0 can be express, each with one function.

$$I = \int_{x_0=0}^{x_1=3} \left(\int_{y_0=-x}^{y_1=x} dy \right) dx$$

3.6 Change of Variables (aka substitution)

In 2-D integration we call change of variable u-substitution :

$$\int f(x)dx = \int f(g(u))\frac{dx}{du}du$$ (remember if definite integral, have to change limit in term of u too!)

In 3-D, we have to use a pair of u and v substitution :

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|du\,dv$$

in 2-D we have to change the 'limit'
in 3-D we have to do so too! → but now instead of just 1-D limit, it's a region (2-D limit)

when solving this part, u and v usually will be that this modulus can be removed (ie everything inside either +ive or -ive)
if it doesn't, have to split region R such that it does! (not in syllabus)

$dxdy = J(u,v)\,du\,dv$

This term is also known as Jacobian:

$$J(u,v) = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|$$

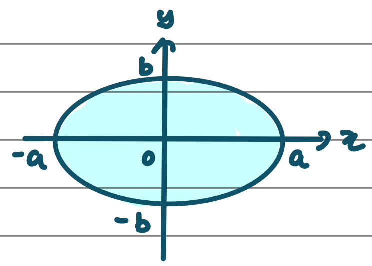
we can also use:

$$dxdy = \frac{du\,dv}{J(x,y)};\ J(x,y) = \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|$$

* when would we need this is when question gives $u=U(x,y)$, $v=V(x,y)$!

eg1. Calculate the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ using change of variable.

Step 1: Draw the integration region R.



just fyi, I in this stage is:

$$I = \int_{y_0=-b}^{y_1=b}\left(\int_{x_0=-a\sqrt{1-y^2/b^2}}^{x_1=a\sqrt{1-y^2/b^2}} dx\right)dy$$ very hard to evaluate.

Step 2: Determine the transformation (x,y) into a pair of new variable.

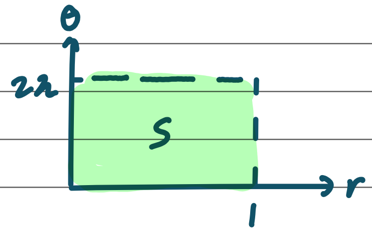
FOR CIRCLE AND ELLIPSE, TRANSFORM TO POLAR COORDS (r,θ)! (ie $x=r\cos\theta$, $y=r\sin\theta$ for circle)
$x = ar\cos\theta$, $y = br\sin\theta$ for ellipse)

$x = ar\cos\theta$, $y = br\sin\theta$

Step 3: Determine the Integration Limit. (usually it's complicated, but for polar coords, it's easy)

$0 \le r \le 1$
$0 \le \theta \le 2\pi$

step 4: Sketch the transformed region S.



step 5: Calculate Jacobian.

$x = ar\cos\theta \qquad y = br\sin\theta$

$\dfrac{\partial x}{\partial r} = a\cos\theta \qquad (-) \qquad \dfrac{\partial y}{\partial r} = b\sin\theta$

$\dfrac{\partial x}{\partial \theta} = -ar\sin\theta \qquad \dfrac{\partial y}{\partial \theta} = br\cos\theta$

$J(r,\theta) = \left| abr\cos^2\theta - (-abr\sin^2\theta) \right|$

$= |abr|$

since $0 \le r \le 1 \rightarrow J(r,\theta) = abr$

$\underbrace{0 \le abr \le 1}_{\text{always true.}}$

Step 6. Evaluate the Integral.

$I = \iint_R dx\,dy$

$= \iint_S J(r,\theta)\,dr\,d\theta$ can do dr or dθ first, in this case both are easy.

$= \int_{\theta_0=0}^{\theta_1=2\pi} \left(\int_{r_0=0}^{r_1=1} abr\,dr \right) d\theta$

$= \int_{\theta_0=0}^{\theta_1=2\pi} \left[\frac{abr^2}{2} \right]_0^1 d\theta$

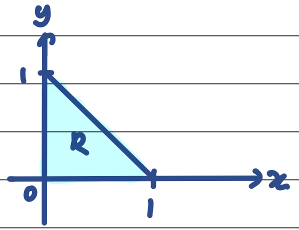
$= \frac{1}{2}ab \int_0^{2\pi} d\theta$

$= \frac{1}{2}ab \left[\theta \right]_0^{2\pi}$

$= \pi ab$ //

eg2. Evaluate $I = \iint_R (x+y)^2 \cos(x^2-y^2)\, dx\, dy$ with transformation: $u = x-y$ and $v = x+y$; R is the region enclosed by $y=0$, $x=0$ and $y = 1-x$.

Step 1: Sketch region R.



Step 2: Determine the transformation (x,y) into a pair of new variable.

$u = x-y$, $v = x+y$

Step 3: Determine the Integration Limit.

At $x=0$ (line): $0 \le y \le 1$

$\left.\begin{array}{l} u = -y \\ v = +y \end{array}\right\} u = -v$

At $y = 0$ (line): $0 \le x \le 1$

$\left.\begin{array}{l} u = x \\ v = x \end{array}\right\} u = v$

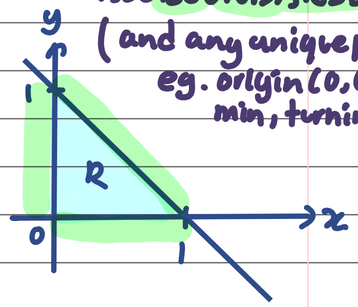
At $y = 1-x$ (line):

$\left.\begin{array}{l} \text{take } x = 0.5,\ y = 0.5 \rightarrow u = 0,\ v = 1 \\ \text{take } x = 0.2,\ y = 0.8 \rightarrow u = -0.6,\ v = 1 \end{array}\right\} v = 1$

At $C(0,0)$:

$\left.\begin{array}{l} u = 0 \\ v = 0 \end{array}\right\}$ pass through origin.

Have to evaluate at ALL BOUNDARIES (and any unique points, eg. origin (0,0), max, min, turning point)



YOU CAN ALSO:

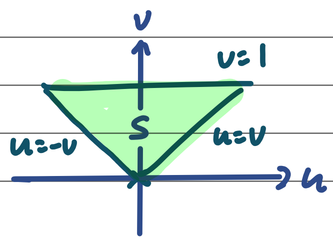
$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

sub $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \rightarrow u = -v$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \rightarrow u = v$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 1-x \end{pmatrix} \rightarrow v = 1$

Step 4: Sketch the transformed region S.



step 5: Calculate Jaccobian.

$u = x - y$; $v = x + y$

$y = x - u$ $\quad v = x + x - u$

$y = \frac{v-u}{2}$ $\quad x = \frac{v+u}{2}$

$\frac{\partial x}{\partial u} = \frac{1}{2}$ $\quad \frac{\partial y}{\partial u} = -\frac{1}{2}$

$\frac{\partial x}{\partial v} = \frac{1}{2}$ $\quad \frac{\partial y}{\partial v} = \frac{1}{2}$

$\partial(u,v) = \left| \frac{1}{4} - \left(-\frac{1}{4}\right) \right|$
$= 1/2$

Step 6. Evaluate the Integral.

$$I = \iint_R (x+y)^2 \cos(x^2-y^2)\, dx\, dy$$

$$= \iint_S v^2 \cos(uv) \frac{1}{2} du\, dv$$

notice that doing du first rather than dv is much easier
(reason: if dv first, v_0 can't be expressed as one function $\to$ split region)

$$= \int_{v_0=0}^{v_1=1} \left(\int_{u_0=-v}^{u_1=v} \frac{1}{2} v^2 \cos(uv)\, du \right) dv$$

$$= \int_0^1 \frac{1}{2} v^2 \left[\frac{1}{v} \sin(uv) \right]_{-v}^{v} dv$$

$$= \int_0^1 \frac{1}{2} v^2 \left(\frac{1}{v} \sin v^2 - \frac{1}{v} \sin(-v^2) \right) dv$$

$\sin(-A) = -\sin A$

$$= \int_0^1 v \left(\sin v^2 + \sin v^2 \right) dv$$

$$= \frac{1}{2} \int_0^1 2v \sin v^2\, dv$$

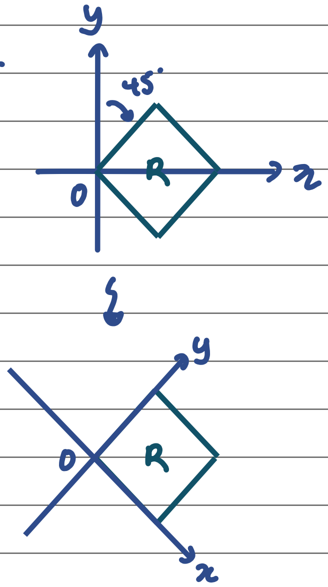
$$= \frac{1}{2} \left[-\cos v^2 \right]_0^1 = \frac{1}{4}(-\cos(1) + 1) = \frac{1-\cos(1)}{2} \quad //$$

How to figure out $u(x,y)$ and $v(x,y)$ to do change of variable (substitution) if it is not given?

$$\begin{pmatrix} u \\ v \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}$$, where M is a 2-D transformation matrix.

What we want to do is to transform our axis (x and y-axis) to align with our region S. To achieve this we multiply our axis with M.

eg.



$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$u = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$$ ($-45°$ indicates $45°$ clockwise rotation)

$$v = -\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y$$

IMPORTANT MATRIX TRANSFORMATION:

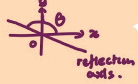
Rotation {θ is anticlockwise about the origin}

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

det = 1

Reflection {θ is measured anticlockwise from positive x-axis:}

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$



det = −1 (reverse orientation)

for this chapter, M choose from this two or a combination of both.

Enlargement

$$k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

det = k^2

Stretching

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ one way stretch parallel to x-axis with s.f. k

$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ one way stretch parallel to y-axis with s.f. k

det = k

Shear

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ shear in dir of x-axis with sf k

$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ shear in dir of y-axis with sf k

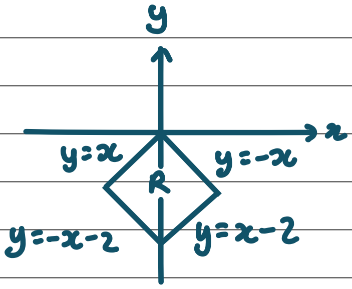
det = 1

more information on CS·2, Further Mathematics 2 Notes.

eg. By an appropriate choice of new variables evaluate the integral

$$I = \iint_R (x^2+y^2)\,dx\,dy$$

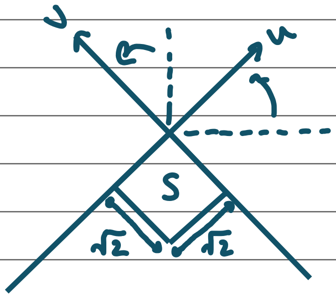
over the interior of the square bounded by $y=\pm x$ and $y=\pm(x-2)$



(anticlockwise rotation $\frac{\pi}{4}$ rad.)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} u &= \frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y \\ v &= \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y \end{aligned}\right\} \text{pair of transformation variable.}$$



from observation, we can already determine the limit of region S:

$u_0 = -\sqrt{2}$, $u_1 = 0$

$v_0 = -\sqrt{2}$, $v_1 = 0$

✿ In vector form:

$$\oint_C \underset{\sim}{F}\cdot d\underset{\sim}{r} = \pm\iint_R (\underset{\sim}{\nabla}\times\underset{\sim}{F})\cdot\underset{\sim}{k}\;dx\,dy$$

where $d\underset{\sim}{r} = dx\,\underset{\sim}{i} + dy\,\underset{\sim}{j}$

3.7 Green's Theorem.

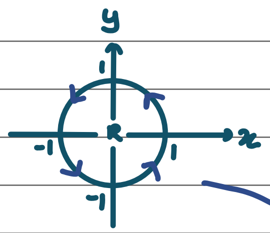
Green's Theorem main purpose is to reduce the order of integration.
(eg. triple integral to double integral, double integral to single integral)

$$\oint_C \left(P(x,y)\,dx + Q(x,y)\,dy \right) = \pm\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

choose +ive if R is on the left of c
choose -ive if R is on the right of c
} since you define your own path for the line integral, just make it such that you don't have to multiply by negative!

eg. Evaluate the Integral $I = \iint_R (x^4 + y^4)\,dx\,dy$, where $R: x^2+y^2 \le 1$
by using Green's Theorem.

step 1: Sketch the region R.



this path don't have to multiplied by (-ive)

step 2: Find Q and P

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy \qquad \iint x^4 + y^4\,dx\,dy.$$

✿ It doesn't have to be this way.
(eg. $\frac{\partial Q}{\partial x} = x^4 + y^4$, $\frac{\partial P}{\partial y} = 0$ will also work, but harder)

$$\frac{\partial Q}{\partial x} = x^4 \qquad\qquad -\frac{\partial P}{\partial y} = y^4$$

$$Q = \int x^4\,dx + \cancel{h(y)}^{\,0} \qquad\qquad P = \int -y^4\,dy + \cancel{g(x)}^{\,0}$$

$$Q = \frac{1}{5}x^5 + \cancel{h(y)}^{\,0} \qquad\qquad P = -\frac{1}{5}y^5 + \cancel{g(x)}^{\,0}$$

} have to ignore h(y) and g(x) even when integrating partial, or else it is not-solvable.

$$\therefore \iint_R (x^4+y^4)\,dx\,dy = \oint_C \left(-\frac{1}{5}y^5\,dx + \frac{1}{5}x^5\,dy \right)$$

(after this is all just basic line integral) but closed loop

$$\iint_R (x^4+y^4)\,dx\,dy = \oint_C \left(-\frac{1}{5}y^5\,dx + \frac{1}{5}x^5\,dy\right)$$

$$= \oint -\frac{1}{5}y^5\,dx + \oint_C \frac{1}{5}x^5\,dy$$

let $x = r\cos\theta$, $y = r\sin\theta$ $(r=1)$

$x = \cos\theta$, $y = \sin\theta$

$\frac{dx}{d\theta} = -\sin\theta$, $\frac{dy}{d\theta} = \cos\theta$

$\left(dx = \frac{dx}{d\theta}d\theta,\ dy = \frac{dy}{d\theta}d\theta\right)$

choose $d\theta$ method as circle question

$$= \oint_C -\frac{1}{5}\sin^5\theta(-\sin\theta\,d\theta) + \oint_C \frac{1}{5}\cos^5\theta(\cos\theta\,d\theta)$$

don't need to split the closed curve cause one loop θ can inc. all

$$= \frac{1}{5}\int_{\theta_0=-\pi}^{\theta_1=\pi} (\sin^6\theta + \cos^6\theta)\,d\theta$$

use reduction formula.

$$= \frac{1}{5}\left(\frac{5\pi}{4}\right)$$

$$= \pi/4 \quad //$$

C4 Ordinary Differential Equation II

4.1 Method of Reduction of Order. (linear ODE, non-constant coeff.)

- reduces the order of n^{th} order ODE to $(n-1)^{th}$ order. (eg 3^{rd} order ODE to 2^{nd} order ODE)
- suppose we have an ODE in this form:

$$\frac{d^2y}{dx^2} + r(x)\frac{dy}{dx} + s(x)y = q(x)$$

step 1: guess **ONE** of the solution to the homogeneous equation (let us call the solution y_0)
* not y_c as it is only part of the complete complimentary function.

step 2: let $y = u(x)y_0(x) = uy_0$ (when find $\frac{dy}{dx}$, remember to do chain rule as u and y_0 are function of x!)

step 3: substitute into the complete (non-homogeneous) ODE (we are straightaway solving y_p, not y_0!)

Clue and Examples	What to Substitute?	Why?
$x^2y'' + axy' + by$ eg. $x^2y'' + 4xy' - 2y = 0$	$y_0 = x^r$	Euler-Cauchy form (will turn into constant coeff. after substitution)
contains $\sqrt{1-x^2}$ or $(1-x^2)$	$y_0 = \sin x$ or $y_0 = \cos x$	$\sin^2(t) = 1 - \cos^2(t)$ $\cos^2(t) = 1 - \sin^2(t)$
contains $\sqrt{1+x^2}$ or $(1+x^2)$	$y_0 = \sinh x$ or $y_0 = \cot x$	$\cosh^2(t) = 1 + \sinh^2(t)$ $\csc^2(t) = 1 + \cot^2(t)$
contains $\sqrt{x^2-1}$ or (x^2-1)	$y_0 = \cosh x$ or $y_0 = \sec x$	$\sinh^2(t) = \cosh^2(t) - 1$ $\tan^2(t) = \sec^2(t) - 1$

eg1 Show that $y = x^2$ is a solution of the homogenous differential equation

$y_0 = x^2$!

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 0.$$

Hence find the general solution to the non-homogenous equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{4}{x^2}y = 1,$$

by seeking a solution of the form $y = x^2u(x)$ and deriving and solving the ODE for u.

Step 1: Find $y_0(x)$

$y_0 = x^2$

$y_0' = 2x$

$y_0'' = 2$

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 2 + \frac{1}{x}(2x) - \frac{4}{x^2}(x^2)$$

$$= 2 + 2 - 4$$

$$= 0 \quad \text{(shown)}$$

Step 2: Set $y = y_0(x)u(x)$

let $y = y_0(x)u(x)$

$y = x^2u$

$y' = x^2u' + 2xu$

$y'' = x^2u'' + 2xu' + 2xu' + 2u$

$= x^2u'' + 4xu' + 2u$

Step 3: Substitute back to the ODE

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 1$$

$$x^2u'' + 4xu' + 2u + \frac{1}{x}(x^2u' + 2xu) - \frac{4}{x^2}(x^2u) = 1$$

$$x^2u'' + 4xu' + \cancel{2u} + xu' + \cancel{2u} - \cancel{4u} = 1$$

$$x^2u'' + 5xu' = 1$$

$$x^2\frac{d^2u}{dx^2} + 5x\frac{du}{dx} = 1$$

Step 4 : Reduce the order of the ODE (we can do this because there is no u, the ODE starts from dy/dx)

let $\alpha = \frac{du}{dx}$,

$\therefore \frac{d\alpha}{dx} = \frac{d^2u}{dx^2}$

"reduced ODE"

$$x^2 \frac{d\alpha}{dx} + 5x\alpha = 1$$

$$\frac{d\alpha}{dx} + \frac{5}{x}\alpha = \frac{1}{x^2} \qquad \frac{dy}{dx} + p(x)y = q(x)$$

$$I.F. = e^{\int \frac{5}{x}dx} = e^{5\ln x} = x^5$$

$$x^5 \frac{d\alpha}{dx} + 5x^4\alpha = x^3$$

$$\frac{d}{dx}(x^5\alpha) = x^3$$

4.2 Method of Substitution

Step 1 : substitute either of the independent variable, x or dependent variable, y into a new function, u.

Step 2 : Differentiate and substitute.

Step 3 : Solve.

so... what to substitute ?

(check next page)

eg. if $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

and let $u = e^x$

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right) \times \frac{du}{dx} \end{aligned}\right\} \text{☆ very important!}$$

☆ IMPORTANT

" " means preferable

Clue and Examples	What to Substitute?	Why?
$x^2y''+axy'+by$ eg. $x^2y''+4xy'-2y=0$	$x=e^t$	Euler-Cauchy form (will turn into constant coeff. after substitution)
coeff. with $\ln(x)$ eg. $\ln(x)y''+y'-2y=0$	$x=e^t$	$\ln(x)=\ln(e^t)=t$ (simplified)
coeff. with $\frac{1}{x}$ eg. $\frac{1}{x}y''+\dots$	$x=e^t$ or $x=\frac{1}{t}$ (we use $x=e^t$ if → Euler-C)	doesn't always work
coeff. with a^x eg. $e^xy''+\dots$	$x=\log_a t$ eg. $x=\ln t$	$a^{\log_a t}=t$ (simplified)
coeff. with $\sqrt{x}$ or $\sqrt[n]{x}$ eg. $\sqrt[3]{x}y''+\dots$	$x=t^2$ or $x=t^n$ eg. $x=t^3$	$\sqrt[n]{x}=\sqrt[n]{t^n}=t$ (simplified)
coeff. with $\sqrt{1-x^2}$	$x=\sin(t)$ or $x=\cos(t)$	$\sin^2(t)=1-\cos^2(t)$ $\cos^2(t)=1-\sin^2(t)$
coeff. with $\sqrt{1+x^2}$	$x=\sinh(t)$ or $x=\cot(t)$	$\cosh^2(t)=1+\sinh^2(t)$ $\csc^2(t)=1+\cot^2(t)$
coeff. with $\sqrt{x^2-1}$	$x=\cosh(t)$ or $x=\sec(t)$	$\sinh^2(t)=\cosh^2(t)-1$ $\tan^2(t)=\sec^2(t)-1$
even power of x eg. $x^2y''+x^4y'+y=0$ x^0 eg. $y''+\cos(x^2)y'+x^{-2}y=0$ eg. $y''+y'+Cy=0$ eg. $4xy''+2y'+y=0$	$x=t^2$	It doesn't always work It sometimes work even if there is only a few even power (sometimes no even power will work also)

The idea behind finding what to substitute is actually straight-forward...

→ find the "most-complicated" coefficient (or usually the highest order one...)

→ simplify it

→ eg: $\sqrt{2-x^2}\,y''+y'-2y=0$

$\sqrt{2-x^2} \to \sqrt{2}\cos t$

let $x=\sqrt{2}\sin t$

4.3 Solving system of ODEs

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n$$

can be rewritten as: $\frac{dx}{dt} = Ax$

solution:

$$\underset{\sim}{x} = C_1 e^{\lambda_1 t}\underset{\sim}{e_1} + C_2 e^{\lambda_2 t}\underset{\sim}{e_2} + \dots + C_n e^{\lambda_n t}\underset{\sim}{e_n}$$

(note that $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$; $\underset{\sim}{e_1}, \underset{\sim}{e_2}, \underset{\sim}{e_n}$ are eigenvector of A; $\lambda_1, \lambda_2, \lambda_n$ are eigenvalue of A)

eg.
$$\frac{dx}{dt} = 2x + z$$
$$\frac{dy}{dt} = 4y$$
$$\frac{dz}{dt} = 5x + 2y + 6z$$

$$\frac{d\underset{\sim}{r}}{dt} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 5 & 2 & 6 \end{pmatrix}\underset{\sim}{r} \quad A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ 5 & 2 & 6 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 2-\lambda & 0 & 1 \\ 0 & 4-\lambda & 0 \\ 5 & 2 & 6-\lambda \end{pmatrix} = 0$$

$$\begin{pmatrix} 5\lambda - 20 \\ -4 + 2\lambda \\ 8 - 6\lambda + \lambda^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 6-\lambda \end{pmatrix} = 0$$

$$5\lambda - 20 + (48 - 36\lambda + 6\lambda^2 - 8\lambda + 6\lambda^2 - \lambda^3) = 0$$

$$-\lambda^3 + 12\lambda^2 - 39\lambda + 28 = 0$$

$$(\lambda - 1)(\lambda - 7)(\lambda - 4) = 0$$

$\lambda = 1$:

$$(A - \lambda I)\underset{\sim}{e_1} = 0$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 5 & 2 & 5 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ \cancel{0} & \cancel{2} & \cancel{0} & 0 \end{array}\right) \quad 5R_1 - R_3$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$3y = 0$ let $z = \mu \in \mathbb{R}$

$y = 0$ $x + z = 0$

$z = -\mu$

$$\underset{\sim}{e_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\lambda_2 = 4$:

$$(A - \lambda I)\underset{\sim}{e_2} = 0$$

$$5R_1 + 2R_3 \sim \left(\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 2 & 2 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 2 & 2 & 0 \end{array}\right)$$

$9z = 0$ let $x = \mu \in \mathbb{R}$

$z = 0$ $5\mu + 2y = 0$

$y = -5/2\,\mu$

$$\underset{\sim}{e_2} = \begin{pmatrix} 1 \\ -5/2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$$

$\lambda_3 = 7$:

$$(A - \lambda I)\underset{\sim}{e_3} = 0$$

$$\left(\begin{array}{ccc|c} -5 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 5 & 2 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -5 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ \cancel{0} & \cancel{2} & \cancel{0} & 0 \end{array}\right)$$

$-3y = 0$ let $x = \mu \in \mathbb{R}$:

$y = 0$ $-5\mu + z = 0$

$z = 5\mu$

$$\underset{\sim}{e_3} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = C_1 e^{t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} + C_3 e^{7t}\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}}$$

4.4 Reducing n^{th} order ODE

$$a\frac{d^n y}{dx^n}+b\frac{d^{(n-1)}y}{dx^{(n-1)}}+\dots+p\frac{d^2y}{dx^2}+q\frac{dy}{dx}+sy=0$$

let $u_1=y$ $u_1'=u_2$
$u_2=y'$ $u_2'=u_3$
$u_3=y''$ $u_3'=u_4$
$\vdots$ $\vdots$
$u_n=y^{(n-1)}$ $u_{n-1}'=u_n$
$u_n'=y^{(n)}=-\left(\frac{b}{a}u_n+\dots+\frac{p}{a}u_3+\frac{r}{a}u_2+\frac{s}{a}u_1\right)$

solve these with 4.3 system of ODEs method!
→ the result you get will be $\begin{pmatrix}u_1\\u_2\\\vdots\\u_n\end{pmatrix}\sim\begin{pmatrix}y\\y'\\\vdots\\y^{(n-1)}\end{pmatrix}$

eg. $2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+y=0$

step 1: write as system of linear ODEs $\left(\frac{dx}{dt}=Ax\right)$

let $u_1=y$ $u_1'=u_2$
$u_2=y'$ $u_2'=y''=\frac{5}{2}y'-\frac{1}{2}y=\frac{5}{2}u_2-\frac{1}{2}u_1$

$$\begin{pmatrix}u_1'\\u_2'\end{pmatrix}=\begin{pmatrix}0&1\\-\frac{1}{2}&\frac{5}{2}\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}\leftarrow\frac{dx}{dt}=Ax\quad(\text{sol.: }x=C_1e^{\lambda_1 t}\underset{\sim}{e_1}+C_2e^{\lambda_2 t}\underset{\sim}{e_2}+\dots)$$

step 2: solve the system of linear ODEs (find λ and $\underset{\sim}{e}$ of A)

$\det(A-\lambda I)=0$

$\det\begin{pmatrix}-\lambda&1\\-\frac{1}{2}&\frac{5}{2}-\lambda\end{pmatrix}=0$

$\lambda^2-\frac{5}{2}\lambda+\frac{1}{2}=0$

$2\lambda^2-5\lambda+1=0$

$\lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{5\pm\sqrt{17}}{4}$

when $\lambda=\frac{5+\sqrt{17}}{4}$,

$(A-\lambda I)\underset{\sim}{e}=\underset{\sim}{0}$

$$\sim\left(\begin{array}{cc|c}-\frac{5+\sqrt{17}}{4}&1&0\\-\frac{1}{2}&\frac{5-\sqrt{17}}{4}&0\end{array}\right)$$

$$\underset{R_2=\frac{1}{2}R_1-\frac{5+\sqrt{17}}{4}R_2}{\sim}\left(\begin{array}{cc|c}-\frac{5+\sqrt{17}}{4}&1&0\\0&0&0\end{array}\right)$$

$-\frac{5+\sqrt{17}}{4}x+y=0$

let $y=\mu\in\mathbb{R}$

$x=\frac{4}{5+\sqrt{17}}\mu\rightarrow\underset{\sim}{e_1}=\begin{pmatrix}\frac{4}{5+\sqrt{17}}\\1\end{pmatrix}\begin{matrix}\times\frac{5+\sqrt{17}}{4}\\\times\frac{5+\sqrt{17}}{4}\end{matrix}\rightarrow\underset{\sim}{e_1}=\begin{pmatrix}1\\\frac{5+\sqrt{17}}{4}\end{pmatrix}$

when $\lambda=\frac{5-\sqrt{17}}{4}$,

same steps

$\underset{\sim}{e_2}=\begin{pmatrix}1\\\frac{5-\sqrt{17}}{4}\end{pmatrix}$

step 3: Found x from $\frac{dx}{dt}=Ax$:

$$\begin{pmatrix}u_1\\u_2\end{pmatrix}=C_1e^{\frac{5+\sqrt{17}}{4}x}\begin{pmatrix}1\\\frac{5+\sqrt{17}}{4}\end{pmatrix}+C_2e^{\frac{5-\sqrt{17}}{4}x}\begin{pmatrix}1\\\frac{5-\sqrt{17}}{4}\end{pmatrix}$$

$$u_1=y=C_1e^{\frac{5+\sqrt{17}}{4}x}+C_2e^{\frac{5-\sqrt{17}}{4}x}$$

C5 Fourier Series

5·1 Intro and Preliminary.

1. periodic function: $f(x+T)=f(x)$

   eg. $\sin x = \sin(x+2\pi)$, $T=2\pi$

   eg2. $f(x)=\underbrace{\cos x}_{\frac{2\pi}{T}=1,\ T=2\pi}+\underbrace{\sin 2x}_{\frac{2\pi}{T}=2,\ T=\pi}+\underbrace{\cos 5x}_{\frac{2\pi}{T}=5,\ T=0.4\pi}$

   $T=\text{LCM}(2\pi,\pi,0.4\pi)=2\pi$!

2. even function: even about $(x=a)$ if $f(x+a)=f(-x+a)$

   eg. $f(x)=(x+2)^2$ is even about $x=-2$ cause:

   $f(x+a)=f(-x+a)$

   $(x+a+2)^2=(-x+a+2)^2$

   $a=-2$

3. odd function: odd about $(x=a)$ if $f(x+a)=-f(-x+a)$

odd × even = odd

even × even = even

odd × odd = even

5·2 Standard Integral Result ☆ MEMORISE THIS!

1. $\int_{-\pi}^{\pi}\cos(mx)\,dx=0$; $m\in\mathbb{Z}^+$

both of these limit from $-L$ to L will work as well

2. $\int_{-\pi}^{\pi}\sin(mx)\,dx=0$; $m\in\mathbb{Z}^+$

3. $\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx=0$; $m,n\in\mathbb{Z}^+$

4. $\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx=\begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}$

5. $\int_{-\pi}^{\pi}\sin(nx)\sin(mx)\,dx=\begin{cases}0 & \text{for } m\neq n\\ \pi & \text{for } m=n\end{cases}$

last two if $\int_{-\pi}^{0}$ or $\int_{0}^{\pi}$ for $m=n$, integral will result in $\pi/2$ instead of π !

5.3 Full Range Fourier Series.

☆ MEMORISE THIS!

Basic Fourier Series	General Range	More General Range
$-\pi \le x \le \pi$	$-L \le x \le L$	$a \le x \le a+2L$ alt: $a-L \le x \le a+L$
$T = 2\pi$	$T = 2L$	$T = 2L$
$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(nx) + B_n\sin(nx)\}$	$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})\}$	$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})\}$
A_0 just put $n=0$! $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ $T/2$ $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$	$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$ $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx$	$A_n = \frac{1}{L}\int_{a}^{a+2L} f(x)\cos(\frac{n\pi x}{L})\,dx$ $B_n = \frac{1}{L}\int_{a}^{a+2L} f(x)\sin(\frac{n\pi x}{L})\,dx$

set $L=\pi$ set $a=-L$

if f is odd: $o \times e = o$ and $\int odd = 0$!

$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ even

if f is even: $e \times o = o$ $\int odd = 0$!

$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$ odd

why?

$A_0 = 0, A_n = 0$ if $f(x)$ is odd

$B_n = 0$ if $f(x)$ is even

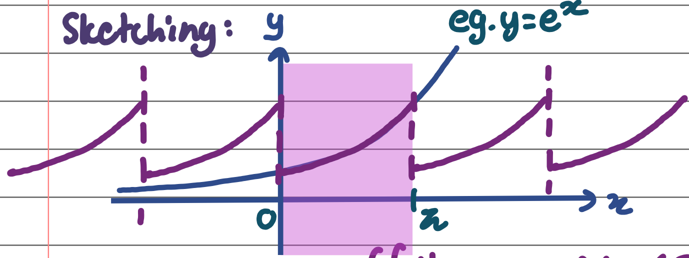
☆ about midpoint!
→ $x=0$ for $-\pi \le x \le \pi$ and $-L \le x \le L$
→ $x = a+L$ for $a \le x \le a+2L$

Parseval's Theorem:

for $f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})\}$ for $a \le x \le a+2L$

then $\frac{1}{L}\int_{a}^{a+2L}(f(x))^2 = \frac{1}{2}A_0^2 + \sum_{n=1}^{\infty}(A_n^2 + B_n^2)$

Sketching:



(full range, $0 \le x \le \pi$, $n \to \infty$)

☆ For full range, it just repeat at boundaries!

5.4 Half Range Fourier Series

save time by not evaluating either A_n or B_n! even if $f(x)$ is not even or odd!

Range are halved, but period of the fourier $f(x)$ still the same!

Full Range	Half Range Cosine	Half Range Sine
$-L \le x \le L$, $T = 2L$	$0 \le x \le L$, $T = 2L$	$0 \le x \le L$
$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left\{A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right\}$	$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$	$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$
$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$A_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$B_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Parseval's Theorem:

for $f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$; $0 \le x \le L$

$$\frac{2}{L}\int_0^L (f(x))^2 dx = \frac{1}{2}A_0^2 + \sum_{n=1}^{\infty} A_n^2$$

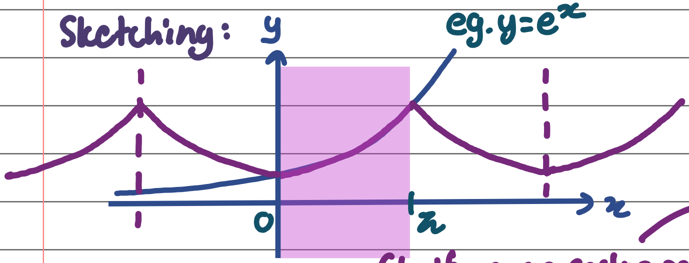
half-range fourier cosine series

for $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$; $0 \le x \le L$

$$\frac{2}{L}\int_0^L (f(x))^2 dx = \sum_{n=1}^{\infty} B_n^2$$

half-range fourier sine series

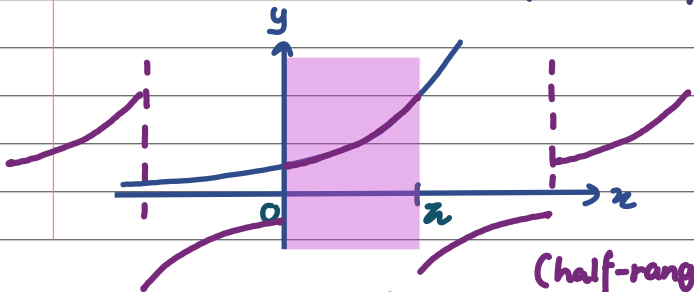
Sketching:



cosine is even function, so should be even at $x=0$!

(half-range cosine series, $0 \le x \le \pi$, $n \to \infty$)

☆ For half range, complete the other half first! $-\pi \le x \le 0$



sine is odd function, so should be odd at $x=0$!

(half-range sine series, $0 \le x \le \pi$, $n \to \infty$)

☆ For half range, complete the other half first! $-\pi \le x \le 0$

☆ if we try to find, let's say from 0 to π ($a=0$, $a+2L=\pi$), we can't $A_0 = A_n = 0$ or $B_n = 0$ cause $f(x) = \sinh(x)$ is not even or odd about $x = \frac{\pi}{2}$!

Examples

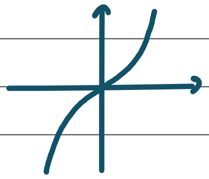
Basic Fourier Series	General Range	More General Range
$-\pi \le x \le \pi$	$-L \le x \le L$	$a \le x \le a+2L$ alt: $a-L \le x \le a+L$
$T = 2\pi$	$T = 2L$	$T = 2L$
$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(nx) + B_n\sin(nx)\}$	$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})\}$	$f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\{A_n\cos(\frac{n\pi x}{L}) + B_n\sin(\frac{n\pi x}{L})\}$
$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ $T/2$ $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$	$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx$ $B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx$	$A_n = \frac{1}{L}\int_{a}^{a+2L} f(x)\cos(\frac{n\pi x}{L})\,dx$ $B_n = \frac{1}{L}\int_{a}^{a+2L} f(x)\sin(\frac{n\pi x}{L})\,dx$

set $L = \pi$ ← ← set $a = -L$

$A_0 = 0$, $A_n = 0$ if $f(x)$ is odd
$B_n = 0$ if $f(x)$ is even

eg. $f(x) = \sinh(x)$

(a) Full Range from $-\pi$ to π (mid: $x=0$)



$\therefore \sinh(x)$ is odd about $x=0$!
$\rightarrow A_0 = 0;\ A_n = 0$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\sinh(x)\sin(nx)\,dx$$

$u = \sin(nx)$ $\quad dv = \sinh(x)\,dx$
$du = n\cos(nx)\,dx$ $\quad v = \cosh(x)$

$$= \frac{1}{\pi}\left\{\left[\cancel{\sin(nx)}^{\,0}\cosh(x)\right]_{-\pi}^{\pi} - n\int_{-\pi}^{\pi}\cos(nx)\cosh(x)\,dx\right\}$$

$u = \cos(nx)$ $\quad dv = \cosh(x)\,dx$
$du = -n\sin(nx)\,dx$ $\quad v = \sinh(x)$

$$= -\frac{n}{\pi}\left\{\left[\cos(nx)\sinh(x)\right]_{-\pi}^{\pi} + n\int_{-\pi}^{\pi}\sin(nx)\sinh(x)\,dx\right\}$$

$$= -\frac{n}{\pi}\left[\cos(n\pi)\sinh\pi + \cos(n\pi)\sinh\pi\right] - \frac{n^2}{\pi}\int_{-\pi}^{\pi}\sin(nx)\sinh(x)\,dx$$

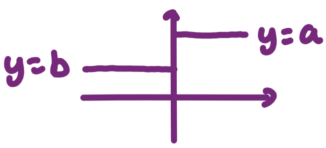
$$\frac{1}{\pi}(1+n^2)\int_{-\pi}^{\pi}\sin(nx)\sinh(x)\,dx = -\frac{2n}{\pi}\cos(n\pi)\sinh\pi$$

$$B_n = -\frac{2n}{\pi(1+n^2)}(-1)^n\sinh\pi$$

$$= \frac{2n}{\pi(1+n^2)}(-1)^{n+1}\sinh\pi$$

$$\therefore f(x) = \sum_{n=1}^{\infty}\frac{2n}{\pi(1+n^2)}(-1)^{n+1}\sinh\pi\,\sin(nx);\quad -\pi \le x \le \pi$$

☆ If there is discontinuity in $f(x)$:



fourier representation, will take the **MEAN** of both sides at discontinuity!
eg. fourier representation at $x=0$ in this case $= \frac{a+b}{2}$!

c6 Partial Differential Equation.

6.1 Introduction.

$\frac{\partial^n y}{\partial x^n} + \frac{\partial^{n-1} y}{\partial x^{n-1}} + y = \sin x$ (this is n^{th} order, we look at highest order)

$xy\frac{\partial^2 u}{\partial x} + 2x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = \sinh(x+y)$ (this is linear)

eg. 2^{nd} order linear general form:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$$

→ notice that non of the 'differential' is multiplied by 'u'

$xyu\frac{\partial u}{\partial x} = x$ (this is non-linear)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (this is homogeneous)

6.2 Classification of PDE (only for 2^{nd} order linear PDE!)

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y)$$

$B^2 - 4AC < 0$ → the PDE is classified as an elliptic PDE
$B^2 - 4AC = 0$ → the PDE is classified as an parabolic PDE
$B^2 - 4AC > 0$ → the PDE is classified as an hyperbolic PDE

IF A, B, and C are function of x and y and are not constant, the PDE can change class based on the current value of x and y!

6.3 Initial and Boundary Conditions

Initial and Boundary conditions are actually the same thing, just to solve PDE.
Initial Condition → describe times
Boundary Condition → describe spaces

eg. $\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$ (heat equation)

require 1 initial condition

require 2 boundary conditions

$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (wave equation)

require 2 initial conditions

require 2 boundary conditions

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace Equation)

require 4 boundary conditions (2 in x and 2 in y)

☆ THIS IS <u>NOT</u> ALWAYS THE PATTERN!
initial condition (time based) and boundary condition (space based) required is NOT always equal to its respective highest order derivative.
→ only if time and space are well seperated (no t and x mixed derivative)
→ less than two order.

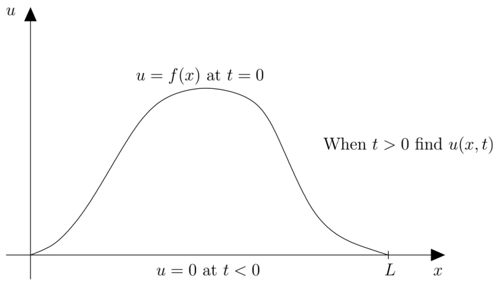
6.4 Seperation of Variable. (Method 1)

Methodology:

Assume the solution is in the form of:

$u(x,y) = X(x)Y(y)$ (basically a function that can seperate both of their variables)

Example 1: An elastic string of length L is placed along the x-axis. The string is clamped firmly in place at either end of the string and at time $t=0$, it is stretched into a shape $f(x)$ and then released from rest. Find the subsequent displacement of the string.



Step 1. Define our PDE.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(requires 2 initial (time) and 2 boundary (space) conditions)

Step 2. Check / Define our Initial and Boundary Conditions.

BOUNDARY CONDITION MUST ALL BE HOMOGENEOUS! (except 1 can be non-homogeneous) if more than 1: DO SUPERPOSITION!

$t=0$: $u(x,0) = f(x)$ for $x \in [0,1]$ IC1

$\frac{\partial}{\partial t} u(x,0) = 0$ IC2

} Initial Condition.

$u=0$: $u(0,t) = 0$ B0

$u(L,t) = 0$ BL

} Boundary condition.

Step 3. Assume solution as $u(x,t) = X(x)T(t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)\frac{d^2 T(t)}{dt^2}$$

(since T is only a function of t, $\frac{\partial^2 T}{\partial t^2}$ can be written as $\frac{d^2 T}{dt^2}$)

$$= X(x)T''(t)$$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$X(x)T''(t) = c^2 X''(x)T(t)$$

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}$$

HOW CAN DIFFERENTIAL OF 't' EQUAL TO 'x'?

→ THE RESULTS OF EACH SIDE WOULD BE CONSTANT!

$$\boxed{\frac{T''(t)}{T(t)} = c^2\frac{X''(x)}{X(x)} = \alpha}$$

☆ IF CAN'T WRITE IN THIS FORM TRY ANOTHER METHOD!

Step 4. Try to solve for $X(x)$ and $T(t)$ for different case of α.

let $\alpha = 0$

$\frac{T''(t)}{T(t)} = 0$ $\qquad$ $\frac{X''(x)}{X(x)} = 0$

$T''(t) = 0$ $\qquad$ $X''(x) = 0$

$T(t) = \iint T''(t)dt$ $\qquad$ $X(x) = \iint X''(x)dx$

$= \int A\,dt$ $\qquad$ $= \int C\,dx$

$= At + B$ $\qquad$ $= Cx + D$

f(x) is not defined use other path...

→ Apply B0: $X(0) = 0$

$C(0) + D = 0$

$D = 0$

? why $X(0)=0$?

B0: $u(0,t) = 0$

$X(0)Y(t) = 0$

$X(0) = 0/Y(t) = 0$!

→ Apply BL: $X(L) = 0$

$C(L) + 0 = 0$

$CL = 0$, since $L \neq 0 \rightarrow C = 0$

? why $X(L)=0$?

BL: $u(L,t) = 0$

$X(L)Y(t) = 0 \rightarrow X(L) = 0$

$\therefore X(x) = 0$ (NULL SOL!)

☆ when one of the seperated function return null solution, the whole thing is null solution,
→ continue with other analysis!

let $\alpha > 0$

$c^2\frac{X''(x)}{X(x)} = \alpha$

$c^2X''(x) = \alpha X(x)$

$X''(x) - \frac{\alpha}{c^2}X(x) = 0$

$\lambda^2 - \frac{\alpha}{c^2} = 0$

$\lambda = \pm\sqrt{\frac{\alpha}{c^2}} = \pm\frac{\sqrt{\alpha}}{c}$

2 if write $X(x) = A\cosh(\frac{\sqrt{\alpha}}{c}x) + B\sinh(\frac{\sqrt{\alpha}}{c}x)$
(hyperbolic form)

$x = 0$:

$X(0) = A\cosh(0) + \cancel{B\sinh(0)}^{\,0}$

$u(0,t) = 0$

$X(0)T(t) = 0$

$X(0) = 0$

$0 = A(1)$

$A = 0$ → $A=0, B=0$ (null sol.!)

$x = L$

$X(L) = \cancel{A\cosh(\frac{\sqrt{\alpha}}{c}L)}^{\,0} + B\sinh(\frac{\sqrt{\alpha}}{c}L)$

$u(L,t) = 0$

$X(L)T(t) = 0$

$X(L) = 0$

$0 = B\underbrace{\sinh(\frac{\sqrt{\alpha}}{c}L)}_{\alpha>0,\text{ means }\alpha\neq 0 \rightarrow B=0!}$

1 $X(x) = Ae^{\frac{\sqrt{\alpha}}{c}x} + Be^{-\frac{\sqrt{\alpha}}{c}x}$

(exponential form)

→ Apply B0: $X(0) = 0$

$A + B = 0 \rightarrow B = -A$

→ Apply BL: $X(L) = 0$

$Ae^{\frac{\sqrt{\alpha}}{c}L} + Be^{-\frac{\sqrt{\alpha}}{c}L} = 0$

$Ae^{\frac{\sqrt{\alpha}}{c}L} - Ae^{-\frac{\sqrt{\alpha}}{c}L} = 0$

$A(e^{\frac{\sqrt{\alpha}}{c}L} - e^{-\frac{\sqrt{\alpha}}{c}L}) = 0$

if $\frac{\sqrt{\alpha}}{c}L \neq 0 \rightarrow e^{\frac{\sqrt{\alpha}}{c}L} \neq e^{-\frac{\sqrt{\alpha}}{c}L}$

$e^{\frac{\sqrt{\alpha}}{c}L} - e^{-\frac{\sqrt{\alpha}}{c}L} \neq 0$

$\rightarrow \underline{A = 0 \rightarrow B = 0}$

NULL SOL!

why?
cause we are doing $\alpha > 0$! so $\alpha \neq 0 \rightarrow \frac{\sqrt{\alpha}}{c}L \neq 0$!

let $\alpha < 0$

$$c^2 \frac{X''(x)}{X(x)} = \alpha$$

$$c^2 X''(x) = \alpha X(x)$$

$$X''(x) - \frac{\alpha}{c^2} X(x) = 0$$

$$\lambda^2 - \frac{\alpha}{c^2} = 0$$

α is negative! taking out sqrt (solving it) will introduce complex numbers!

$$\lambda = \pm\sqrt{\frac{\alpha}{c^2}} = \pm \frac{\sqrt{-\alpha}}{c} i$$

$$X(x) = e^{0x}\left[A\cos\left(\frac{\sqrt{-\alpha}}{c}x\right) + B\sin\left(\frac{\sqrt{-\alpha}}{c}x\right)\right]$$

$$= A\cos\left(\frac{\sqrt{-\alpha}}{c}x\right) + B\sin\left(\frac{\sqrt{-\alpha}}{c}x\right)$$

↳ Apply B0: $X(0) = 0$

$$A\cos(0) + B\sin(0) = 0$$

$$A = 0$$

↳ Apply BL: $X(L) = 0$

$$0\cos\left(\frac{\sqrt{-\alpha}}{c}L\right) + B\sin\left(\frac{\sqrt{-\alpha}}{c}L\right) = 0$$

$$\sin\left(\frac{\sqrt{-\alpha}}{c}L\right) = 0$$

$$R.A. = \sin^{-1}(0) = 0$$

$$\frac{\sqrt{-\alpha}}{c}L = n\pi$$

$$\frac{\sqrt{-\alpha}}{c} = \frac{n\pi}{L}$$

$$\therefore X(x) = B\sin\left(\frac{n\pi}{L}x\right)$$

not a null solution
→ continue with solving $T(t)$!

$$\frac{T''(t)}{T(t)} = \alpha$$

$$T''(t) - \alpha T(t) = 0$$

$$\lambda^2 - \alpha = 0$$

$$\lambda = \pm\sqrt{\alpha} = \pm\sqrt{-\alpha}\, i$$

$$T(t) = C\cos(\sqrt{-\alpha}\, t) + D\sin(\sqrt{-\alpha}\, t)$$

↳ Apply IC2: $\frac{\partial}{\partial t}u(x,0) = 0$

$$T'(t) = -\sqrt{-\alpha}\, C\sin(\sqrt{-\alpha}\, t) + \sqrt{-\alpha}\, D\cos(\sqrt{-\alpha}\, t)$$

$$0 = \sqrt{-\alpha}\, D(1)$$

$$D = 0$$

$$u(x,t) = X(x)T(t)$$

$$\frac{\partial}{\partial t}u(x,0) = X(x)T'(0)$$

$$0 = \underbrace{B\sin\left(\frac{n\pi}{L}x\right)}_{\neq 0}\underbrace{T'(0)}_{=0}$$

Step 5. Find the general solution.

$$u(x,t) = X(x)T(t)$$
$$= B\sin\left(\frac{n\pi}{L}x\right) \times C\cos\left(\sqrt{-\lambda}\,t\right)$$
$$= BC\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{nc\pi}{L}t\right)$$

$$\frac{\sqrt{-\lambda}}{c} = \frac{n\pi}{L}$$
$$\sqrt{-\lambda} = \frac{nc\pi}{L}$$

since BC depends on n : $BC = B_n$

$$u(x,t) = B_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{nc\pi}{L}t\right) \leftarrow \text{INDIVIDUAL SOLUTION.}$$

$$u(x,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{nc\pi}{L}t\right) \leftarrow \text{GENERAL SOLUTION.}$$

Step 6. Find Exact Solution (solve for remaining unknown → B_n)

we still have an initial condition haven't been used,

IC1 : $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\cancel{\frac{nc\pi}{L}(0)}\right)$$

$$f(x) = \sum_{n=1}^{\infty} B_n\underbrace{\sin\left(\frac{n\pi}{L}x\right)}, \text{ for } 0 \le x \le L$$

only sin → can do fourier half-range sine series!
(if only cos → do fourier half-range cosine series!)
(if both sin and cos → do fourier full-range with general range $[a, a+2L]$)

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

then, exact solution will be:

$$u(x,t) = \sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{nc\pi}{L}t\right)$$

6.5 Reduction to Canonical Form (Method 2)

(a) Show that the PDE

$$3u_{xx} + 4u_{xy} + u_{yy} + 2u_x + 2u_y = 0$$

is hyperbolic, transform it to canonical form and hence find the solution in terms of two arbitrary functions.

(b) Find the particular solution that satisfies the boundary conditions:

$$u(x,0) = 1 + xe^{-x} \qquad \frac{\partial u(x,0)}{\partial y} = (x-3)e^{-x}$$

☆ only hyperbolic can use this method!

Step 1. Check whether the PDE is hyperbolic or not.

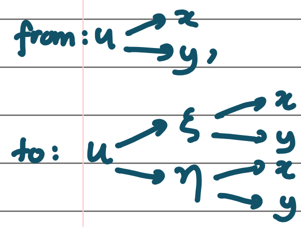
$B^2 - 4AC = 4^2 - 4(3)(1) = 4 > 0 \rightarrow$ PDE is hyperbolic.

Step 2. Define new variable ξ and η AND convert all

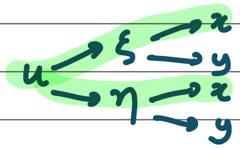
$u_{xx}, u_{xy}, u_{yy}, u_x, u_y$
to
$u_{\xi\xi}, u_{\xi\eta}, u_{\eta\eta}, u_\xi, u_\eta$

$\xi = x + \alpha y \quad ; \quad \eta = x + \beta y \quad \leftarrow$ This is what we sub IF A, B, and C are constant! (for our syllabus, it is the case)

$\frac{\partial \xi}{\partial x} = 1, \ \frac{\partial \xi}{\partial y} = \alpha \ ; \ \frac{\partial \eta}{\partial x} = 1, \ \frac{\partial \eta}{\partial y} = \beta$



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \times \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \times \frac{\partial \eta}{\partial x}$$
$$= \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$



$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

{ partial differential can swap order!

$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)$$
$$= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$
$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \times \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \times \frac{\partial \eta}{\partial y}$$
$$= \alpha \frac{\partial u}{\partial \xi} + \beta \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \alpha^2 \frac{\partial^2 u}{\partial \xi^2} + 2\alpha\beta \frac{\partial^2 u}{\partial \xi \partial \eta} + \beta^2 \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\alpha \frac{\partial u}{\partial \xi} + \beta \frac{\partial u}{\partial \eta}\right)$$
$$= \alpha \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \beta \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right)$$
$$= \alpha \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) + \beta \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$
$$= \alpha \frac{\partial^2 u}{\partial \xi^2} + (\alpha+\beta)\frac{\partial^2 u}{\partial \xi \partial \eta} + \beta \frac{\partial^2 u}{\partial \eta^2}$$

Step 3. Substitute the new partial derivative of ξ and η into the PDE

$$3u_{xx} + 4u_{xy} + u_{yy} + 2u_x + 2u_y = 0$$

$$3(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) + 4(\alpha u_{\xi\xi} + (\alpha+\beta)u_{\xi\eta} + \beta u_{\eta\eta}) + (\alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta})$$
$$+ 2(u_\xi + u_\eta) + 2(\alpha u_\xi + \beta u_\eta) = 0$$

$$(3+4\alpha+\alpha^2)u_{\xi\xi} + (3+4\beta+\beta^2)u_{\eta\eta} + (6+4(\alpha+\beta)+2\alpha\beta)u_{\xi\eta} + (2+2\alpha)u_\xi + (2+2\beta)u_\eta = 0$$

Step 4. Set the coefficient of $u_{\xi\xi}$ and $u_{\eta\eta}$ as 0, and find α and β.

$$\underbrace{3+4\alpha+\alpha^2 = 0 \quad 3+4\beta+\beta^2 = 0}$$

$$3+4\lambda+\lambda^2 = 0$$
$$(\lambda+1)(\lambda+3) = 0$$
$$\lambda = -1,\ \lambda = -3$$
$$\therefore \alpha = -1,\ \beta = -3 \rightarrow \boxed{\begin{array}{l} \xi = x+\alpha y = x-y \\ \eta = x+\beta y = x-3y \end{array}}$$

$$(6+4(-1-3)+2(-1)(-3))u_{\xi\eta} + (2+2(-1))u_\xi + (2+2(-3))u_\eta = 0$$

$$-4u_{\xi\eta} - 4u_\eta = 0$$

$$\boxed{\therefore \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial u}{\partial \eta} = 0}$$

canonical form

derived from: $3\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$

Step 5. Solve the Canonical Form of PDE

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)+\frac{\partial u}{\partial \eta}=0$$

let $\frac{\partial u}{\partial \eta}=\mu$:

$$\frac{\partial}{\partial \xi}\mu+\mu=0$$

$$\int \frac{1}{\mu}\partial\mu=-\int \partial\xi$$

we have ξ, but no η after integrating! add arbitrary function after integrating PDE.

remember that $u(\xi,\eta)$!

both of these must exists after integration!

$$\ln(\mu)=-\xi+h(\eta)$$

$$\mu=e^{-\xi+h(\eta)}$$

$$\mu=e^{h(\eta)}e^{-\xi}$$

$$\frac{\partial u}{\partial \eta}=G(\eta)e^{-\xi}$$

$$\int du=\int G(\eta)e^{-\xi}d\eta$$

we have u,η, but missing ξ

remember that $u(\xi,\eta)$!

both of these must exists after integration!

$$u=e^{-\xi}\left(\int G(\eta)d\eta+H(\xi)\right)$$

let $g(\eta)=\int G(\eta)d\eta$, and $h(\xi)=e^{-\xi}H(\xi)$

$$u=e^{-\xi}g(\eta)+h(\xi)$$

Step 6. Return from η and ξ to x and y!

$$\xi=x-y\ ;\ \eta=x-3y$$

$$\boxed{u=g(x-3y)e^{-(x-y)}+h(x-y)}$$ General Solution

(b) Find the particular solution that satisfies the boundary conditions:

$$u(x,0)=1+xe^{-x} \qquad \frac{\partial u(x,0)}{\partial y}=(x-3)e^{-x}$$

Step 7. Determine the boundary conditions (in this case already given so just check enough or not)

$$3u_{xx}+4u_{xy}+u_{yy}+2u_x+2u_y=0$$

2 boundary (space-based) conditions required.

Step 8. Apply the Boundary Conditions. (keep in mind we try to solve for $g(x-3y)$ and $h(x-y)$!)

$u(x,0) = 1 + xe^{-x} \quad (x=x,\ y=0)$

$g(x-3(0))e^{-(x-0)} + h(x-0) = 1 + xe^{-x}$

$g(x)e^{-x} + h(x) = 1 + xe^{-x}$

$h(x) = 1 + xe^{-x} - g(x)e^{-x}$

{ we want $h(x-y)$

$$\boxed{h(x-y) = 1 + (x-y)e^{-(x-y)} - g(x-y)e^{-(x-y)}}$$

{ find $u(x,y)$ without h first before sub into boundary condition 2.

$$u = g(x-3y)e^{-(x-y)} + 1 + (x-y)e^{-(x-y)} - g(x-y)e^{-(x-y)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial g(x-3y)}{\partial y}e^{-(x-y)} + g(x-3y)e^{-(x-y)} + (x-y)e^{-(x-y)} - e^{-(x-y)}$$
$$- \frac{\partial g(x-y)}{\partial y}e^{-(x-y)} - g(x-y)e^{-(x-y)}$$

DO NOT SUB $y=0$ at this stage! (DO CHAIN RULE FIRST!)

$$\frac{\partial u(x,0)}{\partial y} = (x-3)e^{-x}$$

$$\frac{\partial g(x)}{\partial y}e^{-x} + \cancel{g(x)e^{-x}} + xe^{-x} - e^{-x} - \frac{\partial g(x)}{\partial y}e^{-x} - \cancel{g(x)e^{-x}} = (x-3)e^{-x}$$

THIS IS WRONG!

☆ $g'(A)$ means $\frac{dg(A)}{dA}$!

chain rule!

$$= \frac{\partial g(x-3y)}{\partial(x-3y)} \times \frac{\partial(x-3y)}{\partial y} e^{-(x-y)} + \ldots$$

$$= g'(x-3y) \times -3e^{-(x-y)} + g(x-3y)e^{-(x-y)} + (x-y-1)e^{-(x-y)}$$
$$+ g'(x-y) \times e^{-(x-y)} - g(x-y)e^{-(x-y)}$$

Boundary Condition 2

$$\frac{\partial u(x,0)}{\partial y} = g'(x) \times -3e^{-x} + \cancel{g(x)}e^{-x} + (x-1)e^{-x} + g'(x)e^{-x} - \cancel{g(x)}e^{-x}$$

$$(x-3)e^{-x} = -2g'(x)e^{-x} + (x-1)e^{-x}$$

$2g'(x) = 2$

$g'(x) = 1$

$\therefore g(x) = x + c$

{ we want $g(x-3y)$ { and also $g(x-y)$ which is in $h(x-y)$

$g(x-3y) = x - 3y + c \qquad g(x-y) = x - y + c$

Step 9. Obtain Exact Solution

$u = g(x-3y)e^{-(x-y)} + h(x-y)$

$u = (x-3y+c)e^{-(x-y)} + 1 + (x-y)e^{-(x-y)} - (x-y+c)e^{-(x-y)}$

$\therefore \boxed{u(x,y) = (x-3y)e^{-(x+y)} + 1}$ EXACT SOLUTION.

☆ note that c will ALWAYS cancel out.

6.4 extra (recomended to go through it)

extra complete example on 6.4 seperation of varlable

5. Use the method of separation of variables to find the solution of the Laplace equation

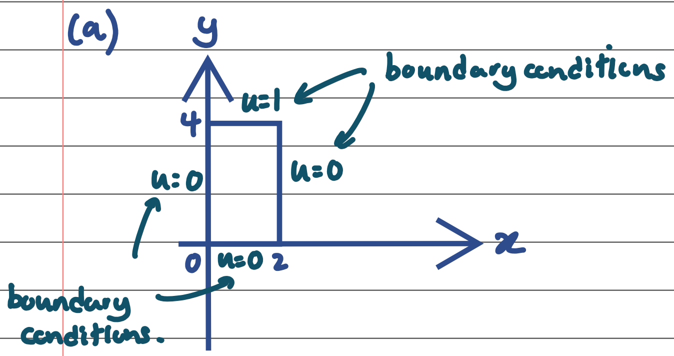
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the rectangle $0 \leqslant x \leqslant 2$, $0 \leqslant y \leqslant 4$, where:

(a) $u = 1$ on the upper side and zero on the other three sides

(b) $u = \sin\left(\frac{\pi x}{2}\right)$ on the upper side and zero on the other three sides

(a)



Step 1. Define PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ (requires 4 boundaries condition) → 2 for x, 2 for y

Step 2. Define Boundaries Conditions.

$u(x,4)=1$; $u(x,0)=0$; $u(0,y)=0$; $u(2,y)=0$

Step 3. Assume solutions as $u(x,y) = X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)Y(y) \qquad \frac{\partial^2 u}{\partial y^2} = X(x)Y''(y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)}$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \alpha$$

Step 4. Solve for different case of α.

for $\alpha=0$:

$$\frac{X''(x)}{X(x)}=0 \qquad -\frac{Y''(y)}{Y(y)}=0$$

$$X''(x)=0 \qquad Y''(y)=0$$

$$X(x)=\iint X''(x)\,dx \qquad Y(y)=\iint Y''(y)\,dy$$

$$=\int A\,dx \qquad =\int C\,dy$$

$$=Ax+B \qquad =Cy+D$$

$u(x,4)=1$; $u(x,0)=0$; $u(0,y)=0$; $u(2,y)=0$

when $x=0$: $\longrightarrow u(0,y)=0$ $\qquad$ $u(2,y)=0 \longleftarrow$ when $x=2$:

$X(0)=A(0)+B$ $\qquad$ $X(0)Y(y)=0$ $\qquad$ $X(2)Y(y)=0$ $\qquad$ $X(2)=A(2)+B$

$0=B \longleftarrow X(0)=0$ $\qquad$ $X(2)=0 \longrightarrow 0=2A+0$

$A=0$

$A=0$; $B=0$

NULL SOLUTION.

for $\alpha>0$:

$$\frac{X''(x)}{X(x)}=\alpha$$

$$X''(x)-\alpha X(x)=0$$

$$\lambda^2-\alpha=0$$

$$\lambda=\pm\sqrt{\alpha}$$

1 Different form (exponential form) $\rightarrow$ $\therefore X(x)=Ae^{\sqrt{\alpha}x}+Be^{-\sqrt{\alpha}x}$

2 if write $X(x)=A\cosh(\sqrt{\alpha}x)+B\sinh(\sqrt{\alpha}x)$
(hyperbolic form)

$x=0$:
$X(0)=A\cosh(0)+B\cancel{\sinh(0)}^{\,0}$
$u(0,y)=0$
$X(0)Y(y)=0$
$X(0)=0$
$0=A(1)$
$A=0 \longrightarrow A=0, B=0$ (null sol.!)

$x=2$:
$X(2)=\cancel{A\cosh(2\sqrt{\alpha})}^{\,0}+B\sinh(2\sqrt{\alpha})$
$u(2,y)=0$
$X(2)Y(y)=0$
$X(2)=0$
$0=B\underbrace{\sinh(2\sqrt{\alpha})}_{\alpha>0,\text{ means }\alpha\neq0\,\rightarrow\, B=0!}$

$u(x,4)=1$; $u(x,0)=0$; $u(0,y)=0$; $u(2,y)=0$

when $x=0$:
$X(0)=A+B$
$0=A+B$
$A=-B$

when $x=2$
$X(2)=Ae^{\sqrt{\alpha}(2)}+Be^{-\sqrt{\alpha}(2)}$
$0=Ae^{2\sqrt{\alpha}}+Be^{-2\sqrt{\alpha}}$
$0=-Be^{2\sqrt{\alpha}}+Be^{-2\sqrt{\alpha}}$
$0=B(-e^{2\sqrt{\alpha}}+e^{-2\sqrt{\alpha}})$
since $\alpha\neq0$
$\pm2\sqrt{\alpha}\neq0$
$e^{2\sqrt{\alpha}}\neq e^{-2\sqrt{\alpha}}$
$-e^{2\sqrt{\alpha}}+e^{-2\sqrt{\alpha}}\neq0 \longrightarrow \underbrace{B=0\rightarrow A=0}$ NULL SOLUTION.

both return null sol.
please note that A and B in both form are NOT the same A and B!

for $a<0$:

$$\frac{X''(x)}{X(x)} = \alpha$$

$$X''(x) - \alpha X(x) = 0$$

$$\lambda^2 - \alpha = 0$$

$$\lambda = \pm\sqrt{\alpha}$$

$$\lambda = \pm\sqrt{-\alpha}\, i$$

$$\therefore X(x) = e^{0x}(A\cos\sqrt{-\alpha}x + B\sin\sqrt{-\alpha}x)$$

$$X(x) = A\cos(\sqrt{-\alpha}x) + B\sin(\sqrt{-\alpha}x)$$

$$\frac{Y''(y)}{Y(y)} = -\alpha$$

$$Y''(y) + \alpha Y(y) = 0$$

$$\lambda^2 + \alpha = 0$$

$$\lambda = \pm\sqrt{-\alpha}$$

$$\therefore Y(y) = Ce^{\sqrt{-\alpha}y} + De^{-\sqrt{-\alpha}y}$$

$$= \underbrace{C\cosh(\sqrt{-\alpha}y) + D\sinh(\sqrt{-\alpha}y)}_{\text{alternative form (recomended)}}$$

$u(x,4)=1$; $u(x,0)=0$; $u(0,y)=0$; $u(2,y)=0$

when $x=0, u=0$:

$$u(0,y)=0$$

$$X(0)Y(y)=0$$

$$X(0)=0$$

$$0 = A\cos(0) + B\sin(0)$$

$$A=0$$

when $x=2, u=0$

$$u(2,y)=0 \longrightarrow u(x,y)=X(x)Y(y)$$

$$X(2)Y(y)=0$$

$$X(2)=0$$

$$0 = A\cos\sqrt{-\alpha}(2) + B\sin\sqrt{-\alpha}(2)$$

$$B\sin(2\sqrt{-\alpha})=0$$

$$\sin(2\sqrt{-\alpha})=0$$

$$R.A = 0$$

$$2\sqrt{-\alpha} = n\pi, \quad n\in\mathbb{Z}$$

$$\sqrt{-\alpha} = \frac{n\pi}{2}$$

$$X(x) = A\cos(\sqrt{-\alpha}x) + B\sin(\sqrt{-\alpha}x)$$

$$\therefore X(x) = B\sin\left(\frac{n\pi}{2}x\right)$$

when $y=0, u=0$:

$$u(x,0)=0$$

$$X(x)Y(0)=0$$

$$Y(0)=0$$

$$C\underbrace{\cancel{\cosh(0)}}_{1} + D\overbrace{\cancel{\sinh(0)}}^{0} = 0$$

$$C=0$$

$$\therefore Y(y) = D\sinh\left(\frac{n\pi}{2}y\right)$$

when $y=4, u=1$:

$$u(x,4)=1$$

$$X(x)Y(4)=1$$

$$\underbrace{B\sin\left(\frac{n\pi}{2}x\right)\Big(D}_{}\sinh(2n\pi)\Big)=1$$

1eqn between B and D → can't solve (for now)

Step 5. Find the general solution.

$u(x,y) = BD\sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}y\right)$ ← individual solution.

$u(x,y) = \sum_{n=1}^{\infty} \underbrace{BD\sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}y\right)}$ ← general solution.

i usually don't combine till want to find exact solution!

Step 6. Find the exact solution

$u(x,4) = 1$

$\sum_{n=1}^{\infty} BD\sin\left(\frac{n\pi}{2}x\right)\sinh(2n\pi) = 1$ for $0 \le x \le 2$

combine coeff.

$\sum_{n=1}^{\infty} B_n\sin\left(\frac{n\pi}{2}x\right) = 1$

☆ to be able to use ☆ half-sine series,

$\sin\left(\underbrace{\frac{n\pi}{2}}x\right)$

this should match with our range!

ie: $0 \le x \le 2$ in this case.

$L=2$

$f(x) = \sum B_n \sin\left(\frac{n\pi x}{L}\right)$

$\sin\left(\frac{n\pi x}{\underbrace{2}}\right)$

matches ✓

Half-range sine:

$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

1. $f(x) = 1$
2. $L = 2$

$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$B_n = \int_0^2 \sin\left(\frac{n\pi}{2}x\right)dx$

$= \left[\frac{2}{n\pi} - \cos\left(\frac{n\pi}{2}x\right)\right]_0^2$

$= \frac{2}{n\pi}\left[-\cos(n\pi) + \cos(0)\right]$

$= \frac{2}{n\pi}\left[1-(-1)^n\right]$

$$\boxed{\therefore u(x,y) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin\left(\frac{n\pi}{2}x\right)}$$

☆ problem about stopping here is no 'y'!

continue...

$BD\sinh(2n\pi) = B_n$

$BD = \frac{1}{\sinh(2n\pi)}\left(\frac{2}{n\pi}\left[1-(-1)^n\right]\right)$

$n=1: 1-(-1)^1 = 2$

$n=2: 1-(-1)^2 = 0$

$n=3: \quad " = 2$

$n=4: \quad " = 0$

$\vdots$

let $n = 2m-1$, $m \in \mathbb{Z}$

$BD = \frac{1}{\sinh((4m-2)\pi)}\left(\frac{4}{(2m-1)\pi}\right)$

$u(x,y) = \sum_{n=1}^{\infty} BD\sin\left(\frac{n\pi}{2}x\right)\sinh\left(\frac{n\pi}{2}y\right)$

Exact solution →

$$= \sum_{m=1}^{\infty} \frac{4}{(2m-1)\pi\sinh((4m-2)\pi)} \times \sin\left(\frac{(2m-1)\pi}{2}x\right)\sinh\left(\frac{(2m-1)\pi}{2}y\right)$$

Step 6. Find the Exact Solution.

$$u(x,4) = \sin\left(\frac{\pi x}{2}\right)$$

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi}{2}(4)\right) = \sin\left(\frac{\pi x}{2}\right)$$

$$B_1 \sin\left(\frac{\pi x}{2}\right)\sinh(2\pi) + B_2 \sin(\pi x)\sinh(4\pi) + \cdots = \sin\left(\frac{\pi x}{2}\right)$$

$$B_1 \sinh(2\pi) = 1$$

$$B_1 = \frac{1}{\sinh(2\pi)}, \quad B_{n\neq 1} = 0$$

$$\therefore u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right) = \frac{\sin\left(\frac{n\pi x}{2}\right)\sinh\left(\frac{n\pi y}{2}\right)}{\sinh(2\pi)}$$

6.5 extra

extra example on 6.5 reduction to canonical form

7. Show that the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic and transform the equation to canonical form. Obtain the general solution of the transformed equation in terms of two arbitrary functions. Deduce the general solution of the original equation and find the particular solution that satisfies the following boundary conditions at $y = 0$:

$$u = 0, \quad \frac{\partial u}{\partial y} = 2xe^{-x^2} \quad \text{for all } x.$$

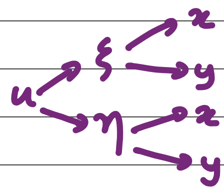
Step 1: Check whether the PDE is hyperbolic or not

$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$

$B^2 - 4AC = (-1)^2 - 4(2)(-1) = 9 > 0$ (the PDE is hyperbolic)

Step 2: Convert to ξ and η

remember this!



$\xi = x + \alpha y$; $\eta = x + \beta y$

$\frac{\partial \xi}{\partial x} = 1$; $\frac{\partial \xi}{\partial y} = \alpha$; $\frac{\partial \eta}{\partial x} = 1$; $\frac{\partial \eta}{\partial y} = \beta$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right) + \frac{\partial u}{\partial \eta}\left(\frac{\partial \eta}{\partial x}\right) = u_\xi + u_\eta$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial x}\right) = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$u_y = \alpha u_\xi + \beta u_\eta$

$u_{yy} = \alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta}$

$u_{xy} = \alpha u_{\xi\xi} + (\alpha + \beta) u_{\xi\eta} + \beta u_{\eta\eta}$

Step 3: Substitute.

$2u_{xx} - u_{xy} - u_{yy} = 0$

$2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - (\alpha u_{\xi\xi} + (\alpha+\beta)u_{\xi\eta} + \beta u_{\eta\eta}) - (\alpha^2 u_{\xi\xi} + 2\alpha\beta u_{\xi\eta} + \beta^2 u_{\eta\eta}) = 0$

$(2 - \alpha - \alpha^2)u_{\xi\xi} + (4 - (\alpha+\beta) - 2\alpha\beta)u_{\xi\eta} + (2 - \beta - \beta^2)u_{\eta\eta} = 0$

Step 4: sub coeff of $u_{\xi\xi}$ and $u_{\eta\eta} = 0$

$2 - \alpha - \alpha^2 = 0 \qquad 2 - \beta - \beta^2 = 0$

$\alpha^2 + \alpha - 2 = 0$

$(\alpha - 1)(\alpha + 2) = 0$

$\alpha = 1, \alpha = -2 \quad \rightarrow \alpha = 1, \beta = -2$

$$[4-(1-2)-2(1)(-2)]u_{\xi\eta}=0$$
$$9u_{\xi\eta}=0$$
$$\frac{\partial^2 u}{\partial\xi\partial\eta}=0 \quad \leftarrow \text{canonical form.}$$

Step 5. Solve the canonical form.

$$\frac{\partial}{\partial\xi}\left(\frac{\partial u}{\partial\eta}\right)=0$$
$$\frac{\partial u}{\partial\eta}=h(\eta)$$
$$\int du=\int h(\eta)\,d\eta$$
$$u=\int h(\eta)\,d\eta+g(\xi)$$

let $f(\eta)=\int h(\eta)\,d\eta$

$$u=f(\eta)+g(\xi)$$

sub back to x and y: $\xi=x+y$; $\eta=x-2y$

$$u(x,y)=f(x-2y)+g(x+y) \quad \leftarrow \text{General Solution.}$$

Step 5: Find Particular Solution

$u(x,0)=0$, $\frac{\partial u}{\partial y}=2xe^{-x^2}$ for $x\in\mathbb{R}, y=0$

$$u(x,0)=0$$
$$f(x)+g(x)=0$$
$$f(x-2y)+g(x-2y)=0$$
$$f(x-2y)=-g(x-2y)$$

$$u(x,y)=g(x+y)-g(x-2y)$$

$$\frac{\partial u}{\partial y}=\frac{\partial g(x+y)}{\partial y}-\frac{\partial g(x-2y)}{\partial y}$$
$$=\frac{\partial g(x+y)}{\partial(x+y)}\times\frac{\partial(x+y)}{\partial y}-\frac{\partial g(x-2y)}{\partial(x-2y)}\times\frac{\partial(x-2y)}{\partial y}$$
$$=g'(x+y)\times 1-g'(x-2y)\times -2$$

$$\frac{\partial u(x,0)}{\partial y}=2xe^{-x^2}$$
$$g'(x)+2g'(x)=2xe^{-x^2}$$
$$g'(x)=\frac{2}{3}xe^{-x^2}$$

$$g(x)=\int\frac{2}{3}xe^{-x^2}dx$$
$$=-\frac{1}{3}\int -2xe^{-x^2}dx$$
$$=-\frac{1}{3}e^{-x^2}+c$$

$$u(x,y) = -\frac{1}{3}e^{-(x+y)^2} + c - \left[-\frac{1}{3}e^{-(x-2y)^2} + c\right]$$

$$= -\frac{1}{3}e^{-(x+y)^2} + \frac{1}{3}e^{-(x-2y)^2}$$

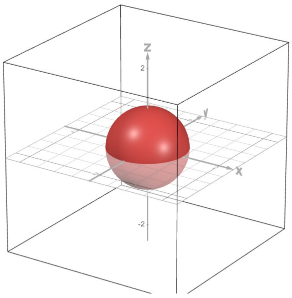
C7. Extras and Appendix

7.1 3-D Surface Equations

1. Spheres with center (a, b, c) and radius r

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

if centered at origin: $x^2+y^2+z^2=r^2$

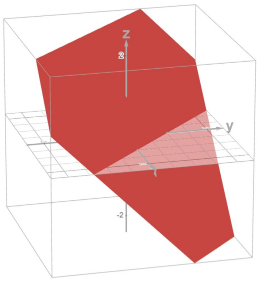


eg. $x^2+y^2+z^2=1$

2. Planes

$$Ax+By+Cz=D$$

recommend to check "Maths Year 1, C13 — Vectors"
to obtain cartesian equation of plane you need to know: $\underset{\sim}{r}\cdot\underset{\sim}{n}=\underset{\sim}{A}\cdot\underset{\sim}{n}$



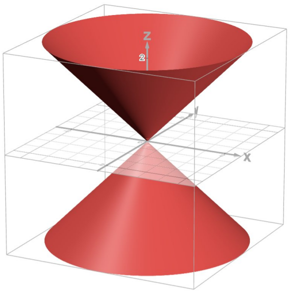
eg. $x+y+z=1$

3. Right circular cone.

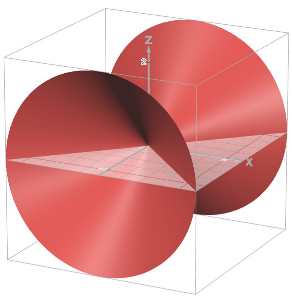
if opening in z-direction: $z^2=k^2(x^2+y^2)$
if opening in y-direction: $y^2=k^2(x^2+z^2)$
if opening in x-direction: $x^2=k^2(y^2+z^2)$

$k=\cot(\alpha)$, where α is the half-angle of the cone (half angle is the angle between cone's central axis to the slanted line)

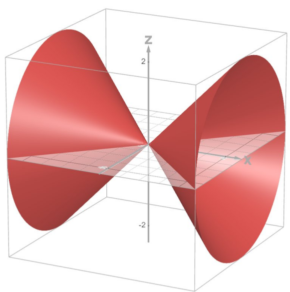
eg if $\alpha=45^\circ$, $k^2=1$
$z^2=x^2+y^2$



$z^2=x^2+y^2$



$y^2=x^2+z^2$



$x^2=y^2+z^2$

4. Cylinders

if it spans infinitely along z-direction, with center along x-y plane at $x=a$ and $y=b$, with radius r:

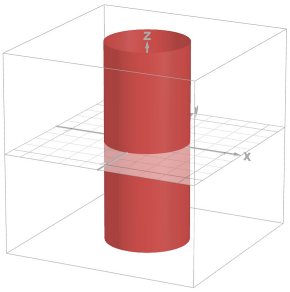
$$(x-a)^2+(y-b)^2=r^2$$ ⟵ looks like equation of circle which it is! just in 3-D it is a cylinder!

(any 2-D equation in x-y plane, if extended to 3-D is just the 3-D version with infinite thickness!)

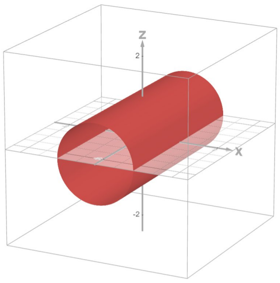
can be also

$$(y-a)^2+(z-b)^2=r^2$$
$$(x-a)^2+(z-b)^2=r^2$$

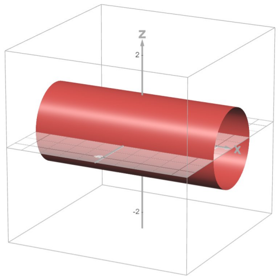
depends on which plane the "circle" is!



$x^2+y^2=1$



$x^2+z^2=1$



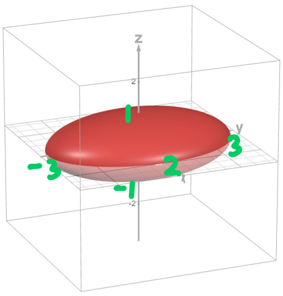
$y^2+z^2=1$

5. Quadric Surfaces (Ellipse version of everything above)

5.1 Ellipsoid (ellipse version of sphere)

$$\frac{(x-a)^2}{m^2}+\frac{(y-b)^2}{n^2}+\frac{(z-c)^2}{p^2}=1$$

⟵ 1 not r^2!

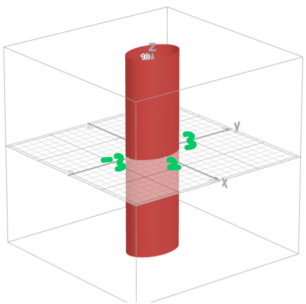


eg. $\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{1^2}=1$

5.2 Elliptical cylinder

if spans infinitely in the z-direction: (can also in x and y direction)

$$\frac{(x-a)^2}{m^2}+\frac{(y-b)^2}{n^2}=1$$



eg. $\frac{x^2}{2^2}+\frac{y^2}{3^2}=1$

7.3 Double Integral Formula

(a) The mass, M, of the region R is given by

$$M = \iint_R \rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

(b) The centre of mass, $(\bar{x}, \bar{y})$, of the region R is given by

$$\bar{x} = \frac{1}{M}\iint_R x\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\bar{y} = \frac{1}{M}\iint_R y\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

(c) The moment of inertia about the x-, y- and z-axis is given by I_x, I_y and I_z respectively

$$I_x = \iint_R y^2\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$I_y = \iint_R x^2\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

$$I_z = \iint_R \left(x^2 + y^2\right)\rho(x, y)\,\mathrm{d}x\,\mathrm{d}y$$