(all the table of content are hyperlinks! Click on it!)

c1. Flexibility Method

c2. Stiffness Method

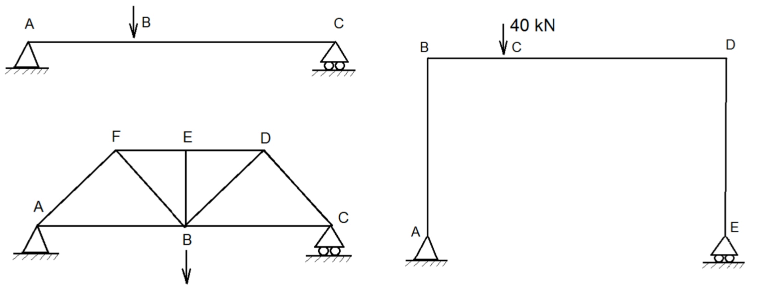
c3. Moment Distribution Method.

c1. Flexibility Method

1.0 Introduction

1. Structural Redundancy (Static indeterminancy)

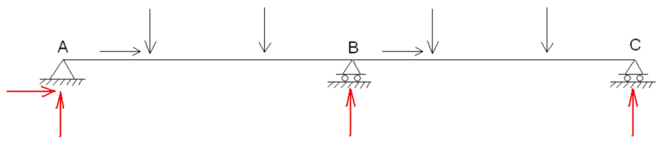
Statically determinate structures (SDS):
- Same number of unknowns (reactions at the support links) than equilibrium equations available
- Only one solution (internal forces) satisfy the equilibrium equations
- The internal forces are independent of the material properties
- We can obtain the unknowns by using only equilibrium equations



Unknowns: 3 (V_A, H_A, $V_{C/E}$)
Equilibrium equations available: 3 (ΣF_H=0; ΣF_V=0; ΣM=0)
Structural redundancy = Unknowns – Number of Equilibrium Equations = 0

Statically indeterminate structures (SIS):
- Larger number of unknowns (reactions at the supports) than equilibrium equations available
- The **structural redundancy**, or static **indeterminacy number (α)**, is the number of extra unknowns in relation to a statically determinate system.
- Infinite number of solutions (internal forces) satisfy the equilibrium equations
- Only one solution satisfies the equilibrium, compatibility and constitutive equations.
- Constitutive and compatibility equations are also required to obtain the solution. We cannot obtain the unknowns by using only the equilibrium equations. The number of extra equations that we need is equal to the redundancy
- The internal forces depend on the mechanical properties

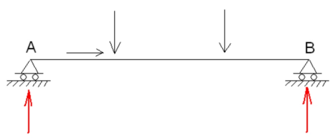
we will look at SIS structures in c1 and c2.
See how to solve them



Unknowns: 4 (V_A, H_A, V_B, V_C)
Equilibrium equations available: 3 (ΣFH=0; ΣFV=0; ΣM=0)
Redundancy= 4 (unknowns) – 3 (eq.) = 1

Mechanisms:
- Smaller number of unknowns (reactions at the supports) than equilibrium equations available
- It is not possible to obtain a solution that satisfies the equilibrium equations. The system is not in equilibrium. It is not a structure; it is a mechanism.

☆ The only certain thing is redundancy < 0 will be mechanism.
That means if redundancy greater or equal to zero it can be mechanism!



Unknowns: 2 (V_A, V_B)
Equilibrium equations available: 3 (ΣF_H=0; ΣF_V=0; ΣM=0)
Redundancy (α) = 2 (unknows) – 3 (eq.) = -1

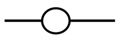
All the above are to find external redundancy.
We have to minus from the total (external) redundancy, the internal redundancy, i.e. minus the "hinges" add 3 × the "cells" and etc.

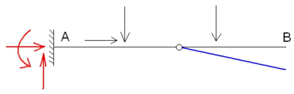
3 rules:

1. minus every "hinges"

Internal redundancy in beams and frames (α) – Reduced by adding hinges

For every internal hinge, there is 1 additional equation available (i.e., the bending moment at the hinge = 0). Internal redundancy = Number of hinges x (-1)

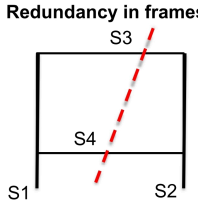




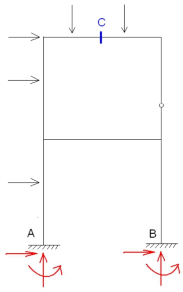
- Unknowns: 3 (V_A, H_A, $M_{A,}$)
- Equilibrium equations available: 3 (ΣF_H=0; ΣF_V=0; ΣM=0)
- Hinges = 1 (additional equation, M_C=0)
- External redundancy (α_e)= 3 (unknowns) – 3 (eq.) =0
- Internal redundancy (α_i) = -1 (1 hinge)
- Redundancy ($\alpha = \alpha_e + \alpha_i$) = 0 -1 = -1
- Statically deficient (hypostatic) or mechanism

2. Add 3 for each "cells" in frames

Redundancy in frames (α) – Increased by adding closed cells



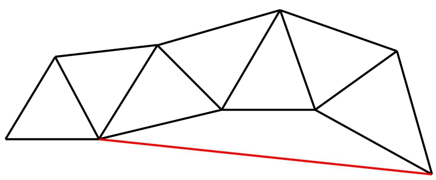
For every closed cell in a frame structure, there are 3 additional unknows (axial force, shear force and bending moment at one location of the cell, either S3 or S4).
For instance, if we know in internal forces at S1 and S2, it is impossible to know the internal forces at the cell cutting a free body, as there would be 3 unknows at S3 and 3 unknows at S4, so 3 additional unknows to the 3 unknows which value could be calculated by the 3 equilibrium equations. Internal redundancy = - Number of hinges + 3x Number of cells



- Unknowns: 6 external (V_A, H_A, M_A, V_B, H_B, M_B)
- Equilibrium equations available: 3 (ΣF_H=0; ΣF_V=0; ΣM=0)
- External redundancy (α_e)= 6 (reactions) – 3 (eq.) = 3
- Hinges =1
- Cells= 1
- Internal redundancy (α_i) = -1+3 =2
- Redundancy ($\alpha = \alpha_e + \alpha_i$) = 3+2=5
- Statically indeterminate (hiperstatic) structure

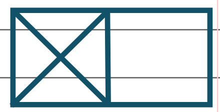
3. Add 1 for each "extra bar" in truss (and minus 1 for each "missing bar")

Internal redundancy in trusses (α_i)



Internal mechanism

quick example:



this is +1, then -1 so total internal redundancy is 0, but it is a mechanism.

Internal redundancy:
- In order to have a truss with an internal redundancy equal to zero we need to start the truss with a triangle, and to create new triangles by connecting two new bars from two existing nodes.
- If a new bar is included (by connecting two existing nodes), then the redundancy increases by 1.
- If one bar is eliminated, the redundancy is reduced by 1. it could also create a mechanism.

2. Principle of Virtual Work (simplified explaination)

Principle of virtual work is just:

"Virtual Internal Forces × Real Deformation = Virtual External Forces × Real Displacement"

In unit load method, we set virtual external force = 1 unit
and then we try to find real displacement.
and hence...
Virtual Internal Forces × Real Deformation = 1 × Real Displacement

Different structures have different sets of "Virtual Internal Forces × Real Deformation"

1. Trusses

Real Axial Force

due to lack of fit or elongation from temperature

Sum of all Virtual Axial Forces (n) × Real Strains $\left(\varepsilon = \frac{N l}{EA} + \Delta L\right)$ = 1 × Real Displacement

2. Beams

VERY IMPORTANT!

real moment

Sum of all Virtual Bending Moment (m) × Real Curvatures $\left(k = \frac{M}{EI}\right)$
+ Sum of all Virtual Shear Forces (v) × Real Distortions (γ) (negligible)
= 1 × Real Displacement

3. Frames

Real Moment

Sum of all Virtual Bending Moment (m) × Real Curvature $\left(k = \frac{M}{EI}\right)$
+ Sum of all Virtual Shear Forces (v) × Real Distortions (γ) (negligible)
+ Sum of all Virtual Axial Forces (n) × Real Strains $\left(\varepsilon = \frac{N l}{EA}\right)$ ☆ sometimes (especially when solving SIS we ignore $n \times \varepsilon$ as it is complicated)
= 1 × Real Displacement

3. Two ways to find the integral of $f_1(x) \times f_2(x)$

1. Simpson's Rule

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

where $f(x) = f_1(x) \times f_2(x)$

2. Table Method

☆ Remember these table provide "$\int f_1 f_2\,dx$" not $\int \frac{f_1 f_2}{EI}\,dx$!

M(x) \ m(x)	b	b	b	b, d
a	Lab	½ Lab	½ Lab	½ La(b+d)
a	½ Lab	⅓ Lab	¼ Lab	⅙ La (2b+d)
a	½ Lab	⅙ Lab	¼ Lab	⅙ La(b+2d)
a	½ Lab	¼ Lab	⅓ Lab	¼ La(b+d)

$\int_0^L M(x)m(x)dx$

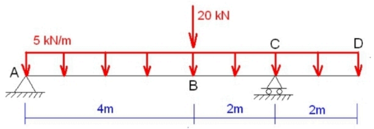
M(x) \ m(x)	b	b	b	b, d
a	⅔ Lab	⅓ Lab	5/12 Lab	⅓ La(b+d)
a, c	½ Lb (a+c)	⅙ Lb (2a+c)	¼ Lb(a+c)	⅙ La(2b+d) + ⅙ Lc(b+2d)
a, ¼a	⅓ Lab	¼ Lab	7/48 Lab	1/12 La(3b+d)
¼a, a	⅓ Lab	1/12 Lab	7/48 Lab	1/12 La(b+3d)
a, c, e; ½L, ½L	⅙ Lb(a+4c+e)	⅙ Lb(a+2c)	1/24 Lb(a+10c+e)	⅙ Lb(a+2c) + ⅙ Ld(2c+e)

☆ This example is for SDS!

usually related to Bending Moment Diagram only.

eg1: How to obtain deflection at any point? (Beam)

Obtain the deflection at D   EI = 20 MN.m²

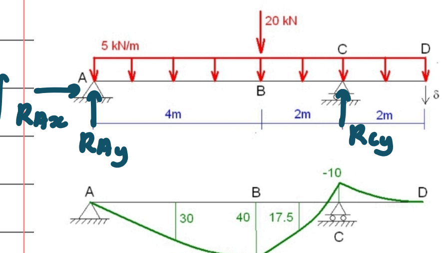


Step 1. create virtual case.

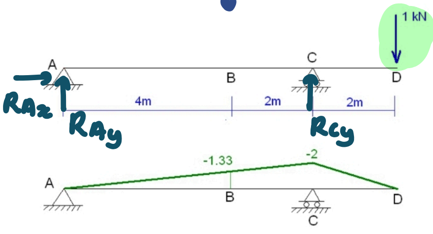
" Virtual Internal Forces × Real Deformation = 1 × Real Displacement "

remove all the forces and add a unit force at the point where we want to find real displacement.

* Reaction Forces needed for BMD



Real structure



Virtual forces

Step 2. Find BMD for both cases

2. Beams

Sum of all Virtual Bending Moment (m) × Real Curvatures ($k = \frac{M}{EI}$) ← real moment

+ Sum of all Virtual Shear Forces (v) × Real Distortions (γ) (negligible)

= 1 × Real Displacement

$$\rightarrow \underbrace{\int m\left(\frac{M}{EI}\right)}dx = 1 \times \delta$$

we need both bending moment distribution for both real and virtual case!

Step 3. Using The Table:

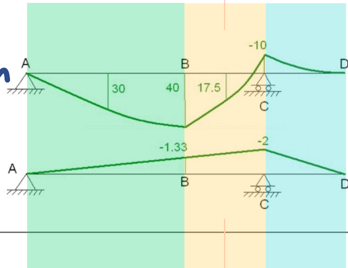
M(x) \ m(x)	b	b	b	b, d
a	Lab	½ Lab	½ Lab	½ La(b+d)
a	½ Lab	⅓ Lab	¼ Lab	⅙ La (2b+d)
a	½ Lab	⅙ Lab	¼ Lab	⅙ La(b+2d)
a	½ Lab	¼ Lab	⅓ Lab	¼ La(b+d)

$\int_0^L M(x)m(x)dx$

☆ why green is not this? cause it should be "◿"!

choose this cause we can b=0!

M(x) \ m(x)	b	b	b	b, d
a	⅔ Lab	⅓ Lab	$^5/_{12}$ Lab	⅓ La(b+d)
a, c	½ Lb (a+c)	⅙ Lb (2a+c)	¼ Lb(a+c)	⅙ La(2b+d) + ⅙ Lc(b+2d)
a, ½a	⅓ Lab	¼ Lab	$^7/_{48}$ Lab	$^1/_{12}$ La(3b+d)
½a, a	⅓ Lab	$^1/_{12}$ Lab	$^7/_{48}$ Lab	$^1/_{12}$ La(b+3d)
a, c, e, ½L, ½L	⅙ Lb(a+4c+e)	⅙ Lb(a+2c)	$^1/_{24}$ Lb(a+10c+e)	⅙ Lb(a+2c) + ⅙ Ld(2c+e)



$$\int_A^B \frac{Mm}{EI}dx = \frac{1}{EI}\int_A^B Mm\,dx = \frac{1}{6EI}Ld(2c+e) = \frac{1}{6EI}(4)(-1.33)(2\times30+40) = -\frac{88.67}{EI}$$

M(x) \ m(x)	b	b	b	b d
a	$^2/_3$ Lab	$^1/_3$ Lab	$^5/_{12}$ Lab	$^1/_3$ La(b+d)
a c	½ Lb (a+c)	$^1/_6$ Lb (2a+c)	¼ Lb(a+c)	$^1/_6$ La(2b+d) + $^1/_6$ Lc(b+2d)
a ½a	$^1/_3$ Lab	¼ Lab	$^7/_{48}$ Lab	$^1/_{12}$ La(3b+d)
½a a	$^1/_3$ Lab	$^1/_{12}$ Lab	$^7/_{48}$ Lab	$^1/_{12}$ La(b+3d)
a c e ½L ½L	$^1/_6$ Lb(a+4c+e)	$^1/_6$ Lb(a+2c)	$^1/_{24}$ Lb(a+10c+e)	$^1/_6$ Lb(a+2c) +$^1/_6$ Ld(2c+e)

$$\int_B^C \frac{Mm}{EI}dx = \frac{1}{6EI}(2)(-1.33)(40+2\times17.5) + \frac{1}{6EI}(2)(-2)(2\times17.5-10) = -\frac{49.92}{EI}$$

$$\int_C^D \frac{Mm}{EI}dx = \frac{1}{4EI}(2)(-10)(-2) = \frac{10}{EI}$$

Step 4.

Virtual Internal Forces × Real Deformation = 1 × Real Displacement

$$-\frac{88.67}{EI} - \frac{49.92}{EI} + \frac{10}{EI} = 1\times\delta$$

$$\delta = -6.4\,mm$$

(negative sign indicates displacement is opposite to our initial "external virtual force")

Alternative method for Step 3: Using Simpson's Rule

$$\int_A^B M_{VF}(x)\frac{M_{REAL}(x)}{EI}dx = \frac{1}{6EI}4\left[0+(-1.33)(40)+4\frac{(-1.33)}{2}(30)\right] = \frac{-88.67}{EI}$$

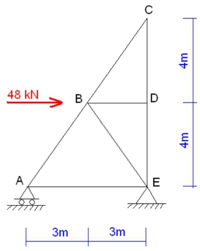
$$\int_B^C M_{VF}(x)\frac{M_{REAL}(x)}{EI}dx = \frac{1}{6EI}2\left[(-1.33)(40)+(-2)(-10)+4\frac{-1.33-2}{2}(17.5)\right] = \frac{-49.92}{EI}$$

$$\int_C^D M_{VF}(x)\frac{M_{REAL}(x)}{EI}dx = \frac{1}{6EI}2\left[(-10)(-2)+0+4\frac{(-10)}{4}\frac{(-2)}{2}\right] = \frac{10}{EI}$$

Related to Axial Force

eg1: How to obtain deflection at any point? (Trusses)

Obtain the horizontal deflection at C

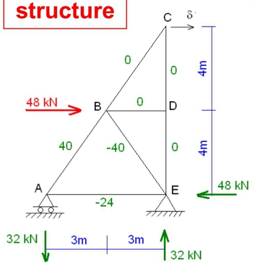


Loading in addition to the applied force of 48 kN:

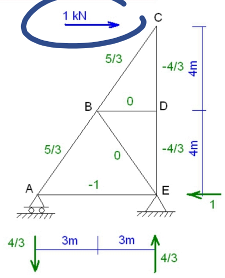
- *Lack of fit:* AB is too long (5 mm) and CD is too short (-5mm)
- *Temperature rise*: AB and BC have a temperature rise of 15 ºC (α=2 10^{-5} /ºC) $\Delta L = \alpha \times \Delta T \times L$
- *Support settlement*: Support A settles vertically 15 mm downwards

Remove every forces and add unit force at the point we want to find the displacement

Real structure



Virtual forces

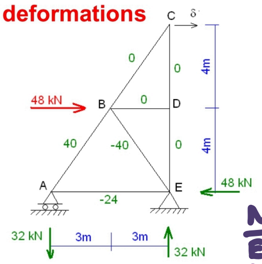


1) We obtain the internal forces (IF) in the structure and in the virtual one (the same structure with a unit force in the direction of the deflection to be obtained)
2) We apply the PVF: First variation of the Complementary Virtual work (δ^1W^*) = First variation of the Complementary Strain energy (δ^1U^*)

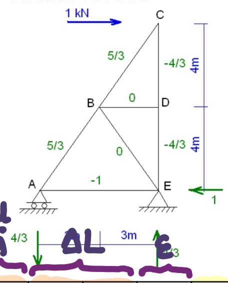
Find all the reaction force first (as needed to find axial force)
then find the axial force of every trusses

Unit-load Theorem (Principle of complementary virtual works / virtual forces)

Real deformations



Virtual forces



In addition to the applied forces:

- *Lack of fit:* AB is too long (5 mm) and CD is too short (- 5 mm)
- *Temperature rise*: AB and BC have a temperature rise of 15 ºC (α=2 10^{-5} /ºC)
- *Support settlement*: Support A settles vertically 15 mm downwards

$\underbrace{\frac{Nl}{EA}}$ $\underbrace{\Delta L}$ $\underbrace{\varepsilon}$

Member	EA (kN)	L (mm)	N_{REAL} (kN)	NL/EA (mm)	$e_{\Delta L}$ (mm)	$e_{\Delta T}$ (mm)	Σe (mm)	N_{VF} (kN)	$N_{VF}\ \Sigma e$ (KN.mm)
AB	25000	5000	40	8	5	1.5	14.5	5/3	24.17
AE	25000	6000	-24	-5.76			-5.76	-1	5.75
BC	25000	5000	0	0		1.5	1.5	5/3	2.5
BD	25000	3000	0	0			0	0	0
BE	25000	5000	-40	-8			-8	0	0
CD	25000	4000	0	0	-5		-5	-4/3	6.67
DE	25000	4000	0	0			0	-4/3	0
								Σ	39.09

1. Trusses
Sum of all Virtual Axial Forces (n) × Real Strains $\left(\varepsilon = \frac{Nl}{EA} + \Delta L\right)$ = 1 × Real Displacement

Real Axial Force

n

$$\left.\begin{aligned}\delta^1W^* &= 1kN\times\delta(mm)+\frac{4}{3}\times 15=\delta+20\,kNmm\\ \delta^1U^* &= 39.09\,kNmm\end{aligned}\right\}\quad \therefore\ \boxed{\delta=19.09mm}$$

$$\varepsilon = \frac{Nl}{EA} + \Delta L$$

End of Introduction

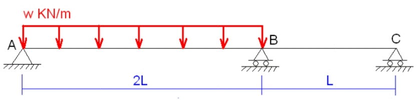
NOTE THAT BOTH EXAMPLE ARE FIND DISPLACEMENT OF A SDS!
IN YEAR 2 WE MAINLY WANT TO SOLVE SIS!

1.1 Solving SIS —— Beam, with redundancy = 1

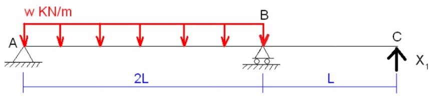
There is two method to solve SIS (that we gonna cover in c1)
i.e. removing support and adding hinge (both will reduce redundancy by 1)

1.1.1 Method 1 — Removing support (and replaced with REACTION FORCE)

Internal forces in this statically indeterminate structure (2 SPAN BEAM) using the flexibility method (removing support conditions)

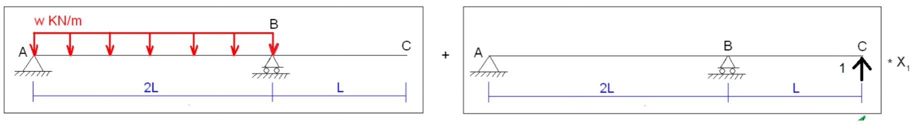


Step 1. Remove one of the support, replace with a reaction force.

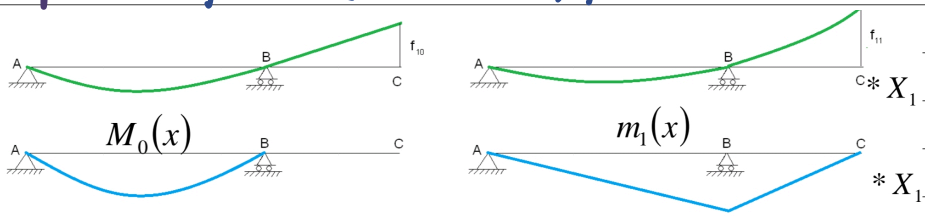


Step 2. Split into two cases:
Case 0 (without reaction force "X_1") + Case 1 (replace "X_1" with unit force) × magnitude of X_1

Case 0: SDS with external load — Case 1: SDS with unit load

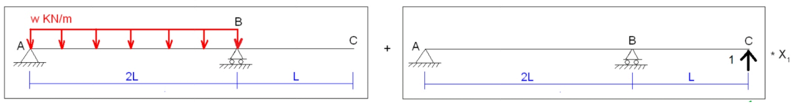


Step 3. Solve for BMD (cause its beam) for both cases.

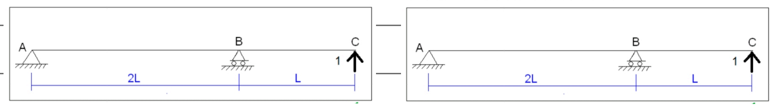


Step 4. Find the displacement at the point where support is removed and replaced

Case 0: SDS with external load — Case 1: SDS with unit load



Case 1: SDS with unit load — Case 1: SDS with unit load



Virtual case for case 0 and case 1

★ Note that Case 1 is the virtual case (needed to find displacement) for both case 0 and case 1! (this is very, very convenient!)

$$f_{10} = \int_s \frac{M_0(x)}{EI} m_1(x)dx = \frac{2L}{6}\left(0 + 4\left(\frac{w(2L)^2}{8}\cdot\frac{L}{2}\right) + 0\right)\frac{1}{EI} = \frac{wL^4}{3EI}$$

$$f_{11} = \int_s \frac{m_1(x)}{EI} m_1(x)dx = \left(\frac{2L}{6}\left(0 + 4\left(\frac{L^2}{4}\right) + L^2\right) + \frac{L}{6}\left(L^2 + 4\left(\frac{L^2}{4}\right) + 0\right)\right)\frac{1}{EI} = \frac{L^3}{EI}$$

'f_{10}' means displacement at '1' for case '0'

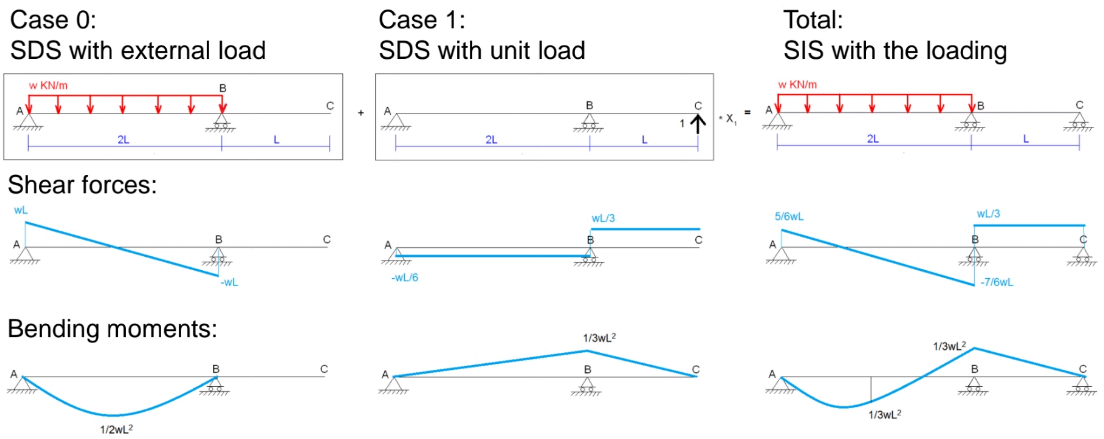
solved with the BMD from step 3. (can use table method or simpson's rule)

Step 5. Displacement at support must equal to zero (using superposition method)

$$f_{10} + X_1 f_{11} = 0 \quad \therefore \quad \frac{wL^4}{3EI} + X_1 \frac{L^3}{EI} = 0 \quad \therefore \quad X_1 = -\frac{wL}{3}$$

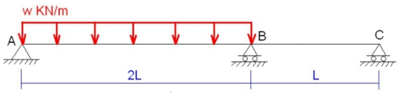
as our original case is splitted by:
" Case 0 + X_1× Case 1 "

☆ With X_1, we can now find everything! (SFBMD and other reaction forces!)

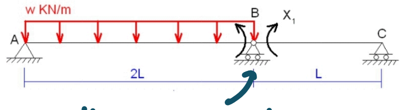


1.1.2 Method 2 — Adding hinges (and add together a couple to balance it out!)

Internal forces in this statically indeterminate structure (2 SPAN BEAM) using the flexibility method (adding hinges)



Step 1. Add a hinge, and a couple of moment to balance it out.



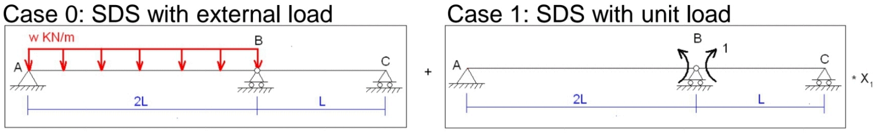
the couple will have opposite direction as the hinge!

At this stage let's assume bottom half of the hinge is in TENSION. (i.e. the BMD at that point originally would be in tension (+ive))

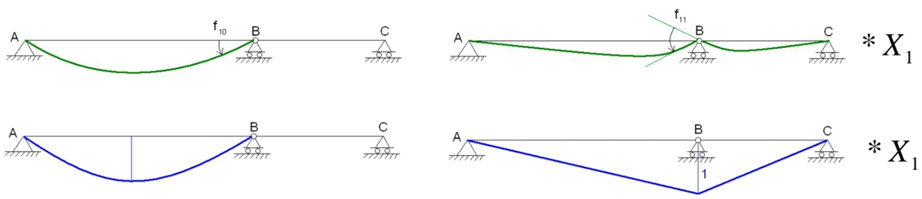
Step 2. Split into two cases:

Case 0 (without moment "X_1") + Case 1 (with unit moment) × magnitude of X_1

Case 0: SDS with external load   Case 1: SDS with unit load

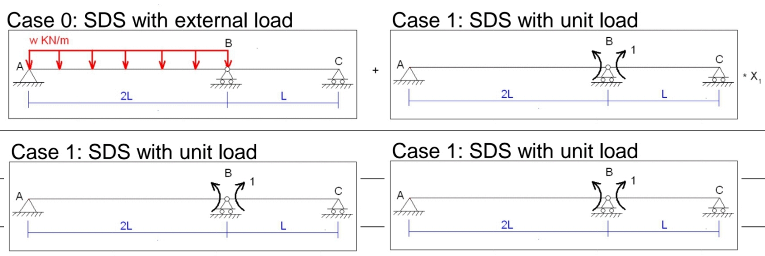


Step 3. Solve for BMD (cause its beam) for both cases.



$M_0(x)$ $m_1(x)$

Step 4. Find the angular displacement at the hinge

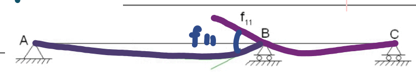


☆ Note that Case 1 is the virtual case (needed to find displacement) for both case 0 and case 1! (this is very, very convenient!)

$$f_{10} = \int_s \frac{M_0(x)}{EI} m_1(x)dx = \frac{2L}{6}\left(0+4\left(\frac{w(2L)^2}{8}0.5\right)+0\right)\frac{1}{EI} = \frac{wL^3}{3EI}$$

$$f_{11} = \int_s \frac{m_1(x)}{EI} m_1(x)dx = \left(\frac{2L}{6}\left(0+4(0.5^2)+1\right)+\frac{L}{6}\left(1+4(0.5^2)+0\right)\right)\frac{1}{EI} = \frac{L}{EI}$$

angular displacement in this sense might be quite hard to visualise: if the beam is bent:

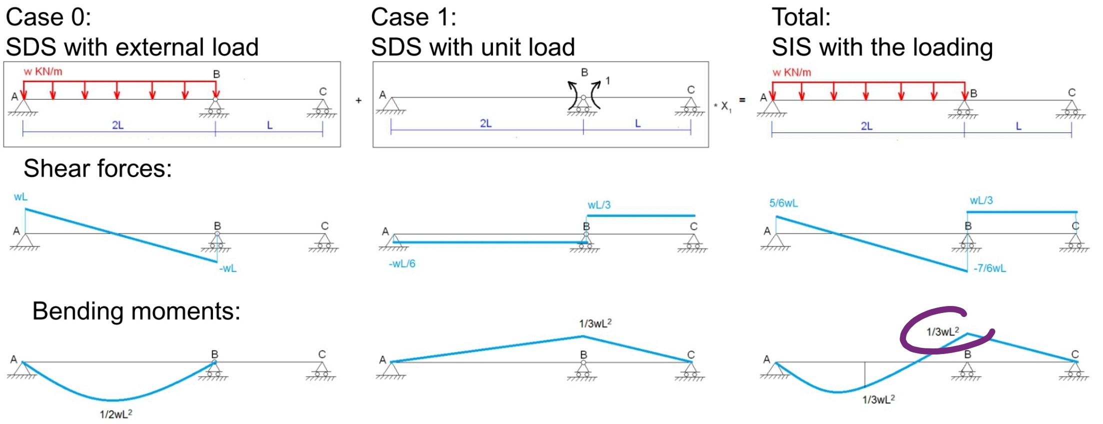


(if can't visualise, memorise it)

Step 5. Angular displacement at support must equal to zero (using superposition method)

$$f_{10} + X_1 f_{11} = 0 \quad \therefore \quad \frac{wL^3}{3EI} + X_1 \frac{L}{EI} = 0 \quad \therefore \quad X_1 = -\frac{wL^2}{3}$$

☆ With X_1, we can now find everything! (SFBMD and other reaction forces!)



☆ NOTE THAT BENDING MOMENT AT THE HINGE, IN THIS CASE AT B, IS EQUAL TO $X_1 = -\frac{1}{3}wL^2$!

1.2 Solving SIS —— Frame, with redundancy = 1

1.2.1 Method 1 — Removing support (and replaced with REACTION FORCE)

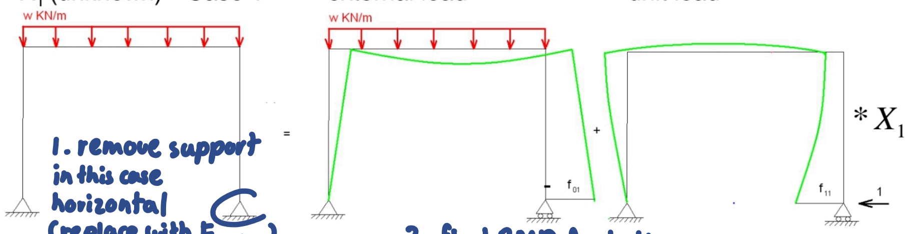
Flexibility method for statically indeterminate frame structures (removing supports)

2. split into two cases.

Structure = Case 0 + X_1 (unknown) * Case 1

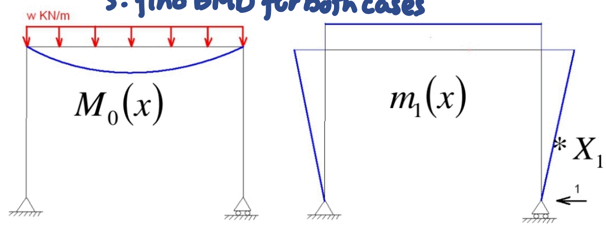
Case 0: external load

Case 1: unit load



1. remove support in this case horizontal (replace with $F_{reaction}$)

3. find BMD for both cases



4. Find displacements for both cases at support

$$f_{10} = \int_s \frac{M_0(x)}{EI} m_1(x)dx = \frac{1}{EI}\frac{2}{3}L\left(\frac{wL^2}{8}\right)(-H) = -\frac{wL^3}{12EI}H$$

$$f_{11} = \int_s \frac{m_1(x)}{EI} m_1(x)dx = \frac{1}{EI}\left(\frac{1}{3}H\cdot(-H)\cdot(-H) + L\cdot(-H)\cdot(-H) + \frac{1}{3}H\cdot(-H)\cdot(-H)\right) =$$

$$= \frac{1}{EI}\left(\frac{2}{3}H^3 + LH^2\right)$$

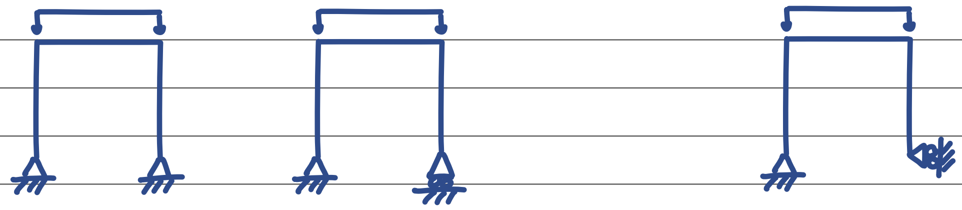
5. Displacement = 0

$$f_{10} + X_1 f_{11} = 0 \rightarrow X_1 = -\frac{f_{10}}{f_{11}} = \frac{\left(\frac{wL^3}{12}\right)}{\left(\frac{2}{3}H^2 + LH\right)}$$

☆ EXTREMELY IMPORTANT !

When removing support, choose wisely !

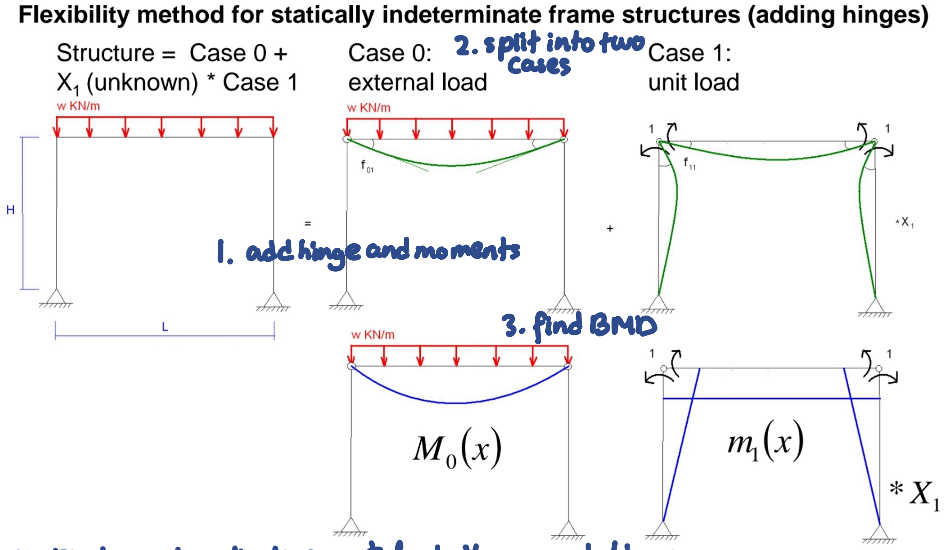
1. Try to avoid creating mechanism.
2. However in the case where mechanism results in easier BMD, we can create mechanism, but in the sense that it can RESIST LOADING that is provided in the question !



(original) (remove horizontal support) becomes SDS ✓ (remove vertical support) becomes mechanism (is alright but...) can't resist loading (NOT OK)

1.2.2 Method 2 — Adding hinges (and add together a couple to balance it out!)

Flexibility method for statically indeterminate frame structures (adding hinges)



4. Find angular displacement for both cases at hinge

$$f_{10}=\int_s \frac{M_0(x)}{EI} m_1(x)dx=\frac{1}{EI}\frac{2}{3}L\left(\frac{wL^2}{8}\right)1=\frac{wL^3}{12EI}$$

$$f_{11}=\int_s \frac{m_1(x)}{EI} m_1(x)dx=\frac{1}{EI}\left(\frac{1}{3}H\cdot 1\cdot 1+L\cdot 1\cdot 1+\frac{1}{3}H\cdot 1\cdot 1\right)=\frac{1}{EI}\left(\frac{2}{3}H+L\right)$$

5. Angular Displacement = 0

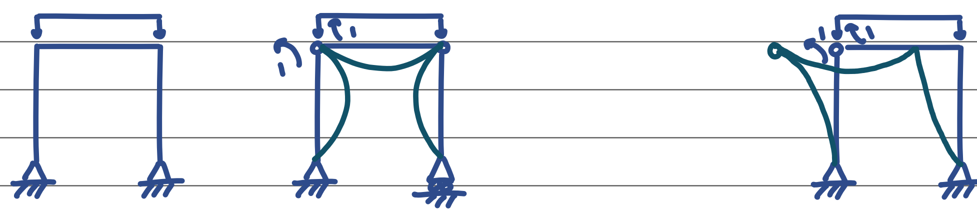
$$f_{10}+X_1 f_{11}=0\rightarrow X_1=-\frac{f_{10}}{f_{11}}=-\frac{\left(\frac{wL^3}{12}\right)}{\left(\frac{2}{3}H+L\right)}$$

this negative means opposite direction to our original assumption. Originally, (↷0↶) tension at bottom hence this mean tension is at top of hinge.

☆ EXTREMELY IMPORTANT!

When adding hinges, choose wisely!

1. Try to avoid creating mechanism.
2. However in the case where mechanism results in easier BMD, we can create mechanism, but in the sense that it can RESIST LOADING that is provided in the question!



(original)

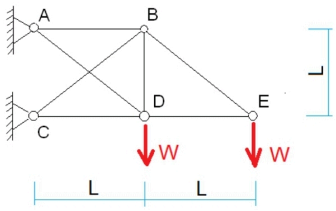
(add two hinges) becomes mechanism but CAN RESIST LOADING
→ if the question has a single horizontal external force, it will not be able to resist it.
If so, use one hinge only.

(add one hinge) become SDS
why we don't do this in this question because two hinge is acceptable (cause able to resist load although it is mechanism)
+ two hinge cases it is symmetry and symmetry is easy!

1.3 Solving SIS —— Trusses, with redundancy = 1

1.3.1 Method 1 — Removing support (and replaced with REACTION FORCE)

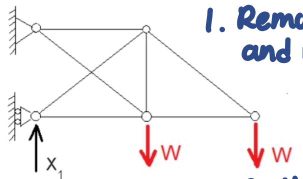
Flexibility method for statically indeterminate trusses (removing supports)



1. Trusses

Real Axial Force

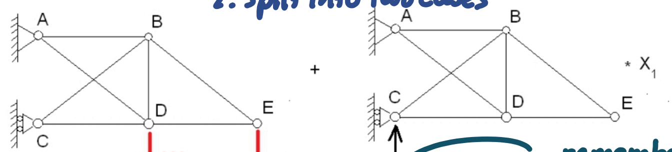
Sum of all Virtual Axial Forces (n) × Real Strains ($\varepsilon = \frac{NL}{EA} + \Delta L$) = 1 × Real Displacement



1. Remove support (in this case vertical support) and replace with reaction force.

2. Split into two cases

3. Find Ns (case 0) and ns (case 1)



Case 0:
SDS with external load

Case 1:
SDS with unit load

remember case 1 is the virtual case for both case 0 and case 1!

4. Find displacement for both case 0 and case 1.

→ this means:

case 0: $\sum n\left(\frac{NL}{EA}\right) = f_{10}$ case 1: $\sum n\left(\frac{nL}{EA}\right) = f_{11}$

Flexibility method for statically indeterminate trusses (α=1)

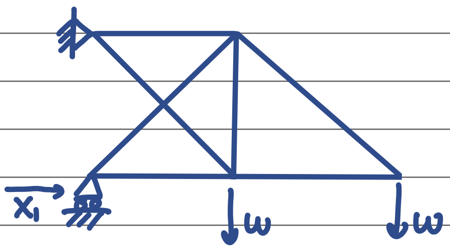
Member	L_k/EA	$N_{0,k}$	$N_{1,k}$	ΔL_k	$(N_{0,k} L_k/EA+\Delta L_k) N_{1,k}$	$N_{1,k}{}^2 L_k/EA$	$N_{1,k}X_1$	$N_{0,k}+N_{1,k}X_1$
AB	L/EA	W	1	0	(WL/EA)	(L/EA)	W	$2W$
AD	$L\sqrt{2}/EA$	$2W\sqrt{2}$	$-\sqrt{2}$	0	$-4\sqrt{2}\,(WL/EA)$	$2\sqrt{2}\,(L/EA)$	$-W\sqrt{2}$	$W\sqrt{2}$
BC	$L\sqrt{2}/EA$	0	$-\sqrt{2}$	0	0	$2\sqrt{2}\,(L/EA)$	$-W\sqrt{2}$	$-W\sqrt{2}$
BD	L/EA	$-W$	1	0	$-(WL/EA)$	(L/EA)	W	0
BE	$L\sqrt{2}/EA$	$W\sqrt{2}$	0	0	0	0	0	$W\sqrt{2}$
CD	L/EA	$-3W$	1	0	$-3\,(WL/EA)$	(L/EA)	W	$-2W$
DE	L/EA	$-W$	0	0	0	0	0	$-W$
				Σ	$(-3-4\sqrt{2})\,(WL/EA)$	$(3+4\sqrt{2})\,(L/EA)$		

$$f_{10} = \left(-3-4\sqrt{2}\right)\frac{WL}{EA} \qquad f_{11} = \left(3+4\sqrt{2}\right)\frac{L}{EA} \qquad f_{10} + X_1 f_{11} = 0 \qquad X_1 = W$$

☆ EXTREMELY IMPORTANT!

In this question

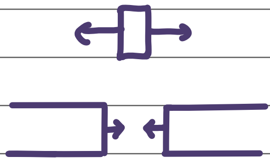
we cannot remove horizontal support as:



is a mechanism, that CANNOT RESIST the loading produce by W.

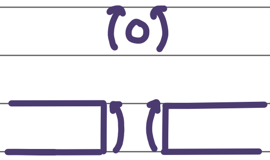
1.3.2 Method 2 —— Cutting Bars (and then add a pair of forces to balance it out)

☆ cutting bars is for trusses ; adding hinges is for beams and frames.



} produces tension throughout the entire length of bar

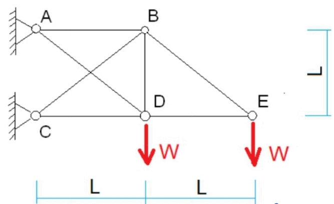
cut bars



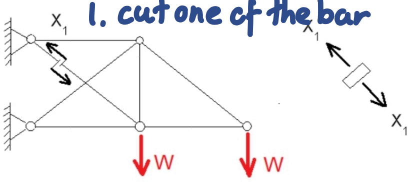
} produces tension on the bottom half of the beam, i.e. sagging

add hinge

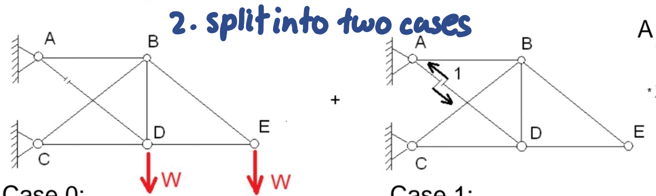
Flexibility method for statically indeterminate trusses (cutting bars)



1. cut one of the bar



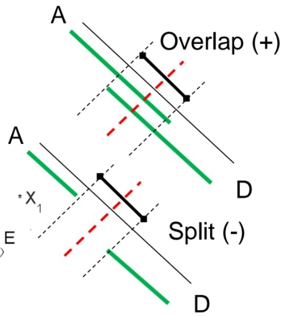
2. split into two cases



Case 0:
SDS with external load

3. Find all N and n

Case 1:
SDS with unit load



} X_1 +ve and -ve meanings. (very NOT important)

4. Find the displacement for both of the cases

Member	L_k/EA	$N_{0,k}$	$N_{1,k}$	ΔL_k	$(N_{0,k} L_k/EA+\Delta L_k) N_{1,k}$	$N_{1,k}{}^2 L_k/EA$	$N_{1,k}X_1$	$N_{0,k}+N_{1,k}X_1$
AB	L/EA	3W	$-1/\sqrt{2}$	0	$-3/\sqrt{2}$ (WL/EA)	1/2(L/EA)	-W	2W
AD	$L\sqrt{2}$ /EA	0	1	0	0	$\sqrt{2}$ (L/EA)	$W\sqrt{2}$	$W\sqrt{2}$
BC	$L\sqrt{2}$ /EA	$-2W\sqrt{2}$	1	0	-4 (WL/EA)	$\sqrt{2}$ (L/EA)	$W\sqrt{2}$	$-W\sqrt{2}$
BD	L/EA	W	$-1/\sqrt{2}$	0	$-1/\sqrt{2}$ (WL/EA)	1/2(L/EA)	-W	0
BE	$L\sqrt{2}$ /EA	$W\sqrt{2}$	0	0	0	0	0	$W\sqrt{2}$
CD	L/EA	-W	$-1/\sqrt{2}$	0	$1/\sqrt{2}$ (WL/EA)	1/2(L/EA)	-W	-2W
DE	L/EA	-W	0	0	0	0	0	-W
				Σ	$(-4-3/\sqrt{2})$ (WL/EA)	$(3/2+2\sqrt{2})$ (L/EA)		

5. Total displacement = 0
solve for X_1

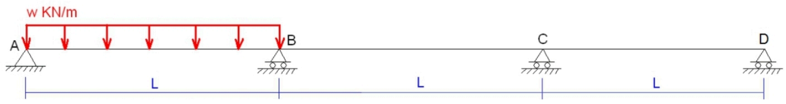
$$f_{10} = \left(-4 - \frac{3}{\sqrt{2}}\right)\frac{WL}{EA} \qquad f_{11} = \left(\frac{3}{2} + 2\sqrt{2}\right)\frac{L}{EA} \qquad f_{10} + X_1 f_{11} = 0 \qquad X_1 = W\sqrt{2}$$

X_1 +ve means the bar we cut is in tension!
(If -ve its compression)

1.4 Solving SIS with higher redundancies

1.4.1 Beams

Flexibility method for statically indeterminate structures (3 SPAN BEAMS)



(This is a redundancy = 5 - 3 = 2 question)

Step 1. Add (redundancy) amount of hinges (or more)

Step 2. Splits into (redundancy + 1) amount of cases

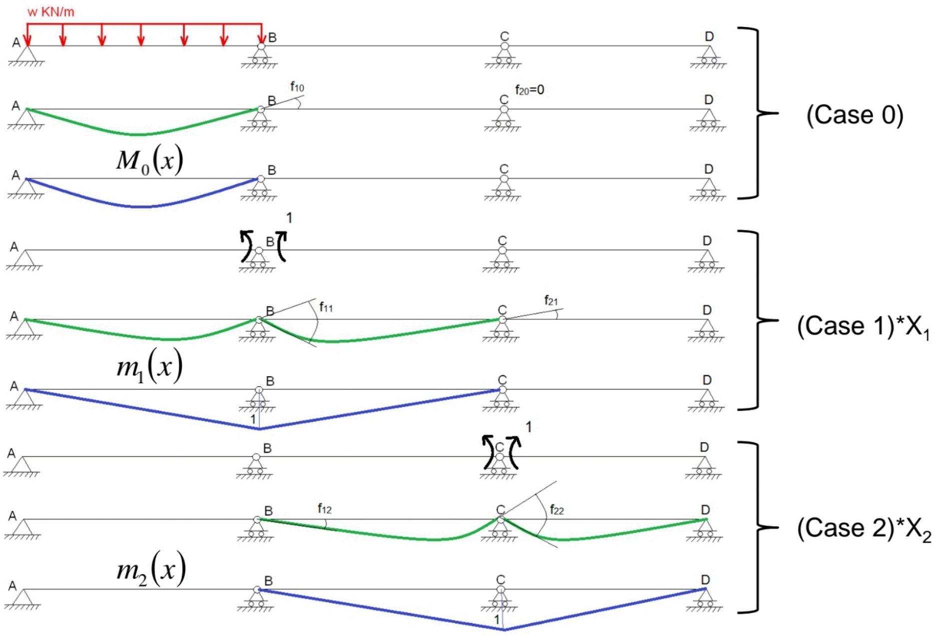
Structure (SIS)=
Case 0 (SDS)+X_1 (unknown)*Case 1(SDS)+X_2 (unknown)*Case 2 (SDS)

Case 0: SDS with external load

Case 1: SDS with unit load

Case 2: SDS with unit load

Step 3. Find the BMD for all cases.



Step 4. Find all the displacements for all cases at all hinges

f_{10} means displacement at hinge '1' for case '0'

$$f_{10}=\int_s \frac{M_0(x)}{EI}m_1(x)dx=\frac{L}{6EI}\left(0+4\frac{wL^2}{8}\frac{1}{2}+0\right)=\frac{wL^3}{24EI}$$

$$f_{11}=\int_s \frac{m_1(x)}{EI}m_1(x)dx=\frac{L}{6EI}\left(0+4\frac{1}{2}\frac{1}{2}+1\right)+\frac{L}{6EI}\left(1+4\frac{1}{2}\frac{1}{2}+0\right)=\frac{2L}{3EI}$$

tips is at which hinge, use which virtual case. eg. f_{21} is at hinge '2' so use 'm_2' ie. virtual case 2

$$f_{12}=\int_s \frac{m_2(x)}{EI}m_1(x)dx=\frac{L}{6EI}\left(0+4\frac{1}{2}\frac{1}{2}+0\right)=\frac{L}{6EI}$$

$$f_{20}=\int_s \frac{M_0(x)}{EI}m_2(x)dx=0$$

notice that $f_{12}=f_{21}$ as $m_1 \times m_2 = m_2 \times m_1$

$$f_{21}=\int_s \frac{m_1(x)}{EI}m_2(x)dx=\frac{L}{6EI}\left(0+4\frac{1}{2}\frac{1}{2}+0\right)=\frac{L}{6EI}$$

$$f_{22}=\int_s \frac{m_2(x)}{EI}m_2(x)dx=\frac{L}{6EI}\left(0+4\frac{1}{2}\frac{1}{2}+1\right)+\frac{L}{6EI}\left(1+4\frac{1}{2}\frac{1}{2}+0\right)=\frac{2L}{3EI}$$

Step 5. Displacement at each hinge equal to 0

$f_{10}+X_1f_{11}+X_2f_{12}=0$ (displacement at hinge '1' equal to 0)

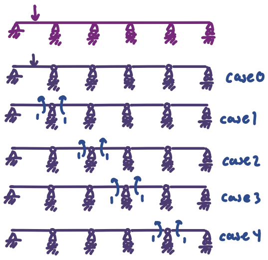
$f_{20}+X_1f_{21}+X_2f_{22}=0$ (displacement at hinge '2' equal to 0)

can be written as:

$$\begin{bmatrix} f_{10} \\ f_{20}\end{bmatrix}+\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{bmatrix}\begin{bmatrix} X_1 \\ X_2\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix}$$

for this example, $\begin{bmatrix} X_1 \\ X_2\end{bmatrix}=\begin{bmatrix} -\frac{wL^2}{15} \\ \frac{wL^2}{60}\end{bmatrix}$

for even higher redundancy, eg.



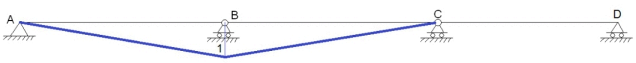
$$\begin{bmatrix} f_{10} \\ f_{20} \\ f_{30} \\ f_{40}\end{bmatrix}+\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44}\end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0\end{bmatrix}$$

only diagonal band will have non zero numbers

These are the reason why we choose to use add hinge instead of remove support as add hinge introduce lots of zeros.

Once the value of the unknowns are determined, the internal forces in the SIS are obtained by superposition of the internal forces in the three SDS (case 0,Case 1 and Case 2):

$M_0(x)$



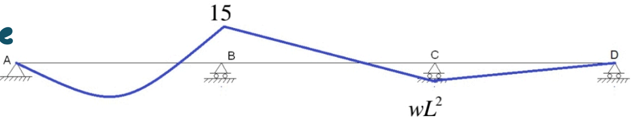
$m_1(x)X_1=-m_1(x)\frac{wL^2}{15}$

these are BMD before multiplying by X_1 and X_2

$m_2(x)X_2=m_2(x)\left(\frac{wL^2}{60}\right)$

after multiplying then we superpose

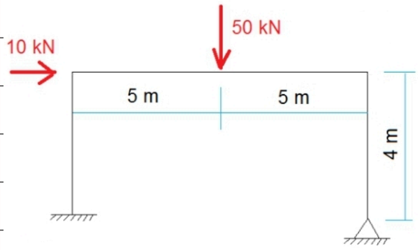
$-\frac{wL^2}{15}$



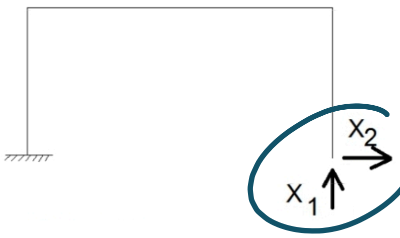
$\frac{wL^2}{60}$

Bending moment diagram of the SIS

1.4.2 Frames

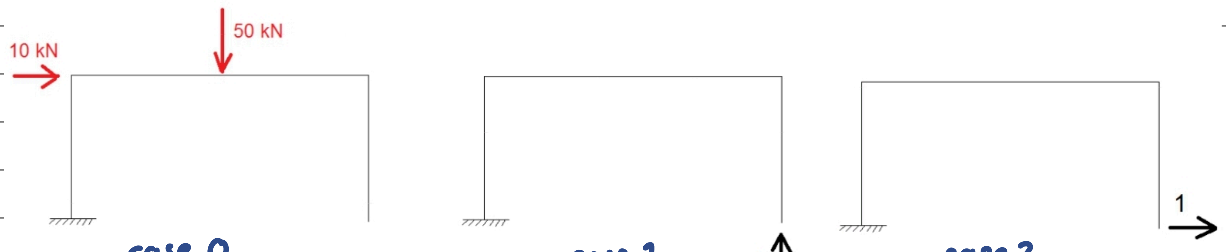


Step 1. Removes (redundancy) amount of supports, and replace with Freaction



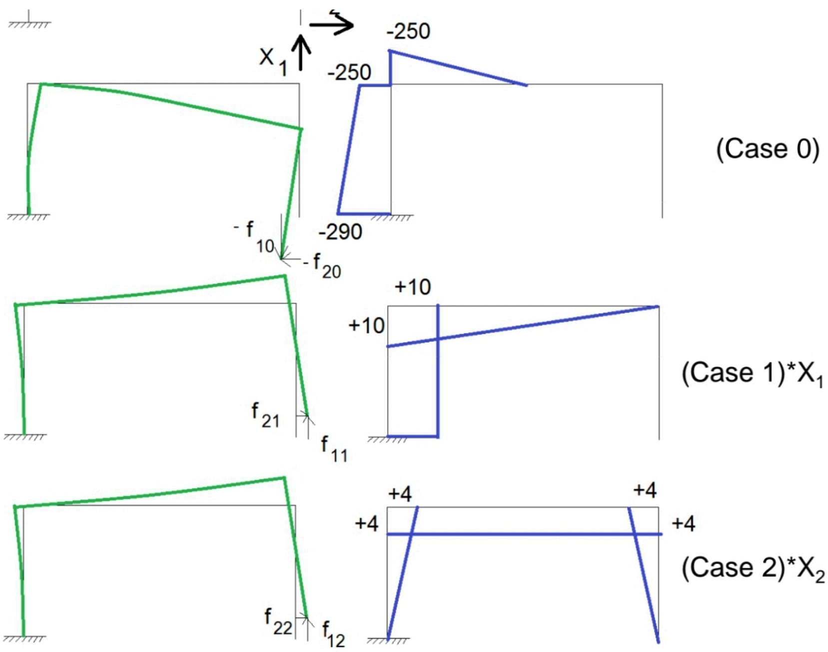
this is freely hanging, it is NOT a mechanism
as the other side is fixed end.

Step 2. Split into (redundancy + 1) amount of cases.



case 0 case 1 case 2

Step 3. Find BMD for all the cases.



Step 4. Find all of the displacements for all cases.

$$f_{10} = \int_s \frac{M_0(x)}{EI} m_1(x)dx = \frac{4}{6\cdot 20000}(-2900-4(2700)-2500)+\frac{5}{6\cdot 40000}(-2500-4(125\cdot 7.5)) = -0.670208m$$

$$f_{20} = \int_s \frac{M_0(x)}{EI} m_2(x)dx = \frac{4}{6\cdot 20000}(0-4(270\cdot 2)-250\cdot 4)+\frac{5}{6\cdot 40000}(-250\cdot 4-4(125\cdot 4)) = -0.167833m$$

$$f_{11} = \int_s \frac{m_1(x)}{EI} m_1(x)dx = \frac{4}{20000}(10\cdot 10)+\frac{10}{6\cdot 40000}(10\cdot 10+4\cdot 5\cdot 5) = 0.028333m/kN$$

$$f_{22} = \int_s \frac{m_2(x)}{EI} m_2(x)dx = \frac{4}{6\cdot 20000}(0+4\cdot 2\cdot 2+4\cdot 4)\cdot 2+\frac{10}{40000}(4\cdot 4) = 0.006133m/kN$$

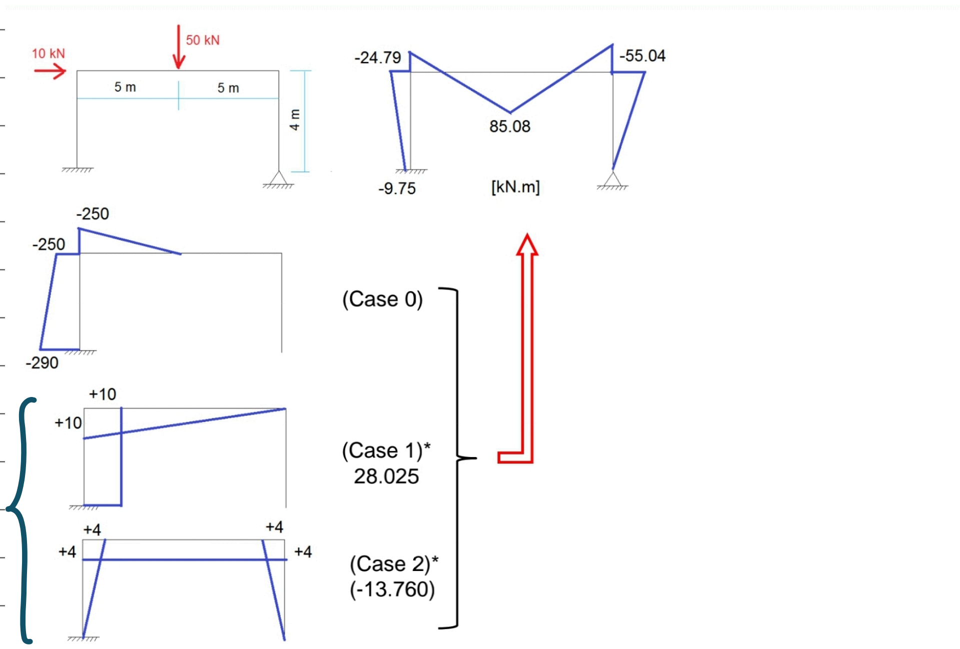
$$f_{12} = f_{21} = \int_s \frac{m_1(x)}{EI} m_2(x)dx =$$
$$\frac{4}{6\cdot 20000}(4\cdot 2\cdot 10+4\cdot 10)+\frac{10}{6\cdot 40000}4(10+4\cdot 5) =$$
$$= 0.009000m/kN$$

Step 5. Displacements at support = 0

$$\begin{bmatrix}-0.670208\\-0.167833\end{bmatrix}+\begin{bmatrix}0.028333 & 0.009000\\0.009000 & 0.006133\end{bmatrix}\begin{bmatrix}X_1\\X_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}X_1\\X_2\end{bmatrix}=\begin{bmatrix}28.025\\-13.760\end{bmatrix}kN$$

after soluing X_1 and X_2, we can superpose all three cases' BMD into final (original)'s BMD



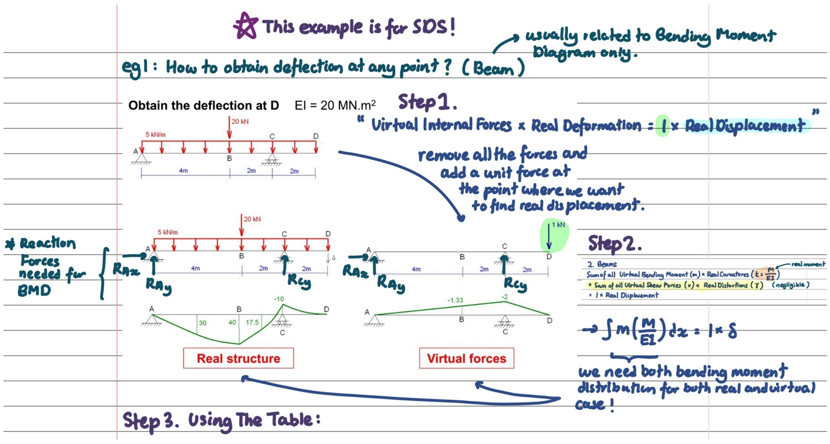
these haven't multiplied by X_1 and X_2. Remember to do so before superposition.

1.5 Finding Displacements in SIS

Quick recap for finding displacements in SDS

notice we create a virtual case where there is a unit force at the point where displacement is asked.

and that virtual case is SDS



☆ This example is for SDS!

eg1: How to obtain deflection at any point? (Beam) → usually related to Bending Moment Diagram only.

Obtain the deflection at D EI = 20 MN.m²

Step 1. " Virtual Internal Forces × Real Deformation = 1 × Real Displacement "

remove all the forces and add a unit force at the point where we want to find real displacement.

* Reaction Forces needed for BMD { R_{Ax}, R_{Ay}, R_{Cy}

Step 2.

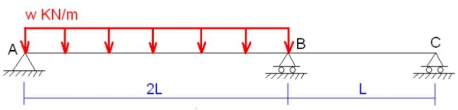
2. Beams
Sum of all Virtual Bending Moment (m) × Real Curvatures ($\kappa = \frac{M}{EI}$) ← real moment
+ Sum of all Virtual Shear Forces (v) × Real Distortions (γ) (negligible)
= 1 × Real Displacement

$\rightarrow \int m\left(\frac{M}{EI}\right)dx = 1 \times \delta$

we need both bending moment distribution for both real and virtual case!

Step 3. Using The Table:

Deflections in Statically Indeterminate Structures:

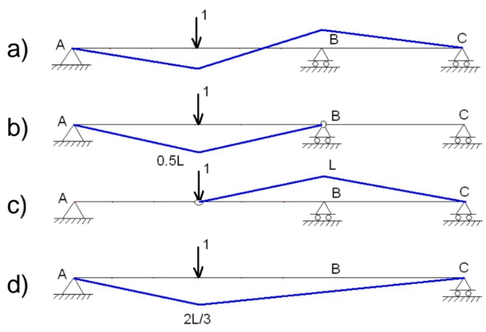


question: determine displacement at $x = L$ from left side (mid-span of first span AB)

Step 1. Create a virtual case

This is the virtual case (ie the case where all external load are removed, replaced with unit force at point where displacement is asked)

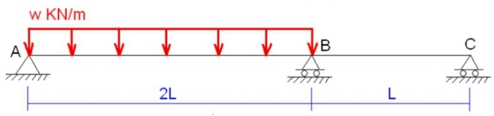
☆ **HOWEVER**, The shortcut is to remove redundancy from this virtual case, make it as simple as possible! (IT CAN BE SDS!) ☆



All four of these possible "virtual" case. (b) is the simplest as you can see. This is a shortcut because if the virtual case is a SIS, we have to solve 2 SIS by case 0, case 1, ... for twice!

(now only one time cause original still SIS)

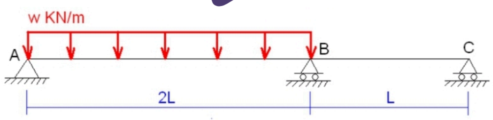
Step 2. Find the BMD for both Real and Virtual cases.



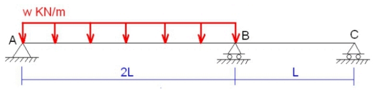
still find BMD for SIS like before

real case

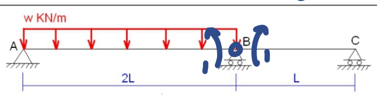
virtual + simplified case.



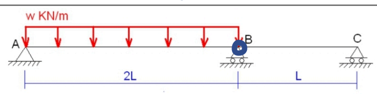
this is still a SIS, to find BMD on SIS, have to do everything like before, case 0 case 1...



↓ add hinge method

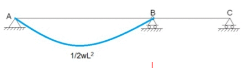


↓ split

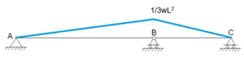


case 0

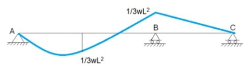
case 1



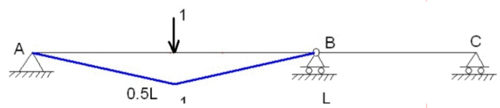
case 0's BMD



case 1 × X_1's BMD (already multiplied)



superposed BMD



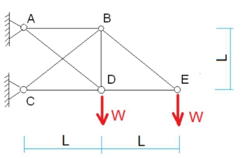
both of these, real and virtual BMD can now be used to find displacement with principle of virtual work.

Step 3. Find displacement.

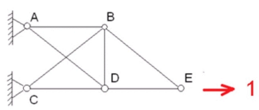
$$\delta = \frac{1}{EI}\left(\frac{5}{12}2L\frac{wL^2}{2}\frac{L}{2} - \frac{1}{4}2L\frac{wL^2}{3}\frac{L}{2}\right) = \frac{1}{EI}wL^4\left(\frac{5}{24}-\frac{1}{12}\right) = \frac{wL^4}{8EI}$$

Example for trusses.

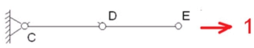
Horizontal displacement at E:



a)



b)



in this case, we prefer (b) cause it is SDS!

to continue solve this we have to find all axial force for both real (SIS) and virtual (SDS) case

use cut bar method

c2. Stiffness Method.

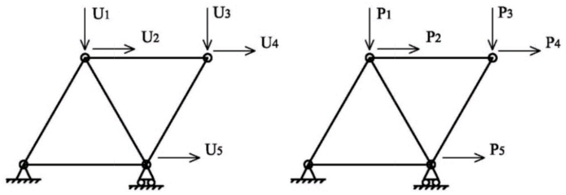
2.0 Introduction

Basically, it's an alternative method to solve SIS, by solving:

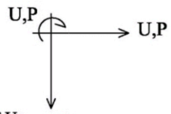
$\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$

where $\underset{\sim}{P}$ is column vector of all the nodal loads (force)
where $\underset{\sim}{U}$ is column vector of all the corresponding nodal displacement
where $\underset{\sim}{K}$ is the stiffness matrix

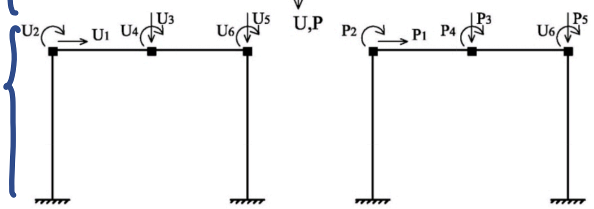
examples of $\underset{\sim}{P}$ and $\underset{\sim}{U}$ in trusses



standard sign convention for this chapter



examples of $\underset{\sim}{P}$ and $\underset{\sim}{U}$ in frames



2.1 Kinematic Indeterminancy. (It's similar to static indeterminancy in c1. Flexibility Method where we always refer as redundancy)

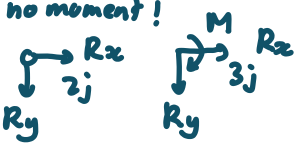
Kinematic Indeterminancy will tell us the dimension of stiffness matrix (It's also like redundancy in c1. which tell us how many case we need to split to solve the SIS)

☆ These are kinematic Indeterminancy of an assembled system!

for planar TRUSSES: $\beta = 2j - R + p - c$

for planar FRAMES and BEAMS: $\beta = 3j - R + p - c$

reason behind 2j and 3j's different is at the joint, trusses are pin joint so there is no moment!



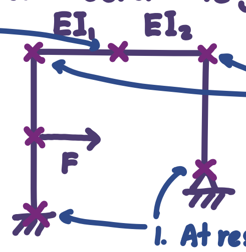
j = number of nodes (joint)
R = number of reactions
p = number of internal releases
c = number of kinematic constraints

☆ There is also a manual method to do this. Check c2.2 worked example 1

How to determine j?

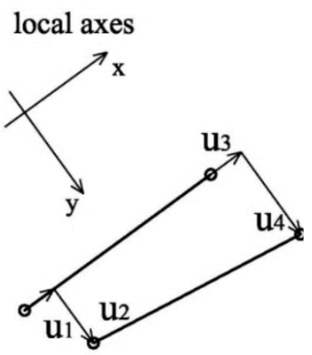
4. At change in material / section

3. At specific point where displacement are asked to calculate



2. At knee joints

1. At restrained points

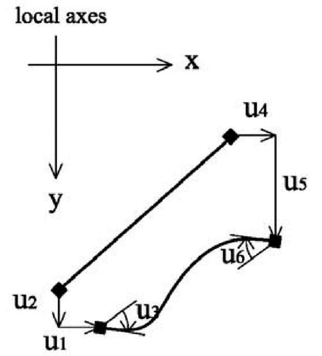


kinematic indeterminancy
for a single bar (that cannot bend)

$\beta = 4$

hence,

$$\underset{\sim}{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$



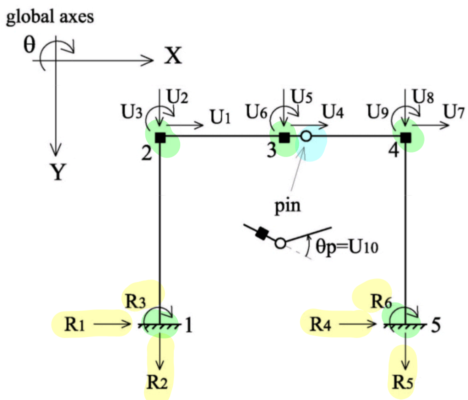
Kinematic indeterminancy
for a single beam (that can bend)

$\beta = 6$

hence,

$$\underset{\sim}{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

eg 1.



It is an assembled system of frames

$\beta = 3j - R + p - c$

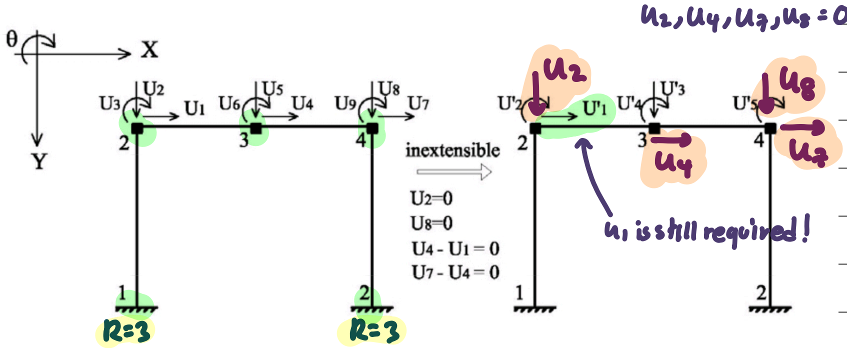
$j = 5$, $R = 6$, $p = 1$, $c = 0$

(p is internal release, in this case a hinge)

$\therefore \beta = 3(5) - 6 + 1 - 0 = 10$

eg 2. case where it was told that the bar is inextensible

$u_2, u_4, u_7, u_8 = 0$ (4 constraints, c)



It is an assembled system of frames

$\beta = 3j - R + p - c$: $j = 5$, $R = 6$, $p = 0$, $c = 4$

$\therefore \beta = 3(5) - 6 + 0 - 4 = 5$

★ HOW DO YOU DETERMINE C?

$\vec{u_1}$ ——— $\vec{u_2}$

if inextensible, the bar can still translate, but not extend. (i.e. $u_2 - u_1 = 0$)

$\vec{u_1}$ ———

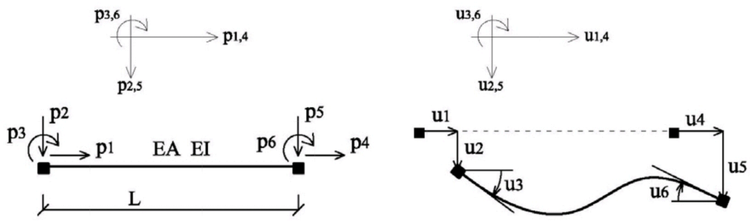
(don't need u_2 anymore as $u_2 = u_1$)

$\downarrow u_1$

if inextensible, $u_1 = 0$ as if $u_1 \neq 0$, the bar gets shorter. As the fixed end is not translatable

2.2 Stiffness Matrix, $\underset{\sim}{K}$ of a uniform elastic beam element (3 cases)

Case 1. Extensible Beam



Member end-forces

Member end-displacements

$\mathbf{p}_e = \{p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6\}^T$

$\mathbf{u}_e = \{u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6\}^T$

$\mathbf{p}_e = \mathbf{k}_e \cdot \mathbf{u}_e$

$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

how to intepret this $\underset{\sim}{K}$?

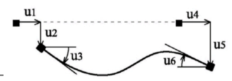
$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

every column is a "mode"

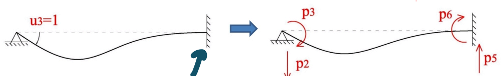
each mode is where its $u=1$ and other $u=0$

eg for mode 3 (ie column 3): $u_3 = 1$, $u_{i\neq 3} = 0$

mode 3:



set all $u=0$ except $u_3 = 1$:



fixed end as $u_6 = 0$

why no P_1 and P_4? cause $u_1 = 0$, $u_4 = 0$

but why got P_2 and P_5? cause it is required to produce u_3

Case 2: Inextensible Beam

$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

Why?
cause we can't have mode 1 and mode 4 where $u_1 = 1$, $u_{i\neq 1} = 0$ and $u_4 = 1$, $u_{i\neq 4} = 0$
which indirectly mean the beam is extending!

$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

$\mathbf{p}_e = \{p_2 \quad p_3 \quad p_5 \quad p_6\}_e^T$

$\mathbf{u}_e = \{u_2 \quad u_3 \quad u_5 \quad u_6\}_e^T$

Case 3. Inextensible beam, modelled with rotational freedom only

$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

why?
if only rotational freedom, means only have u_3 and u_6!
(i.e. mode 3 and mode 6!)

$$\mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

$\mathbf{p}_e = \{p_3 \quad p_6\}_e^T$

$\mathbf{u}_e = \{u_3 \quad u_6\}_e^T$

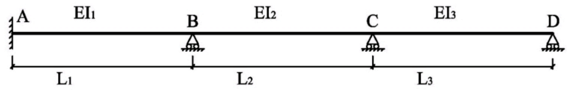
2.3 Stiffness Matrix, $\underset{\sim}{K}$ for an Assembled System

2.3.1 Beams

Worked example 1

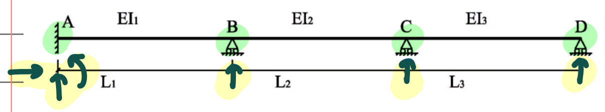
Structural stiffness matrix of an inextensible continuous beam.

Determine the structural stiffness matrix of the inextensible continuous beam shown below, which has a fixed support at A and roller supports at B, C and D.

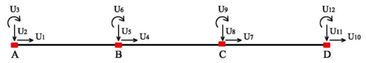


Step 1. Find kinematic indeterminancy (as this let us know the size of $\underset{\sim}{K}$)

There is actually two way to do so.
First method is using formula before.
We use " planar trusses and beams " formula: $\beta = 3j - R + p - c$



$j = 4$, $R = 6$, $p = 0$, $c = 3$

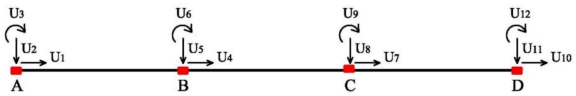


$\therefore \beta = 3(4) - 6 + 0 - 3$

$u_4 - u_1 = 0$
$u_7 - u_4 = 0$
$u_{10} - u_7 = 0$

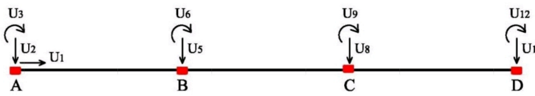
☆ There is actually an alternative way to find β, without formula

1. Draw all the reaction force at all node. (2 for truss, 3 for beam and frames) and then convert to displacement



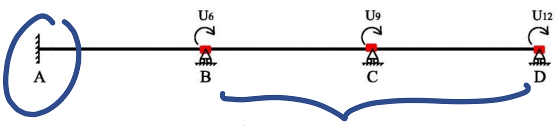
2. Delete everything that is fixed.

This step is equivalent to minus c



$u_4 = u_1$, $u_7 = u_4 = u_1$, $u_{10} = u_7 = u_4 = u_1$ cause inextensible. → eventually all $= 0$ cause $u_1 = 0$

this step is equivalent to minus R



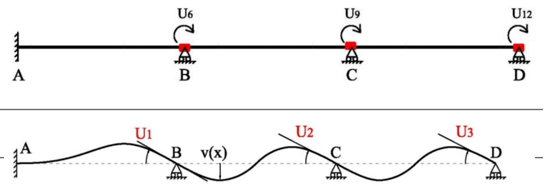
$\boxed{\beta = 3}$

A is a fixed support so $u_1, u_2, u_3 = 0$
(if pinned support u_3 wouldn't be 0)

$u_5, u_8, u_{11} = 0$ because it's a pinned support with vertical displacement restricted

after determining kinematic indeterminancy, $\beta = 3$
we now know that $\underset{\sim}{K}$ for the assembled system is a 3×3 matrix AND the possible displacement is something like this :

IMPORTANT INFORMATION

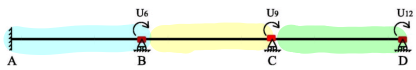


$$\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} \\ K_{2,1} & K_{2,2} & K_{2,3} \\ K_{3,1} & K_{3,2} & K_{3,3} \end{bmatrix}$$

Step 2. "Assemble" the simple beam element stiffness matrix (from c2.1) to form the stiffness matrix for the Assembled System

$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e \quad \mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e \quad \mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

there are three of this, which to choose for this SPECIFIC case?
→ the third 2×2 matrix one, cause...



... when split into 3 beam (AB, BC, CD)
the nodal displacement is described with rotational freedom ONLY

recall that column of the stiffness matrix means "mode"
where the nodal displacement of that mode = 1 and other = 0

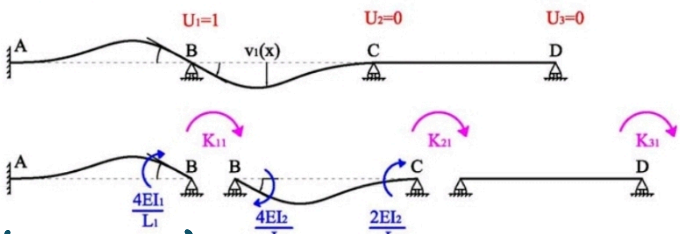
$$\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} \\ K_{2,1} & K_{2,2} & K_{2,3} \\ K_{3,1} & K_{3,2} & K_{3,3} \end{bmatrix}$$

mode 1: $u_1 = 1$, $u_2 = 0$, $u_3 = 0$

mode 2: $u_1 = 0$, $u_2 = 1$, $u_3 = 0$

mode 3: $u_1 = 0$, $u_2 = 0$, $u_3 = 1$

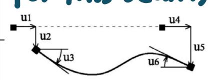
Basic mode 1:



split into three simple beam element

MODE 1 IN ASSEMBLED SYSTEM MIGHT NOT BE MODE 1 IN THE SIMPLE BEAM ELEMENT!

for this beam, $u_3 = 0, u_6 = 1$



(ignore u_1, u_2, u_4, u_5)

mode 3: $u_3 = 1, u_6 = 0$

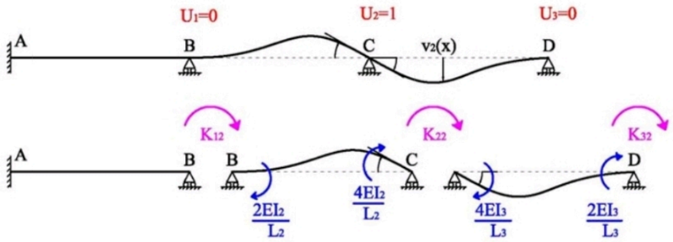
$$\mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

mode 6: $u_3 = 0, u_6 = 1$

we only need $\frac{4EI}{L}$ as that is corresponding to "mode 1-B" of assembled system.

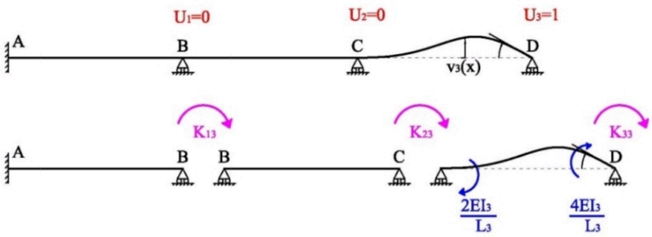
$$K_{1,1} = \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}; \quad K_{2,1} = \frac{2EI_2}{L_2}; \quad K_{3,1} = 0$$

Basic mode 2:



$$K_{1,2}=\frac{2EI_2}{L_2};\quad K_{2,2}=\frac{4EI_2}{L_2}+\frac{4EI_3}{L_3};\quad K_{3,2}=\frac{2EI_3}{L_3}$$

Basic mode 3:



$$K_{1,3}=0;\quad K_{2,3}=\frac{2EI_3}{L_3};\quad K_{3,3}=\frac{4EI_3}{L_3}$$

Thus the structural stiffness matrix is:

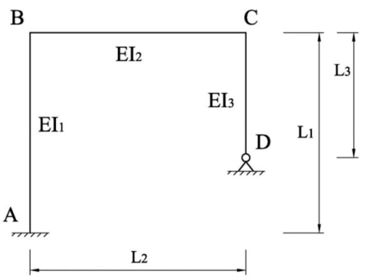
$$\mathbf{K}=\begin{bmatrix}\frac{4EI_1}{L_1}+\frac{4EI_2}{L_2} & \frac{2EI_2}{L_2} & 0\\ \frac{2EI_2}{L_2} & \frac{4EI_2}{L_2}+\frac{4EI_3}{L_3} & \frac{2EI_3}{L_3}\\ 0 & \frac{2EI_3}{L_3} & \frac{4EI_3}{L_3}\end{bmatrix}$$

2.3.2 Frames

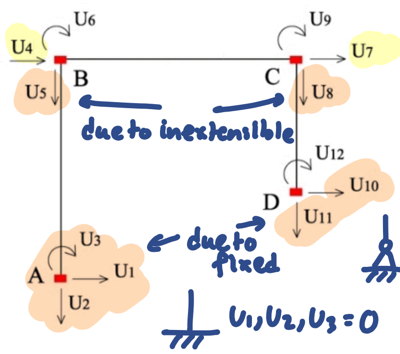
Worked example 2

Structural stiffness matrix of a portal frame with inextensible elements

Determine the structural stiffness matrix for the portal frame with inextensible elements shown below, which has a fixed base at A and a pinned base at D.



Step 1. Find kinematic indeterminancy (as this let us know the size of $\underset{\sim}{K}$)



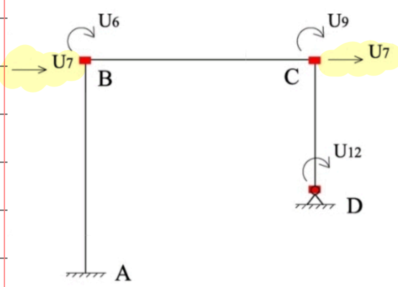
U_4 and U_7 choose one to represent both as $U_4 = U_7$

Inextensible elements:

$U_5 = U_2$; $U_8 = U_{11}$; $U_4 = U_7$

$U_{10}, U_{11} = 0$

$U_{12} \neq 0$ cause at hinge, the bar still free to rotate!



in this example we keep U_7 (we can keep U_4 and answer will be the same)

Accounting for the actual restraint conditions:

at A: $U_1 = 0$; $U_2 = 0$; $U_3 = 0$

at D: $U_{10} = 0$; $U_{11} = 0$

$\therefore$ kinematic indeterminancy, $\beta = 4$

$$\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} \\ K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} \\ K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} \\ K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} \end{bmatrix}$$

$\leftarrow$ Assembled system $\underset{\sim}{K}$ is a 4×4 matrix

Step 2. "Assemble" the simple beam element stiffness matrix (from c2.1) to form the stiffness matrix for the Assembled System

$$\mathbf{k}_e = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

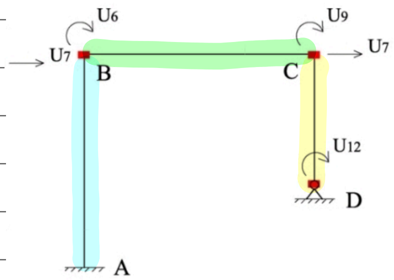
$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

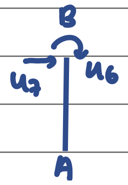
$$\mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

✪ note that when splitting, BC shouldn't be :

cause the bar itself doesn't extend ! it translate, RELATIVE to AB and CD !

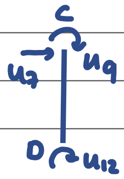
when split the frame into individual bar :





$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

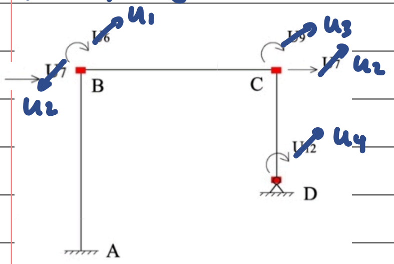
$$\mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$



$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

✪ usually rotational freedom ie the " ↷ " need to include twice but translation freedom ie the " → " only once IF it's inextensible !

for simplicity let's rename the assembled system 'u' (ascending in u)

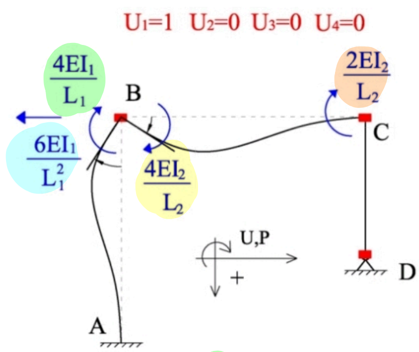


(mode 2) $u_{i=2}=1$, $u_{i\neq 2}=0$ and so on...

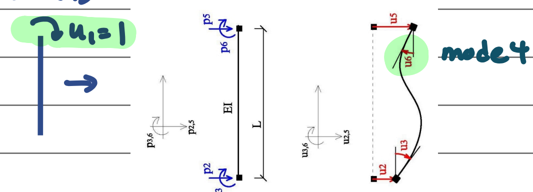
$$\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} & K_{1,3} & K_{1,4} \\ K_{2,1} & K_{2,2} & K_{2,3} & K_{2,4} \\ K_{3,1} & K_{3,2} & K_{3,3} & K_{3,4} \\ K_{4,1} & K_{4,2} & K_{4,3} & K_{4,4} \end{bmatrix}$$

(mode 1) $u_{i=1}=1$, $u_{i\neq 1}=0$

Basic mode 1:



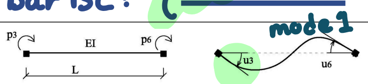
bar AB :



$$\mathbf{k}_e = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e$$

} don't need these two as they are not inc. in the assembled system !

bar BC : $u_1=1$



$$\mathbf{k}_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

(don't need do CD as $u_1=1$ is not in the stiffness matrix of CD)

$$K_{1,1} = \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}$$

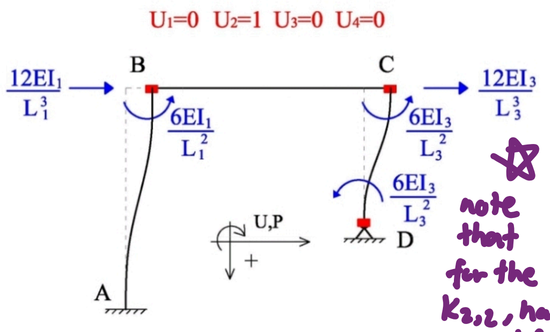
$$K_{2,1} = -\frac{6EI_1}{L_1^2}$$

$$K_{3,1} = \frac{2EI_2}{L_2}$$

$$K_{4,1} = 0$$

($K_{1,1}$ can be referred as " K correspond to U_1, for mode 1 ",
$K_{2,1}$ " " K correspond to U_2, for mode 1 ",...)

Basic mode 2:



$$K_{1,2} = -\frac{6EI_1}{L_1^2}$$

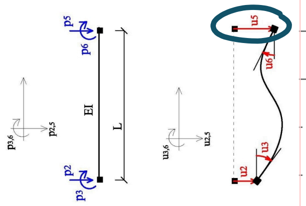
$$K_{2,2} = \frac{12EI_1}{L_1^3} + \frac{12EI_3}{L_3^3}$$

$$K_{3,2} = -\frac{6EI_3}{L_3^2}$$

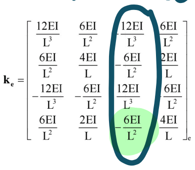
$$K_{4,2} = -\frac{6EI_3}{L_3^2}$$

note that for the $K_{2,2}$, have to account for both AB's and CD's

example: $K_{1,2}$, U_1 for mode 2



mode 2 is this = | which is mode 3 in simple bar $\underset{\sim}{k}$

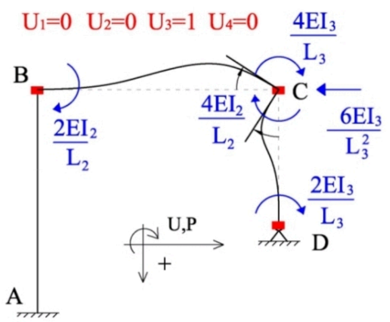


for $K_{1,2}$, U_1 for mode 2

U_1 is U_6 in simple bar

(u_1 in assembled is the rotational freedom on top!)

Basic mode 3:



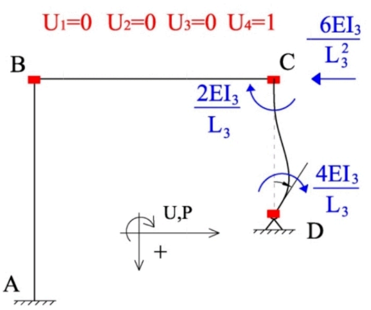
$$K_{1,3} = \frac{2EI_2}{L_2}$$

$$K_{2,3} = -\frac{6EI_3}{L_3^2}$$

$$K_{3,3} = \frac{4EI_2}{L_2} + \frac{4EI_3}{L_3}$$

$$K_{4,3} = \frac{2EI_3}{L_3}$$

Basic mode 4:



$$K_{1,4} = 0$$

$$K_{2,4} = -\frac{6EI_3}{L_3^2}$$

$$K_{3,4} = \frac{2EI_3}{L_3}$$

$$K_{4,4} = \frac{4EI_3}{L_3}$$

Thus the structural stiffness matrix is:

$$\mathbf{K} = \begin{bmatrix} \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2} & -\frac{6EI_1}{L_1^2} & \frac{2EI_2}{L_2} & 0 \\ -\frac{6EI_1}{L_1^2} & \frac{12EI_1}{L_1^3} + \frac{12EI_3}{L_3^3} & -\frac{6EI_3}{L_3^2} & -\frac{6EI_3}{L_3^2} \\ \frac{2EI_2}{L_2} & -\frac{6EI_3}{L_3^2} & \frac{4EI_2}{L_2} + \frac{4EI_3}{L_3} & \frac{2EI_3}{L_3} \\ 0 & -\frac{6EI_3}{L_3^2} & \frac{2EI_3}{L_3} & \frac{4EI_3}{L_3} \end{bmatrix}$$

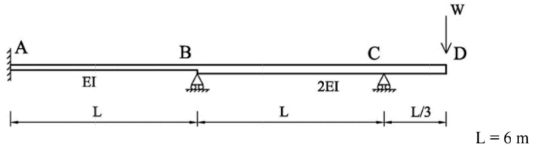
(also known as solving $P = KU$)

2.4 Find SFBMD after determining stiffness matrix, $\underset{\sim}{K}$

eg1. **Worked example 3**

Continuous inextensible beam

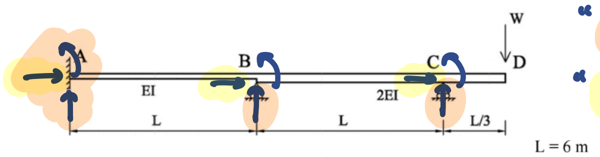
The continuous beam shown below is subjected to a point load W = 30 kN at the free-end D. Treating AB and BC as inextensible elements with flexural rigidity EI = 4 MNm^2 and EI = 8 MNm^2 respectively, calculate the independent nodal displacements and sketch the bending moment distribution.



L = 6 m

Step 1. Find kinematic indeterminancy, β

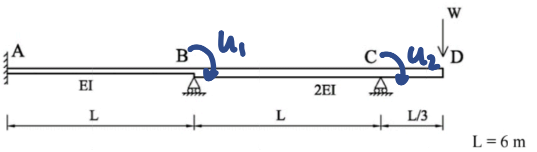
step 1&2 same as before, just to find $\underset{\sim}{K}$



L = 6 m

" " remove R, reaction force.

" " inextensibility, keep 1. (But since the most end already 0 → all 0 → remove all)



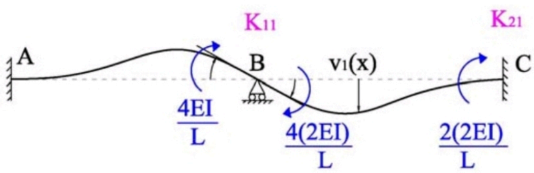
L = 6 m

∴ Structural stiffness matrix: $\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{bmatrix}$

cause $\beta = 2$, $\underset{\sim}{K}: 2\times2$

Step 2. "Assemble" the simple beam element stiffness matrix (from c2.1) to form the stiffness matrix for the Assembled System (i.e. find assembled system stiffness matrix, $\underset{\sim}{K}$)

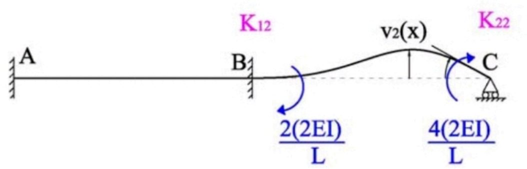
Basic mode 1: $U_1 = 1,\ U_2 = 0$



$$K_{1,1} = \frac{4EI}{L} + \frac{4(2EI)}{L} = \frac{12EI}{6} = 2EI$$

$$K_{2,1} = \frac{2(2EI)}{L} = \frac{2EI}{3}$$

Basic mode 2: $U_1 = 0,\ U_2 = 1$



$$K_{2,1} = \frac{2(2EI)}{L} = \frac{2EI}{3}$$

$$K_{2,2} = \frac{4(2EI)}{L} = \frac{4EI}{3}$$

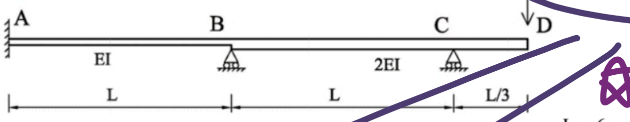
Thus: $\mathbf{K} = \frac{EI}{3}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$

to find u_1 and u_2 from $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$, we need $\underset{\sim}{P}$!

Step 3. Find $\underset{\sim}{P}$

Finding $\underset{\sim}{P}$ can be really hard sometimes. This is the most simple example.

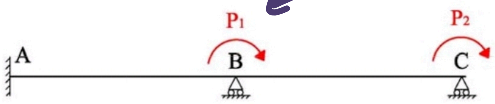
w = 30kN (given in the question)



L = 6 m

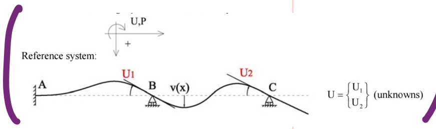
☆ we need to convert any external load to the assembled system nodal load!

Assemble system nodal loads



$$P_2 = 30 \cdot \frac{6}{2} = 60 \text{ kNm}$$

$$P = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \end{Bmatrix} \text{ [KNm]}$$



this is the assembled system after we find kinematic indeterminacy

we can split however we want, but setting one of the P=0 and put the load onto one of them is easier!

Step 4. Solve $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ for $\underset{\sim}{U}$

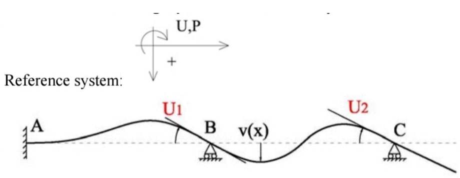
Solve the system stiffness equations for nodal displacements

System stiffness equations: $[K]\{U\} = \{P\}$

$$\frac{EI}{3}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \end{Bmatrix} \rightarrow \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} -18/EI \\ 54/EI \end{Bmatrix}$$

After having $\underset{\sim}{U}$, how do we obtain SFBMD?
(remember in flexibility method, we just superpose the SFBMD for case 0, case 1 · X_1, ...)
For Stiffness Method, it is a bit more complicated.

Step 5. Find SFBMD, by solving $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ (element's $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ NOT assembled system's)



$$\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} -18/EI \\ 54/EI \end{Bmatrix}$$

Element AB:

0 cause only U_1 (θ at B) and U_2 (θ at C) $\neq 0$!

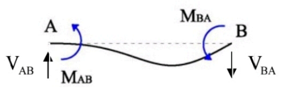
Element end displacements (rotations): $\phi_{AB} = 0$, $\phi_{BA} = U_1 = \frac{-18}{EI}$

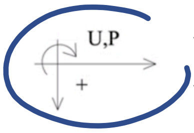
Element stiffness matrix: $K_{AB} = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

This is not assembled system's $\underset{\sim}{K}$! instead, it is element $\underset{\sim}{K}$:

$$k_e = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}_e$$

Element end-moments: $\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = K_{AB}\begin{Bmatrix} \phi_{AB} \\ \phi_{BA} \end{Bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ -18/EI \end{Bmatrix} = \begin{Bmatrix} -6 \\ -12 \end{Bmatrix}$ kNm





M, and V follows this sign convention.

End-shear can be calculated considering the free body diagram as:

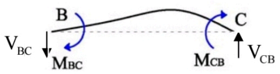
$$V_{AB} = -V_{BA} = (M_{AB} + M_{BA})/L_{AB} = (-6-12)/6 = -3 \text{ kN}$$

Element BC:

Element end displacements (rotations): $\phi_{BC} = U_1 = \frac{-18}{EI}, \quad \phi_{CB} = U_2 = \frac{54}{EI}$

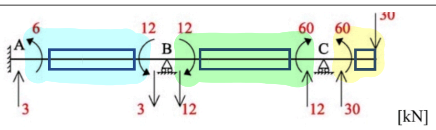
Element stiffness matrix: $K_{BC} = \frac{2EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

Element end-moments: $\begin{Bmatrix} M_{BC} \\ M_{CB} \end{Bmatrix} = K_{BC}\begin{Bmatrix} \phi_{BC} \\ \phi_{CB} \end{Bmatrix} = \frac{EI}{3}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} \frac{-18}{EI} \\ \frac{54}{EI} \end{Bmatrix} = \begin{Bmatrix} 12 \\ 60 \end{Bmatrix} \text{kNm}$

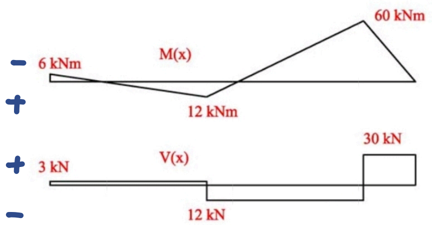


End-shear can be calculated considering the free body diagram as:

$V_{BC} = -V_{CB} = (M_{BC} + M_{CB})/L_{BC} = (12+60)/6 = 12 \text{ kN}$



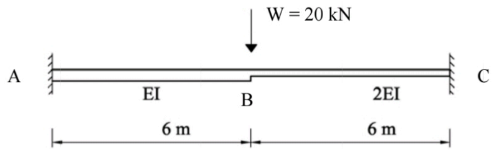
Bending moment and shear force distribution



eg2. **Worked example 4**

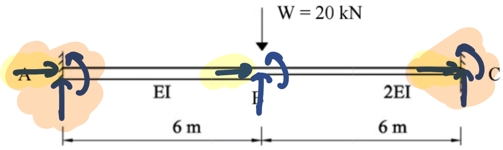
Non- uniform inextensible beam

The fixed-ends beam shown below is subjected to a point load W = 20 kN at mid-span and it has a non-uniform flexural rigidity. Treating the beam as inextensible, calculate the independent nodal displacements and sketch the bending moment distribution.



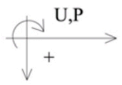
L = 6 m, EI = 4 MNm^2

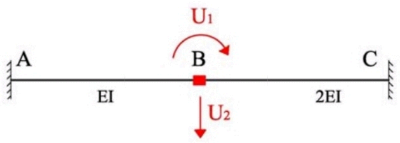
Step 1. Find kinematic Indeterminancy, β



L = 6 m, EI = 4 MNm^2

Reference system:





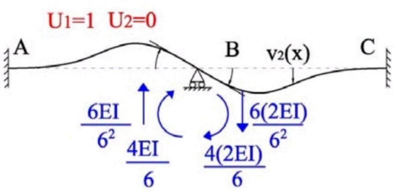
$U = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$ (unknowns)

(U_1 rotational freedom, U_2 sway freedom)

Structural stiffness matrix: $\mathbf{K} = \begin{bmatrix} K_{1,1} & K_{1,2} \\ K_{2,1} & K_{2,2} \end{bmatrix}$

Step 2. "Assemble" the simple beam element stiffness matrix (from c2.1) to form the stiffness matrix for the Assembled System (i.e. find assembled system stiffness matrix, $\underset{\sim}{K}$)

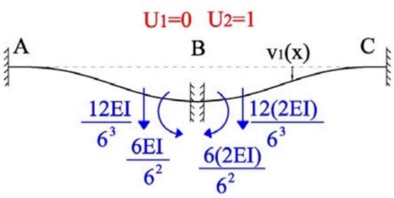
Basic mode 1: $U_1 = 1,\ \ U_2 = 0$



$$K_{11} = \frac{4EI}{6} + \frac{4(2EI)}{6} = 2EI$$

$$K_{2,1} = -\frac{6EI}{6^2} + \frac{6(2EI)}{6^2} = \frac{EI}{6}$$

Basic mode 2: $U_1 = 0,\ \ U_2 = 1$

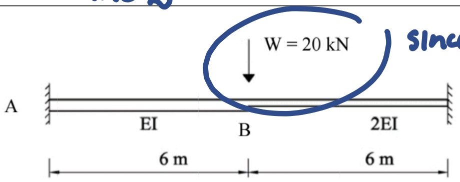


$$K_{2,1} = -\frac{6EI}{6^2} + \frac{6(2EI)}{6^2} = \frac{EI}{6}$$

$$K_{2,2} = \frac{12(2EI)}{6^2} + \frac{12EI}{6^2} = \frac{EI}{6}$$

Thus: $\mathbf{K} = \dfrac{EI}{6}\begin{bmatrix} 12 & 1 \\ 1 & 1 \end{bmatrix}$

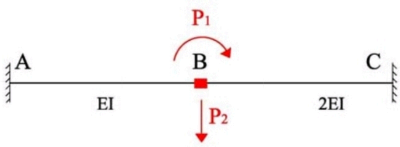
Step 3. Find $\underset{\sim}{P}$



since w on top of B, the only way to do this is $P_2 = w$, $P_1 = 0\,kNm$

L = 6 m, EI = 4 MNm^2

Assemble system nodal loads



$$P = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20\text{ kN} \end{Bmatrix}$$

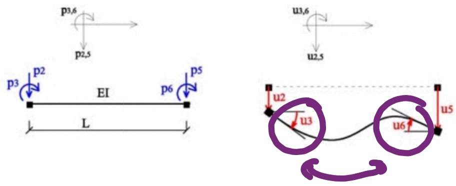
Step 4. Solve $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ for $\underset{\sim}{U}$

Solve the system stiffness equations for nodal displacements

System stiffness equations: $[K]\{U\} = \{P\}$

$$\frac{EI}{6}\begin{bmatrix} 12 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix} \rightarrow \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} -120/11EI \\ 1440/11EI \end{Bmatrix}$$

Step 5. Find SFBMD, by solving $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ (element's $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ NOT assembled system's)



$$\begin{bmatrix} \cancel{p_2} \\ p_3 \\ \cancel{p_5} \\ p_6 \end{bmatrix}_e = \begin{bmatrix} \cancel{\frac{12EI}{L^3}} & \cancel{\frac{6EI}{L^2}} & \cancel{-\frac{12EI}{L^3}} & \cancel{\frac{6EI}{L^2}} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ \cancel{-\frac{12EI}{L^3}} & \cancel{-\frac{6EI}{L^2}} & \cancel{\frac{12EI}{L^3}} & \cancel{-\frac{6EI}{L^2}} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}_e \begin{bmatrix} u_2 \\ u_3 \\ u_5 \\ u_6 \end{bmatrix}_e$$

only u (rotational) is required

☆☆ ☆ REMEMBER for this step, we only need to find moments. for shear we can use "$V = \frac{M_1 + M_2}{2}$" hence for $\underset{\sim}{P} = \underset{\sim}{K}\underset{\sim}{U}$ (element) we can ignore everything except the rotational U.

Element AB:

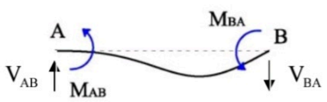
Element end displacements (rotations): $\phi_{AB} = 0,\ \Delta_{AB} = 0$

$$\phi_{BA} = U_1 = \frac{-120}{11EI},\quad \Delta_{BA} = U_2 = \frac{1440}{11EI}$$

Element stiffness matrix: $K_{AB} = \begin{bmatrix} \frac{6EI}{L^2} & \frac{4EI}{L} & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 1 & 4 & -1 & 2 \\ 1 & 2 & -1 & 4 \end{bmatrix}$

Element end-moments: $\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = K_{AB}\begin{Bmatrix} \Delta_{AB} \\ \phi_{AB} \\ \Delta_{BA} \\ \phi_{BA} \end{Bmatrix} = \frac{EI}{6}\begin{bmatrix} 1 & 4 & -1 & 2 \\ 1 & 2 & -1 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ \frac{1440}{11EI} \\ \frac{-120}{11EI} \end{Bmatrix} = \begin{Bmatrix} -25.45 \\ -29.09 \end{Bmatrix}\text{kNm}$

EI = 4 MNm^2



End-shear can be calculated considering the free body diagram as:

$$V_{AB} = -V_{BA} = (M_{AB} + M_{BA})/L_{AB} = (-25.45 - 29.09)/6 = -9.09\text{ kN}$$

Element BC:

Element end displacements (rotations): $\phi_{BC} = U_1 = \dfrac{-120}{11EI}$, $\Delta_{BC} = U_2 = \dfrac{1440}{11EI}$

$\phi_{CB} = 0, \ \ \Delta_{CB} = 0$

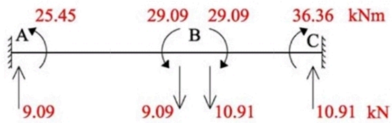
Element stiffness matrix: $K_{BC} = \begin{bmatrix} \dfrac{6(2EI)}{L^2} & \dfrac{4(2EI)}{L} & \dfrac{-6(2EI)}{L^2} & \dfrac{2(2EI)}{L} \\ \dfrac{6(2EI)}{L^2} & \dfrac{2(2EI)}{L} & \dfrac{-6(2EI)}{L^2} & \dfrac{4(2EI)}{L} \end{bmatrix} = \dfrac{EI}{3}\begin{bmatrix} 1 & 4 & -1 & 2 \\ 1 & 2 & -1 & 4 \end{bmatrix}$

Element end-moments: $\begin{Bmatrix} M_{BC} \\ M_{CB} \end{Bmatrix} = K_{BC}\begin{Bmatrix} \Delta_{BC} \\ \phi_{BC} \\ \Delta_{CB} \\ \phi_{CB} \end{Bmatrix} = \dfrac{EI}{3}\begin{bmatrix} 1 & 4 & -1 & 2 \\ 1 & 2 & -1 & 4 \end{bmatrix}\begin{Bmatrix} \dfrac{1440}{11EI} \\ \dfrac{-120}{11EI} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 29.09 \\ 36.36 \end{Bmatrix} \text{kNm}$

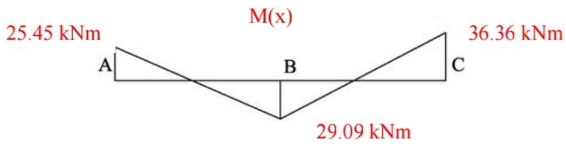
$EI = 8 \ \text{MNm}^2$

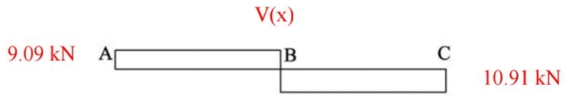
End-shear can be calculated considering the free body diagram as:

$V_{BC} = -V_{CB} = (M_{BC} + M_{CB})/L_{BC} = (29.09 + 36.36)/6 = 10.91 \ \text{kN}$



Bending moment and shear force distribution

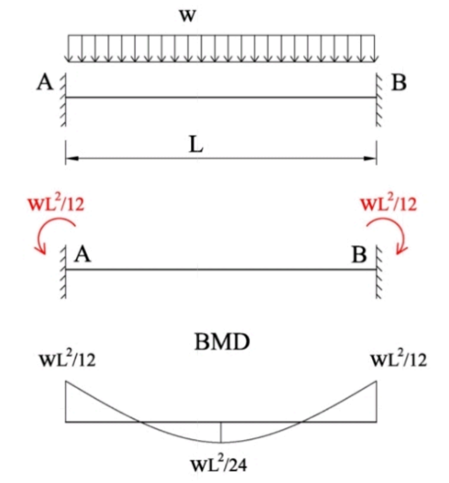




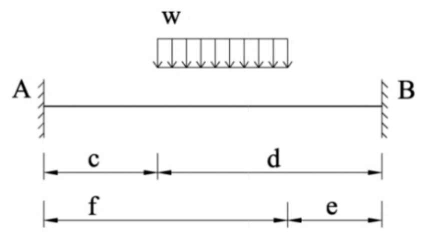
Note that the variation in shear at B is equal to the point load W = 20 kN at B.

☆ given in exams. (harder question sometimes require this method to find P)

2.5 Fixed End Moments and Shears

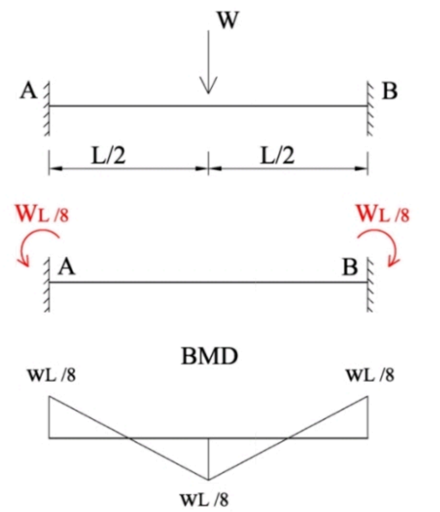


$$M^F_{AB} = -\frac{wL^2}{12} \qquad M^F_{BA} = +\frac{wL^2}{12}$$

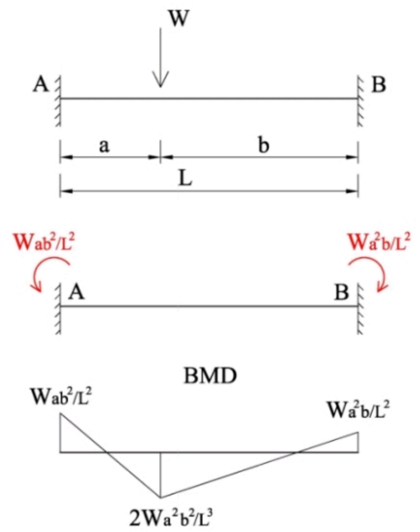


$$M^F_{AB} = -\frac{w}{12L^2}\left[d^3(4L-3d) - e^3(4L-3e)\right]$$

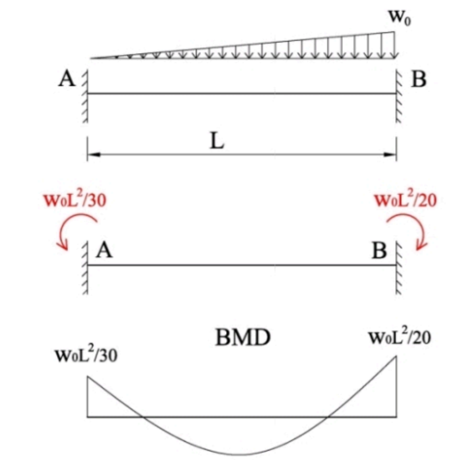
$$M^F_{BA} = \frac{w}{12L^2}\left[f^3(4L-3f) - c^3(4L-3c)\right]$$



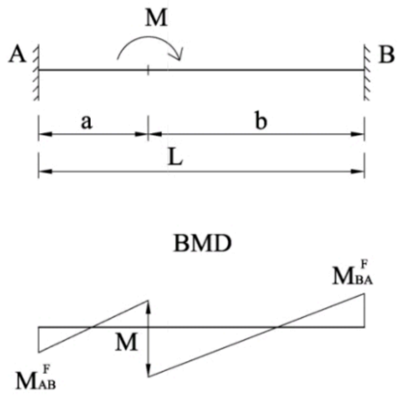
$$M^F_{AB} = -\frac{WL}{8} \qquad M^F_{BA} = +\frac{WL}{8}$$



$$M^F_{AB} = -\frac{Wab^2}{L^2} \qquad M^F_{BA} = +\frac{Wa^2b}{L^2}$$



$$M^F_{AB} = -\frac{w_0L^2}{30} \qquad M^F_{BA} = +\frac{w_0L^2}{20}$$



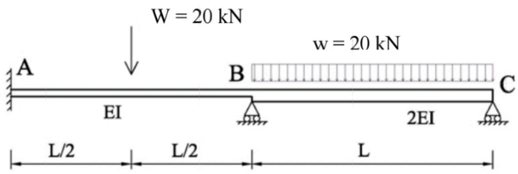
$$M^F_{AB} = +\frac{Mb}{L^2}(2a-b)$$

$$M^F_{BA} = +\frac{Ma}{L^2}(2b-a)$$

eg1. **Worked example 5** ☆ check step3!

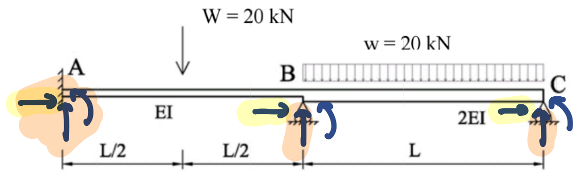
Non-uniform inextensible continuous beam carrying element loads

The continuous beam with uniform flexural rigidity shown below is subjected to a point load W = 16 kN at mid-span of span AB and a load w = 10 kN/m uniformly distributed along span BC. Treating the beam as inextensible, calculate the independent nodal displacements and sketch the bending moment distribution.



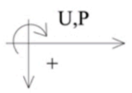
L = 6 m, EI = 4 MNm²

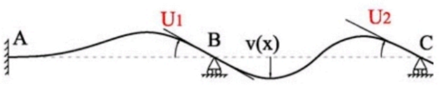
Step 1. Find kinematic Indeterminancy, β



L = 6 m, EI = 4 MNm²

Reference system:

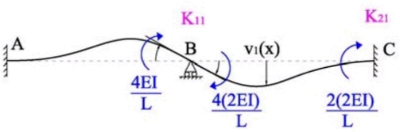




$$U = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \text{ (unknowns)}$$

Step 2. "Assemble" the simple beam element stiffness matrix (from c2.1) to form the stiffness matrix for the Assembled System (i.e. find assembled system stiffness matrix, $\underset{\sim}{K}$)

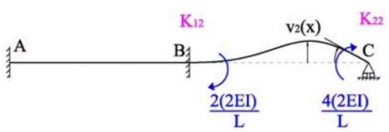
Basic mode 1: $U_1 = 1,\ U_2 = 0$



$$K_{11} = \left(\frac{4EI}{L} + \frac{4(2EI)}{L}\right) = \frac{12EI}{6} = 2EI$$

$$K_{2,1} = \frac{2(2EI)}{L} = \frac{4EI}{6} = \frac{2EI}{3}$$

Basic mode 2: $U_1 = 0,\ U_2 = 1$



$$K_{2,1} = \frac{2(2EI)}{L} = \frac{4EI}{6} = \frac{2EI}{3}$$

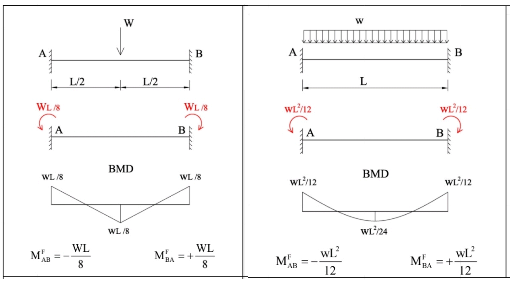
$$K_{2,2} = \frac{4(2EI)}{L} = \frac{8EI}{6} = \frac{4EI}{3}$$

Thus: $\mathbf{K} = \frac{EI}{3}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$

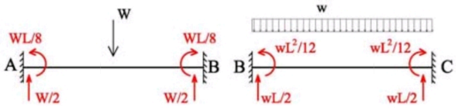
Step 3. Find $\underset{\sim}{P}$

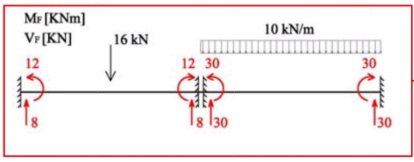
Assemble system nodal loads

Calculation of equivalent nodal loads using fixed-end forces:



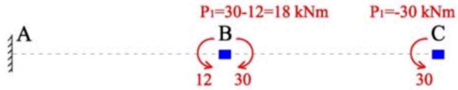
Fixed end forces:





$$\begin{cases} M^F_{AB} = -12 \text{ kNm} \\ M^F_{BA} = 12 \text{ kNm} \\ M^F_{BC} = -30 \text{ kNm} \\ M^F_{CB} = 30 \text{ kNm} \end{cases}$$

Equivalent nodal forces are equal and opposite to fixed-end forces as they represent forces applied from the structure to the nodes.



$$P = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} 18 \\ -30 \end{Bmatrix} \text{ kNm}$$

Step 4. Solve $\underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{U}$ for $\underset{\sim}{U}$

Solve the system stiffness equations for nodal displacements

System stiffness equations: $[K]\{U\} = \{P\}$

$$\frac{EI}{3}\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 18 \\ -30 \end{Bmatrix} \rightarrow \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 19.8/EI \\ -32.4/EI \end{Bmatrix}$$

Step 5. Find SFBMD, by solving $\underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{U}$ (element's $\underset{\sim}{P} = \underset{\sim}{K}\,\underset{\sim}{U}$ NOT assembled system's)

Element AB:

Element end displacements (rotations): $\phi_{AB} = U_1 = \frac{19.8}{EI}$, $\phi_{BA} = 0$

Element stiffness matrix: $K_{AB} = \frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

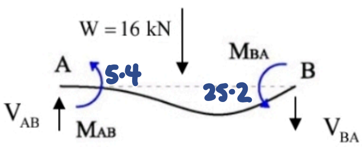
instead of P=KU only this time (for member's stiffness equation),

Element end-moments: $\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = K_{AB}\begin{Bmatrix} \phi_{AB} \\ \phi_{BA} \end{Bmatrix} + \begin{Bmatrix} M^F_{AB} \\ M^F_{BA} \end{Bmatrix} = \frac{EI}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{Bmatrix} 0 \\ 19.8/EI \end{Bmatrix} + \begin{Bmatrix} -12 \\ 12 \end{Bmatrix} = \begin{Bmatrix} -5.4 \\ 25.2 \end{Bmatrix}$ kNm

$P = KU + P_0$

where P_0 is the fixed end moment we had in step 3

this is ONLY NEEDED if we use "fixed end moment" method in step 3.



End-shear can be calculated considering the free body diagram as:

$V_{AB} = (M_{AB} + M_{BA})/L_{AB} - W/2 = (-5.4 + 25.2)/6 - 8 = -4.7$ kN

$V_{BA} = -(M_{AB} + M_{BA})/L_{AB} - W/2 = -(-5.4 + 25.2)/6 - 8 = -11.3$ kN

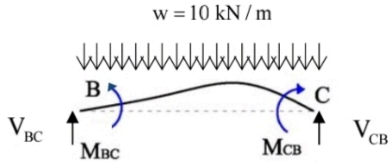
end shear also need to include the external load!

Element BC:

Element end displacements (rotations): $\phi_{BC} = U_1 = \dfrac{19.8}{EI}$, $\phi_{CB} = U_2 = \dfrac{-32.4}{EI}$

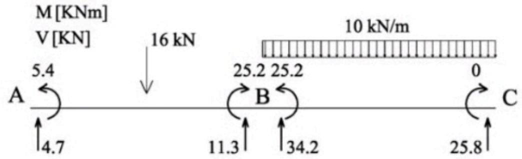
Element stiffness matrix: $K_{BC} = \dfrac{2EI}{L}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix} = \dfrac{EI}{3}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}$

Element end-moments: $\begin{Bmatrix}M_{BC}\\ M_{CB}\end{Bmatrix} = K_{BC}\begin{Bmatrix}\phi_{BC}\\ \phi_{CB}\end{Bmatrix} + \begin{Bmatrix}M^F_{BC}\\ M^F_{CB}\end{Bmatrix} = \dfrac{EI}{3}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}19.8/EI\\ -32.4/EI\end{bmatrix} + \begin{Bmatrix}-30\\ 30\end{Bmatrix} = \begin{Bmatrix}-25.2\\ 0\end{Bmatrix}$ kNm



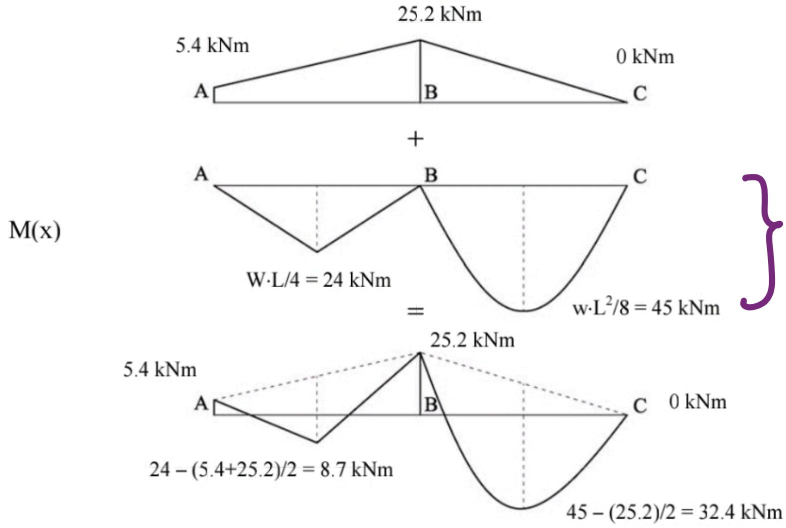
End-shear can be calculated considering the free body diagram as:

$V_{BC} = (M_{BC} + M_{CB})/L_{BC} - wL/2 = (-25.2)/6 - 10\cdot 6/2 = -34.2$ kN
$V_{CB} = -(M_{BC} + M_{CB})/L_{BC} - wL/2 = -(-25.2)/6 - 10\cdot 6/2 = -25.8$ kN

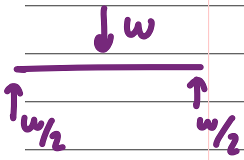


Note that the end moments at B are equal and opposite, representing moment equilibrium at the joint where no external concentrated moment is applied.

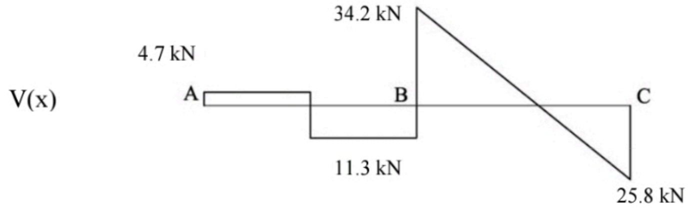
Bending moment and shear force distribution

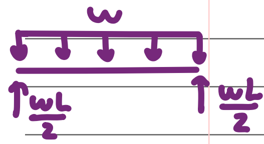


EXTREMELY IMPORTANT
Remember to add the element load:



$\Sigma M_{x=L/2} = 0$
$\frac{w}{2}\left(\frac{L}{2}\right) - M = 0$
$M = \frac{wL}{4}$





$\Sigma M_{x=L/2} = 0$
$\frac{wL}{2}\left(\frac{L}{2}\right) - \frac{wL}{2}\left(\frac{L}{4}\right) - M = 0$
$M = \frac{wL^2}{8}$

eg2.

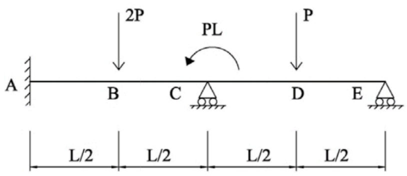
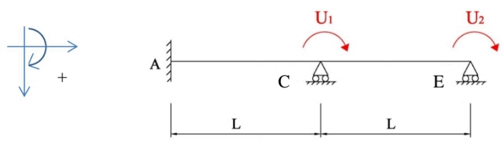


Figure 2

Step 1. Find kinematic Indeterminancy, β

Independent nodal displacements: $\mathbf{U} = \{U_1 \quad U_2\}^T$



Step 2. Find Assembled System's Stiffness Matrix

System stiffness matrix (from Q1): $\mathbf{K} = \begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix}$

(no working as for this example, the focus is on step 3...)

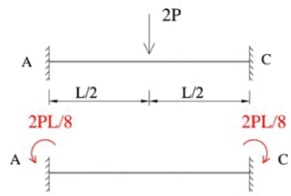
Step 3. Find $\underset{\sim}{P}$

Fixed-end moments:

Element AC:

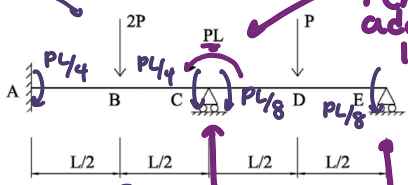
$$M_{AC}^F = -\frac{2PL}{8} = -\frac{PL}{4}$$

$$M_{CA}^F = \frac{2PL}{8} = \frac{PL}{4}$$



Remember to switch the direction!

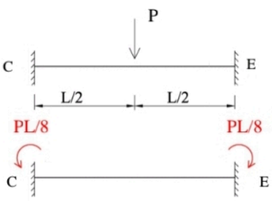
remember to add external load!!! ☆☆



Element CE:

$$M_{CE}^F = -\frac{PL}{8}$$

$$M_{EC}^F = \frac{PL}{8}$$



P at U_1 (ie C)

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

P at U_2 (ie E)

☆ $$\mathbf{P} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} -PL \\ 0 \end{Bmatrix} - \begin{Bmatrix} M_{CA}^F + M_{CE}^F \\ M_{CE}^F \end{Bmatrix} = \begin{Bmatrix} -PL - \frac{PL}{4} + \frac{PL}{8} \\ -\frac{PL}{8} \end{Bmatrix} = \begin{Bmatrix} -\frac{9PL}{8} \\ -\frac{PL}{8} \end{Bmatrix}$$

Step 4. Solve P=KU for U

System of discrete equilibrium: $\{P\} = [K]\{U\} \quad \rightarrow \quad \begin{Bmatrix} -\frac{9PL}{8} \\ -\frac{PL}{8} \end{Bmatrix} = \begin{bmatrix} \frac{8EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$

Solved for U_1 and U_2 $\rightarrow$ $U_1 = -\frac{17PL^2}{112EI}$, $U_2 = \frac{5PL^2}{112EI}$

Step 5. Find SFBMD, by solving $\underset{\sim}{P}=\underset{\sim}{K}\underset{\sim}{U}$ (element's $\underset{\sim}{P}=\underset{\sim}{K}\underset{\sim}{U}$ NOT assembled system's)

Element end-moments:

(Element end-moments = end moments due to nodal displacements/rotations + fixed-end moments)

Element AC:

$$\begin{Bmatrix} M_{AC} \\ M_{CA} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \phi_{AC} \\ \phi_{CA} \end{Bmatrix} + \begin{Bmatrix} M^F_{AC} \\ M^F_{CA} \end{Bmatrix} \qquad \phi_{AC} = 0,\ \phi_{CA} = U_1$$

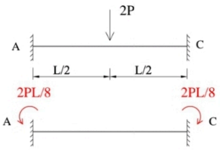
$$\begin{Bmatrix} M_{AC} \\ M_{CA} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} 0 \\ -\frac{17PL^2}{112EI} \end{Bmatrix} + \begin{Bmatrix} -\frac{PL}{4} \\ \frac{PL}{4} \end{Bmatrix} \rightarrow \begin{matrix} M_{AC} = -\frac{31PL}{56} \\ M_{CA} = -\frac{5PL}{14} \end{matrix}$$

only add the fixed end moment. DON'T ADD the external load/moment!

Element AC:

$M^F_{AC} = -\frac{2PL}{8} = -\frac{PL}{4}$

$M^F_{CA} = \frac{2PL}{8} = \frac{PL}{4}$



Element CE:

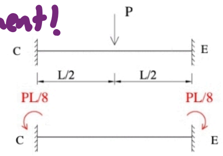
$$\begin{Bmatrix} M_{CE} \\ M_{EC} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} \phi_{CE} \\ \phi_{EC} \end{Bmatrix} + \begin{Bmatrix} M^F_{CE} \\ M^F_{EC} \end{Bmatrix} \qquad \phi_{CE} = U_1,\ \phi_{EC} = U_2$$

$$\begin{Bmatrix} M_{CE} \\ M_{EC} \end{Bmatrix} = \begin{bmatrix} \frac{4EI}{L} & \frac{2EI}{L} \\ \frac{2EI}{L} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} -\frac{17PL^2}{112EI} \\ \frac{5PL^2}{112EI} \end{Bmatrix} + \begin{Bmatrix} -\frac{PL}{8} \\ \frac{PL}{8} \end{Bmatrix} \rightarrow \begin{matrix} M_{CE} = -\frac{9PL}{14} \\ M_{EC} = 0 \end{matrix}$$

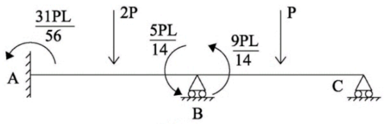
Element CE:

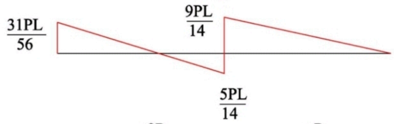
$M^F_{CE} = -\frac{PL}{8}$

$M^F_{EC} = \frac{PL}{8}$



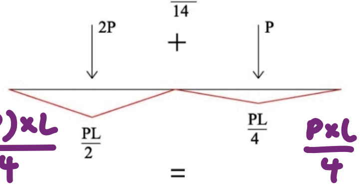
Bending moment distribution (BMD):





BMD due to end-moments

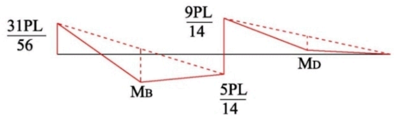
+



$\frac{(2P)\times L}{4}$ $\qquad$ $\frac{P\times L}{4}$

BMD due to element loads in the case of pinned ends

=



BMD

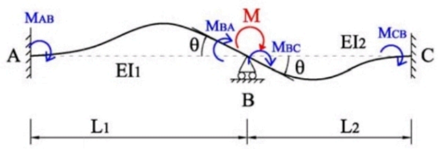
$$M_B = \frac{PL}{2} + \frac{\left(-\frac{31}{56} + \frac{5}{14}\right)}{2} PL = \frac{45PL}{112}$$

$$M_D = \frac{PL}{4} + \frac{\left(-\frac{9}{14} + 0\right)}{2} PL = -\frac{PL}{14}$$

C3. Moment Distribution Method

3.1 Distribution Factor

1. Distribution factors for a two-span continuous fixed-ends beam.



Distribution factor for element AB at B:

$$DF_{BA} = \frac{M_{BA}}{M}$$

Distribution factor for element BC at B:

$$DF_{BC} = \frac{M_{BC}}{M}$$

Solving the stiffness equation: $M = K\theta_B$, with $K = \frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}$ $\rightarrow \theta = \dfrac{M}{\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}}$

Then using the stiffness coefficients for elements AB and BC:

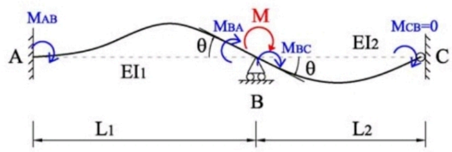
$$\rightarrow M_{AB} = \frac{4EI_1}{L_1}\frac{M}{\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}}, \rightarrow M_{BC} = \frac{4EI_2}{L_2}\frac{M}{\frac{4EI_1}{L_1} + \frac{4EI_2}{L_2}}, \text{ thus}$$

$$\boxed{\begin{aligned} DF_{BA} &= \frac{4EI_1/L_1}{4EI_1/L_1 + 4EI_2/L_2} \\ DF_{BC} &= \frac{4EI_2/L_2}{4EI_1/L_1 + 4EI_2/L_2} \end{aligned}}$$

with $DF_{AB} + DF_{BC} = 1$

just memorise this!

2. Distribution factors for a two-span continuous with one pinned end and one fixed



Distribution factor for element AB at B:

$$DF_{BA} = \frac{M_{BA}}{M}$$

Distribution factor for element BC at B:

$$DF_{BC} = \frac{M_{BC}}{M}$$

Element AB:

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{bmatrix} \frac{4EI_1}{L_1} & \frac{2EI_1}{L_1} \\ \frac{2EI_1}{L_1} & \frac{4EI_1}{L_1} \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix} \rightarrow \begin{matrix} M_{AB} = M_{BA}/2 \\ M_{BA} = \frac{4EI_1}{L_1}\theta_B \end{matrix}$$

$\theta_A = 0$ (fixed-end at A)

Element BC:

$$\begin{Bmatrix} M_{BC} \\ M_{CB} \end{Bmatrix} = \begin{bmatrix} \frac{4EI_2}{L_2} & \frac{2EI_2}{L_2} \\ \frac{2EI_2}{L_2} & \frac{4EI_2}{L_2} \end{bmatrix} \begin{Bmatrix} \theta_B \\ \theta_C \end{Bmatrix} \rightarrow \begin{matrix} 0 = \frac{2EI_2}{L_2}\theta_B + \frac{4EI_2}{L_2}\theta_C \rightarrow \theta_C = -\frac{\theta_B}{2} \\ M_{BC} = \frac{4EI_2}{L_2}\theta_B + \frac{2EI_2}{L_2}\theta_C \rightarrow M_{BC} = \left(\frac{4EI_2}{L_2} - \frac{2EI_2}{2L_2}\right)\theta_B = \frac{3EI_2}{L_2}\theta_B \end{matrix}$$

$M_{CB} = 0$ (pinned end at C)

Solving the stiffness equation: $M = K\theta_B$, with $K = \frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}$

Then using the stiffness coefficients for elements AB and BC:

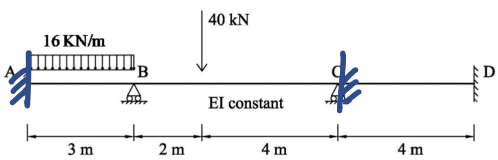
$$\rightarrow M_{AB} = \frac{4EI_1}{L_1}\frac{M}{\frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}}, \rightarrow M_{BC} = \frac{3EI_2}{L_2}\frac{M}{\frac{4EI_1}{L_1} + \frac{3EI_2}{L_2}}, \text{ thus}$$

$$\boxed{\begin{aligned} DF_{BA} &= \frac{4EI_1/L_1}{4EI_1/L_1 + 3EI_2/L_2} \\ DF_{BC} &= \frac{3EI_2/L_2}{4EI_1/L_1 + 3EI_2/L_2} \end{aligned}}$$

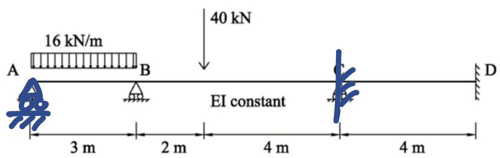
with $DF_{AB} + DF_{BC} = 1$

just memorise this!

☆ Basically it's just a ratio of $\left(\frac{4EI}{L}\right)$ if it's fixed end and $\left(\frac{3EI}{L}\right)$ if it's pinned end.



when looking at node B.
DF_{BA} and DF_{BC} both uses $\frac{4EI}{L}$
(especially at C, although it's pinned but since it's not at very end, rotation is not free!)



for this case, A is $\frac{3EI}{L}$ and C is $\frac{4EI}{L}$!
node A has rotational freedom as it is at very end + pinned support.

3.2 Moment Distribution Method.

Moment Distribution method:

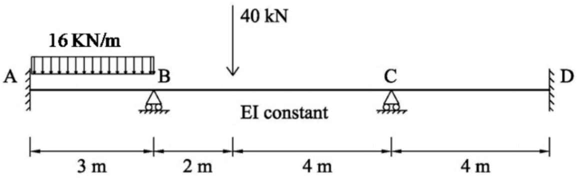
i) Consider all the members as fixed-end beams and evaluate the **distribution factor** at each joint.
ii) Calculate the **fixed-end moments** caused by external loads in each span.
iii) Release the joints in turn, distributing out-of-balance moments to members meeting at a joint, and inserting **carry-overs** as appropriate.
iv) Continue until the effect of relaxation at joints is negligible.

eg 1. All fixed ends

Worked example 6

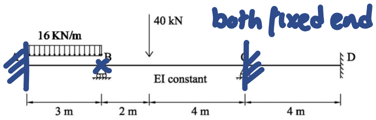
Non-uniform inextensible continuous beam carrying element loads

The continuous beam shown below has uniform flexural rigidity fixed ends. It is subjected to a uniformly distributed load of 16 kN/m along span AB and a point load of 40 KN at 2 m from support B. Use the moment distribution method to calculate end moments.

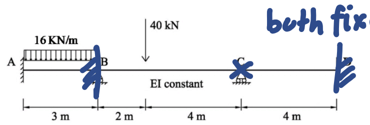


Step 1. Find distribution factor at each joint. (B, C)

both fixed end $\left(\frac{4EI}{L}\right)$



both fixed end $\left(\frac{4EI}{L}\right)$



joint B:

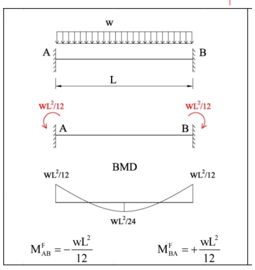
$$DF_{BA}=\frac{\frac{4EI}{3}}{\frac{4EI}{3}+\frac{4EI}{6}}=\frac{2}{3},\quad DF_{BC}=\frac{\frac{4EI}{6}}{\frac{4EI}{3}+\frac{4EI}{6}}=\frac{1}{3}$$

joint C:

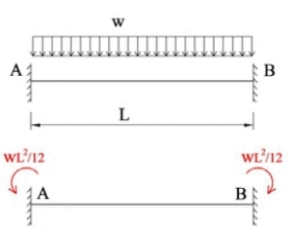
$$DF_{CB}=\frac{\frac{4EI}{6}}{\frac{4EI}{6}+\frac{4EI}{4}}=\frac{2}{5},\quad DF_{CD}=\frac{\frac{4EI}{4}}{\frac{4EI}{6}+\frac{4EI}{4}}=\frac{3}{5}$$

Step 2. Find all the fixed end moment of each span (AB, BC, CD)

(CD has no external load so don't need to find. If there is any, we need to find!)



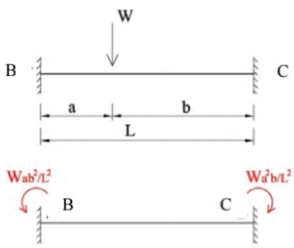
Member AB:

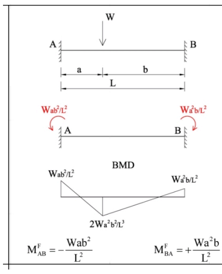


$M^F_{AB}=-wL^2/12=-16\cdot3^2/12=-12$ kNm

$M^F_{BA}=+wL^2/12=+16\cdot3^2/12=+12$ kNm

Member BC:





$M^F_{BC}=-Wab^2/L^2=-40\cdot2\cdot4^2/6^2=-35.56$ kNm

$M^F_{CB}=+Wa^2b/L^2=+40\cdot2^2\cdot4/6^2=+17.78$ kNm

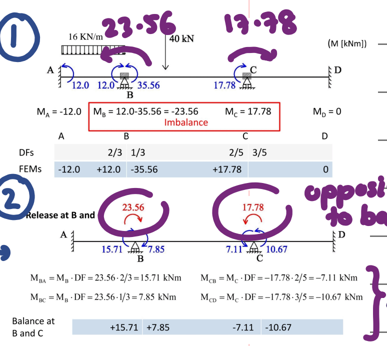
Step 3. Balance, Distibute, Carry-over (BDC)

DFs		2/3	1/3	2/5	3/5	
FEMs	-12.0	+12.0	-35.56	+17.78		
Balance at B and C		+15.71	+7.85	-7.11	-10.67	
Carry-over	+7.86		-3.56	+3.93		-5.34
Balance at B and C		+2.37	+1.19	-1.57	-2.36	
Carry-over	+1.19		-0.79	+0.60		-1.18
Balance at B and C		+0.53	+0.26	-0.24	-0.36	
Carry-over	+0.27		-0.12	+0.13		-0.18
Balance at B and C		+0.08	+0.04	-0.05	-0.08	
Member End-Moment	**-2.68**	**+30.69**	**-30.69**	**+13.47**	**-13.47**	**-6.70**

x0.5 x0.5 x0.5

① ②

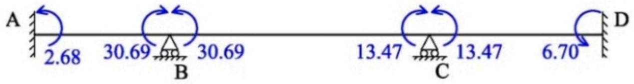
Imbalance



opposite, to balance

distribute

☆ we always stop at final distribution

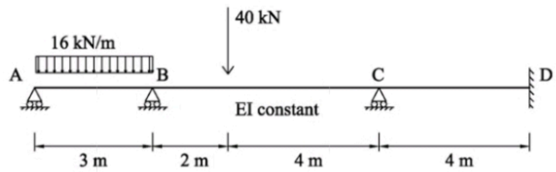


eg2. example with pinned end.

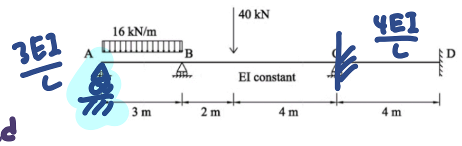
Worked example 7

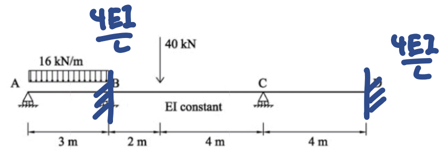
Non-uniform inextensible continuous beam carrying element loads

The continuous beam shown below has uniform flexural rigidity and a pinned end at A and a fixed end at D. It is subjected to a uniformly distributed load of 16 kN/m along span AB and a point load of 40 KN at 2 m from support B. Use the moment distribution method to calculate end moments.



Step 1. Find distribution factor at each joint. (B, C)





☆ It's pinned end and it's at the very end!

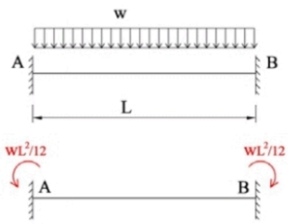
$$DF_{BA} = \frac{\frac{3EI}{3}}{\frac{3EI}{3}+\frac{4EI}{6}} = \frac{3}{5}, \quad DF_{BC} = \frac{\frac{4EI}{6}}{\frac{3EI}{3}+\frac{4EI}{6}} = \frac{2}{5}, \quad DF_{CB} = \frac{\frac{4EI}{6}}{\frac{4EI}{6}+\frac{4EI}{4}} = \frac{2}{5}, \quad DF_{CD} = \frac{\frac{4EI}{4}}{\frac{4EI}{6}+\frac{4EI}{4}} = \frac{3}{5}$$

no load at CD so don't need in this specific case

Step 2. Find all the fixed end moment of each span (AB, BC, CD)

Fixed end moments (FEMs):

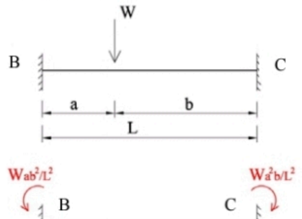
Member AB:



$M^F_{AB} = -wL^2/12 = -16 \cdot 3^2/12 = -12$ kNm

$M^F_{BA} = +wL^2/12 = +16 \cdot 3^2/12 = +12$ kNm

Member BC:



$M^F_{BC} = -Wab^2/L^2 = -40 \cdot 2 \cdot 4^2/6^2 = -35.56$ kNm

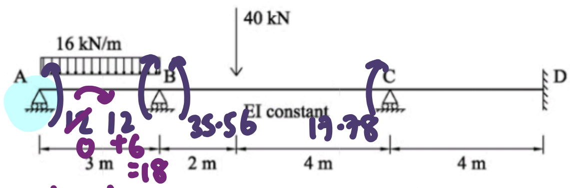
$M^F_{CB} = +Wa^2b/L^2 = +40 \cdot 2^2 \cdot 4/6^2 = +17.78$ kNm

VERY IMPORTANT!

Release at A to account for the pinned joint.

$M^P_{AB} = 0$

$M^P_{BA} = M^F_{BA} - M^F_{AB}/2 = +12 - 0.5(-12) = +18$ kNm



if this is pin, delete the moment and half of it carry over.

Step 3. Balance, Distibute, Carry-over (BDC)

DFs		3/5	2/5	2/5	3/5	
FEMs	(-12.0 0	+12.0) +18.0	-35.56	+17.78		
Balance at B and C		+10.54	+7.02	-7.11	-10.67	
Carry-over			-3.56	+3.51		-5.34
Balance at B and C		+2.14	+1.42	-1.40	-2.11	
Carry-over			-0.70	+0.71		-1.06
Balance at B and C		+0.42	+0.28	-0.28	-0.43	
Carry-over			-0.14	+0.14		-0.22
Balance at B and C		+0.08	+0.06	-0.06	-0.08	
Member End-Moment		**+31.18**	**-31.18**	**+13.29**	**-13.29**	**-6.62**