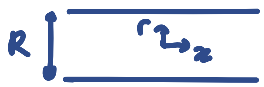
(Velocity Profile / Shear Profile) - laminar
- turbulent.

Tutorial 4: Flow Profiles in Pipes

Question 1

The velocity profile for laminar flow in pipes is given by

$$u(r) = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2),$$



where R is the pipe radius and r the radial coordinate. This is the exact solution (of the Navier-Stokes equations) and often referred to as Hagen-Poiseuille flow. Show that the shear stress distribution is given by

$$\tau = \frac{1}{2}\frac{dp}{dx}r.$$

Using this result, show that the friction factor in laminar pipe flow is given by

$$f = \frac{64}{Re}$$

Explain the significance of this result with respect to the wall roughness. Sketch the velocity profile and the shear stress distribution in the pipe.

The velocity profile for laminar flow in pipes is:

$$u(r) = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2)$$

$f = \frac{8\tau_0}{\rho U^2}$ 1. find τ_0 2. find U

By definition, the shear stress for a Newtonian fluid is determined by

$$\tau = \mu\frac{du}{dr}$$

find $\frac{du}{dr}$

Substituting the expression for the laminar flow velocity profile in the pipe gives:

$$\tau = \frac{1}{2}\frac{dp}{dx}r \qquad \tau_0 = \frac{1}{2}\frac{dp}{dx}R$$

where τ_0 is the wall shear stress. The Reynolds number is associated with the mean velocity of the flow. The mean velocity, in turn, is related to the total pipe discharge. The discharge δQ for a small annular segment between r and $r + \delta r$ is

At this stage, $f = \frac{8\tau_0}{\rho U^2}$, we know we are missing "U"

$$\delta Q = u 2\pi r \delta r \rightarrow Q = 2\pi \int u r \, dr$$

but we cannot just use $Re = \frac{\rho U D}{\mu}$ to get $U = \frac{\mu Re}{\rho D}$

but we need to find $U = \frac{Q}{A} = \frac{2\pi \int u r \, dr}{\pi r^2}$

cause $Re = \frac{\rho U D}{\mu}$ can't get rid of $\frac{dp}{dx}$

Substituting for the velocity profile, $u(r)$, and integrating over the pipe cross section gives the total discharge Q as

$$Q = -\frac{\pi}{2\mu}\frac{dp}{dx}\int_0^R (R^2 r - r^3)dr = -\frac{\pi}{2\mu}\frac{dp}{dx}\left(R^2\frac{R^2}{2} - \frac{R^4}{4}\right) = -\frac{\pi R^4}{8\mu}\frac{dp}{dx}$$

The negative sign indicates that a negative pressure gradient yields a positive discharge. The total discharge may also be expressed in terms of the mean velocity by

$$Q = U\pi R^2$$

The wall shear stress and the friction factor are related by

$$\tau_0 = \frac{f}{8}\rho U^2 = \frac{1}{2}\frac{dp}{dx}R$$

Substituting the above expressions and rearranging gives

$$f = \frac{64\mu}{\rho U D} = \frac{64}{Re}$$

Question 2

For the same conditions as under *Question 1, Tutorial 3*, predict

i. the shear velocity
ii. the Reynolds shear stress at $r = 125$mm
iii. the mean flow velocity profile in the pipe
iv. the viscous shear stress at $r = 125$mm

u_* shear velocity, reynold shear stress, mean flow velocity $\bar{u}$ are all the terms used in turbulent flow

to check if the question is turbulent:

$$Re = \frac{UD}{\nu} = \frac{1.98 \times 10^{-3}}{10^{-6}} = 594\,000$$

$\therefore$ Definitely turbulent

[Ans: i) 0.14 m/s, ii) 16.1 Pa, iv) 0.014 Pa]

Plot (the use of Matlab or Excel is recommended)

i. the mean velocity profile in the pipe
ii. the Reynolds shear stress distribution in the pipe
iii. the viscous shear stress distribution in the pipe

Note: The conditions in *Question 1, Tutorial 3* are as follows: Steady water flow is established in a very long length of 300mm internal diameter pipeline. At section one, the elevation is +90m and the pressure is 275kPa. At section two, 300m distant, the elevation is +75m and the pressure is 345kPa.

From Tutorial 3, question 1:

$\tau_0 = 19.28$ Pa, $f = 0.039$, $U = 1.98$ m/s, $R = 0.15$ m

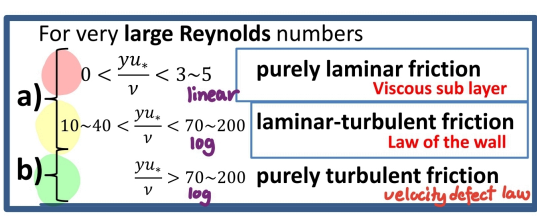
(i) shear velocity, $u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{19.28}{1000}} = 0.139$ m/s

(ii) $-\rho \overline{u'v'} = \tau_0 \frac{r}{R}$

at $r = 125$ mm,

$-\rho \overline{u'v'} = 19.28 \left(\frac{125}{150}\right) = 16.067$ Pa

(iii)



we will be considering this region only cause:
1. viscous sub layer is extremely thin.
2. Law of the wall is complicated.

$$\overline{u(y)} = \frac{u_*}{K} \ln\frac{y}{R} + u_{max}.$$

to find u_{max}:

$$u_{max} = \bar{U} + \frac{3}{2}\frac{u_*}{K} = 1.98 + 1.5\left(\frac{0.139}{0.4}\right) = 2.5$$

$$\therefore \overline{u(y)} = 2.5 + \frac{0.139}{0.4} \ln\left(\frac{y}{150 \times 10^{-3}}\right)$$

(iv) viscous shear stress:

$$\bar{\tau}(y) = \mu \frac{d\bar{u}}{dy} \text{ or } \bar{\tau}(r) = \mu \frac{d\bar{u}}{dr}$$

since $\bar{u}$ is given as a function of y: $\bar{u}(y)$, we choose this.

$$\bar{\tau}(y) = \mu \frac{d}{dy}\left(2.5 + \frac{0.139}{0.4} \ln\left(\frac{y}{150 \times 10^{-3}}\right)\right)$$

$$= \frac{\mu(0.139)}{0.4}\left(\frac{1}{y}\right)$$

when $r = 125$ mm, $y = 25$ mm

$$\tau(y = 25\text{mm}) = \frac{1.005 \times 10^{-3} \times 0.139}{0.4}\left(\frac{1}{0.025}\right)$$

$$= 0.0139 \text{ Pa}$$