14.3.2 The Windowed Fourier Transform

Let $f(t)$ be a *signal* (function). We assume that $\int_{-\infty}^{\infty} |f(t)|^2\,dt$ is finite. This integral is defined to be the *energy* of the signal.

In analyzing a signal, we sometimes want to localize the frequency content with respect to time. We know that $\hat{f}(\omega)$ carries information about the frequencies ω of the signal. However, $\hat{f}(\omega)$ does not particularize this information to specific time intervals, since

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt,$$

and this integration is over all time. From this we can compute the total amplitude spectrum $|\hat{f}(\omega)|$, but cannot look at small time intervals. If we think of $f(t)$ as a piece of music, we have to wait until the entire piece is done before computing this amplitude spectrum.

We can obtain a picture of the frequency content of $f(t)$ within a given time interval by windowing the signal before taking its transform. The idea is to use a *window function* $w(t)$ that is nonzero only on a finite interval, often $[0, T]$ or $[-T, T]$. *Window* $f(t)$ with $w(t)$ by forming the product $w(t)f(t)$, which can be nonzero only on the selected interval. The *windowed Fourier transform* of f, with respect to the particular window function w, is

$$\hat{f}_{\text{win}}(\omega) = \int_{-\infty}^{\infty} w(t)f(t)e^{-i\omega t}\,dt.$$

EXAMPLE 14.11

Let $f(t) = 6e^{-|t|}$. Then

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} 6e^{-|t|}e^{-i\omega t}\,dt = \frac{12}{1+\omega^2}.$$

We will window f with the window function

$$w(t) = \begin{cases} 1 & \text{for } -2 \le t \le 2, \\ 0 & \text{for } |t| > 2. \end{cases}$$

Figures 14.2, 14.3, and 14.4 show, respectively, $f(t)$, the window function $w(t)$, and $w(t)f(t)$. The effect of windowing on this signal is to cut the signal off for times $|t| > 2$. The windowed Fourier transform is therefore an integral only over $[-2, 2]$ instead of the entire real line:

$$\begin{aligned} \hat{f}_{\text{win}}(\omega) &= \int_{-\infty}^{\infty} 6w(t)e^{-|t|}e^{-i\omega t}\,dt \\ &= \int_{-2}^{2} 6e^{-|t|}e^{-i\omega t}\,dt \\ &= \frac{12}{1+\omega^2}\left(-2e^{-2}\cos^2(\omega) + e^{-2} + e^{-2}\omega\sin(2\omega) + 1\right). \quad \blacklozenge \end{aligned}$$

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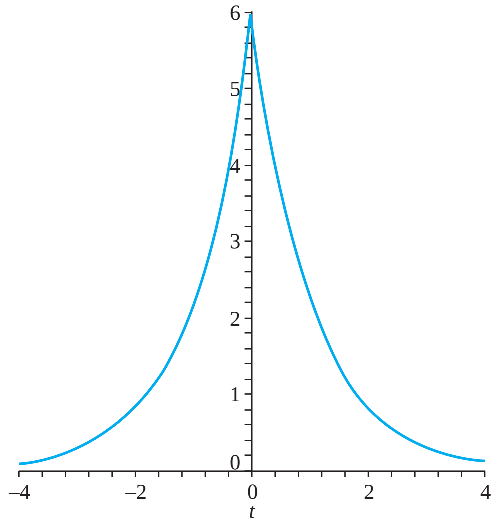

FIGURE 14.2 $f(t) = 6e^{-|t|}$

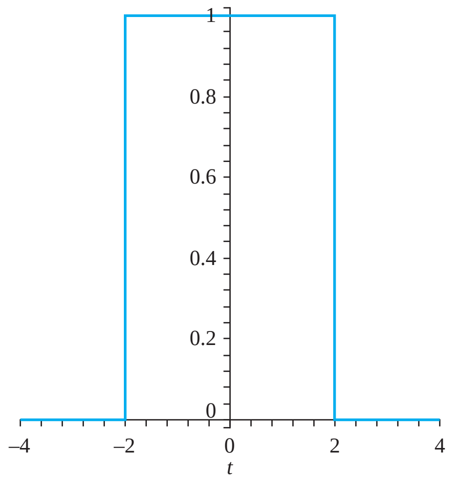

FIGURE 14.3 *Window function* $w(t)$ *in Example 14.11.*

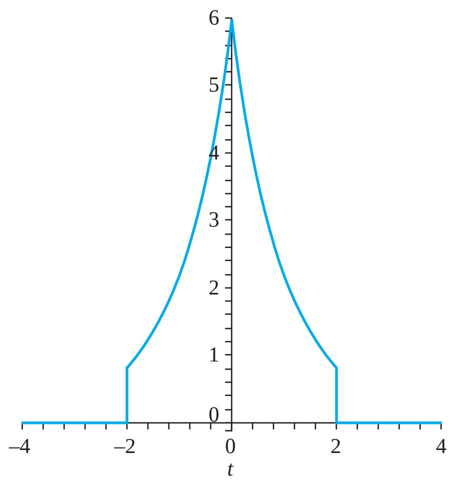

FIGURE 14.4 *Windowed function* $w(t)f(t)$ *in Example 14.11.*

Sometimes we use a *shifted window function*. If $w(t)$ is nonzero only on $[-T, T]$, then the shifted function $w(t - t_0)$ is the graph of $w(t)$ shifted t_0 units to the right and is nonzero only on $[t_0 - T, t_0 + T]$. In this case, the shifted windowed Fourier transform is the transform of $w(t - t_0)f(t)$:

$$\hat{f}_{\text{win}.t_0}(\omega) = \mathcal{F}[w(t - t_0)f(t)](\omega)$$
$$= \int_{t_0 - T}^{t_0 + T} w(t - t_0)f(t)e^{-i\omega t}\,dt.$$

This gives the frequency content of the signal in the time interval $[t_0 - T, t_0 + T]$.

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Engineers refer to the windowing process as *time-frequency localization*. The *center* of the window function w is defined to be

$$t_C = \frac{\int_{-\infty}^{\infty} t|w(t)|^2\,dt}{\int_{-\infty}^{\infty} |w(t)|^2\,dt}.$$

The number

$$t_R = \left(\frac{\int_{-\infty}^{\infty} (t-t_C)^2|w(t)|^2\,dt}{\int_{-\infty}^{\infty} |w(t)|^2\,dt}\right)^{1/2}$$

is the *radius* of the window function. The *width* of the window function is $2t_R$, a number referred to as the RMS *duration of the window*.

Similar terminology applies when we deal with the the Fourier transform of the window function:

$$\text{center of } \hat{w} = \omega_C = \frac{\int_{-\infty}^{\infty} \omega|\hat{w}(\omega)|^2\,d\omega}{\int_{-\infty}^{\infty} |\hat{w}(\omega)|^2\,d\omega}$$

and

$$\text{radius of } \hat{w} = \omega_R = \left(\frac{\int_{-\infty}^{\infty} (\omega-\omega_C)^2|\hat{w}(\omega)|^2\,d\omega}{\int_{-\infty}^{\infty} |\hat{w}(\omega)|^2\,d\omega}\right)^{1/2}.$$

The *width* of $\hat{w}$ is $2\omega_R$, a number referred to as the RMS *bandwidth* of the window function.

14.3.3 The Shannon Sampling Theorem

A signal $f(t)$ is *band-limited* if its Fourier transform $\hat{f}(\omega)$ has nonzero values only on some interval $[-L, L]$. If f is band-limited, the smallest positive L for which this is true is called the *bandwidth* of f. For such L we have

$$\hat{f}(\omega) = 0 \text{ if } |\omega| > L.$$

The total frequency content of such a signal lies in the band $[-L, L]$.

We will show that a band-limited signal can be reconstructed from samples taken at appropriately chosen times. Begin with the integral for the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega.$$

Because f is assumed to have bandwidth L, we actually have

$$f(t) = \frac{1}{2\pi}\int_{-L}^{L} \hat{f}(\omega)e^{i\omega t}\,d\omega. \tag{14.13}$$

Now expand $\hat{f}(\omega)$ in a complex Fourier series on $[-L, L]$:

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{n\pi i\omega/L}, \tag{14.14}$$

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where

$$c_n = \frac{1}{2L}\int_{-L}^{L} \hat{f}(\omega)e^{-n\pi i\omega/L}\,d\omega.$$

Compare c_n with $f(t)$ in equations (14.13) and (14.14) to conclude that

$$c_n = \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right).$$

Substitute this into equation (14.14) to get

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{L} f\left(\frac{-n\pi}{L}\right) e^{n\pi i\omega/L}.$$

Since n takes on all integer values (zero, positive and negative) in this summation, we can replace n with $-n$ without changing the sum:

$$\hat{f}(\omega) = \frac{\pi}{L}\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) e^{-n\pi i\omega/L}.$$

Substitute this expansion of $\hat{f}(\omega)$ into equation (14.13) to get

$$f(t) = \frac{1}{2\pi}\frac{\pi}{L}\int_{-L}^{L} f\left(\frac{n\pi}{L}\right) e^{-n\pi i\omega/L} e^{i\omega t}\,d\omega.$$

Now interchange the summation and the integral and carry out the integration to get

$$\begin{aligned}
f(t) &= \frac{1}{2L}\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\int_{-L}^{L} e^{i\omega(t-n\pi/L)}d\omega \\
&= \frac{1}{2L}\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{1}{i(t-n\pi/L)}\left[e^{i\omega(t-n\pi/L)}\right]_{-L}^{L} \\
&= \frac{1}{2L}\sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{1}{i(t-n\pi/L)}\left(e^{i(Lt-n\pi)} - e^{-i(Lt-n\pi)}\right) \\
&= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{1}{Lt-n\pi}\frac{1}{2i}\left(e^{i(Lt-n\pi)} - e^{-(Lt-n\pi)}\right) \\
&= \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right)\frac{\sin(Lt-n\pi)}{Lt-n\pi}.
\end{aligned}$$

This is the *Shannon sampling theorem*. It says that we know $f(t)$ at all times if we know just the function values $f(n\pi/L)$ for all integers n. An engineer would sample the signal $f(t)$ at times $0, \pm\pi/L, \pm 2\pi/L, \cdots$ and be able to reconstruct the entire signal. This is how engineers convert digital signals to analog signals, with application to technology such as that used in making compact disks.

In the case $L = \pi$ the Shannon sampling theorem is

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin(\pi(t-n))}{\pi(t-n)}.$$

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