

Fundamentals of Math Thought Final Review

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Definitions

1. For integers a and b , define a **divides** b .
 - *Definition*— there exists a k such that $b = a \cdot k$, where $k \in \mathbb{Z}$
2. Define a **proposition**.
 - *Definition*— A **proposition** is a sentence that has exactly one truth value.
3. Define a **conditional sentence**.
 - *Definition*— For propositions P and Q , the **Conditional sentence** $P \implies Q$ is the proposition "If P , then Q ". Proposition P is called the **antecedent** and Q is the **consequent**.
4. The **power set** of a set A .
 - *Definition*— Let A be a set. The **power set** of A is the set whose elements are the subsets of A and is denoted by $\mathcal{P}(A)$. Thus

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

5. The **intersection** of sets A and B .
 - *Definition*— The **intersection of A and B** is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.
6. The **union** of sets A and B .
 - *Definition*— The **union of A and B** is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
7. State one of **Demorgan's Laws** for two sets.
 - *Definition*— $(A \cup B)^c = A^c \cap B^c$. or $(A \cap B)^c = A^c \cup B^c$.
8. An indexed family of sets \mathcal{A} is **pairwise disjoint**.
 - *Definition*— The indexed family $\mathcal{A} = \{A_\alpha : \alpha \in \delta\}$ of sets is **pairwise disjoint** iff for all α and β in δ , either $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta = \emptyset$.
9. State the Principle of Mathematical Induction.
 - (i) $1 \in S$,
 - (ii) for all $n \in \mathbb{N}$, if $n \in S$ then $n + 1 \in S$.

Then $S = \mathbb{N}$

10. State the Well-Ordering Principle.
 - *Definition*— Every nonempty subset of \mathbb{N} has a smallest element.

11. A **relation** from A to B, for sets A and B.
 ◦ *Definition*— R is a relation from a to b iff R is a subset of $A \times B$
12. The **domain** of a relation R from A to B.
 ◦ *Definition*— $\text{Dom}(R) = \{x \in A : \text{there exists } y \in B \text{ such that } xRy\}$
13. The **range** of a relation R from A to B.
 ◦ *Definition*— $\text{Rng}(R) = \{y \in B : \text{there exists } x \in A \text{ such that } xRy\}$
14. The composite $S \circ R$ where R is a relation from A to B and S is a relation from B to C.
 ◦ *Definition*— $S \circ R = \{(a, c) : \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$
15. A relation R on A is **reflexive**.
 ◦ *Definition*— iff for all $x \in A$, xRx
16. A relation R on A is **symmetric**.
 ◦ *Definition*— iff for all $x, y \in A$, xRy then yRx
17. A relation R on A is **transitive**.
 ◦ *Definition*— iff for all $x, y, z \in A$, xRy and yRz , then xRz
18. A **partition** \mathcal{P} of a nonempty set A.
 ◦ *Definition*— \mathcal{P} is a partition of a nonempty set A.
 - (i) if $x \in \mathcal{P}$, then $x \neq \emptyset$
 - (ii) if $x \in \mathcal{P}$ and $y \in \mathcal{P}$, then $x = y$ or $x \cap y = \emptyset$
 - (iii) $\bigcup_{x \in \mathcal{P}} x = A$
19. A **function** from A to B, for sets A and B.
 ◦ *Definition*— is a relation $f : A \rightarrow B$ such that the $\text{Dom}(f) = A$, and if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$
20. $f : A \rightarrow B$ is **surjective**
 ◦ *Definition*— A function $f : A \rightarrow B$ is onto B or **surjective** iff $\text{Rng}(f) = B$
21. $f : A \rightarrow B$ is **injective**
 ◦ *Definition*— A function $f : A \rightarrow B$ is one-to-one or is injective iff when $f(x) = f(y)$, then $x = y$.
22. $f : A \rightarrow B$ and $Y \subseteq B$. Define $f^{-1}(Y)$.
 ◦ *Definition*— $f^{-1}(Y) = \{x \in A : y = f(x) \in Y\}$

True False

1. **True** or **False** - If apples are cube-shaped, bananas are explosive.
2. **True** or **False** - 10 is odd if and only if 13 is odd.
3. **True** or **False** - $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x + y \geq 0)$.
4. **True** or **False** - For any set A, $((A^c)^c)^c = A^c$.
5. **True** or **False** - The sets $\{x \in \mathbb{R} \mid 1 < x < 3\}$ and \mathbb{Z} are disjoint.
6. **True** or **False** - For any family of Sets \mathcal{A} ,

$$\bigcap_{A \in \mathcal{A}} A \subseteq \bigcup_{A \in \mathcal{A}} A$$

7. **True** or **False** - For any sets A,B and C, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B \cap C|$
8. **True** or **False** - $23 \equiv 2 \pmod{7}$
9. **True** or **False** - The relation R from \mathbb{R} to \mathbb{R} , $aRb \iff a = b^3$ is a function.
10. **True** or **False** - Let $f : A \rightarrow B$ and $g : B \rightarrow C$. If $g \circ f$ is injective, g is injective
11. **True** or **False** - Let $f : A \rightarrow B$ and $X \subseteq A$. $f(f^{-1}(f(X))) = f(X)$.

Part 3 Problems

1.

Part 4 Problems

1.

Proofs

1. Prove that $\sqrt{2}$ is irrational.

Proof: Assume $\sqrt{2}$ is rational.

Thus $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and a, b have no common factors.

Now let us write $2 = \frac{a^2}{b^2}$, or $b^2 = 2a^2$.

Thus b^2 is even, the only way this can be true is that b itself is even.

hence b^2 is divisible by 4. This contradicts our assumption that a, b have no common factors

Therefore $\sqrt{2}$ cannot be rationalized. ■

2. Prove that there are infinitely many prime numbers.

Proof: Assume there exists a finite number of primes $\{p_1, p_2, \dots, p_n\}$

Let $N = p_1 \cdot p_2 \cdot p_3 \cdots p_{n+1}$

N is not divisible by any of the known primes since it will leave a remainder of one upon division by any one of them.

Thus, N must be divisible by some other prime not in our list

which contradicts the assumption that there is a finite number of primes.

Therefore There are an infinite number of prime numbers ■

3. Let x be an integer. Prove that if x^2 is not divisible by 4, then x is odd.

Proof: by contrapositive

Assume x is not even.

Then $x = 2k + 1$ for some $k \in \mathbb{Z}$

And if $x = 2k + 1$, it follows that

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

Clearly, 4 does not divide $x^2 = 4(k^2 + k) + 1$, because 4 does not divide 1.

Thus if x is not even $\implies x^2$ is not divisible by 4.

Clearly, if x is even $\implies x^2$ is divisible by 4. ■

4. When any integer n is divided by 3, it has a remainder of 0, 1, or 2 (by the division algorithm). This means that for any integer n

$n = 3k$ for some integer k , or

$n = 3k + 1$ for some integer k , or

$n = 3k + 2$ for some integer. use this fact to prove that for any integer n , 3 divides $n^3 - n$.

Proof: Let $n = 3k$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^3 - n &= (3k)^3 - 3k \\ &= 27k^3 - 3k \\ &= 3(9k^3 - k), 9k^3 - k \in \mathbb{Z}, \text{ and is divisible by 3} \end{aligned}$$

Now let $n = 3k + 1$

$$\begin{aligned}\text{Then, } n^3 - n &= (3k + 1)^3 - (3k + 1) \\ &= (9k^2 + 6k + 1)(3k + 1) - (3k + 1) \\ &= 27k^3 + 18k^2 + 2k + 9k^2 + 6k + 1 - 9k - 1 \\ &= 27k^3 + 27k^2 + 6k \\ &= 3(9k^3 + 9k^2 + 2k) \\ &= (9k^3 + 9k^2 + 2k) \in \mathbb{Z}, \text{ and} \\ &= 3(9k^3 + 9k^2 + 2k) \text{ is divisible by } 3.\end{aligned}$$

Now let $n = 3k + 2$

$$\begin{aligned}\text{Then, } n^3 - n &= (3k + 2)^3 - (3k + 2) \\ &= (9k^2 + 12k + 4)(3k + 2) - (3k + 2) \\ &= (27k^3 + 36k^2 + 12k + 18k^2 + 24k + 8) - 3k - 2 \\ &= 27k^3 + 54k^2 + 33k + 6 \\ &= 3(9k^3 + 18k^2 + 11k + 2) \\ &= (9k^3 + 18k^2 + 11k + 2) \in \mathbb{Z}, \text{ and} \\ &= 3(9k^3 + 18k^2 + 11k + 2) \text{ is divisible by } 3.\end{aligned}$$

Therefore for any integer $n = 3k$, $n = 3k + 1$, or $n = 3k + 2$, n is divisible by 3. ■

5. For integers a, b, c, x and y , prove that if c divides a and c divides b , then c divides $ax + by$.

Proof:

Let a, b, c, x , and $y, \in \mathbb{Z}$,

Let c divide b such that $b = c \cdot k$, where $k \in \mathbb{Z}$, and

Let c divide a such that $a = c \cdot m$, where $m \in \mathbb{Z}$

$$\text{now let } \frac{ax + by}{c} = \frac{(ck)x + (cm)y}{c} = \frac{c}{c}(kx + my) = (kx + my)$$

Therefore c divides $ax + by$ ■

6. For real numbers x and y , prove that if x is rational and y is irrational, then $x + y$ is irrational.

Proof:

Let x be rational, and y be irrational

Let $x + y$ be rational, then $x + y = \frac{a}{b}$. Let $x = \frac{c}{d}$. Then

$$y = \frac{a}{b} - x = \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Thus y is rational which contradicts our first assumption that y is irrational.

Hence it follows that $x + y$ is irrational

7. Let A , B , and C be sets. Prove that if $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, then $A = B$ and $B = C$

Proof:

Let $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$,
and let $x \in A$, then by the definition of \subseteq , $x \in B$, and $x \in C$
Thus $A \subseteq B \subseteq C$.
and in order for $A \subseteq B$, and $B \subseteq C$, and $C \subseteq A$
Therefore $A = B = C$.

8. Let A and B be sets. Prove that $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$

Proof.

Let A and B be sets
Suppose $A = B$
Since $\mathcal{P}(A)$ is the set of all subsets of A and $\mathcal{P}(B)$ is the set of all subsets of B
Then $\mathcal{P}(A) = \mathcal{P}(B)$
Conversally suppose $\mathcal{P}(A) = \mathcal{P}(B)$
 $\mathcal{P}(A)$ is the set of all subsets of A , and $\mathcal{P}(B)$ the set of all subsets of B
Therefore $A = B$

9. Let A and B be sets. Prove that $(A^c \cup B)^c = A \cap B^c$

Proof:

Let A and B be sets
Using Demorgan's Law $(A^c \cap B)^c = (A^c)^c \cup B^c = A \cup B^c$
Thus $(A^c \cap B)^c = A \cup B^c$

10. Let A, B, C , and D be sets. Prove that if $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.

Proof:

11. Use the Principle of Mathematical Induction to prove the following:

For all $n \in \mathbb{N}$

$$1 \cdot 1 + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

Proof: Let $n = 1$

$$\text{so } 1 \cdot 1! = (1+1)! - 1$$

$$1 = 2! - 1$$

$$1 = 1, \text{ this is true!}$$

Now assume truth for any $n \in \mathbb{N}$

we will show proof for $n+1$:

$$\begin{aligned} \text{so, } ((n+1)+1)! - 1 &= 1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! + (n+1) \cdot (n+1)! \\ (n+2)! - 1 &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(1 + (n+1)) - 1 \\ &= (n+1)!(n+2) - 1, \text{ by the definition of factorial} \\ &= (n+2)! - 1 \end{aligned}$$

Thus for all $n \in \mathbb{N}$ $1 \cdot 1 + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$ ■

12. Use the principle of Mathematical Induction to prove the following: For all $n \in \mathbb{N}$:

$$8 \text{ divides } 5^{2n} - 1$$

Proof: Let $n = 1$

$$\text{thus } 5^n - 1 = 24, \text{ which is divisible by } 8$$

Now assume $n \in \mathbb{N}$ is true, we will show $n+1$ is true:

$$\begin{aligned} \text{Thus, } 5^{2(n+1)} - 1 &= 5^{(2n+2)} - 1 \\ &= 5^n \cdot 5^2 - 1 \\ &= 5^n \cdot 25 - 1 \\ &= 5^n \cdot 25 - 25 + 24 \\ &= 25(5^n - 1) + 24 \\ &= 25(5^n - 1) \in \mathbb{N} \text{ and is divisible by } 8 \text{ because } (5^n - 1) \text{ is divisible by } 8 \\ &\text{and, } 24 \text{ is divisible by } 8. \\ \text{Therefor for all } n \in \mathbb{N}, 5^n - 1, &\text{ is divisible by } 8. \end{aligned}$$

■

13. We have the common differentiation formulas $\frac{d}{dx}x = 1$ and $\frac{d}{dx}(fg) = f(\frac{d}{dx}g) + (\frac{d}{dx}f)g$.
Use these formulas and the Principle of Mathematical Induction to prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$.

Proof: Let $n = 1$

$$\frac{d}{dx}x = 1$$

Assume for some $n \in \mathbb{N}$, that $\frac{d}{dx}x^n = nx^{n-1}$
Now let us prove for $n + 1$

$$\begin{aligned} \text{thus, } \frac{d}{dx}x^{n+1} &= (n+1)x^{(n+1)-1} \\ &= (n+1)x^n \\ &= (n+1)nx^{n-1} \\ &= n^2x^{n-1} + nx^{n-1} \\ &= nx^{n-1}(n+1) \end{aligned}$$

■

14. Let $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$, for all $n \leq 1$. Prove that $a_n = 2^n$ for all $n \in \mathbb{N}$

Proof: Let $n=1$, thus $a_1 = 2^1 = 2$

Now assume $a_n = 2^n$

We will now prove $n + 1$

$$\begin{aligned} \text{thus } a_{n+1} &= 5a_n - 6a_{n-1} \\ &= 5 \cdot 2^n - 6 \cdot 2^{n-1} \\ &= 5 \cdot 2^n - 3 \cdot 2 \cdot 2^{n-1} \\ &= 2^n(5 - 3) \\ &= 2^n(2) \\ &= 2^{n+1} \\ a_{n+1} &= 2^{n+1} \end{aligned}$$

Thus by induction of n $a_n = 2^n$

■

15. Let R be a relation from A to B and S be a relation from B to C .
Prove that $\text{Rng}(S \circ R) \subseteq \text{Rng}(S)$.

Proof.

■

16. Prove that if R is a symmetric, transitive relation on A and the domain of R is A , then R is reflexive on A .

Proof:

■

17. Let A be a nonempty set, \mathcal{P} a partition of A , and B be a nonempty subset of A .
Prove that $\mathcal{A} = \{X \cap B \mid X \in \mathcal{P} \text{ and } X \cap B \neq \emptyset\}$ is a partition of B .

Proof: ■

18. Let R be an antisymmetric relation on the set A .
Prove that if R is symmetric and $\text{Dom}(R) = A$, then $R = I_A$

Proof: ■

19. Define a relation R on \mathbb{R} by $aRb \iff a^3 = b^5$. Prove that R is a function from \mathbb{R} to \mathbb{R} .

Proof: ■

20. Prove that, if f and g are functions, then $f \cap g$ is a function by showing that $f = g|_A$ where $A = \{x : g(x) = f(x)\}$.

Proof: ■

21. Prove that if $f : A \xrightarrow{\text{onto}} B$ and $g : B \xrightarrow{\text{onto}} C$, then $g \circ f : A \xrightarrow{\text{onto}} C$.

Proof: ■

22. Let $f : A \rightarrow B$. Prove that if f^{-1} is a function, then f is injective.

Proof: ■

23. Prove that $g : (-\infty, -4) \rightarrow (-\infty, 0)$, defined by $g(x) = -|x + 4|$ is a one-to-one correspondence.

Proof: ■

24. Let $f : A \rightarrow B$ and $Y \subseteq B$. Prove that $f(f^{-1}(Y)) \subseteq Y$.

Proof: ■

25. Let $f : A \rightarrow B$ be injective and $X \subseteq A$.
Prove that $X = f^{-1}(f(X))$

Proof: ■