Fundamentals of Math Thought Final Review

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Definitions

1. For integers a and b, define a **divides** b.
   ◦ *Definition*– there exists a b such that $b = a \cdot k$, where $k \in \mathbb{Z}$

2. Define a **proposition**.
   ◦ *Definition*– A **proposition** is a sentence that has exactly one truth value.

3. Define a **conditional sentence**.
   ◦ *Definition*– For propositions P and Q, the **Conditional sentence** $P \implies Q$ is the proposition "If P, then Q". Proposition P is called the **antecednet** and Q is the **consequent**.

4. The **power set** of a set A.
   ◦ *Definition*– Let A be a set. The **power set** of A is the set whoe elements are the subsets of A and is denoted by $\mathscr{P}(A)$. Thus
   $$\mathscr{P}(A) = \{B : B \subseteq A\}$$

5. The **intersection** of sets A and B.
   ◦ *Definition*– The **intersection of A and B** is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

6. The **union** of sets A and B.
   ◦ *Definition*– The **union of A and B** is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

7. State one of **Demorgan's Laws** for two sets.
   ◦ *Definition*– $(A \cup B)^c = A^c \cap B^c$. or $(A \cap B)^c = A^c \cup B^c$.

8. An indexed family of sets $\mathscr{A}$ is **pairwise disjoint**.
   ◦ *Definition*– The indexed family $\mathscr{A} = \{A_\alpha : \alpha \in \delta\}$ of sets is **pairwise disjoint** iff for all α and β in δ, either $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta = \varnothing$.

9. State the Principle of Mathematical Induction.
   (i) $1 \in S$,
   (ii) for all $n \in \mathbb{N}$, if $n \in S$ then $n + 1 \in S$.

   Then S = $\mathbb{N}$

10. State the Well-Ordering Principle.
    ◦ *Definition*– Every nonempty subset of $\mathbb{N}$ has a smallest element.

11. A **relation** from A to B, for sets A and B.
    ○ $Definition-$ R is a relation from a to b iff R is a subset of A X B

12. The **domain** of a relation R from A to B.
    ○ $Definition-$ Dom(R)=$\{x \in A$: there exists $y \in B$ such that $xRy\}$

13. The **range** of a relation R from A to B.
    ○ $Definition-$ Rng(R)=$\{y \in A$: there exists $x \in B$ such that $xRy\}$

14. The composite $S \circ R$ where R is a relation from A to B and S is a relation from B to C.
    ○ $Definition-$ $S \circ R = \{(a, c)$: there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S\}$

15. A relation R on A is **reflexive**.
    ○ $Definition-$ iff for all $x \in A$, xRx

16. A relation R on A is **symmetric**.
    ○ $Definition-$ iff for all $x, y \in A$, xRy then yRx

17. A relation R on A is **transitive**.
    ○ $Definition-$ iff for all $x, y, z \in A$, xRy and yRz, then xRz

18. A **partition** $\mathscr{P}$ of a nonempty set A.
    ○ $Definition-$ $\mathscr{P}$ is a partition of a nonempty set A.

    (i) if $x \in \mathscr{P}$, then $x \neq \varnothing$
    (ii) if $x \in \mathscr{P}$ and $y \in \mathscr{P}$, then $x = y$ or $x \cap y = \varnothing$
    (iii) $\bigcup_{x=\mathscr{P}} x = A$

19. A **function** from A to B, for sets A and B.
    ○ $Definition-$ is a relation $f : A \to B$ such that the $Dom(f) = A$, and if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$

20. $f : A \to B$ is **surjective**
    ○ $Definition-$ A function $f : A \to B$ is onto B or **surjective** iff $Rng(f) = B$

21. $f : A \to B$ is **injective**
    ○ $Definition-$ A function $f : A \to B$ is one-to-one or is injective iff when $f(x) = f(y)$, then $x = y$.

22. $f : A \to B$ and $Y \subseteq B$. Define $f^{-1}(Y)$.
    ○ $Definition-$ $f^{-1}(Y) = \{x \in A : \ y = f(x) \in Y\}$

True False

1.

Part 3 Problems

1.

Part 4 Problems

1.

Proofs

1. Prove that $\sqrt{2}$ is irrational.

   Proof: Assume $\sqrt{2}$ is rational.
   Thus $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}$ and a, b have no common factors.
   Now let us write $2 = \frac{a^2}{b^2}$, or $b^2 = 2a^2$.
   Thus b^2 is even, the only way this can be true is that b itself is even.
   hence b^2 is divisible by 4. This contradicts our assumption that a,b have no common factors
   Therefore $\sqrt{2}$ cannot be rationalized. ■

2. Prove that there are infinitely many prime numbers.

   Proof: Assume there exists a finite number of primes $\{p_1, p_2, ..., p_n\}$
   Let $N = p_1 \cdot p_2 \cdot p_3 \cdots p_{n+1}$
   N is not divisible by any of the known primes since it will leave a remainder of one upon division by any one of them.
   Thus, N must be a divisible by some other prime not in our list
   which contradicts the assumption that there is a finite number of primes.
   Therefore There are an infinite number of prime numbers ■

3. Let x be an integer. Prove that if x^2 is not divisible by 4, then x is odd.

   Proof: by contrapositive

   Assume x is not even.
   Then $x = 2k + 1$ for some $k \in \mathbb{Z}$
   And if $x = 2k + 1$, it follows that
   $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$
   Clearly, 4 does not divide $x^2 = 4(k^2 + k) + 1$, because 4 does not divide 1.
   Thus if x is not even $\implies$ x^2 is not divisible by 4.
   Clearly, if x is even $\implies$ x^2 is divisible by 4.

   ■

4. When any integer n is divided by 3, it has a remainder of 0, 1, or 2 (by the division algorithm). This means that for any integer n

   $n = 3k$ for some integer k, or
   $n = 3k + 1$ for some integer k, or
   $n = 3k + 2$ for some integer. use this fact to prove that for any integer n, 3 divides $n^3 - n$.

   Proof: Let $n = 3k$ where $k \in \mathbb{Z}$. Then

$$\begin{aligned} n^3 - n &= (3k)^3 - 3k \\ &= 27k^3 - 3k \\ &= 3(9k^3 - k), 9k^3 - k \in \mathbb{Z}, \text{ and is divisible by 3} \end{aligned}$$

Now let $n = 3k + 1$

$$
\begin{aligned}
\text{Then, } n^3 - n &= (3k+1)^3 - (3k+1) \\
&= (9k^2 + 6k + 1)(3k+1) - (3k+1) \\
&= 27k^3 + 18k^2 + 2k + 9k^2 + 6k + 1 - 9k - 1 \\
&= 27k^3 + 27k^2 + 6k \\
&= 3(9k^3 + 9k^2 + 2k) \\
&= (9k^3 + 9k^2 + 2k) \in \mathbb{Z}, \text{ and} \\
&= 3(9k^3 + 9k^2 + 2k) \text{ is divisible by 3.}
\end{aligned}
$$

Now let $n = 3k + 2$

$$
\begin{aligned}
\text{Then, } n^3 - n &= (3k+2)^3 + (3k+2) \\
&= (9k^2 + 12k + 4)(3k+2) - (3k+2) \\
&= (27k^3 + 36k^2 + 12k + 18k^2 + 24k + 9 - 3k - 2 \\
&= 27k^3 + 54k^2 + 33k + 6 \\
&= 3(9k^3 + 18k^2 + 11k + 2) \\
&= (9k^3 + 18k^2 + 11k + 2) \in \mathbb{Z}, \text{ and} \\
&= 3(9k^3 + 18k^2 + 11k + 2) \text{ is divisible by 3.}
\end{aligned}
$$

Therefore for any integer $n = 3k$, $n = 3k + 1$, or $n = 3k + 2$, n is divisible by 3.

■

5. For integers a, b, c, x and y, prove that if c divides a and c divides b, then c divides $ax + by$.

Proof:

Let $a, b, c, x,$ and $y, \in \mathbb{Z}$,
Let c divide b such that $b = c \cdot k$, where $k \in \mathbb{Z}$, and
Let c divide a such that $a = c \cdot m$, where $m \inf \mathbb{Z}$
now let $\dfrac{ax + by}{c} = \dfrac{(ck)x + (cm)y}{c} = \dfrac{c}{c}(kx + my) = (kx + my)$
Therefore c divides $ax + by$

■

6. For real numbers x and y, prove that if x is rational and y is irrational, then $x + y$ is irrational.

Proof:

Let x be rational, and y be irrational
Let $x + y$ be rational, then $x + y = \dfrac{a}{b}$. Let $x = \dfrac{c}{d}$. Then
y $= \dfrac{a}{b} - x = \dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad - bc}{bd}$
Thus y is rational which contradicts our first assumption that y is irrational.
Hence it follows that x + y is irrational

■

7. Let A, B, and C be sets. Prove that if $A \subseteq B$, $B \subseteq C$, and $C \subseteq A$, then $A = B$ and $B = C$

Proof:

Let A $\subseteq B, B \subseteq C$, and $C \subseteq A$,

and let $x \in A$, then by the definition of $\subseteq$, $x \in B$, and $x \in C$

Thus $A \subseteq B \subseteq C$.

and in order for $A \subseteq B$, and $B \subseteq C$, and $C \subseteq A$

Therefore $A = B = C$.

■

8. Let A and B be sets. Prove that A = B if and only if $\mathscr{P}(A) = \mathscr{P}(B)$

Proof.

Let A and B be sets

Suppose A = B

Since $\mathscr{P}(A)$ is the set of all subsets of A and $\mathscr{P}(B)$ is the set of all subsets of B

Then $\mathscr{P}(A) = \mathscr{P}(B)$

Conversally suppose $\mathscr{P}(A) = \mathscr{P}(B)$

$\mathscr{P}(A)$ is the set of all subsets of A, and $\mathscr{P}(B)$ the set of all subsets of B

Therefore $A = B$

■

9. Let A and B be sets. Prove that $(A^c \cup B)^c = A \cup B^c$

Proof:

Let A and B be sets

Using Demorgan's Law $(A^c \cap B)^c = (A^c)^c \cup B^c = A \cup B^c$

Thus $(A^c \cap B)^c = A \cup B^c$

■

10. Let A, B, C, and D be sets. Prove that if $C \subseteq A$ and $D \subseteq B$, then $D - A \subseteq B - C$.

Proof: ■

11. Use the Principle of Mathematical Induction to prove the following:
For all $n \in \mathbb{N}$

$$1 \cdot 1 + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

Proof: Let n = 1

$$\begin{aligned} \text{so } 1 \cdot 1! &= (1+1)! - 1 \\ 1 &= 2! - 1 \\ 1 &= 1, \text{this is true!} \end{aligned}$$

Now assume truth for any n $\in \mathbb{N}$
we will show proof for $n+1$:

$$\begin{aligned} \text{so, } ((n+1)+1)! - 1 &= 1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! + (n+1) \cdot (n+1)! \\ (n+2)! - 1 &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(1 + (n+1)) - 1 \\ &= (n+1)!(n+2) - 1, \text{by the definition of factorial} \\ &= (n+2)! - 1 \end{aligned}$$

Thus for all $n \in \mathbb{N}$ $1 \cdot 1 + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$ ■

12. Use the principle of Mathematical Induction to prove the following: For all $n \in \mathbb{N}$:

$$8 \text{ divides } 5^{2n} - 1$$

Proof: Let n = 1

$$\text{thus } 5^n - 1 = 24, \text{ which is divisible by } 8$$

Now assume $n \in \mathbb{N}$ is true, we will show $n+1$ is true:

$$\begin{aligned} \text{Thus, } 5^{2(n+1)} - 1 &= 5^{(2n+2)} - 1 \\ &= 5^n \cdot 5^2 - 1 \\ &= 5^n \cdot 25 - 1 \\ &= 5^n \cdot 25 - 25 + 24 \\ &= 25(5^n - 1) + 24 \\ &= 25(5^n - 1) \in \mathbb{N} \text{ and is divisible by 8 because } (5^n - 1) \text{ is divisible by 8} \end{aligned}$$

and, 24 is divisible by 8.
Therefor for all $n \in \mathbb{N}$, $5^n - 1$, is divisible by 8.

■

13. We have the common differentiation formulas $\frac{d}{dx}x = 1$ and $\frac{d}{dx}(fg) = f(\frac{d}{dx}g) + (\frac{d}{dx}f)g$.
Use these formulas and the Principle of Mathematical Induction to prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}$.

Proof: ■

14. Let $a_1 = 2$, $a_2 = 4$, and $a_{n+2} = 5a_{n+1} - 6a_n$, for all $n \leq 1$. Prove that $a_n = 2^n$ for all $n \in \mathbb{N}$

    Proof: ■

15. Let R be a relation from A to B and S be a relation from B to C.
    Prove that $Rng(S \circ R) \subseteq Rng(S)$.

    Proof. ■

16. Prove that if R is a symmetric, transitive relation on A and the domain of R is A, then R is reflexive on A.

    Proof: ■

17. Let A be a nonempty set, $\mathscr{P}$ a partition of A, and B be a nonempty subset of A.
    Prove that $\mathscr{A} = \{X \cap B \mid X \in \mathscr{P} \text{ and } X \cap B \neq \varnothing\}$ is a partition of B.

    Proof: ■

18. Let R be an antisymmetric relation on the set A.
    Prove that if R is symmetric and $Dom(R) = A$, then $R = I_A$

    Proof: ■

19. Define a relation R on $\mathbb{R}$ by $aRb \iff a^3 = b^5$. Prove that R is a function from $\mathbb{R}$ to $\mathbb{R}$.

    Proof: ■

20. Prove that, if f and g are functions, then $f \cap g$ is a function by showing that $f = g|_A$ where $A = \{x : g(x) = f(x)\}$.

    Proof: ■

21. Prove that if $f : A \xrightarrow{\text{onto}} B$ and $g : B \xrightarrow{\text{onto}} C$, then $g \circ f : A \xrightarrow{\text{onto}} C$.

    Proof: ■

22. Let $f : A \to B$. Prove that if f^{-1} is a function, then f is injective.

    Proof: ■

23. Prove that $g : (-\infty, -4) \to (-\infty, 0)$, defined by $g(x) = -|x + 4|$ is a one-to-one correspondence.

    Proof: ■

24. Let $f : A \to B$ and $Y \subseteq B$. Prove that $f(f^{-1}(Y)) \subseteq Y$.

Proof: ∎

25. Let $f : A \to B$ be injective and $X \subseteq A$.
Prove that $X = f^{-1}(f(X))$

Proof: ∎