MATH 371

LECTURE

CHAPTER 2

OBJECTIVE

Write linear programming problems in a standard form to which one can apply a solution algorithm known as the Simplex Method.

STANDARD FORM

$$\text{optimize } z = C^T X$$

$$\text{subject to } AX = B$$

$$\text{with } X \geq 0$$

Here,

- z is the objective function, a function of n variables;
- C is the $n \times 1$ cost vector;
- X is the $n \times 1$ vector of independent variables
- A is the $m \times n$ coefficient matrix;
- B is $m \times 1$ vector of the nonnegative right-hand sides of the constraint equations. In addition, we want to be able to find an initial feasible solution $X_0 = B$.

PROCEDURE FOR WRITING A LINEAR PROGRAMMING PROBLEM IN STANDARD FORM

Step 1 Ensure $X \geq 0$: If all variables are already constrained to be nonnegative, this step is done. Otherwise, replace any variable that does not have a nonnegativity constraint with a difference of two variables, each constrained to be nonnegative.

Step 2 Ensure nonnegativity of B: If any constraint has a negative right-hand side, multiply that constraint by -1, reversing the inequality sign.

Step 3 Convert $\leq$ to $=$: For each constraint that has a $\leq$, add a *slack variable* on the left-hand side, and replace the inequality with an equality.

Step 4 Convert $\geq$ to $=$: For each constraint that has a $\geq$, subtract a *surplus variable* on the left-hand side, and replace the inequality with an equality.

Step 5 Ensure the existence of an initial feasible solution: For any constraint that does not have a slack variable, add an *artificial variable* on the left-hand side.

- For a maximization problem, add $-M$ times each artificial variable to the objective function;
- For a minimization problem, add M times each artificial variable to the objective function, where M is understood to be a large positive number.

Step 6 Identify the vectors and matrix: The vector X consists of all of the original independent variables as well as all slack, surplus, and artificial variables. The vector X_0 consists of slack and artificial variables only.

Examples

1. (#2.23)

PROBLEM

Write in matrix standard form: Maximize

$$z = 10x_1 + 11x_2$$

subject to

$$\begin{cases} x_1 + 2x_2 \leq 150 \\ 3x_1 + 4x_2 \leq 200 \\ 6x_1 + x_2 \leq 175 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}.$$

SOLUTION

Step 1 Already done

Step 2 Already done

Step 3 Since the first three constraints all have $\leq$, we add a slack variable to each, converting the inequalities to equalities:

$$\begin{cases} x_1 + 2x_2 + x_3 = 150 \\ 3x_1 + 4x_2 + x_4 = 200 \\ 6x_1 + x_2 + x_5 = 175 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \end{cases}.$$

Step 4 Does not apply.

Step 5 Does not apply.

Step 6 We have

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{bmatrix}^T, C = \begin{bmatrix} 10 & 11 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 150 \\ 200 \\ 175 \end{bmatrix}, \text{ and } X_0 = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}.$$

2. (#2.24)

PROBLEM

Write in matrix standard form: Maximize

$$z = 10x_1 + 11x_2$$

subject to

$$\begin{cases} x_1 + 2x_2 \geq 150 \\ 3x_1 + 4x_2 \geq 200 \\ 6x_1 + x_2 \geq 175 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}.$$

SOLUTION

Step 1 Already done.

Step 2 Already done.

Step 3 Does not apply.

Step 4 Since the first three constraints all have $\geq$, we subtract a surplus variable from each, converting the inequalities to equalities.

$$\begin{cases} x_1 + 2x_2 - x_3 = 150 \\ 3x_1 + 4x_2 - x_4 = 200 \\ 6x_1 + x_2 - x_5 = 175 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0 \end{cases}.$$

Step 5 Since the first three constraints do not have slack variables, add an artificial variable to each. Since this is a maximization problem, subtract M times each of the artificial variables from the objective function. Maximize

$$z = 10x_1 + 11x_2 - Mx_6 - Mx_7 - Mx_8$$

subject to

$$\begin{cases} x_1 + 2x_2 - x_3 + x_6 = 150 \\ 3x_1 + 4x_2 - x_4 + x_7 = 200 \\ 6x_1 + x_2 - x_5 + x_8 = 175 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}.$$

Step 6 We have

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{bmatrix}^T, C = \begin{bmatrix} 10 & 11 & 0 & 0 & 0 & -M & -M & -M \end{bmatrix}^T,$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 0 & 0 & 1 & 0 \\ 6 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 150 \\ 200 \\ 175 \end{bmatrix}, \text{ and } X_0 = \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix}.$$

3. (Extra Problem)

PROBLEM

Write in matrix standard form: Minimize

$$z = 2x_1 - 3x_2$$

subject to

$$\begin{cases} 2x_1 + 5x_2 \geq -100 \\ 5x_1 - 2x_2 \leq -80 \end{cases}.$$

SOLUTION

Step 1 Since neither x_1 nor x_2 is constrained to be nonnegative, write

$$\begin{cases} x_1 = x_3 - x_4 \\ x_2 = x_5 - x_6 \end{cases},$$

with x_3, x_4, x_5, x_6 nonnegative. The problem now says minimize

$$z = 2x_3 - 2x_4 - 3x_5 + 3x_6$$

subject to

$$\begin{cases} 2x_3 - 2x_4 + 5x_5 - 5x_6 \geq -100 \\ 5x_3 - 5x_4 - 2x_5 + 2x_6 \leq -80 \\ x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0 \end{cases}.$$

Step 2 Since the constraints have negative right-hand sides, multiply each by -1, obtaining

$$\begin{cases} -2x_3 + 2x_4 - 5x_5 + 5x_6 \leq 100 \\ -5x_3 + 5x_4 + 2x_5 - 2x_6 \geq 80 \end{cases}.$$

Step 3 Introduce a slack variable in the first constraint:

$$-2x_3 + 2x_4 - 5x_5 + 5x_6 + x_7 = 100$$

Step 4 Introduce a surplus variable in the second constraint:

$$-5x_3 + 5x_4 + 2x_5 - 2x_6 - x_8 = 80$$

Step 5 Introduce an artificial variable in the second constraint: Minimize

$$z = 2x_3 - 2x_4 - 3x_5 + 3x_6 + Mx_9,$$

subject to

$$\begin{cases} -2x_3 + 2x_4 - 5x_5 + 5x_6 + x_7 = 100 \\ -5x_3 + 5x_4 + 2x_5 - 2x_6 - x_8 + x_9 = 80 \\ \text{all variables nonnegative} \end{cases}.$$

Step 6 We have

$$X = \begin{bmatrix} x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \end{bmatrix}^T, C = \begin{bmatrix} 2 & -2 & -3 & 3 & 0 & 0 & M \end{bmatrix}^T,$$

$$A = \begin{bmatrix} -2 & 2 & -5 & 5 & 1 & 0 & 0 \\ -5 & 5 & 2 & -6 & 0 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 100 \\ 80 \end{bmatrix}, \text{ and } X_0 = \begin{bmatrix} x_7 \\ x_9 \end{bmatrix}.$$