MATH 371

LECTURE

Chapter 8 - Integer Programming: The Transportation Algorithm

1. **What is a Transportation Problem?** A transportation problem involves m sources, each of which has available a_i units of a homogeneous product, and n destinations, each of which requires b_j units of this product. The numbers a_i and b_j are positive integers. The cost $c_{i,j}$ of transporting one unit of product from the ith source to the jth destination is given for each i and j. The objective is to develop an integral transportation schedule that meets all demands from current inventory at a minimum total shipping cost. It is assumed that total supply and total demand are equal; that is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j.$$

This is guaranteed by creating either a fictitious destination with a demand equal to the surplus if total demand is less than total supply, or a fictitious source with a supply equal to the shortage if total demand exceeds total supply.

2. **Standard Form:** Let x_{ij} represent the (unknown) number of units to be shipped from source i to destination j. Then the standard mathematical model for this problem is:

$$\begin{array}{ll} \text{Minimize} & z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} = a_i \ (i = 1, ..., m) \\ & \sum_{i=1}^{m} x_{ij} = b_j \ (j = 1, ..., n) \\ \text{with} & \text{all } x_{ij} \text{ nonnegative and integral} \end{array}$$

3. **The Transportation Algorithm**: The transportation algorithm is the simplex method specialized to transportation problems. As with the standard simplex method, it involves

   - finding an initial, basic feasible solution;
   - testing the solution for optimality;
   - improving the solution when it is not optimal; and
   - repeating the previous two steps until an optimal solution is obtained

   (a) An initial basic solution

   - Northwest corner rule: Beginning with the $(1, 1)$-cell, allocate to x_{11} as many units as possible without violating the constraints. This will be the smaller of a_1 and b_1. Thereafter, continue by moving one cell to the right, if some supply remains, or, if not, one cell down. At each step, allocate as much as possible to the cell under consideration without violating the constraints. The allocation may be zero.
   - Vogel's method: This is a more complicated alternative which takes into account the unit shipping costs, thereby often resulting in a closer-to-optimal starting solution.

   (b) Testing for optimality: Initially, the basic variables are the ones to which we have assigned values. Assign one of the u_i or v_j the value zero (a good choice is the one in the row or column with the most basic variables), and calculate the remaining u_i and v_j so that for each basic variable, $u_i + v_j = c_{ij}$. Then, for each nonbasic variable, calculate the quantity $c_{ij} - u_i - v_j$. If all of these latter quantities are nonnegative, the current solution is optimal; otherwise it is not.

(c) Improving the solution

- Definition: A loop is a sequence of cells in the tableau such that
  - each pair of consecutive cells lie in either the same row or the same column;
  - no three consecutive cells lie in the same row or column;
  - the first and last cells of the sequence lie in the same row or column;
  - no cell appears more than once in the sequence.
- Process: Consider the nonbasic variable corresponding to the most negative of the quantities $c_{ij} - u_i - v_j$ calculated in the test for optimality; it is made the incoming variable. Construct a loop consisting exclusively of this incoming variable and current basic variables. Then allocate to the incoming cell as many units as possible such that, after appropriate adjustments have been made to the other cells in the loop, the supply and demand constraints are not violated, all allocations remain nonnegative, and one of the old basic variables has been reduced to zero (whereupon it ceases to be basic).

(d) Degeneracy: Only $n + m - 1$ of the constraint equation are independent. Thus, a nondegenerate basic feasible solution will be characterized by positive values for exactly $n+m-1$ basic variables. If the process of improving the current basic solution results in two or more current basic variables being reduced to zero simultaneously, one one is allowed to become nonbasic (with preferably the variable with the largest unit shipping cost). The other variables remain basic, but with a zero allocation, rendering the new basic solution degenerate. The northwest corner rule can yield a degenerate solution. Improving a degenerate solution may result in replacing one basic variable having a zero value by another such. This is necessary for the algorithm to proceed.