MATH 371 - Nonlinear Optimization

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a) Calculus:

$$f(x,y) = 4x^2 - 4xy + 2y^2$$

$$f_x = 8x - 4y$$
$$f_y = 4y - 4x$$

$$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$$

$$f_{xx} = 8$$
$$f_{yy} = 4$$
$$f_{xy} = -4$$

$$\begin{cases} 8x - 4y = 0 \\ 4y - 4x = 0 \end{cases}$$

Point	$f_{xx} f_{yy} - f_{xy}^2$	Classification
$(0,0)$	$(8)(4) - (-4)^2 = 16$	Relative Minimum

Steepest Descent:

$$f(x,y) = 4x^2 - 4xy + 2y^2$$

$$X_{k+1} = x_k + \lambda_k \nabla f \big|_{x_k}$$

$$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$$

k	X	f(X)	grad(X)	lambda
0	<1, 1>	2	<4, 0>	l: -0.125
1	<0.5, 1>	1	<0, 2>	l: -0.25
2	<0.5, 0.5>	0.5	<2, 0>	l: -0.125
3	<0.25, 0.5>	0.25	<0, 1>	l: -0.25
4	<0.25, 0.25>	0.125	<1, 0>	l: -0.125
5	<0.125, 0.25>	0.0625	<0, 0.5>	l: -0.25
6	<0.125, 0.125>	0.03125	<0.5, 0>	l: -0.125
7	<0.0625, 0.125>	0.015625	<0, 0.25>	l: -0.25
8	<0.0625, 0.0625>	0.0078125	<0.25, 0>	l: -0.125
9	<0.03125, 0.0625>	0.00390625	<0, 0.125>	l: -0.25
10	<0.03125, 0.03125>	0.001953125	<0.125, 0>	l: -0.125

Newton-Raphson:

$$f(x,y) = 4x^2 - 4xy + 2y^2$$

$$H = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

$$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$$

$$X_{k+1} = X_k - \left(H\big|_{X_k}\right)^{-1} \vec{\nabla} f \big|_{X_k}$$

k	X	f(X)	grad(X)
0	<1, 1>	2	<4, 0>
1	<0, 0>	0	<0, 0>
2	<0, 0>	0	<0, 0>

b) Calculus:

$$g(x,y) = -xy(x-2)(y+3)$$

$$\vec{\nabla} g = \langle -y(y+3)(2x-2),\ -x(x-2)(2y+3) \rangle$$

$$f_{xx} = -2y(y+3)$$
$$f_{yy} = -2x(x-2)$$
$$f_{xy} = -(2x-2)(2y+3)$$

$$\begin{cases} -y(y+3)(2x-2) = 0 \\ -x(x-2)(2y+3) = 0 \end{cases}$$

Point	$f_{xx}f_{yy} - f_{xy}^2$	Classification
$(0,0)$	$0 \cdot 0 - 6^2 = -36$	Saddle Point
$(2,0)$	$0 \cdot 0 - (-6)^2 = -36$	Saddle Point
$(0,-3)$	$0 \cdot 0 - (-6)^2 = -36$	Saddle Point
$(2,-3)$	$0 \cdot 0 - 6^2 = -36$	Saddle Point
$(1,-3/2)$	$9/2 \cdot 2 - 0^2 = 9$	Relative Minimum

Steepest Descent:

$$g(x,y) = -xy(x-2)(y+3)$$

$$X_{k+1} = x_k + \lambda_k \nabla g\Big|_{x_k}$$

$$\vec{\nabla} g = \langle -y(y+3)(2x-2),\ -x(x-2)(2y+3) \rangle$$

k	X	f(X)	grad(X)	lambda
0	<0.5, 0.5>	1.3125	<1.75, 3>	l: -0.25
1	<0.0625, -0.25>	-0.08325195	<-1.28906, 0.30273>	l: -0.85
2	<1.1582, -0.50732>	-1.23294423	<0.40013, 1.93566>	l: -0.5
3	<0.95814, -1.47516>	-2.24544128	<-0.18832, 0.0496>	l: -0.25
4	<1.00522, -1.48756>	-2.24978385	<0.02349, 0.02489>	l: -0.3
5	<0.99817, -1.49502>	-2.24996772	<-0.00822, 0.00996>	l: -0.35
6	<1.00105, -1.49851>	-2.24999529	<0.00473, 0.00299>	l: -0.25
7	<0.99987, -1.49925>	-2.2499994	<-0.00059, 0.00149>	l: -0.45
8	<1.00013, -1.49993>	-2.24999995	<0.00061, 0.00015>	l: -0.25
9	<0.99998, -1.49996>	-2.25	<-8E-05, 7E-05>	l: -0.3
10	<1.00001, -1.49999>	-2.25	<3E-05, 3E-05>	l: -0.3

Newton-Raphson:

$$g(x,y) = -xy(x-2)(y+3)$$

$$\vec{\nabla} g = \langle -y(y+3)(2x-2),\ -x(x-2)(2y+3) \rangle$$

$$H = \begin{bmatrix} -2y(y+3) & -(2x-2)(2y+3) \\ -(2x-2)(2y+3) & -2x(x-2) \end{bmatrix}$$

$$X_{k+1} = X_k - \left(H\Big|_{X_k}\right)^{-1} \vec{\nabla} g\Big|_{X_k}$$

k	X	f(X)	grad(X)
0	<1, 1>	4	<0, 5>
1	<1, -1.5>	-2.25	<0, 0>
2	<1, -1.5>	-2.25	<0, 0>

c. Calculus:

$$h(x,y) = (1-y)^2 + 100(x-y^2)^2$$

$$\vec{\nabla} h = \langle 200(x-y^2),\ -2(1-y) - 400y(x-y^2) \rangle$$

$$f_{xx} = 200$$
$$f_{yy} = 2 - 400(x-3y^2)$$
$$f_{xy} = -400y$$

$$\begin{cases} 200(x-y^2) = 0 \\ -2(1-y) - 400y(x-y^2) = 0 \end{cases}$$

Point	$f_{xx}f_{yy} - f_{xy}^2$	Classification
(1, 1)	$200 \cdot 802 - (-400)^2 = 400$	Relative Minimum

$$\begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix} + \lambda \begin{bmatrix} 18 \\ -32.6 \end{bmatrix}$$

$$0.9 + 18\lambda,\ 0.9 - 32.6\lambda$$

Steepest Descent:

$$h(x,y) = (1-y)^2 + 100(x-y^2)^2$$

$$X_{k+1} = x_k + \lambda_k \nabla h \big|_{x_k}$$

$$\vec{\nabla} h = \langle 200(x-y^2),\ -2(1-y) - 400y(x-y^2) \rangle$$

k	X	f(X)	grad(X)	lambda
0	<1.1, 1.1>	1.22	<-22, 48.6>	l: -0.00087
1	<1.11914, 1.05772>	0.00334525	<0.07453, -0.04222>	l: -0.00137
2	<1.11904, 1.05778>	0.00334024	<0.02963, 0.05286>	l: -0.00271
3	<1.11896, 1.05763>	0.00333527	<0.07418, -0.04164>	l: -0.00137
4	<1.11886, 1.05769>	0.0033303	<0.02972, 0.05252>	l: -0.00274
5	<1.11877, 1.05755>	0.00332532	<0.07431, -0.04207>	l: -0.00137
6	<1.11867, 1.0576>	0.00332034	<0.02956, 0.05267>	l: -0.00271
7	<1.11859, 1.05746>	0.00331539	<0.07392, -0.04142>	l: -0.00138
8	<1.11849, 1.05752>	0.00331045	<0.02934, 0.05298>	l: -0.00267
9	<1.11841, 1.05738>	0.00330555	<0.0735, -0.04069>	l: -0.00139
10	<1.11831, 1.05743>	0.00330066	<0.02915, 0.05322>	l: -0.00264

Newton-Raphson:

$$h(x,y) = (1-y)^2 + 100(x-y^2)^2$$

$$\vec{\nabla} h = \langle 200(x-y^2),\ -2(1-y) - 400y(x-y^2) \rangle$$

$$H = \begin{bmatrix} 200 & -400y \\ -400y & 2 - 400(x-3y^2) \end{bmatrix}$$

$$X_{k+1} = X_k - \left(H\big|_{X_k}\right)^{-1} \vec{\nabla} h \big|_{X_k}$$

k	X	f(X)	grad(X)
0	<0, 0>	1	<0, -2>
1	<0, 1>	100	<-200, 400>
2	<1, 1>	0	<0, 0>
3	<1, 1>	0	<0, 0>

Problem 2

We used two scripts for problem two found here.

(A) Three-Point Method

Minima of $x^3 - x + 1$ on the Interval $[0, 1.28]$											
Index	a	x1	x2	x3	b	f(a)	f(x1)	f(x2)	f(x3)	f(b)	Min Point	Min Value
1	0.000	0.320	0.640	0.960	1.280	1.000	0.713	0.622	0.925	1.817	0.640	0.622
2	0.320	0.480	0.640	0.800	0.960	0.713	0.631	0.622	0.712	0.925	0.640	0.622
3	0.480	0.560	0.640	0.720	0.800	0.631	0.616	0.622	0.653	0.712	0.560	0.616
4	0.480	0.520	0.560	0.600	0.640	0.631	0.621	0.616	0.616	0.622	0.560	0.616
5	0.520	0.540	0.560	0.580	0.600	0.621	0.617	0.616	0.615	0.616	0.580	0.615
6	0.560	0.570	0.580	0.590	0.600	0.616	0.615	0.615	0.615	0.616	0.580	0.615
7	0.570	0.575	0.580	0.585	0.590	0.615	0.615	0.615	0.615	0.615	0.575	0.615
8	0.570	0.573	0.575	0.577	0.580	0.615	0.615	0.615	0.615	0.615	0.577	0.615
9	0.575	0.576	0.577	0.579	0.580	0.615	0.615	0.615	0.615	0.615	0.577	0.615

Minimum Point: 0.577
Minimum Value: 0.615

(B) Fibonacci Search

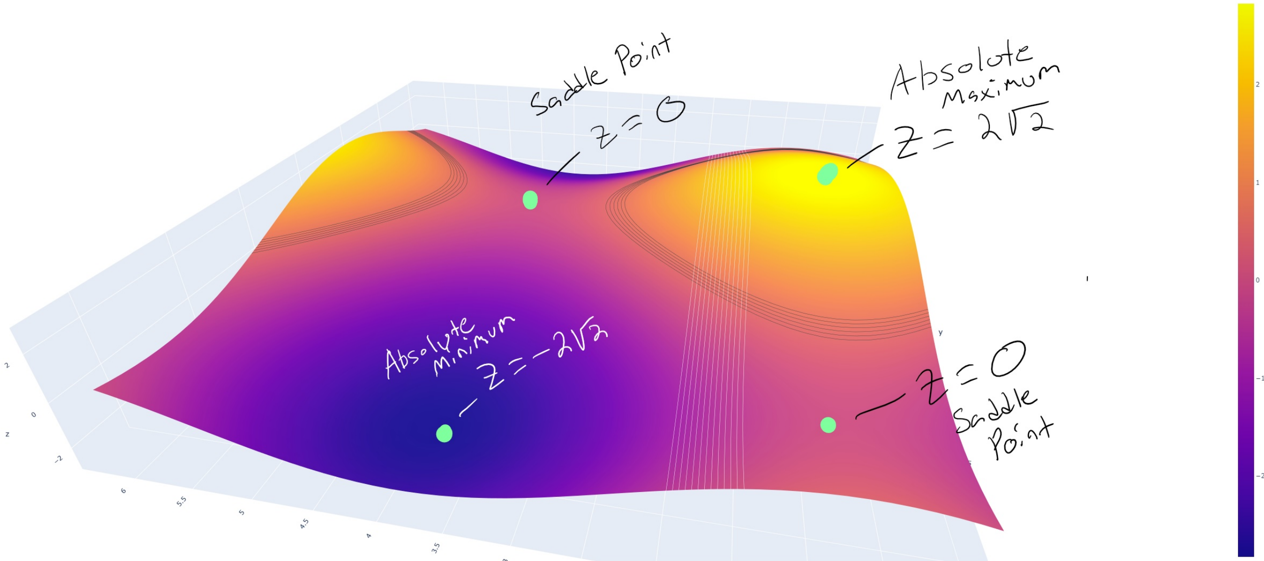
Minima of $x^3 - x + 1$ on the Interval $[0, 1.28]$									
Index	a	b	fib[N]	x_1	x_2	$f(x_1)$	$f(x_2)$	mid-point	mid-point value
0	0.000	1.280	144	N/A	N/A	N/A	N/A	N/A	N/A
1	0.000	0.791	144	0.791	0.489	0.704	0.628	0.640	0.622
2	0.302	0.791	89	0.489	0.302	0.628	0.725	0.396	0.666
3	0.489	0.791	55	0.604	0.489	0.616	0.628	0.547	0.617
4	0.489	0.676	34	0.676	0.604	0.633	0.616	0.640	0.622
5	0.489	0.604	21	0.604	0.560	0.616	0.616	0.582	0.615
6	0.533	0.604	13	0.560	0.533	0.616	0.618	0.547	0.617
7	0.560	0.604	8	0.578	0.560	0.615	0.616	0.569	0.615
8	0.560	0.587	5	0.587	0.578	0.615	0.615	0.582	0.615
9	0.569	0.587	3	0.578	0.569	0.615	0.615	0.573	0.615
10	0.569	0.578	2	0.578	0.578	0.615	0.615	0.578	0.615

Minimum Point: 0.578
Minimum Value: 0.615

Problem 3

For problem three we created this script to find the critical points and maximum and minimum values of $g(x, y) = sin(x)+cos(x)+sin(y)-cos(y)$ on the region $[0, 2\pi]$ x $[0, 2\pi$ of the $(x, y)-$plane.

$g(x, y) = sin(x) + cos(x) + sin(y) - cos(y)$		
Points	Values	Classification
$\frac{5\pi}{4}, \frac{3\pi}{4}$	0	saddle point
$\frac{5\pi}{4}, \frac{7\pi}{4}$	$-2\sqrt{2}$	rel. min.
$\frac{\pi}{4}, \frac{3\pi}{4}$	$2\sqrt{2}$	rel. max.
$\frac{\pi}{4}, \frac{7\pi}{4}$	0	saddle point



Problem 4

For problem four we created these scripts for ...

(A) Finding the gradient of F.

$$\nabla F(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 2xy^2 + 2xz^2 + 2x - 2yz + 2y - 2z - 2 \\ 2x^2y - 2xz + 2x + 2yz^2 + 2y - 2z + 2 \\ 2x^2z - 2xy - 2x + 2y^2z - 2y + 2z - 2 \end{bmatrix}$$

(B) Finding the Hessian matrix of F

$$H_f = \begin{bmatrix} \frac{\partial f^2}{\partial xx} & \frac{\partial f^2}{\partial xy} & \frac{\partial f^2}{\partial xz} \\ \frac{\partial f^2}{\partial yx} & \frac{\partial f^2}{\partial yy} & \frac{\partial f^2}{\partial yz} \\ \frac{\partial f^2}{\partial zx} & \frac{\partial f^2}{\partial zy} & \frac{\partial f^2}{\partial zz} \end{bmatrix} = \begin{bmatrix} (2y^2 + 2z^2 + 2) & (4xy - 2z + 2) & (4xz - 2y - 2) \\ (4xy - 2z + 2) & (2x^2 + 2z^2 + 2) & (-2x + 4yz - 2) \\ (4xz - 2y - 2) & (-2x + 4yz - 2) & (2x^2 + 2y^2 + 2) \end{bmatrix}$$

(C) using the method of steepest descent for $x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Steepest descent		
k	x	λ
0	0	-0.30108
1	-2.41925189	-0.2472
2	-2.89866381	-0.12236
3	-2.97600187	-0.25493
4	-2.99421551	-0.1096
5	-2.99851808	-0.25413

(D) Using our custom Newton-Rhapson script

Newton-Rhapson		
k	x	f(x)
0	$< 2, -1, 1 >$	5
1	$< 2.333333, -0.333333, 0.333333 >$	-1.48148148
2	$< 1.260253, -0.431333, 0.431333 >$	-2.99323589
3	$< 1.207582, -0.452788, 0.452788 >$	-2.99999305
4	$< 1.205633, -0.453391, 0.453391 >$	-2.99999999
5	$< 1.205571, -0.453397, 0.453397 >$	-3
6	$< 1.205569, -0.453398, 0.453398 >$	-3
7	$< 1.205569, -0.453398, 0.453398 >$	-3

So it appears that our function F is converging to -3.

Problem 5

For problem five we created this script, considering the functions...

$$
\begin{aligned}
f(x,y,z) &= xyz \\
g(x,y,z) &= x^2+y^2+z^2-12 \\
h(x,y,z) &= x+y+z-4
\end{aligned}
$$

(A) Using the LaGrange multipliers, we ended up with several points that gave similar answers: these points are one of six sets. We used several scripts written from scratch as shown here.

Lagrange Multipliers		
Variable	Approximate	Exact
x	2.38742588672...	$\frac{\sqrt{10}}{3}+\frac{4}{3}$
y	2.38742588672...	$\frac{\sqrt{10}}{3}+\frac{4}{3}$
z	-0.7748517734...	$\frac{4}{3}-\frac{2\sqrt{10}}{3}$
λ	1.1937129433...	$\frac{\sqrt{10}}{6}+\frac{2}{3}$
μ	-3.8499011822...	$\frac{-22}{9}-\frac{4\sqrt{10}}{9}$

Min. & Max. values	
type	value
Max. value	0.268353822346948
Min. value	-4.41650197049510

(B) using the Newton-Rhapson we got...

Minima of x^3-x+1 on the Interval $[0,1.28]$						
Index	x	y	z	L	M	L(x, y, z, L, M)
1	-3.000	3.500	3.500	-10.000	12.000	12.000
2	-1.346	2.673	2.673	-4.485	12.627	178.250
3	-0.836	2.418	2.418	-1.809	5.490	8.783
4	-0.776	2.388	2.388	-1.211	3.895	-4.179
5	-0.775	2.387	2.387	-1.194	3.850	-4.416

Global Extremum: -4.416