MATH 371

PROJECT 4

MULTIVARIABLE NONLINEAR OPTIMIZATION

SYNOPSIS

This project is designed to give you practice in finding extrema of nonlinear functions of one and two variables, both analytically and numerically.

INSTRUCTIONS

- Provide neatly-written, easy-to-follow solutions to the following.
- For the analytical methods, show your work clearly.
- For the numerical methods, find, where possible, a way to at least partially automate the process without simply using a package that contains the algorithm.

PROBLEMS

1. For each of the three functions listed below, do the following.
   - Use calculus to find and classify all critical points.
   - Use the method of steepest descent to find all local minima.
   - Use the Newton-Raphson method to find all local minima.

   (a) $f(x,y) = 4x^2 - 4xy + 2y^2$

   (b) $g(x,y) = -xy(x-2)(y+3)$

   (c) $h(x,y) = (1-y)^2 + 100\left(x-y^2\right)^2$

   - For the calculus solutions, include
     - the gradient, $\nabla f(x,y)$
     - the system of equations you are solving in order to find the extrema
     - the critical points and their classification (local max, local min, saddle point). If there is more than one critical point, arrange this information in a table.
   - For the numerical solutions, include
     - the gradient
     - the generic iterative equation used in the algorithm
     - a table showing the successive x_k's along with the value of the function and the gradient at each. In the case of steepest descent, also include the value of λ at each.

2. Consider the function $f(x) = x^3 - x + 1$. Find the minimum value of f and its location on the interval $[0, 1.28]$ to within 0.01 by each of the following methods.

   (a) three-point method (bisection method)

   (b) Fibonacci search

3. Consider the function

$$g(x, y) = \sin x + \cos x + \sin y - \cos y$$

on the region $[0, 2\pi] \times [0, 2\pi]$ of the (x, y)-plane.

(a) Determine the critical points of g.

(b) Determine the absolute maximum and minimum values of g and where they occur. Be sure to consider the boundary values.

4. Consider the function

$$\begin{aligned} F(x, y, z) &= \left(y^2 z^2 + z^2 x^2 + x^2 y^2\right) + \left(x^2 + y^2 + z^2\right) - 2xyz \\ &\quad -2(z + x - y) - 2(yz + zx - xy). \end{aligned}$$

(a) Find the gradient of F.

(b) Find the Hessian matrix of F.

(c) Starting with $X_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, apply the method of steepest descent to approximate a (global) minimum of F. Continue through X_5.

(d) Starting with $X_0 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, apply the Newton-Raphson method to approximate a (global) minimum of F. Continue through X_7.

5. Consider the functions

$$\begin{aligned} f(x, y, z) &= xyz \\ g(x, y, z) &= x^2 + y^2 + z^2 - 12 \\ h(x, y, z) &= x + y + z - 4. \end{aligned}$$

(a) Using the method of Lagrange Multipliers, determine the absolute maxima and minima of $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$ and where those extrema occur.

(b) By considering the function

$$L(x, y, z, \lambda, \mu) = f(x, y, z) - \lambda \cdot g(x, y, z) - \mu \cdot h(x, y, z),$$

apply the Newton-Raphson method to approximate a global extremum of $f(x, y, z)$ subject to $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Use $Z_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.