LECTURE

Chapter 3 - The Simplex Method

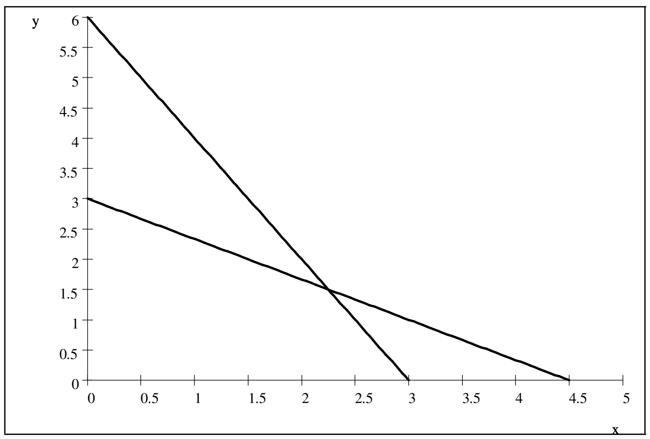
Example (3.16): Maximize

$$z = 3x_1 + 4x_2,$$

subject to

$$\begin{cases} 2x_1 + x_2 \le 6 \\ 2x_1 + 3x_2 \le 9 \\ x_1, x_2 \ge 0 \end{cases}.$$

1. Graphical solution: As usual, we draw the feasible region, find the corner points, and evaluate the objective function at each.



Point	Coordinates	Value of $z = 3x_1 + 4x_2$	
A	$(0,0)$	0	
B	$(3,0)$	9	
C	$\left(\frac{9}{4}, \frac{3}{2}\right)$	$\frac{51}{4}$	maximum
D	$(0,3)$	12	

2. Simplex method solution:

   (a) We begin by writing the problem in standard form. We introduce slack variables x_3 and x_4, to write the problem as: Maximize

$$z = 3x_1 + 4x_2$$

   subject to

$$\begin{cases} 2x_1 + x_2 + x_3 = 6 \\ 2x_1 + 3x_2 + x_4 = 9 \\ \text{all variables nonnegative} \end{cases}.$$

   The corresponding matrices are

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, X_0 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}, C_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T, C = \begin{bmatrix} 3 & 4 & 0 & 0 \end{bmatrix}.$$

   (b) We now set up the **initial simplex tableau**. For a maximization problem with no artificial variables, it is

$$\begin{array}{c|c|c} & X^T & \\ \hline X_0 & A & B \\ \hline & C_0^T A - C^T & C_0^T B \end{array}.$$

   In this particular example, we have

$$\begin{array}{c|rrrr|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline x_3 & 2 & 1 & 1 & 0 & 6 \\ x_4 & 2 & 3 & 0 & 1 & 9 \\ \hline & -3 & -4 & 0 & 0 & 0 \end{array}$$

   The corresponding solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 6 \\ x_4 = 9 \end{cases} \text{and } z = 0.$$

   Note that this is point A in the graphical solution.

   (c) We begin the simplex method. Note: In numbering the rows and columns, I'll ignore the top row and the left column.

   - The most negative element in the bottom row is the -4. Dividing the entries in the last column by the corresponding entries in the second column, we find that (exclusive of the last row) the smallest positive ratio occurs in the second row. Thus, the pivot element is the 3 in the $(2,2)$-spot. We now get a 1 in this spot and zeros in the rest of that column by first multiplying row 2 by $1/3$ and then replacing row 1 and row 3, respectively, by $r1 - r2$ and $r3 + 4r2$. We then replace the pivot row variable by the pivot column variable. We get

$$\begin{array}{c|rrrr|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline x_3 & 2 & 1 & 1 & 0 & 6 \\ x_4 & \frac{2}{3} & 1 & 0 & \frac{1}{3} & 3 \\ \hline & -3 & -4 & 0 & 0 & 0 \end{array} \text{ and then } \begin{array}{c|rrrr|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline x_3 & \frac{4}{3} & 0 & 1 & -\frac{1}{3} & 3 \\ x_2 & \frac{2}{3} & 1 & 0 & \frac{1}{3} & 3 \\ \hline & -\frac{1}{3} & 0 & 0 & \frac{4}{3} & 12 \end{array}.$$

     The corresponding solution is

$$\begin{cases} x_1 = 0 \\ x_2 = 3 \\ x_3 = 3 \\ x_4 = 0 \end{cases} \text{and } z = 12.$$

     Note that this is point D in the graphical solution.

- Since there is still a negative entry in the bottom row (exclusive of the last column), we repeat the process. This time, the pivot element is the $\frac{4}{3}$ in the $(1,1)$-spot. We have

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \\
\hline
x_3 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{9}{4} \\
x_2 & \frac{2}{3} & 1 & 0 & \frac{1}{3} & 3 \\
\hline
 & -\frac{1}{3} & 0 & 0 & \frac{4}{3} & 12
\end{array}
\quad \text{and then} \quad
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \\
\hline
x_1 & 1 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{9}{4} \\
x_2 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
\hline
 & 0 & 0 & \frac{1}{4} & \frac{5}{4} & \frac{51}{4}
\end{array}.
$$

  The corresponding solution is

$$
\begin{cases}
x_1 = \frac{9}{4} \\
x_2 = \frac{3}{2} \\
x_3 = 0 \\
x_4 = 0
\end{cases}
\quad \text{and } z = \frac{51}{4}.
$$

  Note that this is point C in the graphical solution. Since there is no longer a negative entry in the bottom row, the maximum has been reached.

Example (3.19) Maximize

$$z = 2x_1 + 3x_2 + 4x_3$$

subject to

$$\begin{cases} x_1 + x_2 + x_3 \leq 1 \\ x_1 + x_2 + 2x_3 = 2 \\ 3x_1 + 2x_2 + x_3 \geq 4 \\ \text{all variables nonnegative} \end{cases}.$$

1. Begin by writing this in standard form: Maximize

$$z = 2x_1 + 3x_2 + 4x_3 - Mx_6 - Mx_7$$

subject to

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + x_2 + 2x_3 + x_6 = 2 \\ 3x_1 + 2x_2 + x_3 - x_5 + x_7 = 4 \\ \text{all variables nonnegative} \end{cases}.$$

The corresponding matrices are

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, X_0 = \begin{bmatrix} x_4 \\ x_6 \\ x_7 \end{bmatrix}, C_0 = \begin{bmatrix} 0 \\ -M \\ -M \end{bmatrix}$$

and

$$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}^T, C = \begin{bmatrix} 2 & 3 & 4 & 0 & 0 & -M & -M \end{bmatrix}^T.$$

2. Set up the initial simplex tableau: As before, it is

$$\begin{array}{c|l|l} & X^T & \\ \hline X_0 & A & B \\ \hline & C_0^T A - C^T & C_0^T B \end{array},$$

except that now, the last row is split into two rows, the first consisting of the terms not involving M, the second consisting of the terms involving M. Thus we have

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline x_4 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ x_6 & 1 & 1 & 2 & 0 & 0 & 1 & 0 & 2 \\ x_7 & 3 & 2 & 1 & 0 & -1 & 0 & 1 & 4 \\ \hline & -2 & -3 & -4 & 0 & 0 & 0 & 0 & 0 \\ & -4 & -3 & -3 & 0 & 1 & 0 & 0 & \text{-6} \end{array}.$$

3. We begin a modified version of the simplex method.

- The first pivot element is the 1 in the $(1, 1)$-spot. We obtain

$$\begin{array}{c|ccccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline x_1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ x_6 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 \\ x_7 & 0 & -1 & -2 & -3 & -1 & 0 & 1 & 1 \\ \hline & 0 & -1 & -2 & 2 & 0 & 0 & 0 & 2 \\ & 0 & 1 & 1 & 4 & 1 & 0 & 0 & -2 \end{array}.$$

- The last row has no negative elements exclusive of the last column, and the next-to-last row contains no negative element above a zero element, so we are done, but nonzero artificial variables remain, so there is no solution.

Example (3.28) (This is Problem 1.18.) Maximize

$$z = 6x_1 + 4x_2 + 6x_3 + 8x_4$$

subject to

$$\begin{cases} 3x_1 + 2x_2 + 2x_3 + 4x_4 \leq 480 \\ x_1 + x_2 + 2x_3 + x_4 \leq 400 \\ 2x_1 + x_2 + 2x_3 + x_4 \leq 400 \\ x_1 \geq 50 \\ x_2 + x_3 \geq 100 \\ x_4 \leq 25 \\ \text{all variables nonnegative} \end{cases}$$

1. Begin by writing this in standard form. For the $\leq$'s we introduce slack variables; for the $\geq$'s we introduce surplus and artificial variables. This gives us the following problem: Maximize

$$z = 6x_1 + 4x_2 + 6x_3 + 8x_4 - Mx_{11} - Mx_{12}$$

   subject to

$$\begin{cases} 3x_1 + 2x_2 + 2x_3 + 4x_4 + x_5 = 480 \\ x_1 + x_2 + 2x_3 + x_4 + x_6 = 400 \\ 2x_1 + x_2 + 2x_3 + x_4 + x_7 = 400 \\ x_1 - x_9 + x_{11} = 50 \\ x_2 + x_3 - x_{10} + x_{12} = 100 \\ x_4 + x_8 = 25 \\ \text{all variables nonnegative} \end{cases},$$

   so that

$$A = \begin{bmatrix} 3 & 2 & 2 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 480 \\ 400 \\ 400 \\ 50 \\ 100 \\ 25 \end{bmatrix}, X_0 = \begin{bmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_{11} \\ x_{12} \end{bmatrix}, C_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -M \\ -M \end{bmatrix},$$

   and

$$\begin{aligned} X &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} \end{bmatrix}^T, \\ C &= \begin{bmatrix} 6 & 4 & 6 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & -M & -M \end{bmatrix}^T. \end{aligned}$$

2. Now set up the initial simplex tableau: As in the previous example, we split the last row in two. We have

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	
x_5	3	2	2	4	1	0	0	0	0	0	0	0	480
x_6	1	1	2	1	0	1	0	0	0	0	0	0	400
x_7	2	1	2	1	0	0	1	0	0	0	0	0	400
x_8	1	0	0	0	0	0	0	0	−1	0	1	1	50
x_{11}	0	1	1	0	0	0	0	0	0	−1	0	1	100
x_{12}	0	0	0	1	0	0	0	1	0	0	0	0	25
	−6	−4	−6	−8	0	0	0	0	0	0	0	0	0
	0	−1	−1	−1	0	0	0	−1	0	1	1	0	−125

3. We begin the simplex method (the two-phase method).

- For the initial pivot, we use the 1 in the (5,2)-spot. Once we get 0's in the rest of this column, we have

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	
x_5	3	0	0	4	1	0	0	0	0	2	0	-2	280
x_6	1	0	1	1	0	1	0	0	0	1	0	-1	300
x_7	2	0	1	1	0	0	1	0	0	1	0	-1	300
x_8	1	0	0	0	0	0	0	0	-1	0	1	1	50
x_2	0	1	1	0	0	0	0	0	0	-1	0	1	100
x_{12}	0	0	0	1	0	0	0	1	0	0	0	0	25
	-6	0	-2	-8	0	0	0	0	0	-4	0	4	400
	0	0	0	-1	0	0	0	-1	0	0	1	1	-25

- For the next pivot, we use the 1 in the (6,4)-spot. We get

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	
x_5	3	0	0	0	1	0	0	-4	0	2	0	-2	180
x_6	1	0	1	0	0	1	0	-1	0	1	0	-1	275
x_7	2	0	1	0	0	0	1	-1	0	1	0	-1	275
x_8	1	0	0	0	0	0	0	0	-1	0	1	1	50
x_2	0	1	1	0	0	0	0	0	0	-1	0	1	100
x_4	0	0	0	1	0	0	0	1	0	0	0	0	25
	-6	0	-2	0	0	0	0	8	0	-4	0	4	600
	0	0	0	0	0	0	0	0	0	0	1	1	0

- We can now delete the x_{11} and x_{12} columns, and thus the last row.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_5	3	0	0	0	1	0	0	-4	0	2	180
x_6	1	0	1	0	0	1	0	-1	0	1	275
x_7	2	0	1	0	0	0	1	-1	0	1	275
x_8	1	0	0	0	0	0	0	0	-1	0	50
x_2	0	1	1	0	0	0	0	0	0	-1	100
x_4	0	0	0	1	0	0	0	1	0	0	25
	-6	0	-2	0	0	0	0	8	0	-4	600

- For the next pivot, we use the 1 in the (4,1)-spot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_5	0	0	0	0	1	0	0	-4	3	2	30
x_6	0	0	1	0	0	1	0	-1	1	1	225
x_7	0	0	1	0	0	0	1	-1	2	1	175
x_1	1	0	0	0	0	0	0	0	-1	0	50
x_2	0	1	1	0	0	0	0	0	0	-1	100
x_4	0	0	0	1	0	0	0	1	0	0	25
	0	0	-2	0	0	0	0	8	-6	-4	900

- For the next pivot, we use the 3 in the (1,9)-spot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_9	0	0	0	0	$1/3$	0	0	$-4/3$	1	$2/3$	10
x_6	0	0	1	0	$-1/3$	1	0	$1/3$	0	$1/3$	215
x_7	0	0	1	0	$-2/3$	0	1	$5/3$	0	$-1/3$	155
x_1	1	0	0	0	$1/3$	0	0	$-4/3$	0	$2/3$	60
x_2	0	1	1	0	0	0	0	0	0	-1	100
x_4	0	0	0	1	0	0	0	1	0	0	25
	0	0	-2	0	2	0	0	0	0	0	960

- For the next pivot, we use the 1 in the (5,3)-spot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_9	0	0	0	0	$1/3$	0	0	$-4/3$	1	$2/3$	10
x_6	0	-1	0	0	$-1/3$	1	0	$1/3$	0	$4/3$	115
x_7	0	-1	0	0	$-2/3$	0	1	$5/3$	0	$2/3$	55
x_1	1	0	0	0	$1/3$	0	0	$-4/3$	0	$2/3$	60
x_3	0	1	1	0	0	0	0	0	0	-1	100
x_4	0	0	0	1	0	0	0	1	0	0	25
	0	2	0	0	2	0	0	0	0	-2	1160

- For the next pivot, we use the $2/3$ in the (1,10)-spot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_{10}	0	0	0	0	$1/2$	0	0	-2	$3/2$	1	15
x_6	0	-1	0	0	-1	1	0	3	-2	0	95
x_7	0	-1	0	0	-1	0	1	3	-1	0	45
x_1	1	0	0	0	0	0	0	0	-1	0	50
x_3	0	1	1	0	$1/2$	0	0	-2	$3/2$	0	115
x_4	0	0	0	1	0	0	0	1	0	0	25
	0	2	0	0	3	0	0	-4	3	0	1190

- For the next pivot, we use the 3 in the (3,8)-spot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	
x_{10}	0	$-2/3$	0	0	$-1/6$	0	$2/3$	0	$5/6$	1	45
x_6	0	0	0	0	0	1	-1	0	-1	0	50
x_8	0	$-1/3$	0	0	$-1/3$	0	$1/3$	1	$-1/3$	0	15
x_1	1	0	0	0	0	0	0	0	-1	0	50
x_3	0	$1/3$	1	0	$-1/6$	0	$2/3$	0	$5/6$	0	145
x_4	0	$1/3$	0	1	$1/3$	0	$-1/3$	0	$1/3$	0	10
	0	$2/3$	0	0	$5/3$	0	$4/3$	0	$5/3$	0	1250

- Since no negative entries remain in the last row, we are done.

4. We read off the solution:

$x_1 = 50$	$x_5 = 0$	$x_9 = 0$	$z = 1250$
$x_2 = 0$	$x_6 = 50$	$x_{10} = 45$	
$x_3 = 145$	$x_7 = 0$	$x_{11} = 0$	
$x_4 = 10$	$x_8 = 15$	$x_{12} = 0$	

.

Example (3.33) Minimize

$$z = x_1 + 2x_2 + 3x_3$$

subject to

$$\begin{cases} x_1 - 2x_2 + x_3 \geq 1 \\ 3x_1 - 4x_2 + 6x_3 \leq 8 \\ 2x_1 - 4x_2 + 2x_3 \leq 2 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

1. We begin by writing the problem in standard form. We introduce slack variables x_4 and x_5, surplus variable x_6, and artificial variable x_7, to write the problem as:

   Minimize

   $$z = x_1 + 2x_2 + 3x_3 + Mx_7,$$

   subject to

   $$\begin{cases} x_1 - 2x_2 + x_3 - x_6 + x_7 = 1 \\ 3x_1 - 4x_2 + 6x_3 + x_4 = 8 \\ 2x_1 - 4x_2 + 2x_3 + x_5 = 2 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{cases}.$$

   The corresponding matrices are

   $$A = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & -1 & 1 \\ 3 & -4 & 6 & 1 & 0 & 0 & 0 \\ 2 & -4 & 2 & 0 & 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix},\ X_0 = \begin{bmatrix} x_4 \\ x_5 \\ x_7 \end{bmatrix},\ C_0 = \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}$$

   $$X = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}^T,\ C = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & M \end{bmatrix}^T.$$

2. We now set up the **initial simplex tableau**. For a minimization problem, it is

   $$\begin{array}{c|c|c} & X^T & \\ \hline X_0 & A & B \\ \hline & C^T - C_0^T A & -C_0^T B \end{array}.$$

   In this particular example, we have

   $$\begin{array}{cccccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ x_4 & 1 & -2 & 1 & 0 & 0 & -1 & 1 & 1 \\ x_5 & 3 & -4 & 6 & 1 & 0 & 0 & 0 & 8 \\ x_7 & 2 & -4 & 2 & 0 & 1 & 0 & 0 & 2 \\ & -2M+1 & 4M+2 & -2M+3 & 0 & -M & 0 & M & -2M \end{array}.$$

   We split the second row into two rows:

   $$\begin{array}{cccccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ x_4 & 1 & -2 & 1 & 0 & 0 & -1 & 1 & 1 \\ x_5 & 3 & -4 & 6 & 1 & 0 & 0 & 0 & 8 \\ x_7 & 2 & -4 & 2 & 0 & 1 & 0 & 0 & 2 \\ & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ & -2 & 4 & -2 & 0 & -1 & 0 & 1 & -2 \end{array}.$$

3. Alternatively, we can use the dual simplex method. We write all constraints as $\leq$ and add slack variables.

   Minimize
$$z = x_1 + 2x_2 + 3x_3,$$
   subject to
$$\begin{cases} -x_1 + 2x_2 - x_3 \leq -1 \\ 3x_1 - 4x_2 + 6x_3 \leq 8 \\ 2x_1 - 4x_2 + 2x_3 \leq 2 \\ x_1, x_2, x_3 \geq 0 \end{cases}.$$
   Minimize
$$z = x_1 + 2x_2 + 3x_3,$$
   subject to
$$\begin{cases} -x_1 + 2x_2 - x_3 + x_4 = -1 \\ 3x_1 - 4x_2 + 6x_3 + x_5 = 8 \\ 2x_1 - 4x_2 + 2x_3 + x_6 = 2 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{cases}.$$

4. The initial dual simplex tableau is
$$\begin{array}{cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ x_4 & -1 & 2 & -1 & 1 & 0 & 0 & -1 \\ x_5 & 3 & -4 & 6 & 0 & 1 & 0 & 8 \\ x_6 & 2 & -4 & 2 & 0 & 0 & 1 & 2 \\ & 1 & 2 & 3 & 0 & 0 & 0 & 0 \end{array}.$$

5. We begin the dual simplex method. The most negative element basic variable (entry in the last column excluding the last entry) becomes the departing variable. Thus x_4 is the departing variable. The row containing x_4 is called the work row. Divide the entries in the last row (excluding the last entry) by the corresponding negative entries in the work row. The nonbasic variable with the smallest absolute ratio becomes the entering variable. In this case, that is x_1. Hence, the pivot element is the -1 in the $(1, 1)$-spot. Use row operations to convert the pivot element to a 1 and the rest of the entries in the pivot column to 0's. We get
$$\begin{array}{cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ x_4 & 1 & -2 & 1 & -1 & 0 & 0 & 1 \\ x_5 & 3 & -4 & 6 & 0 & 1 & 0 & 8 \\ x_6 & 2 & -4 & 2 & 0 & 0 & 1 & 2 \\ & 1 & 2 & 3 & 0 & 0 & 0 & 0 \end{array} \text{ and then } \begin{array}{cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ x_1 & 1 & -2 & 1 & -1 & 0 & 0 & 1 \\ x_5 & 0 & 2 & 3 & 3 & 1 & 0 & 5 \\ x_6 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ & 0 & 4 & 2 & 1 & 0 & 0 & -1 \end{array}.$$
   Since there is no further negative value for a basic variable, we are done. The corresponding solution is
$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 0 \end{cases} \text{ and } z = 1.$$