MATH 371 - Nonlinear Optimization

Justin Overstreet, Austin Spurlin, and Jonas Smith

Spring 2020

a) Calculus:

$f(x,y) = 4x^2 - 4xy + 2y^2$

$f_x = 8x - 4y$

$f_y = 4y - 4x$

$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$

$f_{xx} = 8$

$f_{yy} = 4$

$f_{xy} = -4$

$$\begin{cases} 8x - 4y = 0 \\ 4y - 4x = 0 \end{cases}$$

Point	$f_{xx}f_{yy} - f_{xy}^2$	Classification
(0,0)	$(8)(4) - (-4)^2 = 16$	Relative Minimum

Steepest Descent:

$f(x,y) = 4x^2 - 4xy + 2y^2$

$X_{k+1} = x_k + \lambda_k \nabla f \big|_{x_k}$

$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$

k	X	f(X)	grad(X)	lambda
0	<1, 1>	2	<4, 0>	l: -0.125
1	<0.5, 1>	1	<0, 2>	l: -0.25
2	<0.5, 0.5>	0.5	<2, 0>	l: -0.125
3	<0.25, 0.5>	0.25	<0, 1>	l: -0.25
4	<0.25, 0.25>	0.125	<1, 0>	l: -0.125
5	<0.125, 0.25>	0.0625	<0, 0.5>	l: -0.25
6	<0.125, 0.125>	0.03125	<0.5, 0>	l: -0.125
7	<0.0625, 0.125>	0.015625	<0, 0.25>	l: -0.25
8	<0.0625, 0.0625>	0.0078125	<0.25, 0>	l: -0.125
9	<0.03125, 0.0625>	0.00390625	<0, 0.125>	l: -0.25
10	<0.03125, 0.03125>	0.001953125	<0.125, 0>	l: -0.125

Newton-Raphson:

$f(x,y) = 4x^2 - 4xy + 2y^2$

$$H = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix}$$

$\vec{\nabla} f = \langle 8x - 4y,\ 4y - 4x \rangle$

$$X_{k+1} = X_k - \left(H\big|_{x_k}\right)^{-1} \vec{\nabla} f \big|_{x_k}$$

k	X	f(X)	grad(X)
0	<1, 1>	2	<4, 0>
1	<0, 0>	0	<0, 0>
2	<0, 0>	0	<0, 0>

b) Calculus:

$$g(x,y) = -xy(x-2)(y+3)$$

$$\vec{\nabla}g = \langle -y(y+3)(2x-2), -x(x-2)(2y+3) \rangle$$

$$f_{xx} = -2y(y+3)$$
$$f_{yy} = -2x(x-2)$$
$$f_{xy} = -(2x-2)(2y+3)$$

$$\begin{cases} -y(y+3)(2x-2) = 0 \\ -x(x-2)(2y+3) = 0 \end{cases}$$

Point	$f_{xx}f_{yy} - f_{xy}^2$	Classification
$(0,0)$	$0 \cdot 0 - 6^2 = -36$	Saddle Point
$(2,0)$	$0 \cdot 0 - (-6)^2 = -36$	Saddle Point
$(0,-3)$	$0 \cdot 0 - (-6)^2 = -36$	Saddle Point
$(2,-3)$	$0 \cdot 0 - 6^2 = -36$	Saddle Point
$(1, -3/2)$	$9/2 \cdot 2 - 0^2 = 9$	Relative Minimum

Steepest Descent:

$$g(x,y) = -xy(x-2)(y+3)$$

$$X_{k+1} = x_k + \lambda_k \nabla g \big|_{x_k}$$

$$\vec{\nabla}g = \langle -y(y+3)(2x-2), -x(x-2)(2y+3) \rangle$$

k	X	f(X)	grad(X)	lambda
0	<0.5, 0.5>	1.3125	<1.75, 3>	l: -0.25
1	<0.0625, -0.25>	-0.08325195	<-1.28906, 0.30273>	l: -0.85
2	<1.1582, -0.50732>	-1.23294423	<0.40013, 1.93566>	l: -0.5
3	<0.95814, -1.47516>	-2.24544128	<-0.18832, 0.0496>	l: -0.25
4	<1.00522, -1.48756>	-2.24978385	<0.02349, 0.02489>	l: -0.3
5	<0.99817, -1.49502>	-2.24996772	<-0.00822, 0.00996>	l: -0.35
6	<1.00105, -1.49851>	-2.24999529	<0.00473, 0.00299>	l: -0.25
7	<0.99987, -1.49925>	-2.2499994	<-0.00059, 0.00149>	l: -0.45
8	<1.00013, -1.49993>	-2.24999995	<0.00061, 0.00015>	l: -0.25
9	<0.99998, -1.49996>	-2.25	<-8E-05, 7E-05>	l: -0.3
10	<1.00001, -1.49999>	-2.25	<3E-05, 3E-05>	l: -0.3

Newton-Raphson:

$$g(x,y) = -xy(x-2)(y+3)$$

$$H = \begin{bmatrix} -2y(y+3) & -(2x-2)(2y+3) \\ -(2x-2)(2y+3) & -2x(x-2) \end{bmatrix}$$

$$\vec{\nabla}g = \langle -y(y+3)(2x-2), -x(x-2)(2y+3) \rangle$$

$$X_{k+1} = X_k - \left(H\big|_{X_k}\right)^{-1} \vec{\nabla}g \big|_{X_k}$$

k	X	f(X)	grad(X)
0	<1, 1>	4	<0, 5>
1	<1, -1.5>	-2.25	<0, 0>
2	<1, -1.5>	-2.25	<0, 0>

C. Calculus:

$h(x,y) = (1-y)^2 + 100(x-y^2)^2$

$f_{xx} = 200$

$f_{yy} = 2 - 400(x-3y^2)$

$f_{xy} = -400y$

$\vec{\nabla} h = \langle 200(x-y^2), -2(1-y) - 400y(x-y^2) \rangle$

$$\begin{cases} 200(x-y^2) = 0 \\ -2(1-y) - 400y(x-y^2) = 0 \end{cases}$$

Point	$f_{xx} f_{yy} - f_{xy}^2$	Classification
(1,1)	$200 \cdot 802 - (-400)^2 = 400$	Relative Minimum

$$\begin{bmatrix} 0.9 \\ 0.9 \end{bmatrix} + \lambda \begin{bmatrix} 18 \\ -32.6 \end{bmatrix}$$

$0.9 + 18\lambda, \; 0.9 - 32.6\lambda$

Steepest Descent:

$h(x,y) = (1-y)^2 + 100(x-y^2)^2$

$X_{k+1} = x_k + \lambda_k \nabla h|_{x_k}$

$\vec{\nabla} h = \langle 200(x-y^2), -2(1-y) - 400y(x-y^2) \rangle$

k	X	f(X)	grad(X)	lambda
0	<1.1, 1.1>	1.22	<-22, 48.6>	l: -0.00087
1	<1.11914, 1.05772>	0.00334525	<0.07453, -0.04222>	l: -0.00137
2	<1.11904, 1.05778>	0.00334024	<0.02963, 0.05286>	l: -0.00271
3	<1.11896, 1.05763>	0.00333527	<0.07418, -0.04164>	l: -0.00137
4	<1.11886, 1.05769>	0.0033303	<0.02972, 0.05252>	l: -0.00274
5	<1.11877, 1.05755>	0.00332532	<0.07431, -0.04207>	l: -0.00137
6	<1.11867, 1.0576>	0.00332034	<0.02956, 0.05267>	l: -0.00271
7	<1.11859, 1.05746>	0.00331539	<0.07392, -0.04142>	l: -0.00138
8	<1.11849, 1.05752>	0.00331045	<0.02934, 0.05298>	l: -0.00267
9	<1.11841, 1.05738>	0.00330555	<0.0735, -0.04069>	l: -0.00139
10	<1.11831, 1.05743>	0.00330066	<0.02915, 0.05322>	l: -0.00264

Newton-Raphson:

$h(x,y) = (1-y)^2 + 100(x-y^2)^2$

$$H = \begin{bmatrix} 200 & -400y \\ -400y & 2 - 400(x-3y^2) \end{bmatrix}$$

$\vec{\nabla} h = \langle 200(x-y^2), -2(1-y) - 400y(x-y^2) \rangle$

$X_{k+1} = X_k - (H|_{X_k})^{-1} \vec{\nabla} h|_{X_k}$

k	X	f(X)	grad(X)
0	<0, 0>	1	<0, -2>
1	<0, 1>	100	<-200, 400>
2	<1, 1>	0	<0, 0>
3	<1, 1>	0	<0, 0>

Problem 2

Problem 3

For problem three we used this script to find the critical points and maximum and minimum values of $g(x, y) = sin(x) + cos(x) + sin(y) - cos(y)$ on the region $[0, 2\pi]$ x $[0, 2\pi$ of the $(x, y)-$plane.

$g(x, y) = sin(x) + cos(x) + sin(y) - cos(y)$		
Points	Values	Classification
$\frac{5\pi}{4}, \frac{3\pi}{4}$	0	saddle point
$\frac{5\pi}{4}, \frac{7\pi}{4}$	$-2\sqrt{2}$	rel. min.
$\frac{\pi}{4}, \frac{3\pi}{4}$	$2\sqrt{2}$	rel. max.
$\frac{\pi}{4}, \frac{7\pi}{4}$	0	saddle point

Problem 4

For problem three we used this script to find the critical points and maximum and minimum values of $g(x, y) = sin(x) + cos(x) + sin(y) - cos(y)$ on the region $[0, 2\pi]$ x $[0, 2\pi$ of the $(x, y)-$plane.

Problem 5

For problem five we used this script, considering the functions...

$$\begin{aligned} f(x,y,z) &= xyz \\ g(x,y,z) &= x^2+y^2+z^2-12 \\ h(x,y,z) &= x+y+z-4 \end{aligned}$$

(A) we found the Lagrange multipliers after six iterations giving...

Lagrange Multipliers		
Variable	Approximate	Exact
x	2.38742588672...	$\frac{\sqrt{10}}{3}+\frac{4}{3}$
y	2.38742588672...	$\frac{\sqrt{10}}{3}+\frac{4}{3}$
z	-0.7748517734...	$\frac{4}{3}-\frac{2\sqrt{10}}{3}$
λ	1.1937129433...	$\frac{\sqrt{10}}{6}+\frac{2}{3}$
μ	-3.8499011822...	$\frac{-22}{9}-\frac{4\sqrt{10}}{9}$

(B) using the Newton-Rhapson we got...

Newton-Raphson values	
Type	Value
Min. Value	0.268353822346948
Max. Value	-4.41650197049510