Gravitational system at 1,2 and 3 bodies. From earth to Jupiter and Saturn. Asteroid of Troy

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Contents

1 Introduction

The purpose of this exercise is to observe the movement of N ($N \leq 3$)body subjected to gravitational forces. At the course level, the aim is to introduce a high order numerical scheme and an automatically adjustable integration step algorithm (more information: Section 1.1). The numerical scheme of Runge-Kutta order 4 will therefore be used in this exercise. This scheme is used because it allows to obtain results with high precision with few time steps. This is due to the fact that, since the order of the scheme is high, convergence is very fast. In addition, this exercise introduces a very interesting new concept: the Lagrange points. When two bodies are in circular orbits, the Lagrange points represent the points where a third body would remain stationary relative to the other two.
In order to study all this, one is looking firstly at the transfer orbit from the Earth to Jupiter, the goal is to launch a rocket from around the Earth into an orbit in the solar system to reach the orbit of Jupiter. Secondly, one is considering only two bodies : Jupiter and the Sun. The goal is to analytically find the radius and the angular velocity in order to have circular uniforms trajectories and to then verify physical proprieties of the movement. Thirdly, one is considering three bodies : Jupiter, the Sun and the rocket. The main purpose is to understand how these bodies interact with each others. Finally, the goal is to study Lagrange point L4. The aim is to show that it is possible to place the rocket in an orbit that keeps constant distances from the sun and Jupiter. Than, the stability of this point is studied.

1.1 Fixed and adaptive time step

For a clear and precise explanation of the adaptive time step, this section briefly summarizes chapter 2.5.2 of the course [1] P.60-62.

The moment when two celestial bodies are close to each other is the one where their velocities and accelerations are the highest. To be able to describe the movements correctly, a very small Δt should be taken and this would be very expensive in computing time. This implies that the simulations would be very long . Yet, in most of the time, a big enough time step is enough to have a certain precision. So it would be interesting to have a time step Δt variable over time that adapts to the need. The adaptive time-step scheme responds exactly to this need by adapting itself according to the need to guarantee a level of precision called ϵ imposed by the programmer.

The idea of this scheme is, at the time t_i, to do two different estimations of the solution at the next time $t_{i+1} = t_i + \Delta t$. The first estimation $y_{i+1}^{(1)}$ is obtained by making a whole time step to go from t_i to $t_i + \Delta t$. The second estimation $y_{i+1}^{(2)}$ is obtained by making a half step to go from t_i to $t_i + \frac{\Delta t}{2}$ and then another half step to go from $t_i + \frac{\Delta t}{2}$ to $t_i + \Delta t$. After that, the idea is to test the absolute value of the difference between $y_{i+1}^{(2)}$ and $y_{i+1}^{(1)}$. If this difference is smaller than a certain precision ϵ, which is chosen by the programmer, than it is possible to go to the next step and the scheme will try to increase Δt in order to reduce the computation time. If this difference is smaller than ϵ, than the scheme will reduce Δt and remake the estimations until the difference becomes smaller than ϵ.

An adaptive time steps scheme allows to reduce the computation time and increase the precision. This point is discussed in Section 2.3.

2 Orbit transfer from Earth to Jupiter

2.1 Analytical calculation

In this section, the goal is to calculate the components of the velocity of the rocket at $t = 0$s written v_0 and v_{min} . The orbit that the rocket follow is the one shown on Fig.1. The period of planet's T orbit is also determined.
The only force on the system is the gravitational force : $\mathbf{F_G} = \mathbf{F_{i,j}} = -\frac{G \cdot m_i \cdot m_i}{r^2} \cdot \mathbf{e_r}$. $\mathbf{F_G}$ derives from a potential. Hence, one can use the con-

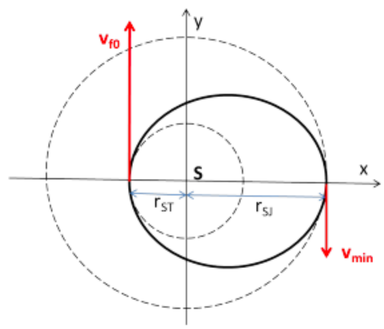


Figure (1) Orbit transfer between Earth and Jupiter [2]

servation of the mechanical energy eq.1 :

$$E_{mec} = E_{cin} + E_{pot} = \frac{1}{2} m_f \cdot v_0^2 - \frac{G m_f m_s}{r_{st}} = \frac{1}{2} m_f \cdot v_{min}^2 - \frac{G m_f m_s}{r_{sj}} \tag{1}$$

As second equation, the law of conservation of momentum is used. Indeed, $\mathbf{M_s} = \mathbf{F_G} \times \mathbf{r} = 0$ because $\mathbf{F_G}$ and $\mathbf{r}$ are along the same axes. Knowing that $\frac{dL_s}{dt} = M_s$ one conclude that $L_s = cst$. Therefore, one can write :

$$- r_{st} \cdot v_0 \mathbf{e_y} = r_{sj} \cdot v_{min} \mathbf{e_y} \tag{2}$$

Using eq.(2) one can isolate v_{min} :

$$v_{min} \mathbf{e_y} = -\frac{v_0 \mathbf{e_y}}{r_{sj}} \tag{3}$$

Inserting eq.(3) in eq.(1) one get the expression of v_0 :

$$v_0 = \sqrt{\frac{2 G m_s (\frac{1}{r_{st}} - \frac{1}{r_{sj}})}{1 - \frac{r_{st}^2}{r_{sj}^2}}} \simeq 38582.6 \mathrm{m} \cdot \mathrm{s}^{-1} \tag{4}$$

This implies :

$$v_{min} = -7414.21 \mathrm{m} \cdot \mathrm{s}^{-1} \tag{5}$$

In order to determine the period of the rocket, T_f, Kepler's third law is used.

$$T_f = 2\pi \cdot \sqrt{\frac{(r_{sj} + r_{st})^3}{8G(m_s + m_f)}} \simeq 5.464 \mathrm{years} \tag{6}$$

2.2 Trajectory of the rocket

In this subsection, the trajectory of the rocket is studied when the rocket is placed in the vicinity of the earth $(x_f, y_f) = (-149.6 \cdot 10^9\text{m}, 0.0\text{m})$. The sun is placed at $(x_s, y_s) = (0, 0)$ and Jupiter is placed at the position $(x_j, y_j) = (778.5 \cdot 10^9\text{m}, 0.0)$. The mass of Jupiter is neglected. The trajectory of the rocket is studied over a period of two periods of revolution $t_{final} = 2T_f$. A fixed time-step is used $\Delta t = 2000\text{s}$.

Fig.2 shows that the trajectory of the rocket over a period of two periods of revolution is an ellipse. This is the expected trajectory. To check that the rocket is doing two turns, a zoom in a randomly selected location of Fig.2 has been done (Fig. 3). This figure shows immediately that the rocket is doing two turns. Analytically, after two turns, the rocket should return to its initial position. Numerically, it does approximately returns to its initial position with an error of $|x_{finitial} - x_{ffinal}| = 0.0120\text{m}$.

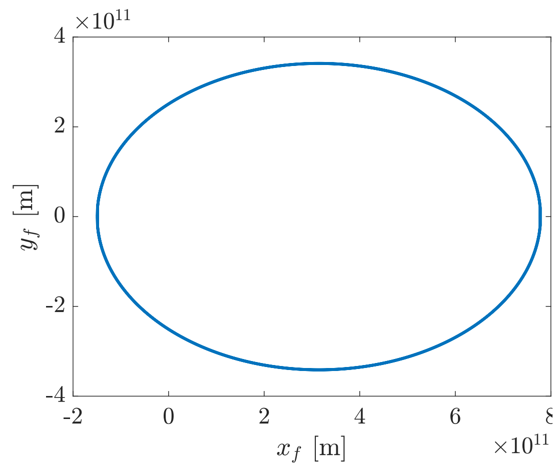


Figure (2) Trajectory of the rocket under the conditions described above

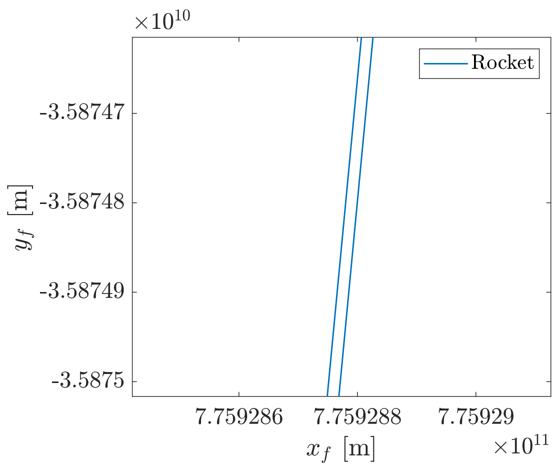


Figure (3) Zoom in on a random spot in the Fig.2

2.3 Convergence rate

In this subsection, convergence studies are made on

- The final position of the rocket.

- The maximum distance between the rocket and the sun $\max(|\overrightarrow{r_{fs}}|)$.
- The minimum rocket speed $\min(|\overrightarrow{v_f}|)$.

Each convergence study is performed twice. Once with a fixed time step and once with an adaptive time step. A comparison is made between these two methods. The conditions used are the same as in the previous section (Section 2.2) (Recall: $t_{final} = 2T_f$).

In this case, the analytical solutions for the final position of the rocket at the final time, the maximum distance between the rocket and the sun and the minimum rocket speed are known. So it is possible to compute $Error = Solution_{Analytical}$ - $Solution_{Numerical}$. In a graphic which represents the error as a function of $Nsteps$ (the number of steps in a simulation) with a log-log scale, the slope represents the opposite of the convergence rate (if a is the slope, $-a$ is the convergence rate).

Fig.4, Fig.5 and Fig.6 represent the convergence studies for the final position, the maximum distance between the rocket and the sun and the minimum rocket speed respectively.

The linear fit of each curve of all figure give that the convergence rate is 4 for the Runge-Kutta 4 method, which is the result predicted in the theory. The linear fit of these curves are represented in the Annex (8) by Fig.29, Fig.30 and Fig31 for the final position, the maximum distance between the rocket and the sun and the minimum rocket speed respectively.
It is important to note that Fig.4, Fig.5 and Fig.6 show clearly that the order of convergence of the method does not depend on the type of time step. However, it is possible to see that the curves corresponding to the adaptive time step are always below the curves corresponding to the fixed time step. This means that the error on the analyzed data is always smaller when an adaptive time step is used. Particularly, to obtain a result on the final position for a given precision, it is possible to see that it is necessary to do less step with the adaptive time step than with the fixed time step. This can be shown by observing Fig.7, the gap between the two lines clearly shows that more steps must be taken with the fixed time step and that we have the relationship: $Nsteps_{fixed} \simeq Nsteps_{adaptive} \cdot 6$ to have a given precision. Where $Nsteps_{fixed}$ is the number of steps with a fixed time stp and $Nsteps_{adaptive}$ is the number of steps with an adaptive time step.

The method that uses an adaptive time step is more accurate than the one that uses a fixed time step for a certain number of steps because the time step is modified according to the need (as it is described in Section 1.1). Fig.8 shows clearly that the time step is modified over time and when the required accuracy ϵ becomes smaller, the more the time step is modified. As it was said before, the adaptive time step allows to have a more accurate simulation which lasts less than a simulation with a fixed time step because the time step adapts itself to ensure a given precision.

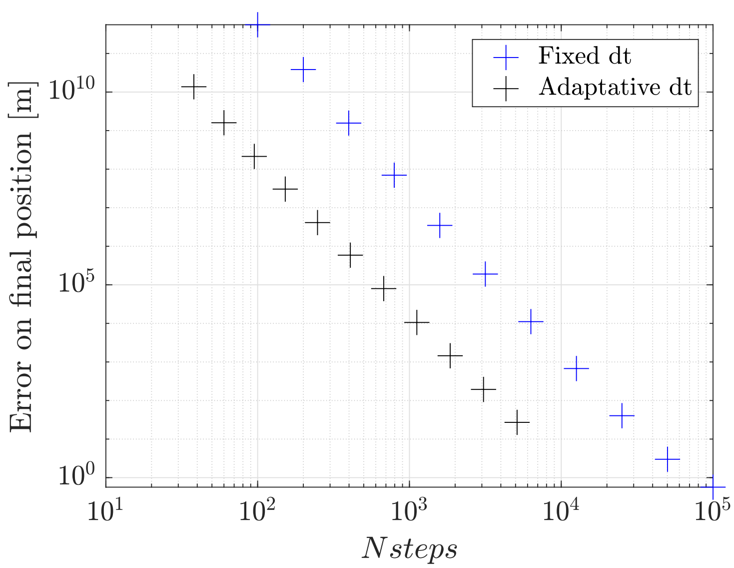


Figure (4) Convergence study on the final position with the Runge-Kutta 4 method for 11 simulations with a log-log scale. Blue: Fixed timestep , Black: adaptative time step

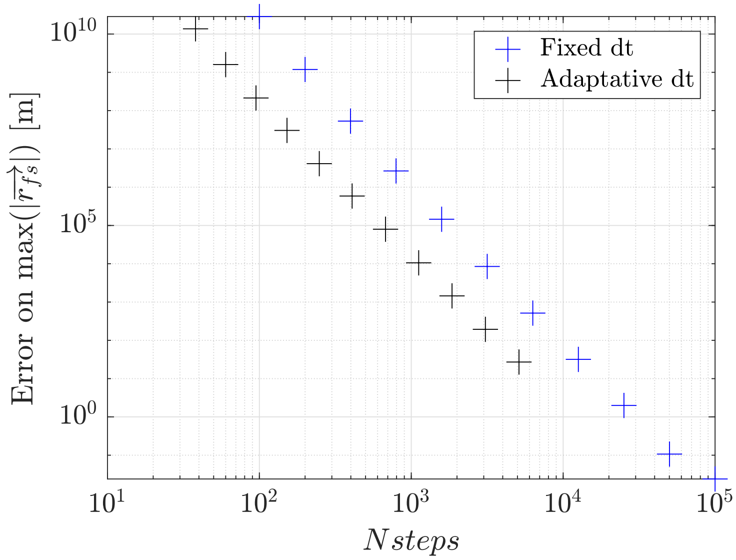


Figure (5) Convergence study on the maximum distance between the sun and the rocket with the Runge-Kutta 4 method for 11 simulations with a log-log scale. Blue: Fixed time step , Black: adaptative timestep

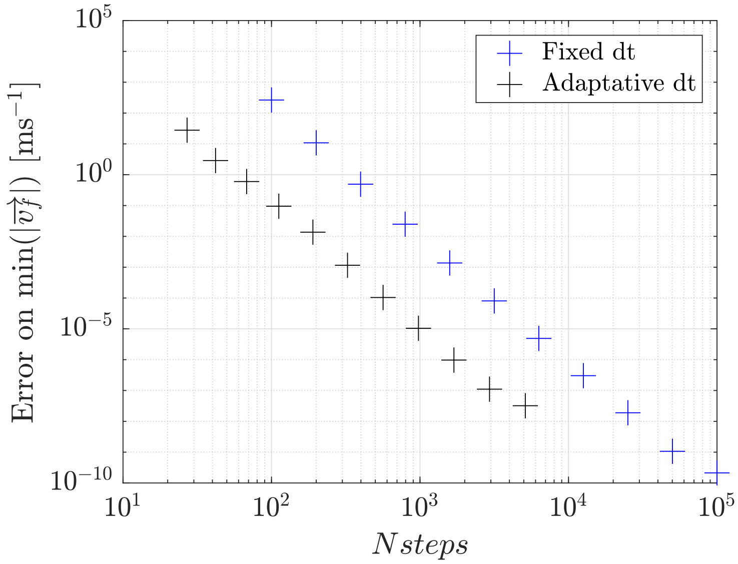


Figure (6) Convergence study on the minimum speed of the rocket with the Runge-Kutta 4 method for 11 simulations with a log-log scale. Blue: Fixed time step , Black: adaptative timestep

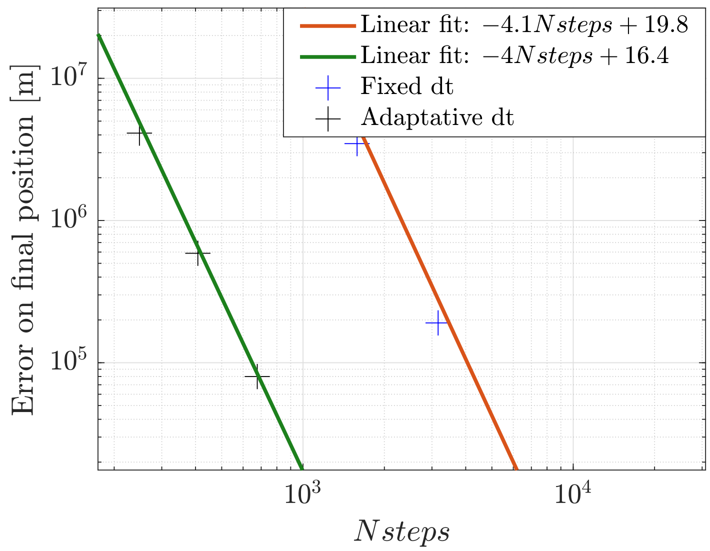


Figure (7) Zoom on Fig.29 in order to determine how Convergence study on the final position with the Runge-Kutta 4 method for 11 simulations with a log-log scale. Blue: Fixed timestep , Black: adaptative time step.

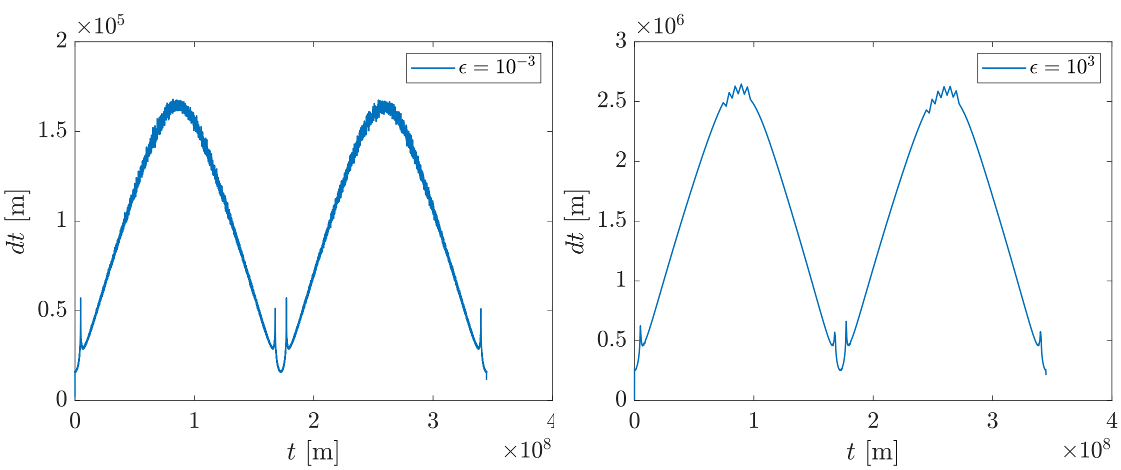


Figure (8) Evolution of dt over time for two different ϵ. Left: $\epsilon = 10^{-3}$m, Right: $\epsilon = 10^3$m

3 Two bodies : Sun and Jupiter

3.1 Analytical calculation : Radius of the trajectory of Sun and Jupiter and angular velocity

In this part, one consider uniform circular trajectory in the reference of the center of mass of Jupiter and the sun: R_{CM}. The distance between Sun and Jupiter is called $d = 778.5 \times 10^{-6}$km. The goal is to determine analytically the radius of the trajectory of the Sun and Jupiter and their angular velocity ω.
Therefore there are four unknowns : the radius of the sun R_s, the radius of the Jupiter R_j, the angular velocity of the Sun ω_s and the angular velocity of Jupiter ω_j. The radius here corresponds to the distance to the centre of mass. One needs than a system of four equations.

$$d = R_s + R_j \Leftrightarrow R_s = d - R_j \tag{7}$$

The center of mass theorem is used. Noticing that $R_{CM} = 0$ one can then simplify the equation.

$$R_{CM} = \frac{m_s R_s - m_j R_j}{m_s + m_j} \Leftrightarrow m_s R_s = m_j R_j \tag{8}$$

One use then Newton's law : $\sum \mathbf{F} = m\mathbf{a}$ where the only force is the gravitational one. And using the fact that the trajectories are uniform circular, one know that the acceleration is centripetal. Therefore one can write $\mathbf{a} = \frac{\mathbf{v}^2}{R}$. Hence one get :

$$\frac{G m_s m_j}{d^2} = \frac{v_s^2 m_s}{R_s} \tag{9}$$

$$\frac{G m_s m_j}{d^2} = \frac{v_j^2 m_j}{R_j} \tag{10}$$

Inserting eq.(7) in eq.(8) one gets the values of R_j and R_s.

$$R_j = \frac{d \cdot m_s}{(m_s + m_j)} \simeq 777757825,9\text{km} \Leftrightarrow \mathrm{R_s = d - R_j} \simeq 742174,1\text{km} \tag{11}$$

Inserting the values found in eq.(11) in eq.(9) and in eq.(10) one get the velocity of the Sun and Jupiter. Besides using the known formula : $\omega = \frac{v}{R}$,

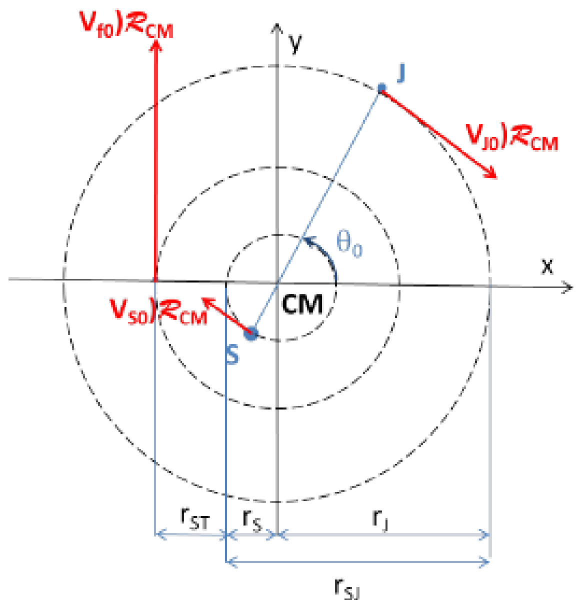


Figure (9) Initial condition for the Sun, Jupiter and the rocket [2]

one can calculate the angular velocities.

$$v_s = \sqrt{\frac{R_s G m_j}{d^2}} \simeq 12.45\text{m} \cdot \text{s}^{-1} \Leftrightarrow \omega = -1.68 \times 10^{-8}\text{rad} \cdot \text{s}^{-1} \tag{12}$$

$$v_j = \sqrt{\frac{R_j G m_s}{d^2}} \simeq 13051,9\text{m} \cdot \text{s}^{-1} \Leftrightarrow \omega = -1.68 \times 10^{-8}\text{rad} \cdot \text{s}^{-1} \tag{13}$$

3.2 Verification of the physical properties of the movement

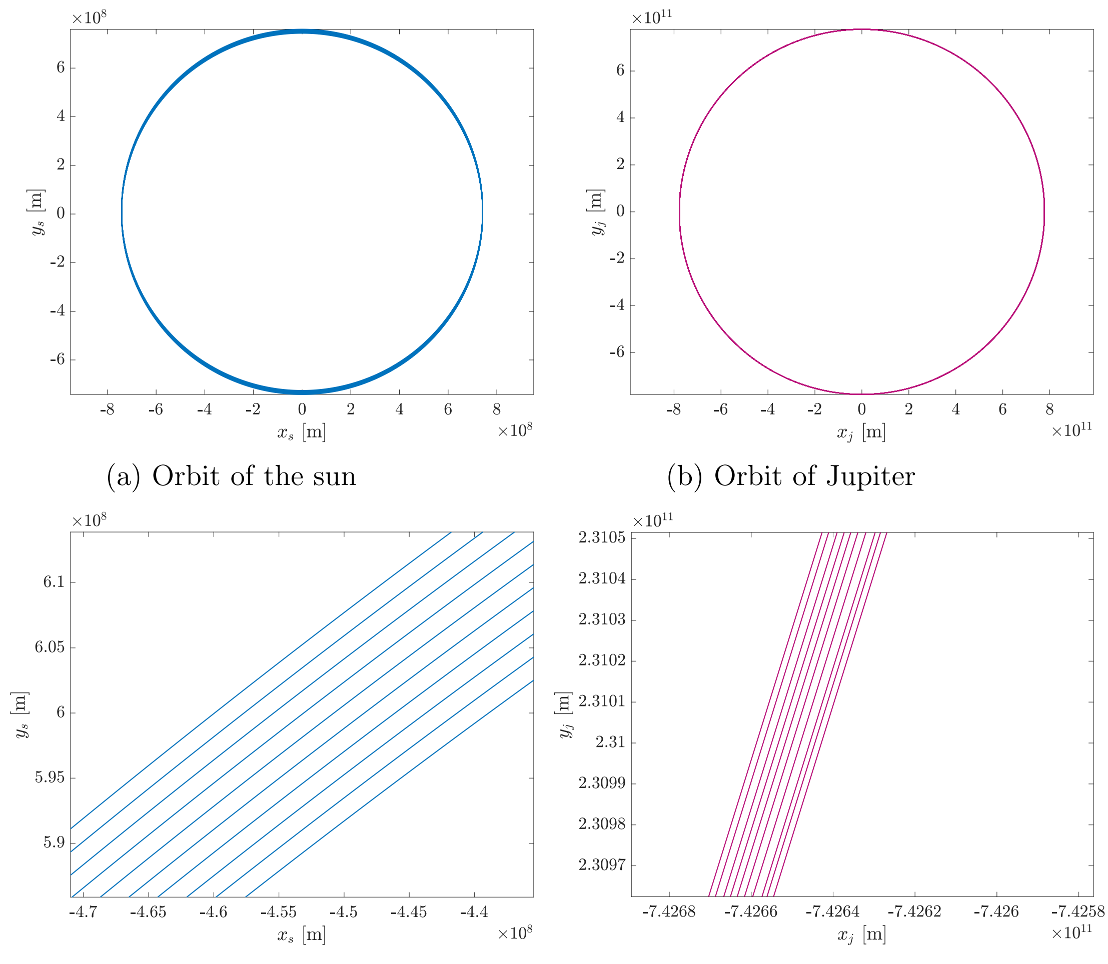
In this section some simulations with initial conditions compatible with the analytical calculation above are done. The main purpose is to verify that the physical properties of the movement are satisfied. Three proprieties are then verified :

- The conservation of the Mechanical Energy
- The conservation of the momentum

- The conservation of the distance d between the Sun and Jupiter

In order to make the simulation, one state that $\theta_0 = 0$. Hence one consider the initial conditions presented in Tab.1.

One simulates 10 periods of revolution using eq.(12) or eq.(13), one gets $10T = \frac{20\pi}{\omega} \simeq 3.74 \times 10^9$s. The simulations were done using a fixed Δt and using the adaptive step of time.



(a) Orbit of the sun

(b) Orbit of Jupiter

(c) Zoom on the orbit of the Sun

(d) Zoom on the orbit of the Jupiter

Figure (10) Circular orbits of the Sun and Jupiter over 10 periods of revolution

	x_0[m]	y_0[m]	$vx_0[\mathrm{m} \cdot \mathrm{s}^{-1}]$	$vy_0[\mathrm{m} \cdot \mathrm{s}^{-1}]$
Sun	-742174.1×10^3	0.0	0.0	-12.45
Jupiter	777757825.9×10^3	0.0	0.0	13051.9

Table (1) Initials conditions used for the simulation

On Fig.10a and on Fig.10b one can see that as expected the orbits are circular. Fig.10c and Fig.10d represents a zoom on the orbits. This enables one to verify that exactly 10 periods are simulated.

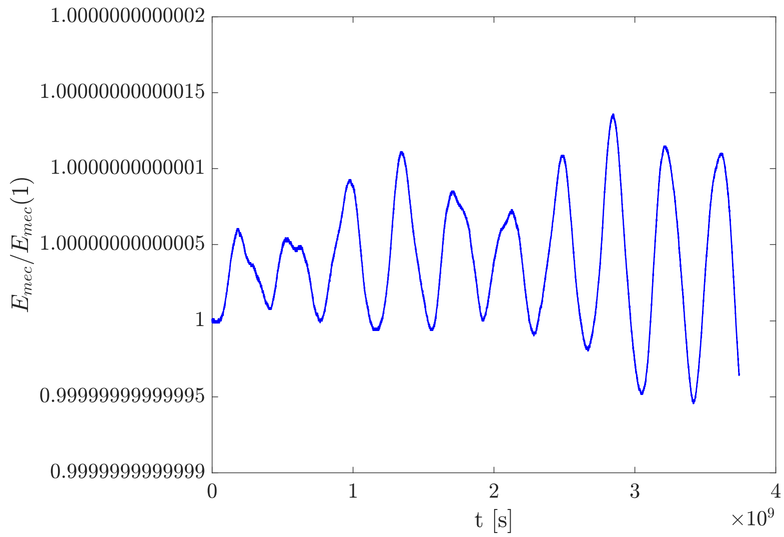


Figure (11) $\frac{E_{mec}}{E_{mec}(1)}$ as a function of time during 10 periods

As the Fig.11 represents $\frac{E_{mec}}{E_{mec_0}}$ the curve should be linear an equal to 1 for all time. Nevertheless, one can see that the E_{mec} oscillates around the same position through time and that the values are really close to 1. Therefore, looking at Fig.11 one can conclude that the E_{mec} of the system (composed of the Sun and Jupiter) is conserved. Concerning the conservation of momentum one can make exactly the same observation. Hence Fig.12 shows that the momentum of the system is conserved.

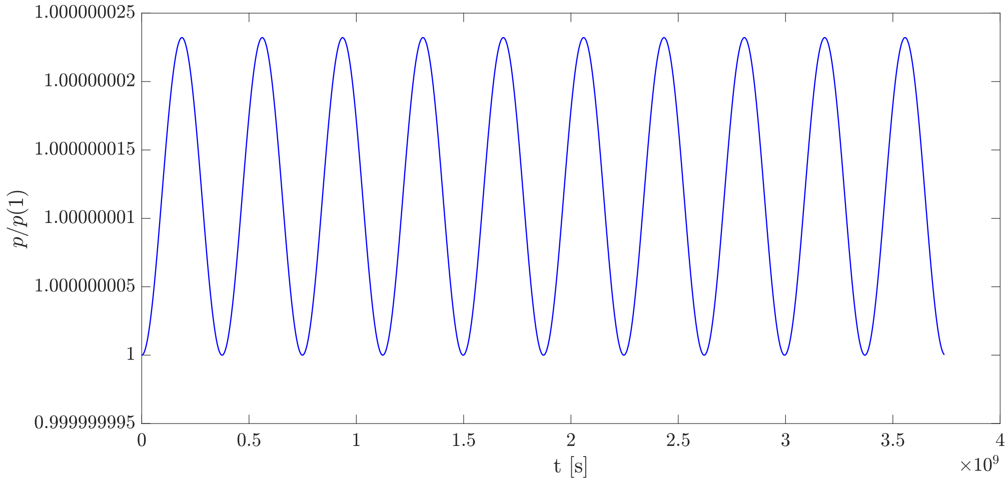


Figure (12) $\frac{p}{p(1)}$ as a function of time during 10 periods

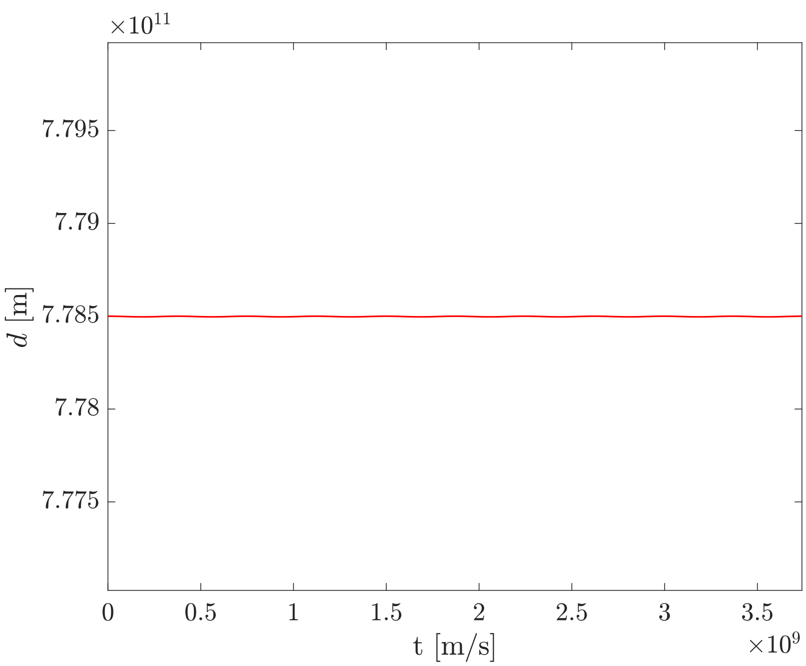


Figure (13) Distance d between the sun and Jupiter

Regarding the conservation of the distance between the Sun and Jupiter the result is really satisfying. Indeed, Fig.13 shows that the distance is almost perfectly conserved and stays at $d = 778.5 \times 10^9$m. Indeed, the standard deviation is $\sigma = 0.040115858165945$.Â§

4 Three bodies : Sun, Jupiter and Rocket

4.1 Initial condition of the bodies as in Fig.9

In this part the goal is to place the bodies as in Fig.9. Concerning the rocket the same initial position and velocity that the one calculated in Section3 are taken. In order to find the initial position of the rocket one has to determine the angle $\theta(0)$ between the initial axis Sun-Jupiter and the x-axis (see Fig.9). By statement one knows that at $t = \tau = \frac{T_f}{2}$ Jupiter is on the x-axis, $x > 0$. By using the equation of a MCU one gets the expression of $\theta(t)$:

$$\theta(t) = \omega t + \theta(0) \tag{14}$$

where the value of ω is given by eq.12. Hence using the information given in the statement one gets the following equation : at $t = \tau$

$$\theta(\tau) = 0 = \omega t + \theta(0) \Leftrightarrow \theta(0) = -\omega t \simeq 1.44\text{rad}. \tag{15}$$

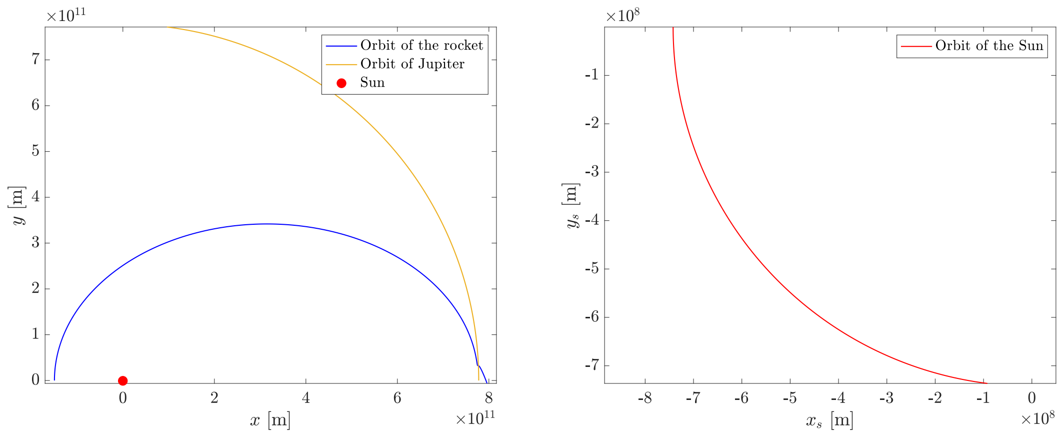
Know using the equation of the positions and velocities in a MCU one can compute the initials positions and velocities of Jupiter.

$$\begin{cases} x_j(t) = R_j \cos(\omega t + \theta(0)) \Leftrightarrow x_j(0) = R_j \cos(\theta(0)) \simeq 9.643 \times 10^{10}\text{m} \\ y_j(t) = R_j \sin(\omega t + \theta(0)) \Leftrightarrow y_j(0) = R_j \sin(\theta(0)) \simeq 7.718 \times 10^{11}\text{m} \\ vx_j(t) = -R_j \omega \sin(\omega t + \theta(0)) \Leftrightarrow vx_j(0) = -\omega R \sin(\theta(0)) \simeq 1.295 \times 10^4\text{m} \cdot \text{s}^{-1} \\ vy_j(t) = R_j \omega \cos(\omega t + \theta(0)) \Leftrightarrow vy_j(0) = \omega R \cos(\theta(0)) \simeq -1.618 \times 10^3\text{m} \cdot \text{s}^{-1} \end{cases}$$

By Proceeding exactly the same way one can find the initials conditions for the Sun.

$$\begin{cases} x_s(0) \simeq -9.202 \times 10^7\text{m} \\ y_s(0) \simeq -7.364 \times 10^8\text{m} \\ vx_s(0) \simeq -12.359\text{m} \cdot \text{s}^{-1} \\ vy_s(0) \simeq 1.544\text{m} \cdot \text{s}^{-1} \end{cases}$$

Fig.14a shows that if $t_f = \tau = \frac{Tf}{2}$ Jupiter is on the x-axis. The orbit of the sun is far more smaller than the orbit of Jupiter and the orbit of rocket. Therefore, when these orbits are represented on the same plot the sun is represented by a point. Fig.14brepresents the orbit of the Sun under the same condition as the one in Fig.14a.



(a) Orbit of the rocket and of Jupiter

(b) Orbit of the Sun

Figure (14) Orbits of the three bodies at $t_f = \tau$ and with the initials conditions when $\theta(0) \simeq 1.44\text{rad} \cdot \text{s}^{-1}$

4.2 Study of the mechanical energy of the rocket by varying τ

Here, the simulations are made using the adaptive time step. The initial conditions of Jupiter and of the Sun are modified by considering different values of time τ. This signifies that one is changing the angle θ_0. The goal is to observe how the mechanical energy of the rocket is changing.
The constants used to make these simulations are recorded in Tab.2. In order to simplify the lecture, the Tab.3 record the different $\theta(0)$ used in the simulations. Again, in the following figures, the sun is represented by a point, one can see on Fig.32 (in Annex 8) the orbit of the sun when $\theta(0) = \theta_1$.

t_{final}[s]	dt[s]	ϵ	$x_f(0)$[m]	$y_f(0)$[m]	$v_{xf}(0)[\text{m} \cdot \text{s}^{-1}]$	$v_{yf}(0)[\text{m} \cdot \text{s}^{-1}]$
3.45×10^9	1×10^{-7}	0.001	-149.6×10^9	0.0	0.0	3.8582×10^4

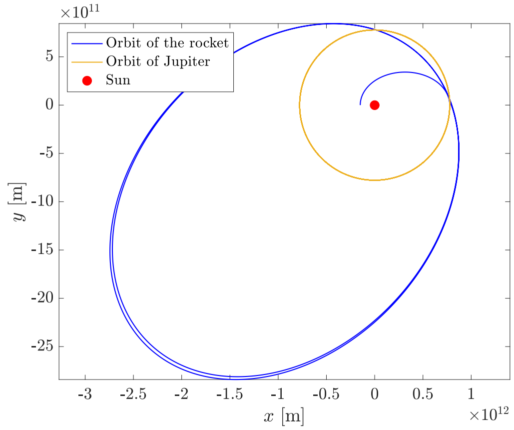
Table (2) Constants used for the following simulations

θ_1[rad]	θ_2[rad]	θ_3[rad]	θ_4[rad]	θ_5[rad]
1.44649263404215	1.48265494989321	1.49711987623363	0.7	1.485105373759913

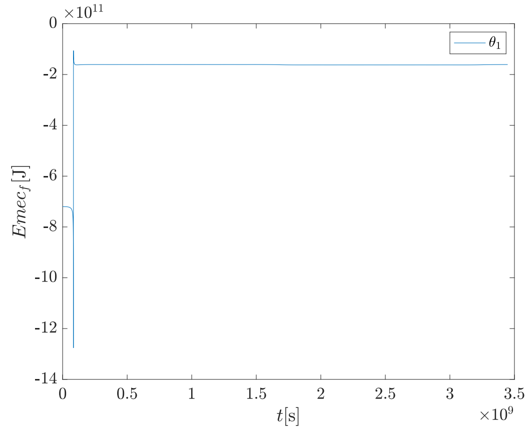
Table (3) Different values of $\theta(0)$ used in the following simulations

In order to observe the mechanical energy of the rocket, four different values of $\theta(0)$ have been studied. The first three values ($\theta_1, \theta_2$and θ_3) are taken voluntarily close to each other. Indeed, $\theta_2 = 1.025 \cdot \theta_1$ and $\theta_3 = 1.035 \cdot \theta_1$. Hence, on Fig.15, one is looking at the same regime where Jupiter interfere with the trajectory of the rocket. Looking at Fig.15 one remarks that there is the same pic of mechanical energy at the beginning of the simulation. After that one can see that E_{mec} is constant. Nevertheless, looking at Fig.16b one can see that this pic is not exactly the same for each angle. This pic corresponds at the situation where the rocket is close to the sun and so its energy is really big. Depending on $\theta(0)$ the E_{mec} of the rocket is not the same (see Fig.16a). In order to analyse better the energy of the rocket Fig.17a and Fig.17b shows respectively the potential and the kinetic energy of the rocket. Looking at Fig.17a and at Fig.17b one can see that when $\theta(0) = \theta_1$ the amplitude and frequency of respectively E_{pot} end E_{cin} are smaller than when θ_2 and θ_3 are used as initial angle. By looking at Fig.17 it seems that the bigger $\theta(0)$ is [1], bigger is the frequency and the amplitude of the mechanical energy of the rocket. In each case, the orbit of the rocket is bigger than the orbit of Jupiter.

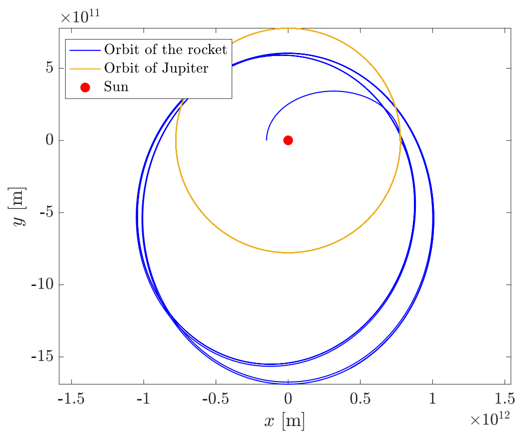
[1] This is true if we stay in the regime explained above



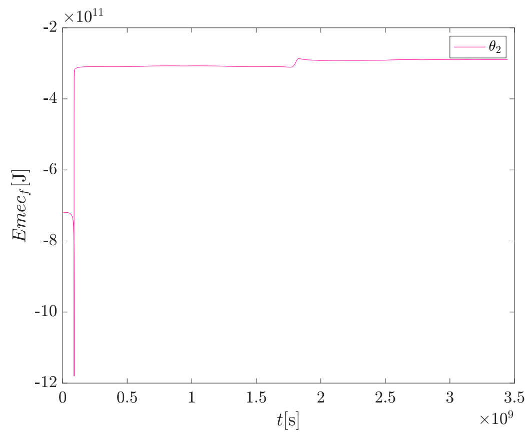
(a) Orbits of the 3 bodies with $\theta(0) = \theta_1$



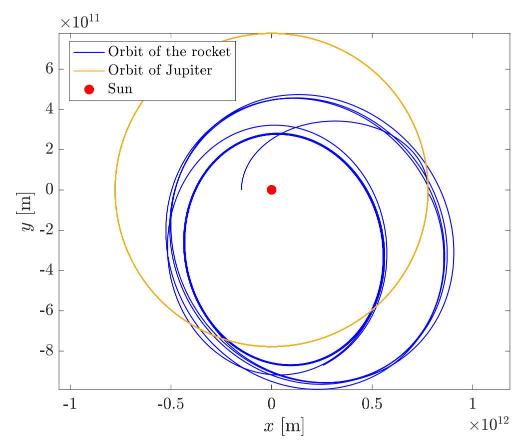
(b) Mechanical energy of the rocket with $\theta(0) = \theta_1$



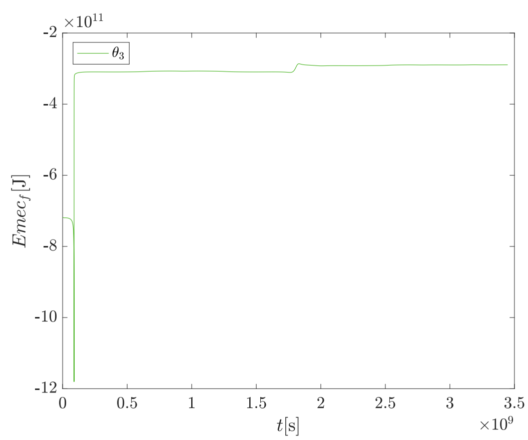
(c) Orbits of the 3 bodies with $\theta(0) = \theta_2$



(d) Mechanical energy of the rocket with $\theta(0) = \theta_2$

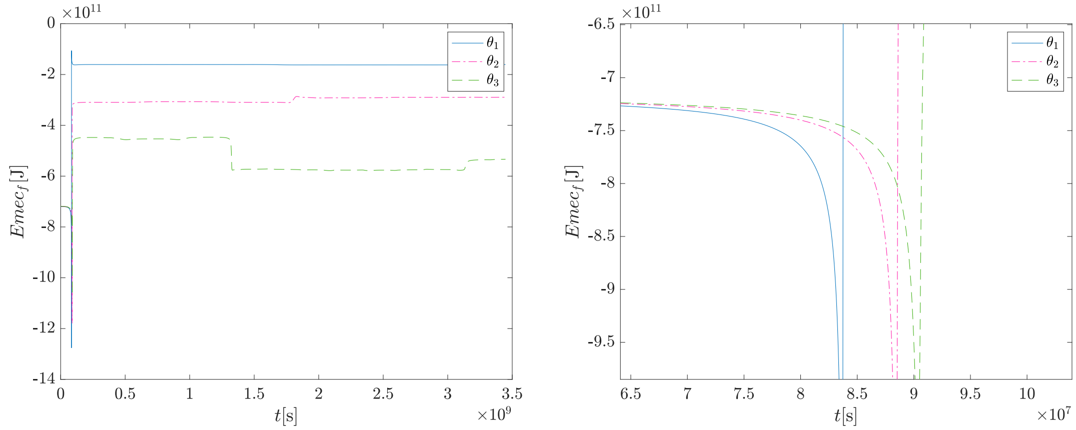


(e) Orbits of the 3 bodies with $\theta(0) = \theta_3$



(f) Mechanical energy of the rocket with $\theta(0) = \theta_3$

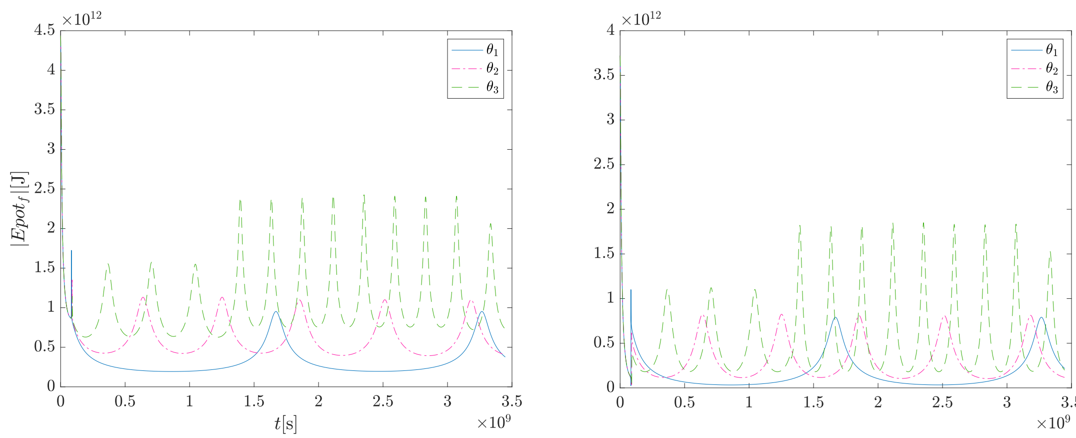
Figure (15) Orbits of the three bodies and mechanical energy of the rocket for different $\theta(0)$ over t$_{final}$



(a) Mechanical energy of the rocket in function of time

(b) Zoom on the pic of the mechanical energy

Figure (16) Mechanical energy of the rocket in function of time with different initials angles θ_1, θ_2 and θ_3 over t$_{final}$

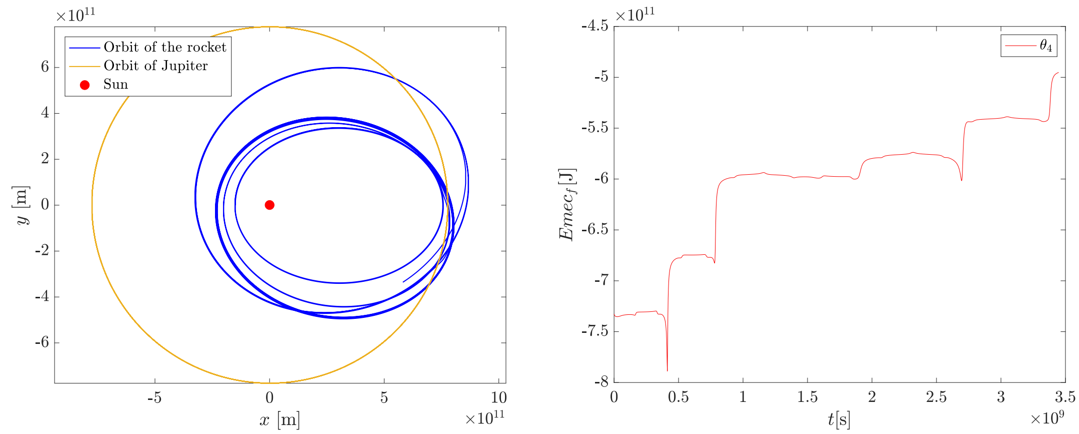


(a) Absolute value of the potential energy of the rocket in function of time

(b) Absolute value of the kinetic energy of the rocket in function of time

Figure (17) Potential and kinetic energy of the rocket in function of time with different initials angles θ_1, θ_2 and θ_3 over t$_{final}$

It is interesting to look at another case, Fig.18, where $\theta(0) = \theta_4$ is far smaller than the angles studied above. Jupiter does not interfere with the trajectory of the rocket and so the rocket is never close to the sun. This is why there is not the same pic of E_{mec} at the beginning . By looking at Fig.19 one can see that the potential and kinetic energy oscillate and decrease with time. On Fig.18a one can remark that the orbit of the rocket is in the orbit of Jupiter. This proves that Jupiter does not interfere with the trajectory of the rocket.



(a) Orbits of the 3 bodies (b) Mechanical energy of the rocket

Figure (18) Orbits of the three bodies and mechanical energy of the rocket for different θ_4 over t$_{final}$

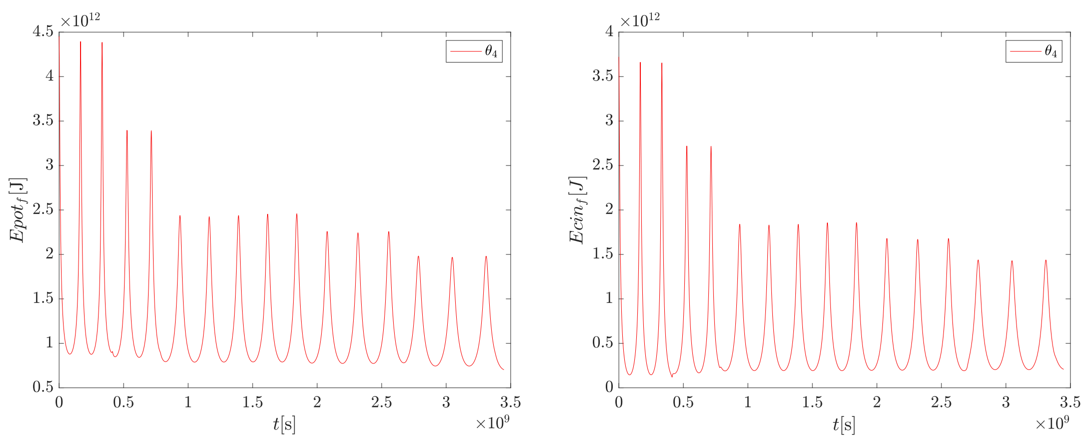
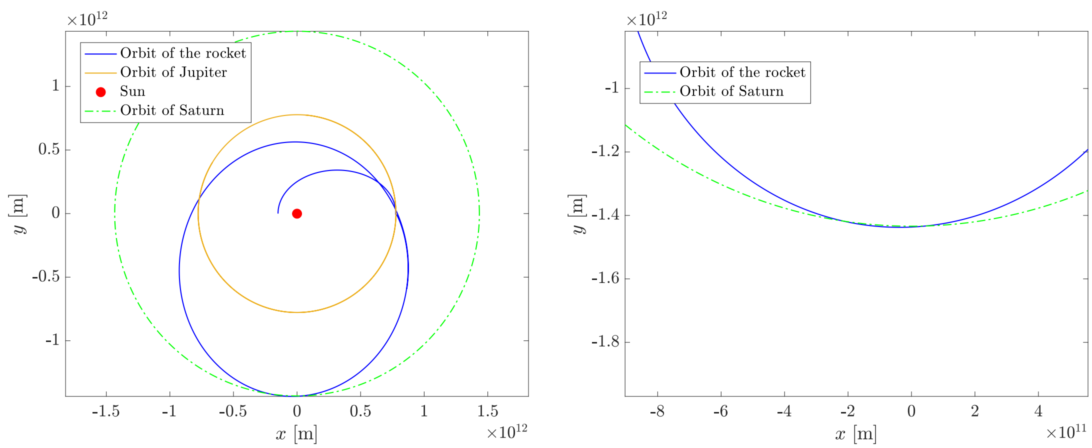


Figure (19) Absolute values of the potential and kinetic energy of the rocket in function of time with initial angle θ_4 over t$_{final}$

4.2.1 Rocket tangent to Saturn

Here, the main purpose was to find a certain τ or θ_0 so that the trajectory of the rocket after passing near Jupiter is tangent to Saturn. The initial angle one was looking for had to be close to the one found in eq.(15). Fig.20a shows the orbit of rocket when its trajectory is tangent to Saturn. This happens when $\theta(0) = \theta_5$ (see Tab.3). Fig.20b demonstrates that the trajectory of the rocket is tangent to Saturn. The final time used for the simulation is $t_{fin} = 3.96T_f = 6.826714287324355 \times 10^8$s.



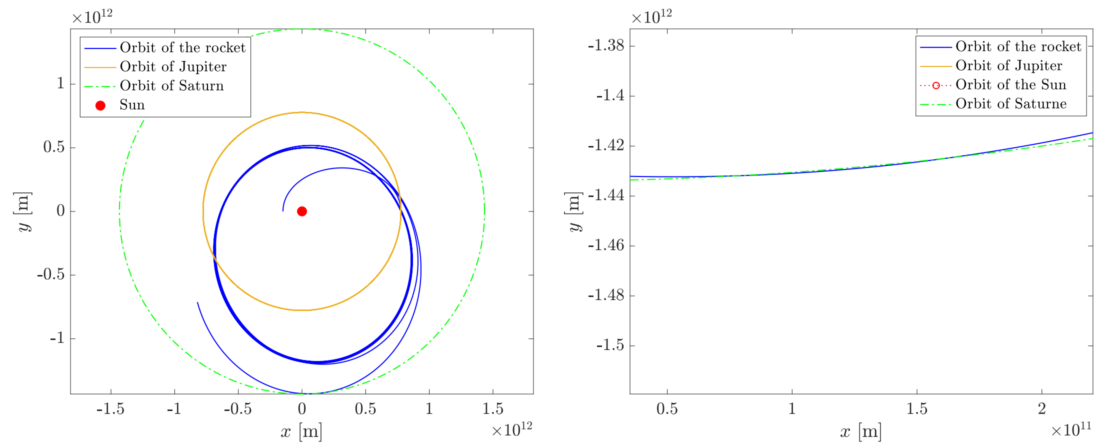
(a) Orbits of the 3 bodies (b) Zoom of the orbits

Figure (20) Orbits of the three bodies and zoom for different $\theta(0) = \theta_5$ over t_{fin}

There exist other initials angles which create initials conditions that make the trajectory of the rocket tangent to Saturn as one can see on Fig.21a. Nevertheless this does not happen after only one orbit of the rocket.

5 Three bodies : Sun, Jupiter and Rocket - Lagrange points

The purpose of this section is to prove that it is possible to place the rocket in an orbit that keeps constant distances from the sun and from Jupiter such that the triangle made by the rocket, the sun and Jupiter is equilateral for all time. The point where the rocket is placed for that is called Lagrange



(a) Orbits of the 3 bodies

(b) Zoom of the orbits

Figure (21) Orbits of the three bodies and mechanical energy of the rocket for different $\theta(0) = 1.4910777777777777$ over t_{final}

point L4 (Lagrange point L5 has the same properties but this work is not interested in Lagrange point L5)

5.1 Analytical calculations

The goal of these calculations is to find the position of the Lagrange point L4 to be able to place the rocket at this point. Then, one has to find the initial velocity of the rocket at this point. For analytical calculations, one must place oneself in the reference frame rotating with the Sun and Jupiter, with the mass center of the system as the rotation axis Fig.22. For the calculation, the Lagrange point L4 is called P_0

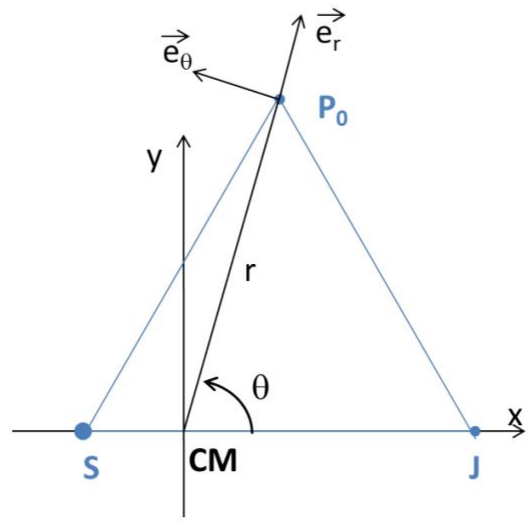


Figure (22) Repository for the analytical calculations, P_0 represents the Lagrange point L4 [2]

As the triangle formed by the sun, Jupiter and P_0 is equilateral, one has the relation:

$$||\mathbf{P}_0\mathbf{J}|| = ||\mathbf{P}_0\mathbf{S}|| = ||\mathbf{SJ}|| \tag{16}$$

where $||\mathbf{P}_0\mathbf{J}||$ is the norm of the vector that connects P_0 and J. That is, the length between these two points.
Let A be the point between the sun and Jupiter. One has $||\mathbf{AJ}|| = \frac{||\mathbf{SJ}||}{2}$.
As the median of an equilateral triangle is also its mediatrix, the triangle formed by P_0, A and Jupiter is rectangle. Applying Pythagoras's theorem, one has:

$$||\mathbf{P}_0\mathbf{A}|| = \sqrt{||\mathbf{P}_0\mathbf{J}||^2 - ||\mathbf{AJ}||^2} = \sqrt{||\mathbf{SJ}||^2 - ||\mathbf{AJ}||^2} \simeq 6.742 \cdot 10^8\text{km} \tag{17}$$

Using eq.(11), one gets that $||\mathbf{CMJ}|| \simeq 777757825.9\text{km}$. This implies that

$$||\mathbf{CMA}|| = ||\mathbf{CMJ}|| - ||\mathbf{AJ}|| \simeq 3.885078 \cdot 10^8\text{km} \tag{18}$$

By projecting the point P_0 on the x and y axis, one finds that

$$\mathrm{L4} = \mathrm{P}_0 = (\mathrm{P}_{0\mathrm{x}}, \mathrm{P}_{0\mathrm{y}}) = (||\mathbf{CMA}||, ||\mathbf{P}_0\mathbf{A}||) \tag{19}$$

That means that $(x_f(0), y_f(0) = P_0 = (3.885078 \cdot 10^8\text{km}, 6.742 \cdot 10^8\text{km})$

The initial speed of the rocket is given by

$$\mathbf{v}_f(0) = \Omega r \mathbf{e}_\theta \tag{20}$$

(given by the notice) where Ω is the angular velocity of the system Sun-Jupiter given by eq.(12),(r, θ) are the the polar coordinates centered at CM.

As the triangle formed by CM, A and $P_0 = (x_f(0), y_f(0)$ is rectangle, one has

$$\tan(\theta) = \frac{y_f(0)}{x_f(0)} \Leftrightarrow \theta = \arctan(\frac{y_f(0)}{x_f(0)}) \simeq 60.05° \tag{21}$$

and using Pythagoras's theorem,

$$r = \sqrt{x_f(0)^2 + y_f(0)^2} \simeq 7.78 \cdot 10^8 \text{km} \tag{22}$$

By projecting $\mathbf{e}_\theta$ onto the x and y axes, one gets:

$$\mathbf{e}_\theta = -\sin(\theta)\mathbf{e_x} + \cos(\theta)\mathbf{e_y} \tag{23}$$

So the initial speed od the rocket can be expressed as

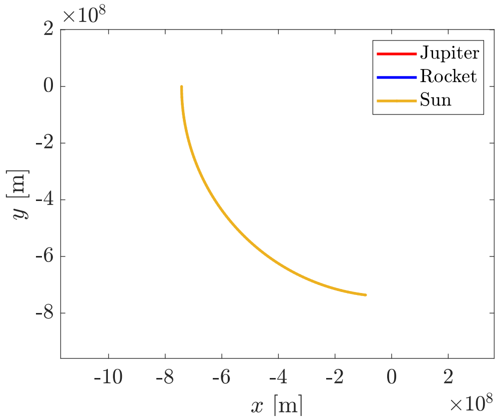
$$\mathbf{v}_f(0) = (v_{fx}(0), v_{fy}(0)) = \Omega r(-\sin(\theta)\mathbf{e_x} + \cos(\theta)\mathbf{e_y}) \simeq (-11319\text{ms}^{-1}, 6523\text{ms}^{-1}) \tag{24}$$

5.2 Rocket placed at the Lagrange point L4

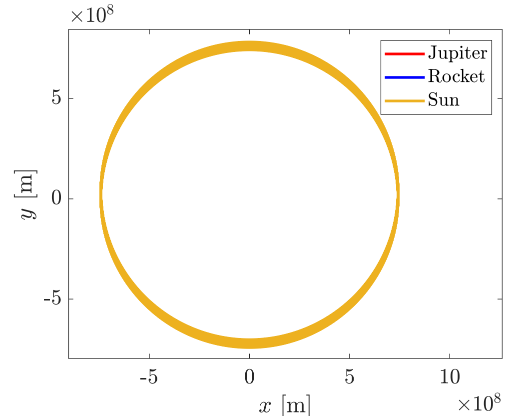
The goal of this subsection is to show that it is possible to place the rocket in an orbit that keeps constant distances from the sun and from Jupiter such that the triangle made by the rocket, the sun and Jupiter is equilateral for all time.

The rocket is placed at the Lagrange point L4 with his initial velocity which were found in subsection 5.1. For the first simulations, one must place oneself in the referential of the mass center.

In order to make the graphs more understandable, it was decided to represent the sun as a ball centered in its initial position. Comparing Fig.23a and 23b to Fig. 24a and 24b, it is possible to see that the trajectory of the sun is far smaller than the one of Jupiter and the rocket. So the trajectory of the sun was not visible when it was plotted with the other trajectories. That is why a ball was taken to represent the sun.



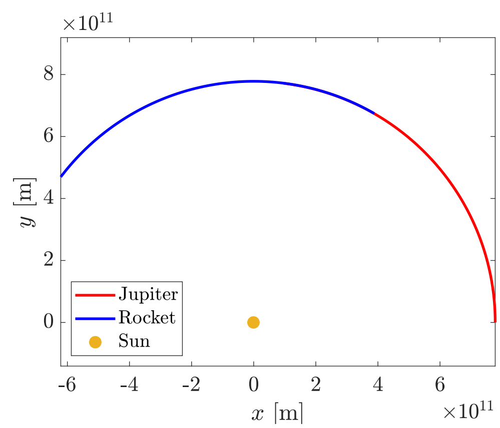
(a) $t_{final} = 0.5T_f$



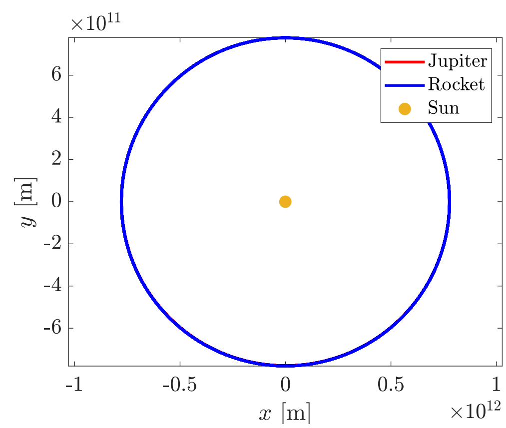
(b) $t_{final} = 50T_f$

Figure (23) Orbits of the sun for different t_{final}

To begin with, it is possible to observe that the equilateral triangle formed by the rocket, Jupiter and the sun is preserved. To observe it clearly, a short simulation is made. Fig.24a shows a simulation with $t_{final} = 0.5T_f$ This figure shows that the equilateral triangle formed by the sun, the rocket and Jupiter is preserved. Fig.24b shows that the rocket and Jupiter remain in the same orbit for a very long time ($t_{final} = 50T_f$) (because Jupiter is hidden by the Rocket here). But these are not rigorous proof to show that the distances are conserved.



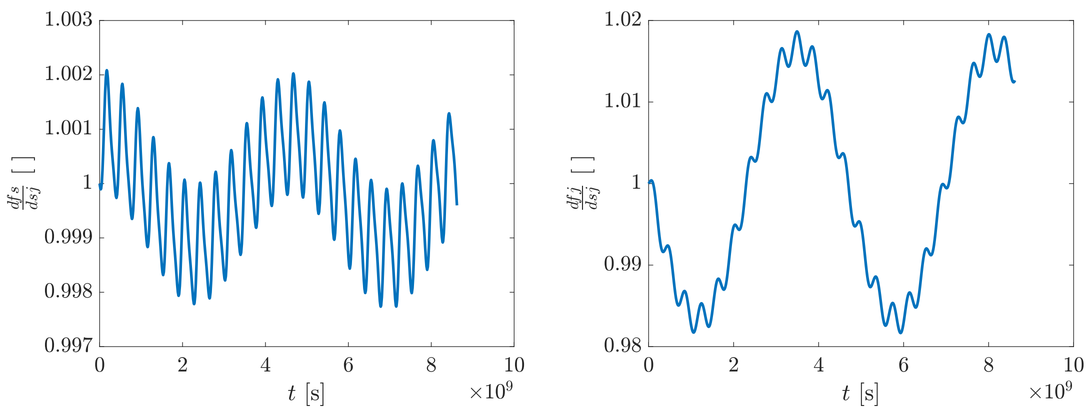
(a) $t_{final} = 0.5T_f$



(b) $t_{final} = 50T_f$

Figure (24) Orbits of the rocket and Jupiter for different t_{final}

In order to show that the distance between the sun and the rocket dfs and the distance between Jupiter and the rocket dfj are preserved. It is possible to plot the relative error $\frac{dfs}{djs}$ and $\frac{dfj}{djs}$ as a function of the time for a very long time. djs is the distance between the sun and Jupiter. As dfs and dfj should be equal to dfs at all time, $\frac{dfs}{djs}$ and $\frac{dfj}{djs}$ should be equal to 1 for all time. Fig.25a and 25b show that this distance is conserved. The reason why the distance is not perfectly conserved is explained later.



(a) Distance between the sun and the rocket

(b) distance between Jupiter and the rocket

Figure (25) Distance between the rocket and the sun and distance between upiter and the rocket $t_{final} = 50T_f$

Another way to show that the distance between the sun and the rocket dfs and the distance between Jupiter and the rocket dfj are preserved to get into the referential rotating with the sun and Jupiter, with the mass center of the system as the rotation axis (Fig.22) and observe the trajectory of the rocket in that referential. To express the position of the rocket in this repository, it is necessary to subtract the change induced from the rotation of the system to the position of the rocket. To do this, it is easier to express the position of the rocket (x_f, y_f) in polar coordinates (r, θ). Using a bit of trigonometry, one get:

$$\begin{cases} \theta(t) = \arctan(\frac{y_f(t)}{x_f(t)}) \\ r(t) = \sqrt{x_f^2(t) + y_f^2(t)} \end{cases}$$

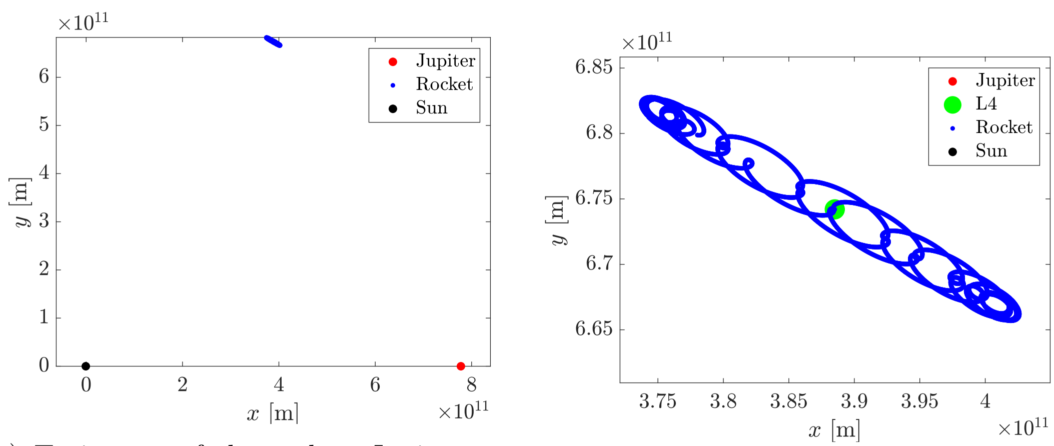
To move into the rotating referential(θ', r'), one needs to subtract the rotation speed of the system:

$$\begin{cases} \theta'(t) = \theta(t) - \Omega t \\ r'(t) = r(t) \end{cases}$$

Now that the rotation speed has been subtracted, one can return to the Cartesian coordinates but in the new rotating reference frame:

$$\begin{cases} x'_f(t) = r'(t)\cos(\theta'(t)) \\ y'_f(t) = r'(t)\sin(\theta'(t)) \end{cases}$$

Concerning Jupiter and the sun, the influence of the mass of the rocket is so low that if one places oneself in the rotating referential, Jupiter and the sun does not move. So $(x_s(t), y_s(t)) \simeq (x_s(0), y_s(0))$ and $(x_j(t), y_j(t)) \simeq (x_j(0), y_j(0))$ for all time t.
Fig.26a shows the trajectory of the sun, the rocket and Jupiter in this new referential for a long simulation $t_{final} = 50T_f$. Fig.26a clearly shows that the equilateral triangle formed by the rocket, Jupiter and the sun is conserved over time, so the distance between the rocket and Jupiter and the distance between the rocket and the sun are conserved. However, a zoom on the trajectory of the rocket (Fig.26b) shows that the rocket moves around the Lagrange point L4. This is due to the fact that the Coriolis force acts on the rocket and make it turn around L4.



(a) Trajectory of the rocket, Jupiter and the sun in the referential rotating with the sun and Jupiter

(b) Zoom on the trajectory of the rocket on Fig.26a

Figure (26) $t_{final} = 50T_f$

5.3 Rocket placed next to the Lagrange point L4

In this subsection, the movements of the rocket are studied when the latter is placed next to this point of Lagrange. To do that, one has to place oneself in the reference frame rotating with the Sun and Jupiter, with the mass center of the system as the rotation axis Fig.22. To do that, see the end of Section5.2.

Fig.27 shows the trajectory of the rocket in the rotating referential for different initial positions $(x_f(0), y_f(0))$. In this figure, Rocket + δ_i means that $(x_f(0), y_f(0)) = (\mathrm{L4_x}, \mathrm{L4_y} + \delta_\mathrm{i})$ where $\delta_0 = 0$, $\delta_1 = 1 \cdot 10^6$km, $\delta_2 = 5 \cdot 10^6$km, $\delta_3 = 10 \cdot 10^6$km. The initials velocities were recalculated for each point using Ã©q(20). Fig.27 shows clearly that the Lagrange point L4 is a stable position using Ã©q. This means that, even if the rocket is placed next to this point, its trajectory will remain around this point.

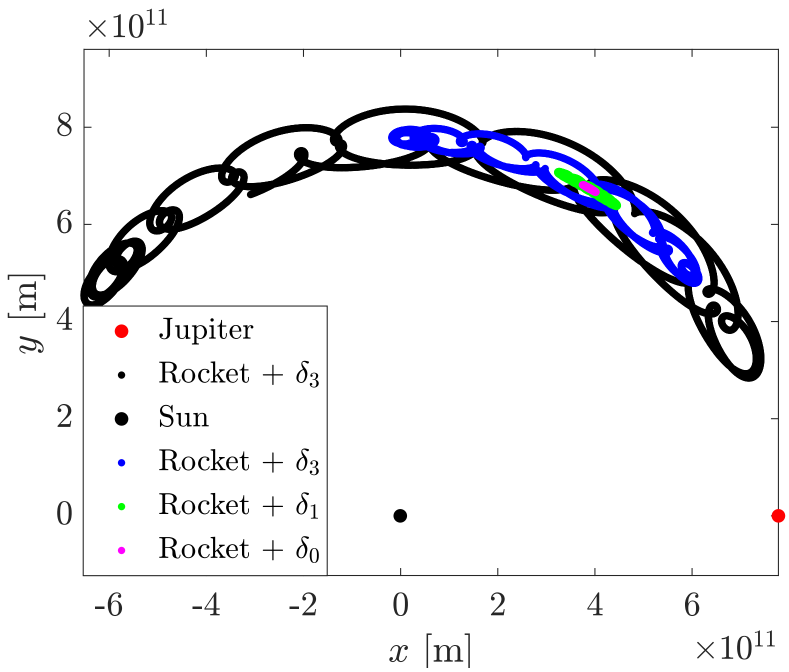


Figure (27) Trajectory of the rocket in the rotating referential for different initial positions next to L4

6 Optional

In the section the aim was to find a certain τ or θ_0 so that the trajectory of the rocket after passing near Jupiter leaves the solar system. This happens under the conditions given in Tab.4.

θ_6[rad]	τ[s]
1.80811579259278	$T_f/1.6 = 1.077448593327707 \times 10^8$

Table (4) Conditions in order to make the rocket leave the solar system

Fig.28 shows the rocket leaving the solar system. In order to represent this, the orbit of Pluto has been drawn. Indeed, Pluto is the last planet of the solar system.

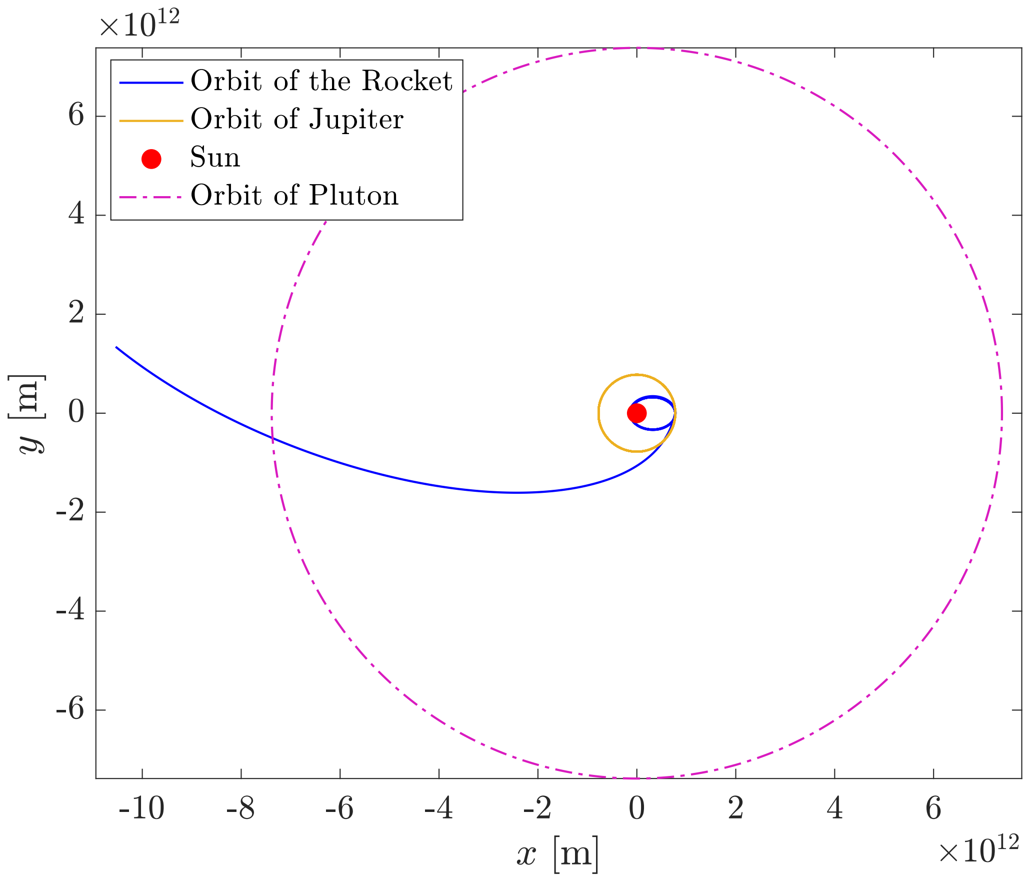


Figure (28) Trajectory of the rocket when it leaves the solar system with $\theta(0) = \theta_6$

7 Conclusion

On one side, at the course level, this work has made it possible to introduce a high convergence order scheme: the scheme of Runge-Kutta order 4. The convergence studies carried out made it possible to verify that this scheme converges order 4. In addition, this work made it possible to become familiar with the adaptive time-step method. It has been seen that this scheme makes it possible to be more accurate for a number of step than with a fixed time step diagram and it reduces the computing time.
On the other side, this exercise allowed to observe the movement of N ($N \leq$

3) bodies subjected to gravitational forces. It was first seen that it was possible to launch the rocket in the vicinity of the Earth to reach the orbit of Jupiter. This point made it possible to check that the Runge-Kutta4 scheme converges order 4 and the proprieties of the adaptive time step method. It was really interesting to simulate a system with three bodies. This enables us to see that Jupiter has a strong influence on the rocket and that if the rocket passes close to Jupiter, its orbit is totally disturb. Finally, it was possible to see that there exist initial conditions for the rocket such that the rocket is in an orbit that keeps constant distances from the sun and from Jupiter. The point where the rocket has to be placed for that is called Lagrange point L4. Moreover, it has been seen that this point is stable

8 Annexe

8.1 Linear fit for the convergence rate

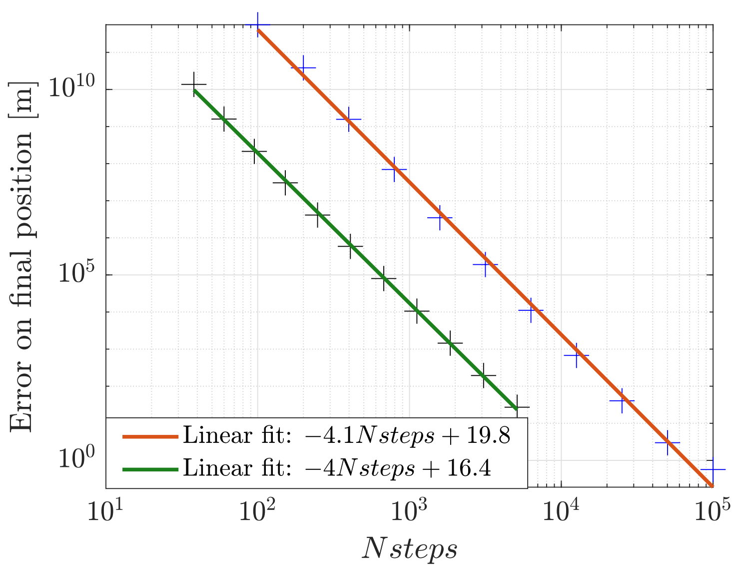


Figure (29) Linear fit of the convergence study for the position (Fig.4). Red line: Linear fit for the fixed time step, Green line: Linear fit for the adaptive time step

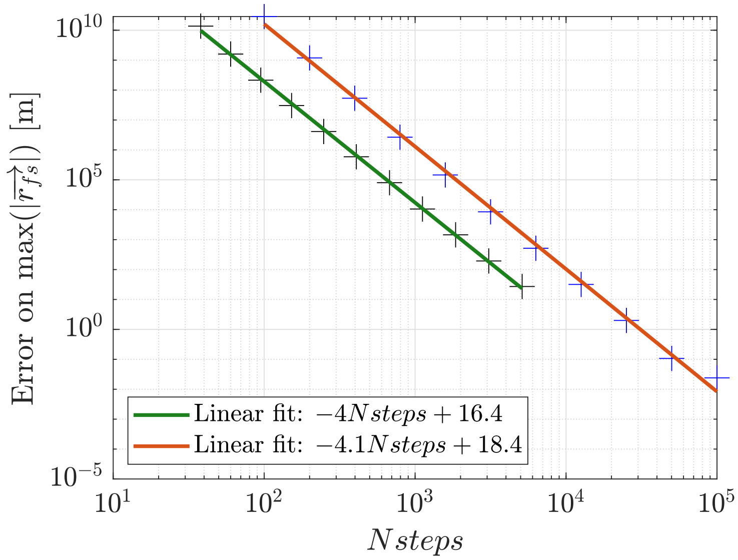


Figure (30) Linear fit of the convergence study for the maximum distance between the rocket and the sun (Fig.5). Red line: Linear fit for the fixed time step, Green line: Linear fit for the adaptive time step

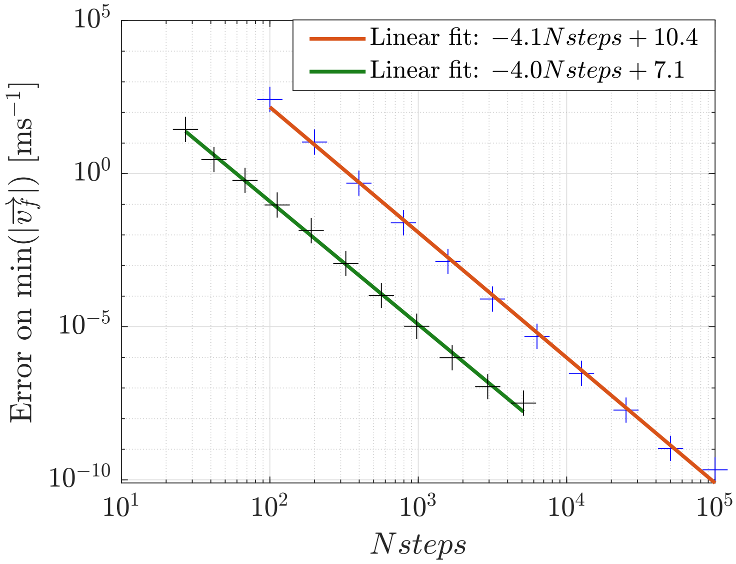


Figure (31) Linear fit of the convergence study for the minimum speed of the rocket (Fig.6). Red line: Linear fit for the fixed time step, Green line: Linear fit for the adaptive time step

8.2 Orbit of the sun when $\theta(0) = \theta_1$

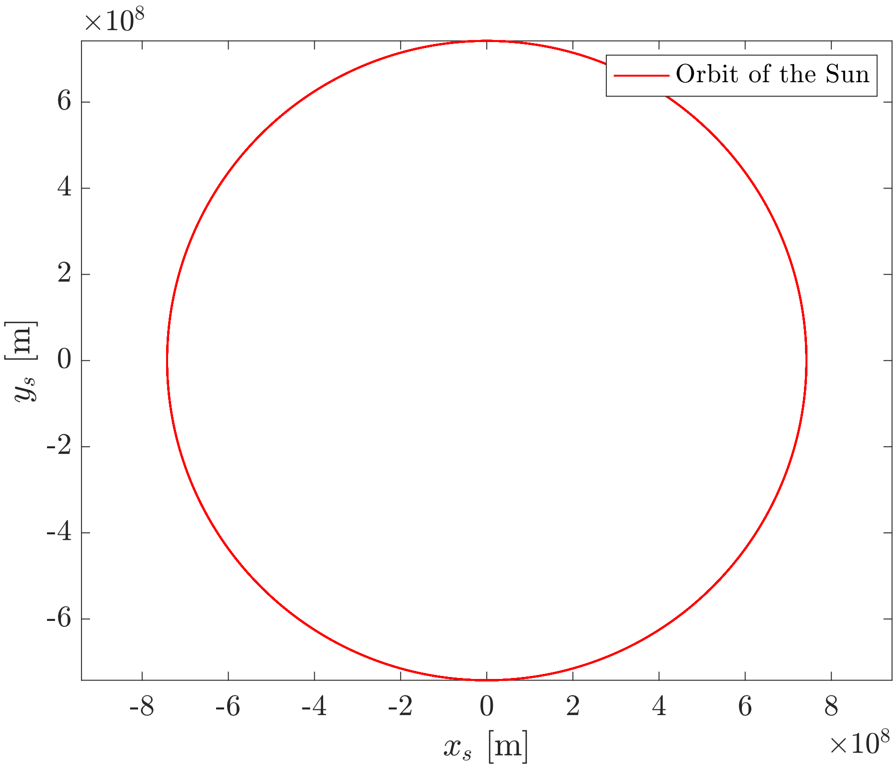


Figure (32) Orbit of the sun when $\theta(0) = \theta_1$

References

[1] Physique numérique I-II Laurent Villard

[2] Instruction sheet