Trajectory of a ball in fluid, without and with gravitation

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Contents

1 Introduction

In physics, the problems that can be solved "exactly"(analytical methods) are very limited, in the sense that they are often based on the idealization of reality. Nevertheless reality is way more complex, the numerical approach allows to take into account several effects traditionally neglected.

In this study, one is looking at the trajectory of a ball in a fluid.
In the first part, one studies the trajectory without gravitation. An analytical approach is first used and then a numerical one using four different methods : Euler, Euler-Cromer, Runge-Kutta and Boris-Buneman. One is interested in the convergence of the numerical solution compared to the analytical one. The non-conservation of mechanical energy is also discussed.
In the second part, the gravitation is taken into account. The same analysis as the one from the first part is done. In addition to this a comparison between the analytical and numerical solution for the frequency of a ball so that the trajectory is tangent to the ground is also done.
These analysis enable us to understand the differences between the numerical methods and which one suits better our problem.

2 Analytical calculations

2.1 Differential equation system of movement

The different parameters used in this problem are :

m	0.5	kg	Mass of the ball
R	0.1	m	Radius of the ball
ρ_M	1.2	Kg m^{-3}	Density of the ball
μ	6		Geometric coefficient

Table (1) Constant parameters used in the exercise

Two forces act on the ball : $\mathbf{F_p} = \mu R^3 \rho \omega \wedge \mathbf{v}$ and $\mathbf{F_g} = m\mathbf{g}$. Where
By the statement we know that $\mathbf{F_g} = -F_g \mathbf{e_z}$, $\mathbf{v} = (v_x, 0, v_z)$ and $\mathbf{w} = w\mathbf{e_y}$.

This implies :

$$\mathbf{F_p} = \mu R^3 \rho \omega \wedge \mathbf{v} = \mu R^3 \rho \begin{pmatrix} 0 \\ w \\ 0 \end{pmatrix} \wedge \begin{pmatrix} v_x \\ w \\ v_z \end{pmatrix} = \mu R^3 \rho \begin{pmatrix} w v_z \\ w \\ -w v_x \end{pmatrix} \tag{1}$$

The forces decomposition can now be written and use Newton's law is used.
Along the x-Axis : $\sum \mathbf{F_x} = \mu R^3 \rho \omega v_z = m\mathbf{a_x}$.
Along the z-Axis : $\sum \mathbf{F_z} = -\mu R^3 \rho \omega v_x - m\mathbf{g} = m\mathbf{a_z}$.
The movement equation are written under the form $\frac{d\mathbf{y}}{dt} = f(\mathbf{y})$, for $\mathbf{y} = (x, z, v_x, v_{z)}$.

$$\begin{pmatrix} f(x) = \frac{dx}{dt} = v_x \\ f(z) = \frac{dz}{dt} = v_z \\ f(v_x) = \frac{dv_x}{dt} = a_x = \frac{\mu R^3 \rho \omega v_z}{m} \\ f(v_z) = \frac{dv_z}{dt} = a_z = -(\frac{\mu R^3 \rho \omega v_x + g)}{m} \end{pmatrix} \tag{2}$$

2.2 Mechanical energy and proof of conservation

The mechanical energy is given by :

$$E_{mec} = E_{cin} + E_{pot} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2 + mgz \tag{3}$$

where $E_{cin} = E_{translation} + E_{rotation} = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$.
To prove the conservation, we have to show that $\frac{dE_{mec}}{dt} = 0$. By statement, $\omega = cst$ and $I = cst$ so $E_{rotation} = cst$.

$$\frac{dE_{mec}}{dt} = \frac{dE_{cin}}{dt} + \frac{dE_{pot}}{dt} = v_x \mu R^3 \rho \omega v_z - v_z(\mu R^3 \rho \omega v_x + m\mathbf{g}) + m\mathbf{g} v_z = 0 \tag{4}$$

2.3 Case $\mathbf{g} = 0$ and $\omega \neq 0$, with initial condition $\mathbf{x}(0) = 0$ and $\mathbf{v}(0) = v_0\mathbf{e_z}$

g	0	ms$^(2)$	Gravity
ω	4π	rps	Rotation velocity of the ball
$\mathbf{v_0}$	$10\mathbf{e_z}$	ms^{-1}	Initial velocity of the ball
$\mathbf{x}(0)$	0	m	Initial position of the ball
t_{final}	200	s	Final time used for the simulations

Table (2) Values of the parameters used in the simulations

With $\mathbf{g} = 0$ we can rewrite the system of equation as :
Along the x-Axis : $\sum \mathbf{F_x} = \mu R^3 \rho\omega v_z = m\mathbf{a_x}$.
Along the z-Axis : $\sum \mathbf{F_z} = -\mu R^3 \rho\omega v_x = m\mathbf{a_z}$.
We resolve the system using matrix (linear algebra) :

$$\frac{d}{dt}\begin{pmatrix} v_x \\ v_z \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mu R^3\rho\omega}{m} \\ -\frac{\mu R^3\rho\omega}{m} & 0 \end{pmatrix}\begin{pmatrix} v_x \\ v_z \end{pmatrix} \tag{5}$$

The solution of this kind of equation is of the form $x(\mathrm{t}) = \exp(\mathrm{At})\mathrm{B}$. Where $A = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}$, using the following notation $\Omega = \frac{\mu R^3\rho\omega}{m}$ and $B = \begin{pmatrix} 0 \\ v_0 \end{pmatrix}$.
In order to resolve our differential system we now calculate the exponential of the matrix above.

$$\exp\begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & cos(\Omega t) \end{pmatrix} \tag{6}$$

The solution can now be computed :

$$\begin{pmatrix} v_x \\ v_z \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & cos(\Omega t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_0 \end{pmatrix} = \begin{pmatrix} \sin(\Omega t)v_0 \\ \cos(\Omega t))v_0 \end{pmatrix} \tag{7}$$

An integration of the system, using the initial condition gives the solution for x and z.
$x = \int v_x dt = -\cos(\Omega t)\frac{v_0}{\Omega} + C_1$ $z = \int v_z dt = \sin(\Omega t)\frac{v_0}{\Omega} + C_2$. With $C_1 = \frac{v_0}{\Omega}$ and $C_2 = 0$.
Which gives the following answer (see (8)):

$$\begin{cases} \mathrm{x} = & -\cos(\Omega t)\frac{v_0}{\Omega} + \frac{v_0}{\Omega} \simeq 51.79\mathrm{m} \\ \mathrm{z} = & sin(\Omega t)\frac{v_0}{\Omega} \simeq -55.15\mathrm{m} \end{cases} \tag{8}$$

3 Numerical simulation : case without gravitation but with rotation

3.1 Convergence rate

As it is shown in Fig 1, the method which has the biggest error on z is the Euler method. Indeed,as it is possible to see on Fig 2, the trajectory of the ball according to this method, diverges from the prediction which was a circle. The trajectory according to Boris-Buneman and Euler-Cromer and Runge-Kutta2 are very close to the theoretical one. Contrary to Euler's method, Euler-Cromer's method has the smallest error on z (Fig 1).

In this case, the analytical solution for z at the final time is known, so it is possible to compute $Errorz = z_{Analytical} - z_{Numerical}$. In a graphic which represents the error as a function of Δt with a log-log scale, the slope represents the convergence rate. So the linear fit of each curve in Fig 1 (which is represented by Fig 18) gives that the convergence rate is 1 for Euler method and 2 for the others. The convergence order of Euler-Cromer's method is two in this case but this method comes from a limited expansion of order one. That is because Euler-Cromer is a "brutal" algorithm which works in every cases and here (without gravity), the differential system is very simple. That is why it works very well here and we have a convergence of order two.

3.2 Non-conservation of mechanical energy

Analytically, the mechanical energy has to be conserved. Indeed, it is proved in section2.2. Numerically, it is not conserved because of the approximation and the quality of the methods. In this section, the stability of the mechanical energy of the different methods is analysed with the graphics of the non-conservation of mechanical energy.

A method is said to be unstable if the numerical error grows exponentially as a function of time for all Δt. In the case the mechanical energy, Euler and Runge-Kutta2 methods are unstable. It is possible to see it in Fig 3 and 4 respectively.

In Fig 3 and 4, the error grows exponentially in all cases. The Δt were chosen close enough so that the graphs are understandable (a too big Δt implies that only one curve is visible). It's also possible to see that the growth rate is proportional to Δt.

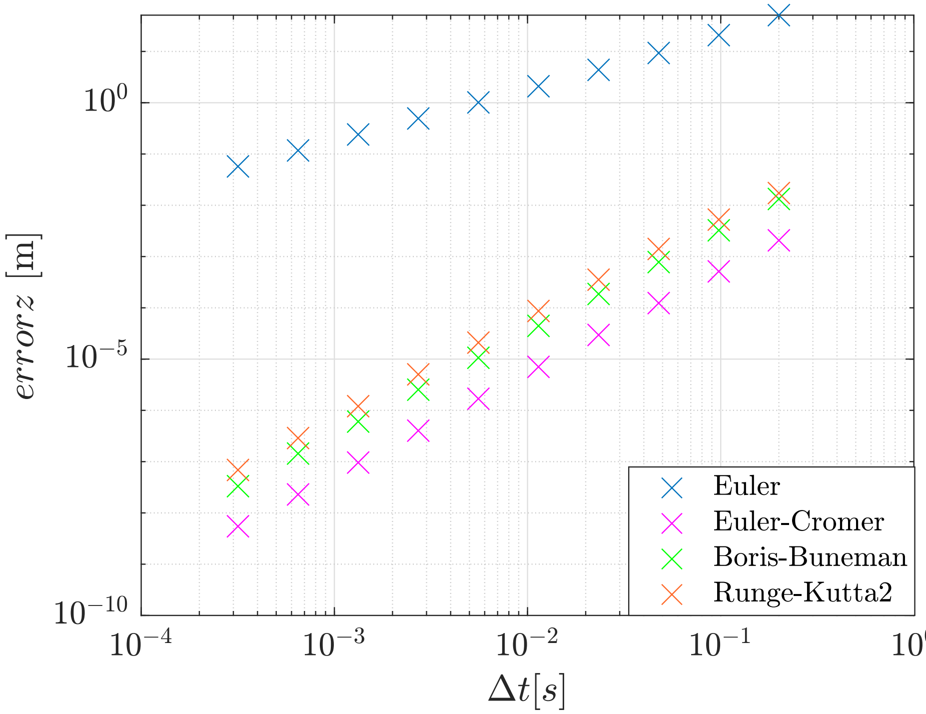
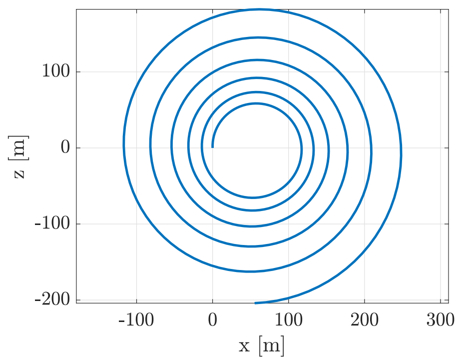
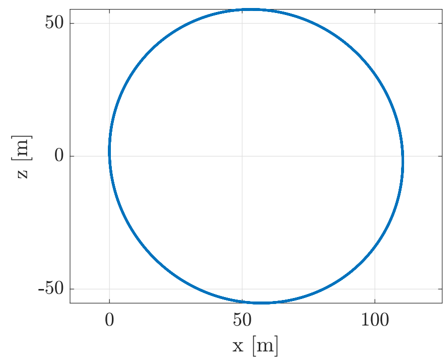


Figure (1) Convergence rate of the Euler, Euler-Cromer, Runge-kutta order 2 and Boris-Buneman methods for 10 simulations with a log-log scale.

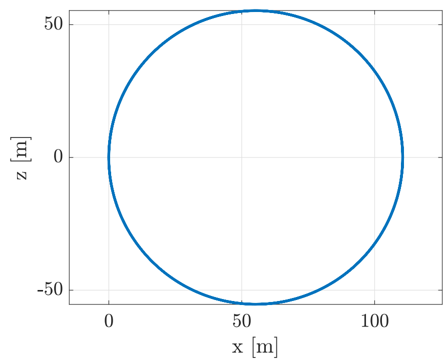
Concerning the Euler-Cromer's method, it is stable for all Δt (Fig 5 and Fig 6) but it does not converge. It is possible to see that increasing nsteps make the oscillations closer to the analytical solution. Both of the graphs oscillate around the analytical solution for the energy.



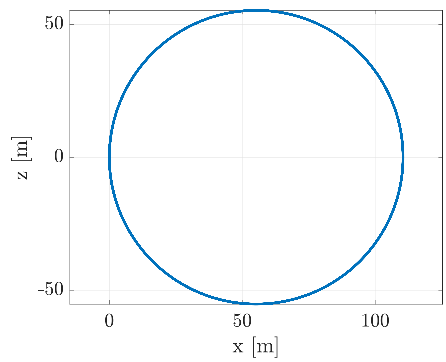
(a) Euler's method



(b) Euler-Cromer's method



(c) Runge-Kutta2's method



(d) Boris-Buneman's method

Figure (2) Trajectory of the ball without gravity with nsteps = 500

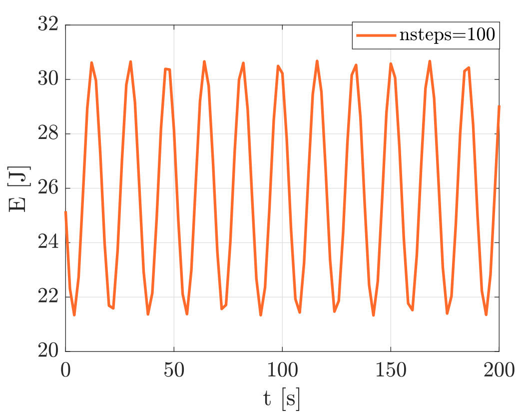


Figure (5) Non-conservation of mechanical energy with Euler-Cromer's method for nsteps = 100

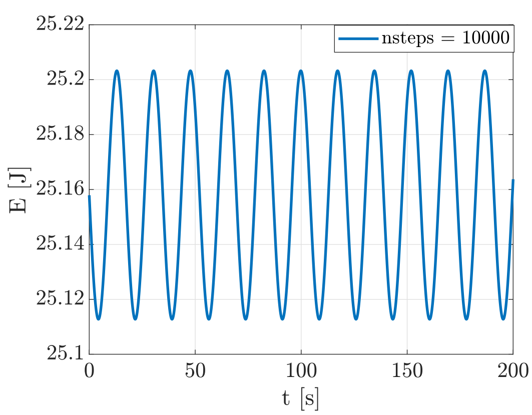


Figure (6) Non-conservation of mechanical energy with Euler-Cromer's method for nsteps = 10000

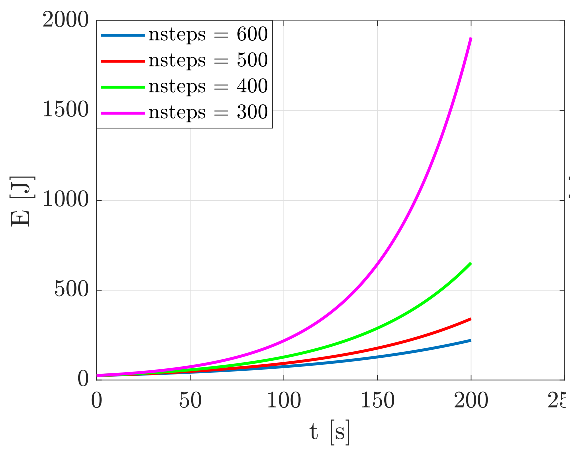


Figure (3) Non-conservation of mechanical energy with Euler method for different Δt.

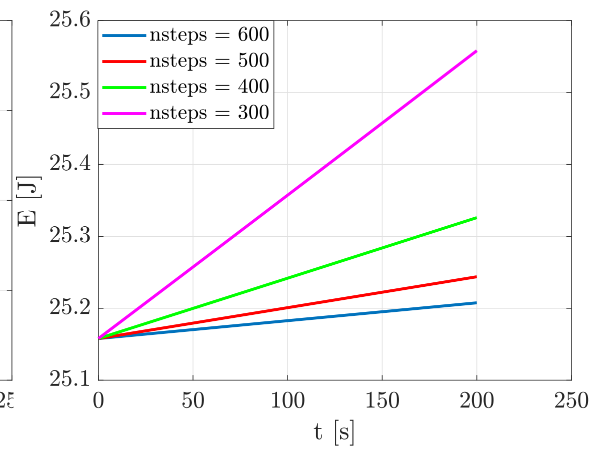


Figure (4) Non-conservation of mechanical energy with Runge-Kutta2 method for different Δt.

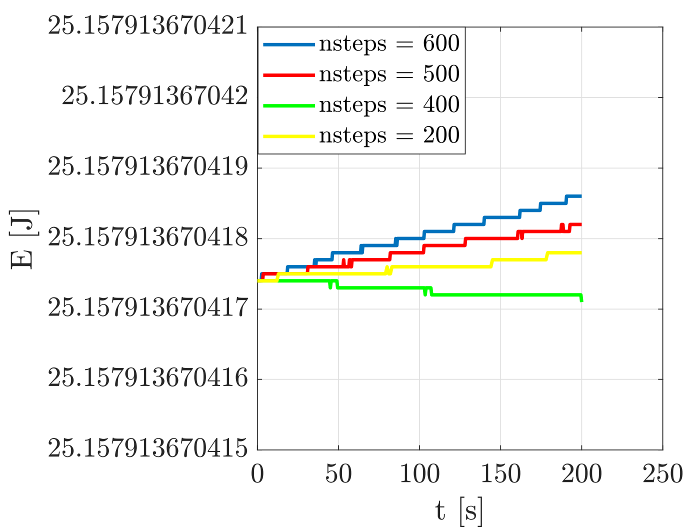


Figure (7) Non-conservation of mechanical energy according to Boris-Buneman's method for differentât.

Boris-Buneman's method approximates mechanical energy best. As it is possible to see in Fig 7, all the curves are very near to the analytical value of energy and the shape of them let think that the only problems are the rounding errors. Numerical representation for real numbers uses a finite number of bits. So all real numbers are rounded to their nearest binary representation. This leads to a significant error in this case.

4 Case with gravitation

4.1 Motion of a ball in a reference in translation at velocity $v_m = \frac{\omega \wedge m\mathbf{g}}{\mu R^3 \rho \omega^2}$

g	9.81	ms(2)	Gravity
ω	4π	rps	Rotation velocity of the ball
$\mathbf{v}(0)$	0	ms^{-1}	Initial velocity of the ball
x_0	0	m	Initial position on x
z_0	160	m	Initial position on z
t_{final}	500	s	Final time used for the simulations

Table (3) Values of the parameters used in the simulations

Let's show that the motion of a ball in a reference in translation at velocity $v_m = \frac{\omega \wedge m\mathbf{g}}{\mu R^3 \rho \omega^2}$ is uniform circular.

The same forces as in 2.1 act on the ball. Using the Newton's law the sum of forces is written as : $\sum F = ma'$. The velocity is given by (9) :

$$\mathbf{v}_{obj-ref} = \mathbf{v}_{obj-earth} + \mathbf{v}_{earth-ref} = \begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix} + \begin{pmatrix} -\frac{m\mathbf{g}}{\mu R^3 \rho \omega} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v'_x \\ 0 \\ v_z \end{pmatrix} \tag{9}$$

The forces decomposition is written as :
Along the x-Axis : $\sum \mathbf{F_x} = \mu R^3 \rho \omega v_z = m\mathbf{a_x}$.
Along the z-Axis : $\sum \mathbf{F_z} = -\mu R^3 \rho \omega v_x - m\mathbf{g} = m\mathbf{a_z}$.
Hence,

$$\begin{cases} m\frac{d}{dt}v_x = \mu R^3 \rho \omega v_z \\ m\frac{d}{dt}v_z = -\mu R^3 \rho \omega v_x - mg \end{cases} \tag{10}$$

Using $v'_z = v_z$ and $v'_x = v_x - \frac{m\mathbf{g}}{\mu R^3 \rho\omega}$ the system (10) is rewritten as

$$\left\{ \begin{array}{l} m\frac{d}{dt}v'_x = \mu R^3 \rho\omega v'_z \\ m\frac{d}{dt}v'_z = -\mu R^3 \rho\omega v'_x \end{array} \right. \tag{11}$$

and under the matrix form as :

$$\frac{d}{dt}\begin{pmatrix} v'_x \\ v'_z \end{pmatrix} = \begin{pmatrix} 0 & \frac{\mu R^3 \rho\omega}{m} \\ -\frac{\mu R^3 \rho\omega}{m} & 0 \end{pmatrix}\begin{pmatrix} v'_x \\ v'_z \end{pmatrix} \tag{12}$$

Using the following notation $\Omega = \frac{\mu R^3 \rho\omega}{m}$ and (6) the solution can be computed.

$$\begin{pmatrix} v'_x \\ v'_z \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & cos(\Omega t) \end{pmatrix} \cdot \begin{pmatrix} \frac{\mathbf{g}}{\Omega} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\Omega t)\frac{\mathbf{g}}{\Omega} \\ -\sin(\Omega t))\frac{\mathbf{g}}{\Omega} \end{pmatrix} \tag{13}$$

An integration of the system, using the initial condition gives the solution for x' and y'. The initial condition are $x(0) = 0$; $z(0) = 160$; $x'(0) = x(0)v_m t$; $z'(0) = 0$.

$x' = \int_0^t v'_x \, \mathrm{d}t = \int_0^t \cos(\Omega t)\frac{\mathbf{g}}{\Omega^2} \, \mathrm{d}t = x'(t) - x'(0) = sin(\Omega t)\frac{\mathbf{g}}{\Omega^2}$.

$z' = \int_0^t v'_z \, \mathrm{d}t = -\int_0^t \sin(\Omega t)\frac{\mathbf{g}}{\Omega^2} \, \mathrm{d}t = z'(t) - z'(0) = cos(\Omega t)\frac{\mathbf{g}}{\Omega^2} - \frac{\mathbf{g}}{\Omega^2}$.

Which gives the following answer (see (14)):

$$\left\{ \begin{array}{ll} \mathrm{x} = & v_m t + x' = sin(\Omega t)\frac{\mathbf{g}}{\Omega^2} - \frac{\mathbf{g}}{\Omega t} \\ \mathrm{z} = & z' = cos(\Omega t)\frac{\mathbf{g}}{\Omega^2} - \frac{\mathbf{g}}{\Omega^2} + 160 \end{array} \right. \tag{14}$$

This proves that the motion of the ball is uniform circular with Ω as angular velocity.

As the terms $\frac{\mathbf{g}}{\Omega t}$ and -$\frac{\mathbf{g}}{\Omega^2} + 160$ have no influence on the radius, R .

$$R = \frac{\mathbf{g}}{\Omega^2} \tag{15}$$

Moreover, the frequency is given by :

$$f = \frac{\Omega}{2\pi} \tag{16}$$

4.2 Convergence rate

As it can be seen in Fig 8, the Euler method remains the worst and the Euler-Cromer method remains better than Euler method concerning the error on z. It's obvious to see in Fig 10 that Euler is the only method which diverges and does not respect the theory. Again, the linear fit of each curve in Fig 9 (which is represented by Fig 19) gives the convergence. The convergence order is 1 for Euler's and Euler-Cromer's method and 2 for Boris-Buneman and Runge-Kutta2. It's possible to see that the convergence rate according Euler-Cromer method has changed from one in the case without gravity to two here. That's because the algorithm works worst than before because the differential system is more complicated.
The Magnus effect that acts on the ball gives the impression that the ball oscillates in a reference frame in translation according to x (Fig 9).

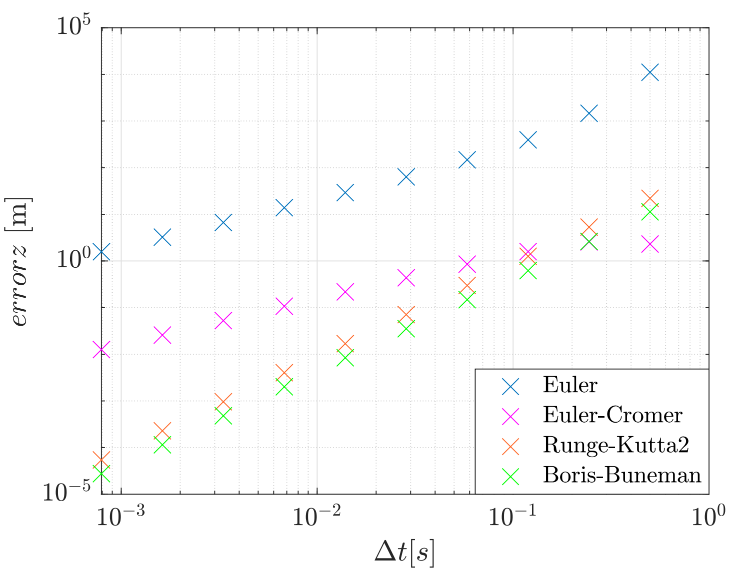


Figure (8) Convergence rate of the Euler, Euler-Cromer, Runge-Kutta order 2 and Boris-Buneman methods for 10 simulations with a log-log scale.

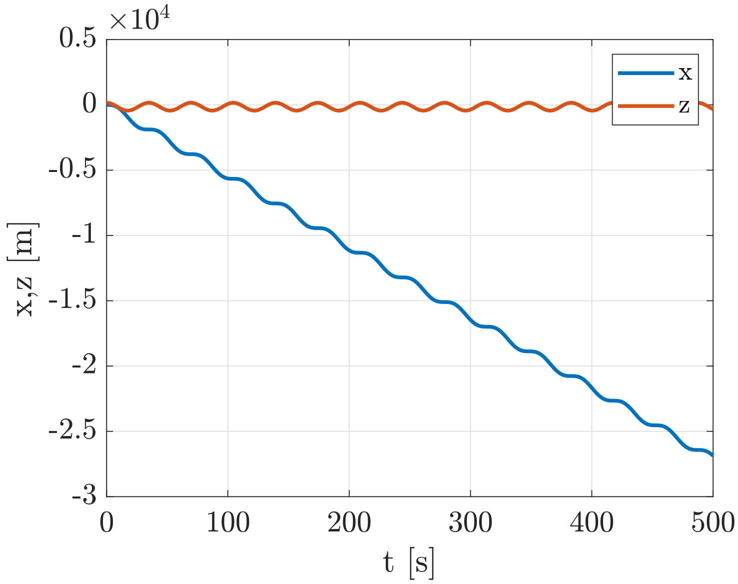
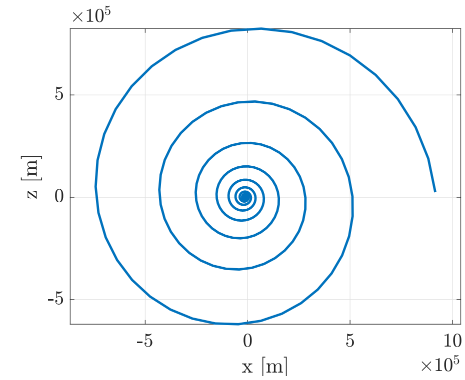


Figure (9) Trajectory of the ball according to Boris-Buneman method with nsteps = 500.

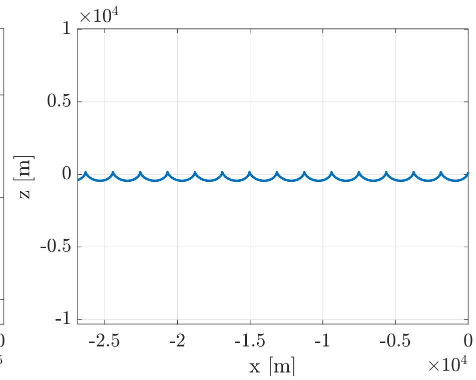
4.3 Non-conservation of mechanical energy

In this case, the analytical value of mechanical energy is also conserved but again, its numerical value is not conserved. The methods which are unstable are the same as before see 3.2.

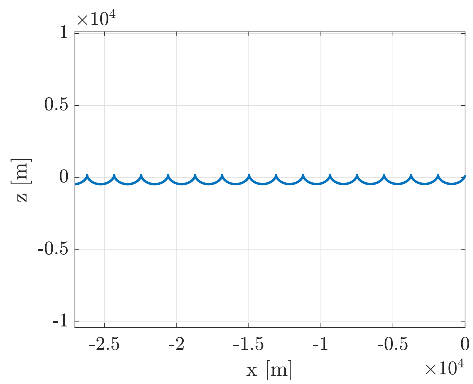
As it is possible to see on Fig 11 and Fig 12, Euler's and Runge-Kutta2's method grow up exponentially and proportionally to Δt.
Again, Euler-Cromer's method does not converge (Fig 13) and oscillates around the analytical solution for the mechanical energy for all Δt.



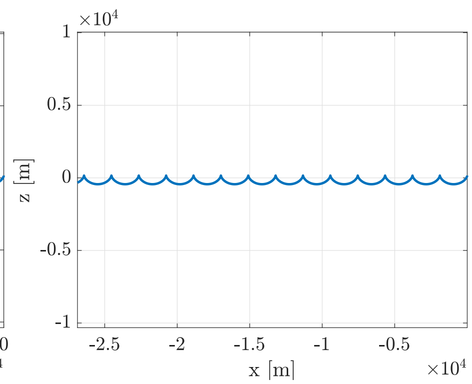
(a) Euler's method



(b) Euler-Cromer's method



(c) Runge-Kutta2's method



(d) Boris-Buneman's method

Figure (10) Trajectory of the ball for each method with gravitation and nsteps = 500

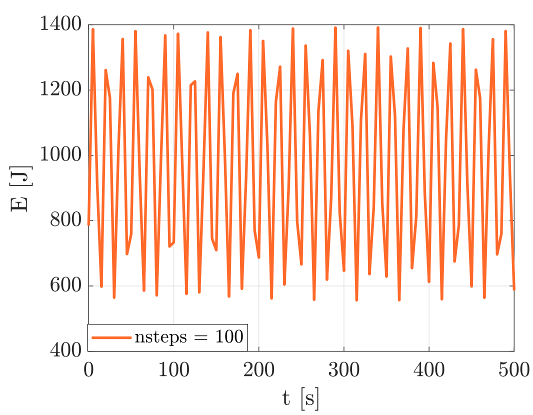


Figure (13) Non-conservation of mechanical energy with Euler-Cromer method for nsteps = 100

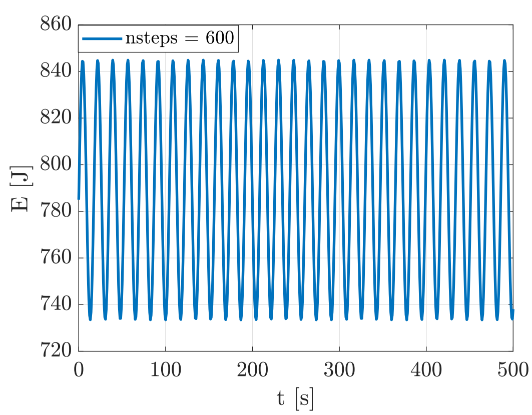


Figure (14) Non-conservation of mechanical energy with Euler-Cromer method for nsteps = 600

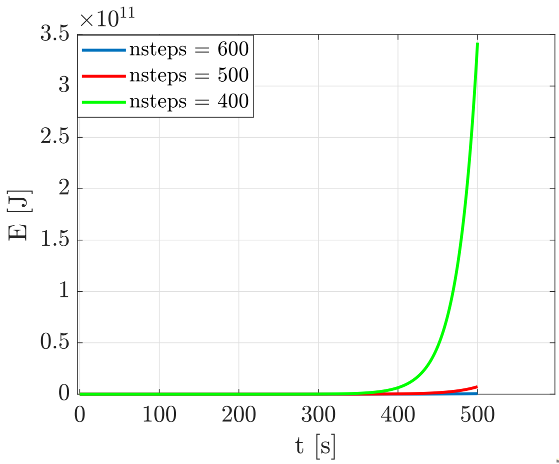


Figure (11) Non-conservation of mechanical energy with Euler method for different Δt.

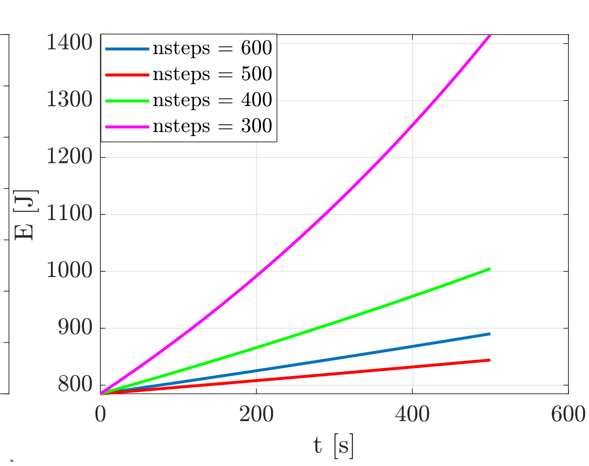


Figure (12) Non-conservation of mechanical energy with Runge-Kutta2 method for different Δt.

Boris-Buneman gives the best approximation of the mechanical energy again (Figure 15). As explained in section 3.2, the only problem seems to be the rounding errors.

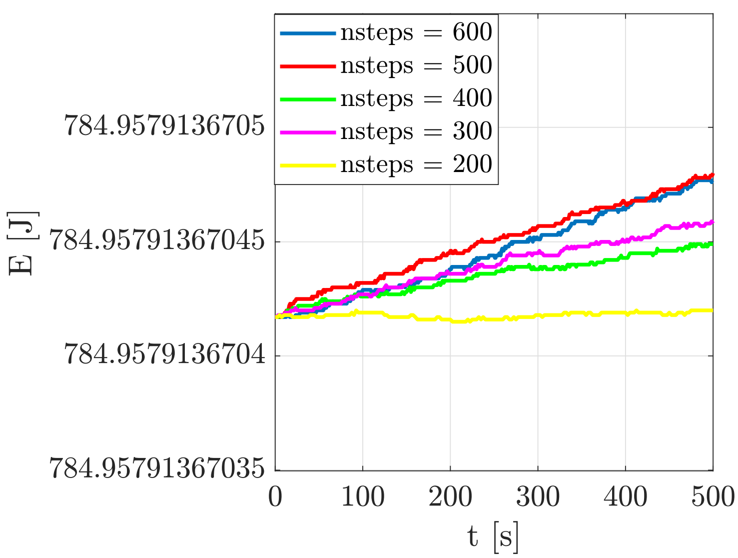


Figure (15) Non-conservation of mechanical energy according to Boris-Buneman's method for different Δt.

4.4 Rotation frequency of the ball so that the trajectory is tangent to the ground and numerical verification

When the trajectory is tangent to the ground $z = 0$ and $v_z = 0$. Using (13) the time t where $v_z(t) = 0$ is found. $\sin(\Omega t) = 0$. Which implies :

$$t = \frac{\pi}{\Omega} \tag{17}$$

Inserting (17) in (14) we get $\Omega = \sqrt{\frac{\mathbf{g}}{80}}$

$$\omega = \frac{m\Omega}{\mu R^3 \rho} \simeq 24,318\text{rads}^{-1} \tag{18}$$

The position of the drop point is given by $x(\pi)$ using (14).

$$x(\pi) = -\frac{\mathbf{g}}{\Omega^2} = -80\pi \tag{19}$$

To check which method approximates the best the drop point, one has to check all the graphics at the point which is shown in Fig 16 and see where is the curve when it should be zero. This point represents where the ball should hit the ground analytically. As it is possible to see in Fig 17, the worst approximation is made by Euler's method with an absolute error of $\approx 2m$. Boris-Buneman, Runge-Kutta2 and Euler-Cromer are very good with a very small absolute error of $\approx 3 \cdot 10^{-3}m$ (which means a relative error in tis case of $\approx 0.001.\%$!).

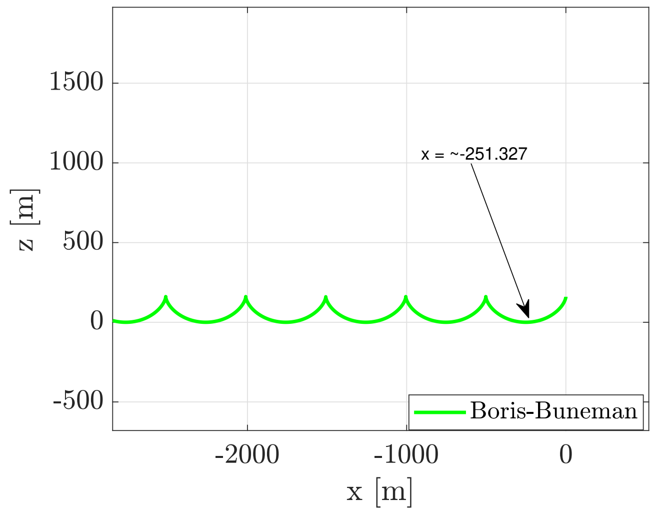


Figure (16) Trajectory of the ball according to Boris-Buneman's method with $\omega = \approx 24,318 \text{rads}^{-1}$ and nsteps $= 10000$

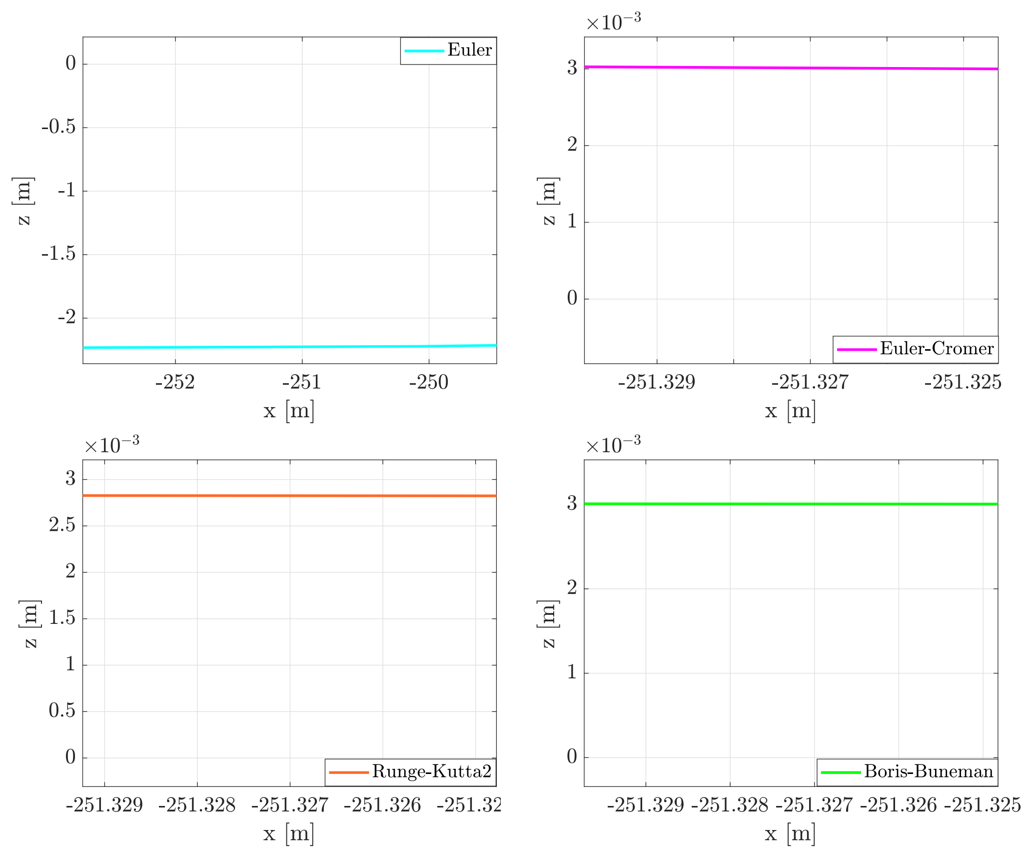


Figure (17) Trajectory of the ball for each method with $\omega = \approx 24,318 \text{rads}^{-1}$ and nsteps $= 10000$

5 Conclusion

This exercise is a good introduction to numerical physics. It has clearly highlighted the differences between analytical and numerical solutions. Moreover, the concepts of stability and convergence were introduced.Four numerical methods were analysed: Euler, Euler-Cromer Runge-Kutta2 and Boris-Buneman.

It has been seen that, unlike the others, the Euler method differs from the analytical solution. Analysis of the non-conservation of mechanical energy showed that the methods of Euler and Runge-Kutta are unstable, while Boris-Buneman and Euler-Cromer are stable. This shows that, even Euler and Euler-Cromer are very close to each other, a small change in the algorithm causes both methods to behave completely differently.

Finally, this work made it possible to distinguish the advantages and disadvantages of each method. For example, it is possible to see that Euler's method is the easiest to implement but it differs from the analytical solution and is not stable. Runge-Kutta, which uses intermediate times steps, is very accurate concerning the trajectory of the ball but does not conserve energy. Euler-Cromer conserves mechanical energy on average over time. It is stable but does not converge. Boris-Buneman is quite difficult to implement but it is the best, it conserve mechanical energy (the only problem is the rounding error) and is very accurate for all Δt.

6 Annex

References

[1] Mimeo of the course of digital physics

[2] Statement of the exercise 2 : Trajectory of a ball in fluid, without and with gravitation

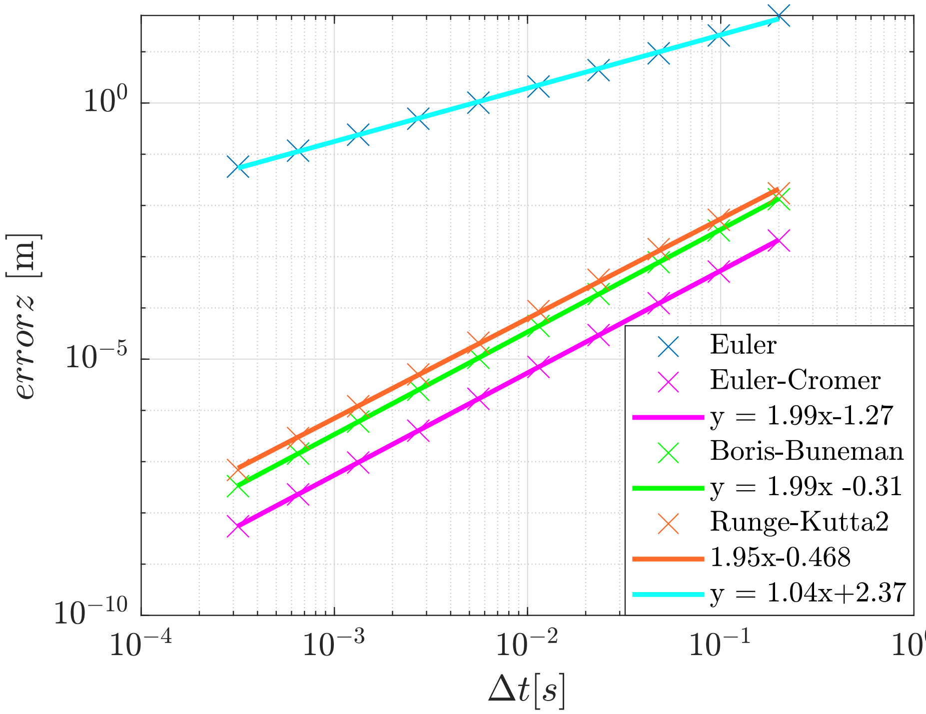


Figure (18) Convergence rate of the Euler, Euler-Cromer, Runge-kutta order 2 and Boris-Buneman methods for 10 simulations with the fitting curves (case without gravity).

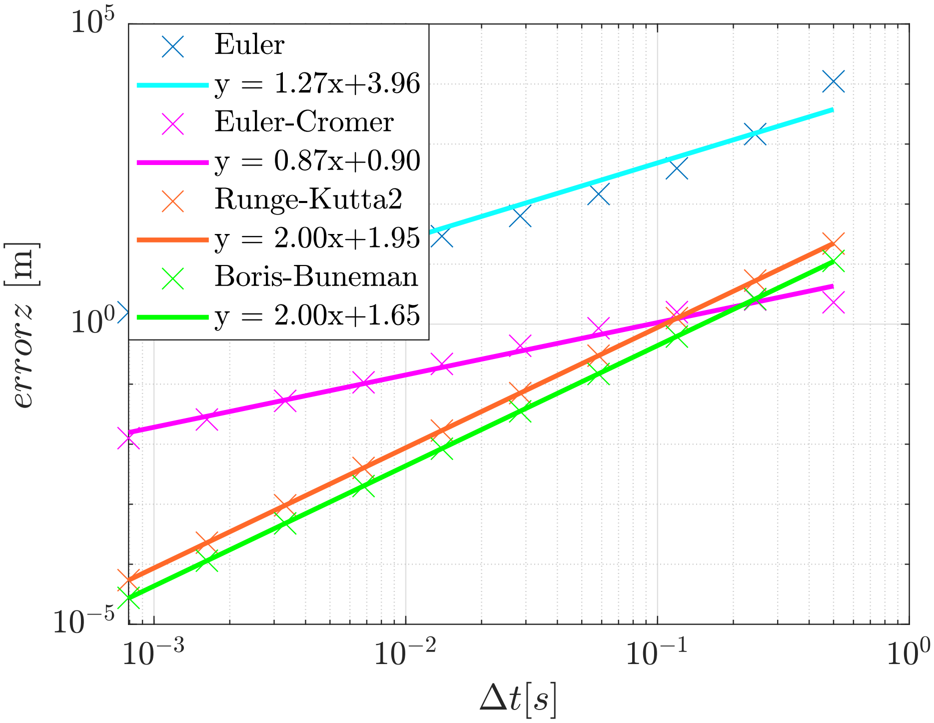


Figure (19) Convergence rate of the Euler, Euler-Cromer, Runge-kutta order 2 and Boris-Buneman methods for 10 simulations with the fitting curves (case with gravity).