Derivation of the induced velocities of an infinite and finite line vortex

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1 Goal

This report aims at providing a step-by-step and comfortable-to-follow derivation of the induced velocities of an infinite and finite line vortex. Information about the final induced velocities of an infinite line vortex is readily available online, however, no thorough derivation (not for the infinite nor the finite case) could be found by the author.

First, some prerequisites are stated. Then, the induced velocity from an infinite straight-line vortex is derived. Based on that, the induced velocity for a finite straight-line vortex is derived. In the end, an example Python code is provided. The report is lengthy as all choices and thoughts are tried to be explained.

2 Prerequisites

Throughout the derivation, boldface $\boldsymbol{r}$ denote vectors, two successive vectors $\boldsymbol{r_1 r_2}$ mean a dot product, and $|\cdot|$ mean the Euclidian norm. Besides fundamental analysis, two concepts and an additional equation play an important role in the derivation. These are:

- **Dot product as directional filter:** The dot product of a vector $\boldsymbol{r}$ and a unit vector $\boldsymbol{n}$, $|\boldsymbol{n}| = 1$, yields the distance $\boldsymbol{r}$ is pointing in the same direction as $\boldsymbol{n}$.
- **Cross product as area of parallelogram:** The magnitude of the cross product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ equals the area A of a parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. $A = |\boldsymbol{a} \times \boldsymbol{b}|$
- **The area of a parallelogram:** The area of a parallelogram is equal to the base length times its height, see fig. 1 with $A = ht$.

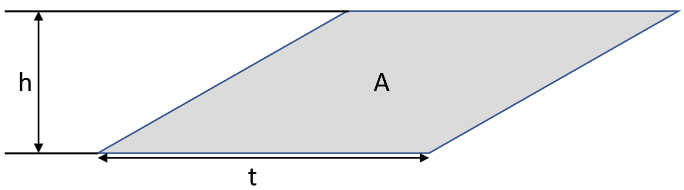

Figure 1: Area of a parallelogram

3 Infinite straight-line vortex

The induced velocity $\boldsymbol{v}_i$ of any vortex with constant circulation Γ can be calculated using the Biot-Savart law as

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi} \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3} \tag{1}$$

With an infinitesimal vortex element $\boldsymbol{dl}$ and the vector $\boldsymbol{r}$ pointing from the core of the infinitesimal vortex element (if it is a line vortex) to the induction point. The capital V over which is integrated denotes the whole vortex. Assuming a straight-line vortex, the sketch from fig. 2 can be done. It contains the induction point $\boldsymbol{x_p}$, the orange vortex, the normal distance h between the vortex and the induction point, a vortex vector element $\boldsymbol{dl}$, a vector $\boldsymbol{r}$ pointing from the middle of the vortex vector element to the induction point, the distance l between the middle of the vortex vector element and the intersection with the h line, the angle φ between the h line and the vector $\boldsymbol{r}$[1], and an angle $d\varphi$ spanning the vortex vector element. This view is normal to the plane spanned by the vortex line and the control point. The idea to introduce an angle comes from an MIT lecture slide on fluid mechanics: https://web.mit.edu/16.unified/www/SPRING/fluids/Spring2008/LectureNotes/f06.pdf. However, this source lacks the finite case as well as an easy-to-follow derivation.

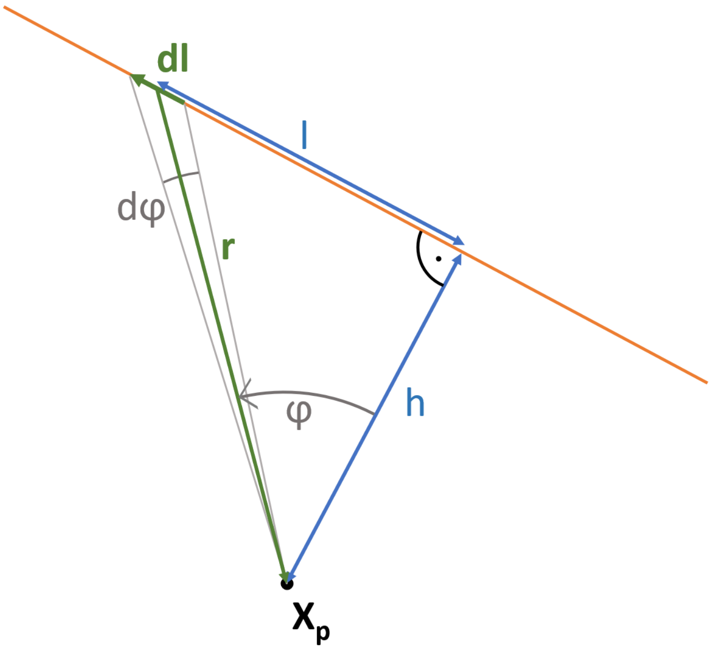

Figure 2: A vortex element on a straight vortex line

First, the equation for the induction of an infinite straight-line vortex will be derived. Hence, the focus lies on the integral of eq. (1), as the factor in front of it is known.

$$\boldsymbol{v_i} \propto \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \tag{2}$$

Since $\boldsymbol{dl}$ can be split into its length and direction as $\boldsymbol{dl} = |\boldsymbol{dl}|\, \boldsymbol{e_v}$ with the unit vector $\boldsymbol{e_v}$ that is pointing in the direction of the vortex[2], there are 3 remaining unknowns in eq. (2) after assuming

[1] Positive as it is drawn, if $\boldsymbol{r}$ was pointing to the other side of h, φ would be negative

[2] This means in the same direction as the vorticity vector of that vortex.

the vortex' direction is known. These are the length $|\boldsymbol{dl}|$, the cross-product $\boldsymbol{dl} \times \boldsymbol{r}$, and the distance $|\boldsymbol{r}|^3$. One could argue that only $\boldsymbol{dl}$ and $\boldsymbol{r}$ are unknown, but starting the derivation based on those two turned messy. However, $|\boldsymbol{r}|^3$ can be readily derived and the cross-product's geometric meaning eases the derivation.

Therefore, let us first examine $|\boldsymbol{dl}|$. Assuming (for now, derivation will follow later) a known h, the right triangle of fig. 2 brings

$$l = h \tan(\varphi) \tag{3}$$

The same logic can be used for $|\boldsymbol{dl}|$. The tip of $\boldsymbol{dl}$ is at the angle $\varphi + d\varphi/2$, while the root of $\boldsymbol{dl}$ is at the angle $\varphi - d\varphi/2$. The length from the line h to the tip of $\boldsymbol{dl}$ minus the length from the line h to the root of $\boldsymbol{dl}$, which is exactly $|\boldsymbol{dl}|$, becomes

$$\begin{aligned} |\boldsymbol{dl}| &= h \tan\left(\varphi + \frac{d\varphi}{2}\right) - h \tan\left(\varphi - \frac{d\varphi}{2}\right) && (4) \\ &= h \left(\tan\left(\varphi + \frac{d\varphi}{2}\right) - \tan\left(\varphi - \frac{d\varphi}{2}\right)\right) && (5) \end{aligned}$$

Using basic trigonometric angle sum identities for $\tan()$, $\cos()$, and $\sin()$ shows

$$\tan(\alpha + \beta) = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \tag{6}$$

which can now be applied to eq. (5):

$$|\boldsymbol{dl}| = h \left(\frac{\sin(\varphi)\cos\left(\frac{d\varphi}{2}\right) + \cos(\varphi)\sin\left(\frac{d\varphi}{2}\right)}{\cos(\varphi)\cos\left(\frac{d\varphi}{2}\right) - \sin(\varphi)\sin\left(\frac{d\varphi}{2}\right)} - \frac{\sin(\varphi)\cos\left(\frac{d\varphi}{2}\right) - \cos(\varphi)\sin\left(\frac{d\varphi}{2}\right)}{\cos(\varphi)\cos\left(\frac{d\varphi}{2}\right) + \sin(\varphi)\sin\left(\frac{d\varphi}{2}\right)}\right) \tag{7}$$

Due to $d\varphi \to 0$, $\cos\left(\frac{d\varphi}{2}\right) \to 1$ and $\sin(d\varphi/2) \to d\varphi/2$ follow. Using this simplifies our equation to

$$|\boldsymbol{dl}| = h \left(\frac{\sin(\varphi) + \cos(\varphi)\frac{d\varphi}{2}}{\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}} - \frac{\sin(\varphi) - \cos(\varphi)\frac{d\varphi}{2}}{\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}}\right) \tag{8}$$

Bringing both summands inside the parenthesis on the same denominator yields

$$|\boldsymbol{dl}| = h \left(\frac{\left(\sin(\varphi) + \cos(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}\right) - \left(\sin(\varphi) - \cos(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}\right)}{\left(\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}\right)}\right) \tag{9}$$

The terms in eq. (9) now have to be multiplied by one another. While doing so, terms with $d\varphi^2$ can be neglected. The result will be

$$|\boldsymbol{dl}| = h \left(\frac{\sin(\varphi)\cos(\varphi) + \sin^2(\varphi)\frac{d\varphi}{2} + \cos^2(\varphi)\frac{d\varphi}{2} - \sin(\varphi)\cos(\varphi) + \sin^2(\varphi)\frac{d\varphi}{2} + \cos^2(\varphi)\frac{d\varphi}{2}}{\cos^2(\varphi) + \sin(\varphi)\cos(\varphi)\frac{d\varphi}{2} - \sin(\varphi)\cos(\varphi)\frac{d\varphi}{2}}\right) \tag{10}$$

$$= h \left(\frac{\left(\sin^2(\varphi) + \cos^2(\varphi)\right) d\varphi}{\cos^2(\varphi)}\right) \tag{11}$$

$$|\boldsymbol{dl}| = \frac{h}{\cos^2(\varphi)} d\varphi \tag{12}$$

With that, $\boldsymbol{dl}$ is known by $\boldsymbol{dl} = |\boldsymbol{dl}|\, \boldsymbol{e_v}$ (again with the unit vector of the direction of the vortex $\boldsymbol{e_v}$).

Now we turn onto $\boldsymbol{dl} \times \boldsymbol{r}$. First, rewrite it into

$$\boldsymbol{dl} \times \boldsymbol{r} = |\boldsymbol{dl}|\, \boldsymbol{e_v} \times \boldsymbol{r} = \frac{h}{\cos^2(\varphi)} d\varphi \, (\boldsymbol{e_v} \times \boldsymbol{r}) \tag{13}$$

Leaving us with an unknown term $\boldsymbol{e_v} \times \boldsymbol{r}$. However, the magnitude of the resulting vector of that cross-product is the area of the parallelogram spanned by $\boldsymbol{e_v}$ and $\boldsymbol{r}$. Keeping fig. 2 in mind, the height of that parallelogram is h while the base length is $|\boldsymbol{e_v}| = 1$. Therefore, the parallelogram's area is always h, regardless of where the point $\boldsymbol{r}$ is based on the vortex. Furthermore, since both $\boldsymbol{r}$ and $\boldsymbol{e_v}$ always lie in the plane that we are looking at, the direction of the vector $\boldsymbol{e_v} \times \boldsymbol{r}$ is always the same: normal to the plane (in this case coming out of the plane since $\boldsymbol{x_p}$ always lies on the left of the vortex when following $\boldsymbol{dl}$). In summary, we've just shown that the resulting vector from $\boldsymbol{e_v} \times \boldsymbol{r}$ has a constant magnitude h (=area of the parallelogram) and a constant direction (normal to the plane):

$$\boldsymbol{e_v} \times \boldsymbol{r} = h\boldsymbol{e_i} \tag{14}$$

The unit vector $\boldsymbol{e_i}$ has an important meaning: it is the direction of the induced velocities at point $\boldsymbol{x_p}$. It is currently unknown to us but just like for h, we will find an expression for it later.

The last thing from eq. (2) we have not yet looked at is $|\boldsymbol{r}|^3$. From the right triangle, it quickly follows that

$$r = \frac{h}{\cos(\varphi)} \tag{15}$$

and with that the integral becomes

$$\boldsymbol{dl} \times \boldsymbol{r} = |\boldsymbol{dl}|\, \boldsymbol{e_v} \times \boldsymbol{r} = \frac{h}{\cos^2(\varphi)} d\varphi h \boldsymbol{e_i} = \frac{h^2}{\cos^2(\varphi)} \boldsymbol{e_i} d\varphi \tag{16}$$

$$r^3 = \left(\frac{h}{\cos(\varphi)}\right)^3 \tag{17}$$

$$\boldsymbol{v_i} \propto \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3} = \int\limits_{\varphi_V} \frac{\cos(\varphi)}{h} d\varphi \boldsymbol{e_i} \tag{18}$$

With φ_V being all angles φ belonging to the vortex. Equation (18) can be readily integrated, noting that neither h nor $\boldsymbol{e_i}$ are dependent on φ.

$$\boldsymbol{v_i} \propto \frac{1}{h} \sin(\varphi)\Big|_{\varphi_V} \boldsymbol{e_i} = \frac{1}{h}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) \boldsymbol{e_i} = \frac{2}{h} \boldsymbol{e_i} \tag{19}$$

For an infinite vortex line, φ_V spans from $\varphi_{Vs} = -\pi/2$ to $\varphi_{Ve} = \pi/2$ with "Vs" and "Ve" denoting the start and end of the vortex.

So far, we have only reduced the number of unknowns by 1 (we know $|\boldsymbol{r}|$). The new two unknowns are h and $\boldsymbol{e_i}$. As stated and used above, h stands in direct connection to the area of the parallelogram spanned by a vector $\boldsymbol{r}$ between the vortex and the induction point, and a vector along the vortex (earlier that was $\boldsymbol{dl}$). We can therefore choose these two vectors such that we know them and recalculate the area. Introducing a point $\boldsymbol{x_{vs}}$ that lies in the vortex core, we get a new vector from the vortex to the induction point

$$\boldsymbol{r_s} = \boldsymbol{x_p} - \boldsymbol{x_{vs}} \tag{20}$$

And instead of $\boldsymbol{dl}$ we can use any vector $\boldsymbol{r_v}$ along the vortex core which points in the same direction as the vorticity vector.[3] Hence, we take a second point $\boldsymbol{x_{ve}}$ on the vortex core (following the just stated condition) and get

$$\boldsymbol{r_v} = \boldsymbol{x_{ve}} - \boldsymbol{x_{vs}}\,, \quad \boldsymbol{r_v} = |\boldsymbol{r_v}|\, \boldsymbol{e_v} \tag{21}$$

[3] If it was the opposite direction, the induced velocity vector $\boldsymbol{v_i}$ would be negative of its true values

With the unit vector of the direction of the vortex $\boldsymbol{e_v}$ and the length $|\boldsymbol{r_v}|$ that $\boldsymbol{r_v}$ is pointing along the vortex. The area A of the parallelogram spanned by $\boldsymbol{r_v}$ ($\boldsymbol{r_v}$ again represents the base of the parallelogram) and $\boldsymbol{r_s}$ can now be calculated in two different ways

$$A = |\boldsymbol{r_v} \times \boldsymbol{r_s}| \tag{22}$$

$$A = h\,|\boldsymbol{r_v}| \tag{23}$$

Combining both equations yields

$$h = \frac{|\boldsymbol{r_v} \times \boldsymbol{r_s}|}{|\boldsymbol{r_v}|} \tag{24}$$

Again, h is the normal distance from the vortex core to the control point. Hence, it can be used to figure out whether the control point lies directly on the extension of the vortex core or very close to it, where a solid body rotation might be a better assumption than an irrotational vortex. Now, $\boldsymbol{e_i}$ (the direction of the induced velocities) is straightforward. Equation (14) already shows the necessary relationship. Instead of $\boldsymbol{r}$ we can use $\boldsymbol{r_s}$ because, again, $\boldsymbol{r}$ has no influence on the result of the cross product as long as it is pointing from the vortex core to the induction point. Recall the paragraph above eq. (14) for that. In the end, we take the same equation and modify it in a way that leaves us with only $\boldsymbol{e_i}$ as an unknown.

$$\boldsymbol{e_v} \times \boldsymbol{r_s} = h\boldsymbol{e_i} \quad \Big| \cdot |\boldsymbol{r_v}| \tag{25}$$

$$|\boldsymbol{r_v}|\,\boldsymbol{e_v} \times \boldsymbol{r} = |\boldsymbol{r_v}|\,h\boldsymbol{e_i} \tag{26}$$

$$\boldsymbol{r_v} \times \boldsymbol{r_s} = |\boldsymbol{r_v}|\,h\boldsymbol{e_i} \tag{27}$$

$$\boldsymbol{e_i} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_v}|\,h} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_v}|\,\frac{|\boldsymbol{r_v} \times \boldsymbol{r_s}|}{|\boldsymbol{r_v}|}} \tag{28}$$

$$\boldsymbol{e_i} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_v} \times \boldsymbol{r_s}|} \tag{29}$$

Where eq. (28) used eq. (24) and with that, everything is known for the induced velocities of the infinite straight-line vortex if one knows two points on the vortex and the induction point.

4 Finite straight-line vortex

So far, we have not done unnecessary work in terms of finding an expression for the induction of the finite straight-line vortex. The only changes now are concerning φ_V from eq. (19). For that, consider fig. 3. The solid orange line shows the finite vortex, which is bound by the start of the vortex $\boldsymbol{x_{vs}}$ and the end of the vortex $\boldsymbol{x_{ve}}$. These names are not by coincidence the same as above because we can use the expressions for $\boldsymbol{e_i}$ and h from the infinite vortex case. Furthermore, we have a normal distance l_s from the h line to the start of the vortex and a normal distance l_e to the end of the vortex. Lastly, there are the vectors $\boldsymbol{r_s}$ and $\boldsymbol{r_e}$ pointing from the start and end of the vortex to the induction point, respectively. The vortex vector $\boldsymbol{r_v}$ pointing from $\boldsymbol{x_{vs}}$ to $\boldsymbol{x_{ve}}$ is not drawn due to clarity reasons. Mind again that φ is defined as positive in the clockwise direction and it would turn negative was $\boldsymbol{r_e}$ and $\boldsymbol{r_s}$ on the other side of h. Now, we want to know the angles φ_1 and φ_2 (which we already knew for the infinite vortex case).

There are many ways of calculating the φ's since we already know two sides and one angle in the right triangles of fig. 3, rendering the triangles fully determined. However, one idea is to find equations for which the φ's are in $\sin()$ functions, as we could then use that expression directly

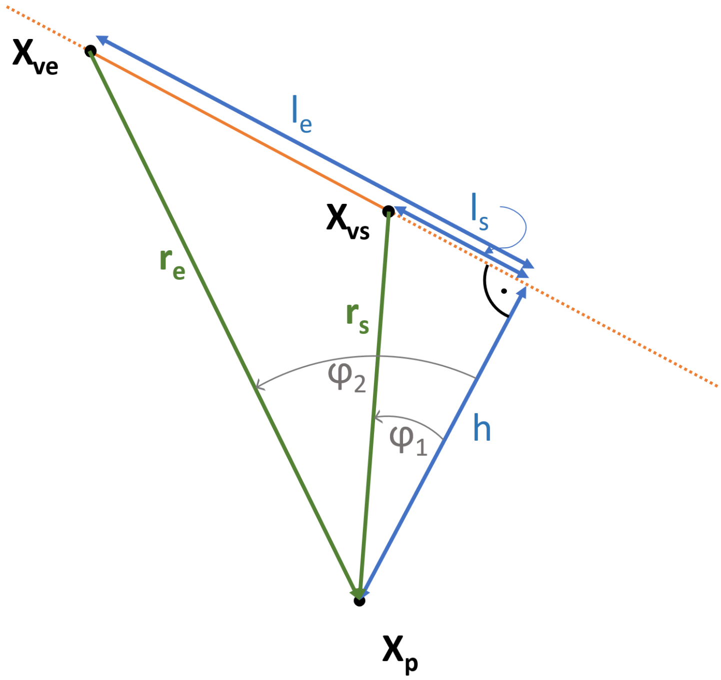

Figure 3: Sketch of a finite straight-line vortex

in eq. (19). Looking at the right triangles we find

$$\sin(\varphi_1) = \frac{l_s}{|\boldsymbol{r_s}|} \tag{30}$$

$$\sin(\varphi_2) = \frac{l_e}{|\boldsymbol{r_e}|} \tag{31}$$

Which leaves l_s and l_e open for calculation. Again, trigonometric functions or Pythagoras' theorem could be applied, but there exists a computationally easier solution. This solution is solely based on multiplication and division, not on evaluating square roots or trigonometric functions. The length l_s is the negative distance of the part of $\boldsymbol{r_s}$ pointing parallel to the vortex. The same is true for l_e with $\boldsymbol{r_e}$. The "negative" arises because we defined φ to be positive for the situation in fig. 3. This means that l_s has to be positive because of eq. (30) and the limitations $-\pi/2 < \varphi < \pi/2$. However, when looking along the vortex' axis, $\boldsymbol{r_s}$ is facing the opposite direction of the vortex vector $\boldsymbol{r_v}$. A dot product, which we will have to use here, would thus return a negative length. Hence, we are correcting with another minus sign. Concluding: if we had the unit vector $\boldsymbol{e_v}$ pointing in the direction of the vortex, we would get

$$l_s = -\boldsymbol{r_s}\boldsymbol{e_v} \tag{32}$$

$$l_e = -\boldsymbol{r_e}\boldsymbol{e_v} \tag{33}$$

We can multiply both sides with the length of the vortex $|\boldsymbol{r_v}|$ to get rid of the unit vector $\boldsymbol{e_v}$

$$l_s\,|\boldsymbol{r_v}| = -\boldsymbol{r_s}\boldsymbol{e_v}\,|\boldsymbol{r_v}| = -\boldsymbol{r_s}\boldsymbol{r_v} \tag{34}$$

$$l_e\,|\boldsymbol{r_v}| = -\boldsymbol{r_e}\boldsymbol{e_v}\,|\boldsymbol{r_v}| = -\boldsymbol{r_e}\boldsymbol{r_v} \tag{35}$$

Which can be inserted into eqs. (30) and (31) to get

$$\sin(\varphi_1) = -\frac{\boldsymbol{r_s}\boldsymbol{r_v}}{|\boldsymbol{r_s}|\,|\boldsymbol{r_v}|} \tag{36}$$

$$\sin(\varphi_2) = -\frac{\boldsymbol{r_e}\boldsymbol{r_v}}{|\boldsymbol{r_e}|\,|\boldsymbol{r_v}|} \tag{37}$$

And with that the induced velocities of the finite straight-line vortex are derived:

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi} \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3} \tag{38}$$

$$= \frac{\Gamma}{4\pi h} \sin(\varphi) \Big|_{\varphi_1}^{\varphi_2} \boldsymbol{e_i} \tag{39}$$

$$= \frac{\Gamma}{4\pi h} \left(\sin(\varphi_2) - \sin(\varphi_1)\right) \boldsymbol{e_i} \tag{40}$$

$$= \frac{\Gamma}{4\pi h} \left(- \frac{\boldsymbol{r_e} \boldsymbol{r_v}}{|\boldsymbol{r_e}| \, |\boldsymbol{r_v}|} + \frac{\boldsymbol{r_s} \boldsymbol{r_v}}{|\boldsymbol{r_s}| \, |\boldsymbol{r_v}|} \right) \boldsymbol{e_i} \tag{41}$$

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi h \, |\boldsymbol{r_v}|} \left(\boldsymbol{r_v} \left(\frac{\boldsymbol{r_s}}{|\boldsymbol{r_s}|} - \frac{\boldsymbol{r_e}}{|\boldsymbol{r_e}|} \right) \right) \boldsymbol{e_i} \tag{42}$$

With

$$h = \frac{|\boldsymbol{r_v} \times \boldsymbol{r_s}|}{|\boldsymbol{r_v}|} \qquad \boldsymbol{e_i} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{h \, |\boldsymbol{r_v}|} \tag{43}$$

5 Example python code

The code can also be found with comment on GitHub.

```python
import numpy as np

def vortex_induction_factor(vortex_start: np.ndarray,
                            vortex_end: np.ndarray,
                            induction_point: np.ndarray) -> np.ndarray:
    """
    This function calculates the induction at a point 'induction_point' from a
    straight vortex line between the two points 'vortex_start' and 'vortex_end'
    for a unity circulation. The returned value is a vector of induced velocities.
    :param vortex_start: numpy array of size (3,)
    :param vortex_end: numpy array of size (3,)
    :param induction_point: numpy array of size (3,)
    :return: numpy array of size(3,)"""
    r_s = induction_point-vortex_start # vector from the start of the vortex to
                                       # the induction point
    r_e = induction_point-vortex_end # vector from the end of the vortex to the
                                     # induction point
    r_v = vortex_end-vortex_start # vector from the start of the vortex to the
                                  # end of the vortex

    l_s = np.linalg.norm(r_s) # distance between the induction point and the
                              # start of the vortex
    l_e = np.linalg.norm(r_e) # distance between the induction point and the
                              # end of the vortex
    l_v = np.linalg.norm(r_v) # length of the vortex

    h = np.linalg.norm(np.cross(r_v, r_s))/l_v # shortest distance between the
    # control point and an infinite extension of the vortex filament
    if h <= 1e-10: # the control point lies too close (for stability reasons)
                   # normal to the vortex line
        # todo incorporate solid body rotation
        return np.zeros(3) # for now assume there is no induction
    e_i = np.cross(r_v, r_s)/(h*l_v) # unit vector of the direction of induced
                                     #velocity
    return e_i/(4*np.pi*h*l_v)*(np.dot(r_v, (r_s/l_s-r_e/l_e)))
```