Derivation of the induced velocities of an infinite and finite line vortex

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1 Goal

This report aims at providing a step-by-step derivation of the induced velocities of an infinite and finite line vortex. Information about the final induced velocities of an infinite line vortex is readily available online, however, no thorough derivation (not for the infinite nor the finite case) could be found by the author.

First, some prerequisites are stated. Then, the induced velocity from an infinite straight-line vortex is derived. Based on that, the induced velocity for a finite straight-line vortex is derived. In the end, an example Python code is provided.

2 Prerequisites

Throughout the derivation, boldface $\boldsymbol{r}$ denote vectors, two successive vectors $\boldsymbol{r_1 r_2}$ mean a dot product, and $|\cdot|$ mean the Euclidian norm. Besides fundamental analysis, two concepts and an additional equation play an important role in the derivation. These are:

- **Dot product as directional filter:** The dot product of vector $\boldsymbol{r}$ and a unit vector $\boldsymbol{n}$, $|\boldsymbol{n}| = 1$, yields the distance $\boldsymbol{r}$ is pointing in the same direction as $\boldsymbol{n}$.
- **Cross product as area of parallelogram:** The magnitude of the cross product of two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ equals the area A of a parallelogram spanned by $\boldsymbol{a}$ and $\boldsymbol{b}$. $A = \boldsymbol{a} \times \boldsymbol{b}$
- **The area of a parallelogram:** The area of a parallelogram is equal to the base length times its height, see fig. 1 with $A = ht$.

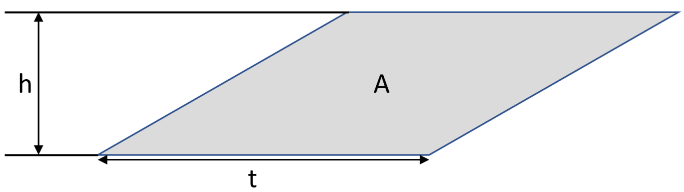

Figure 1: Area of a parallelogram

3 Infinite straight-line vortex

The induced velocity $\boldsymbol{v}_i$ of any vortex with constant circulation Γ can be calculated using the Biot-Savart law as

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi} \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3} \tag{1}$$

With an infinitesimal vortex element $\boldsymbol{dl}$ and the vector $\boldsymbol{r}$ pointing from the core of the infinitesimal vortex element (if it is a line vortex) to the induction point. The capital V over which is integrated denotes the whole vortex. Assuming a straight-line vortex, the sketch from fig. 2 can be done. It contains the induction point $\boldsymbol{x_p}$, the orange vortex, the normal distance h between the vortex and the induction point, a vortex vector element $\boldsymbol{dl}$, a vector $\boldsymbol{r}$ pointing from the middle of the vortex vector element to the induction point, the distance l between the middle of the vortex vector element and the intersection with the h line, the angle φ between the h line and the vector $\boldsymbol{r}$, and an angle $d\varphi$ spanning the vortex vector element. This view is normal to the plane spanned by the vortex line and the control point. The idea to introduce an angle comes from an MIT lecture slide on fluid mechanics: https://web.mit.edu/16.unified/www/SPRING/fluids/Spring2008/LectureNotes/f06.pdf. However, this source lacks the finite case as well as an easy-to-follow derivation.

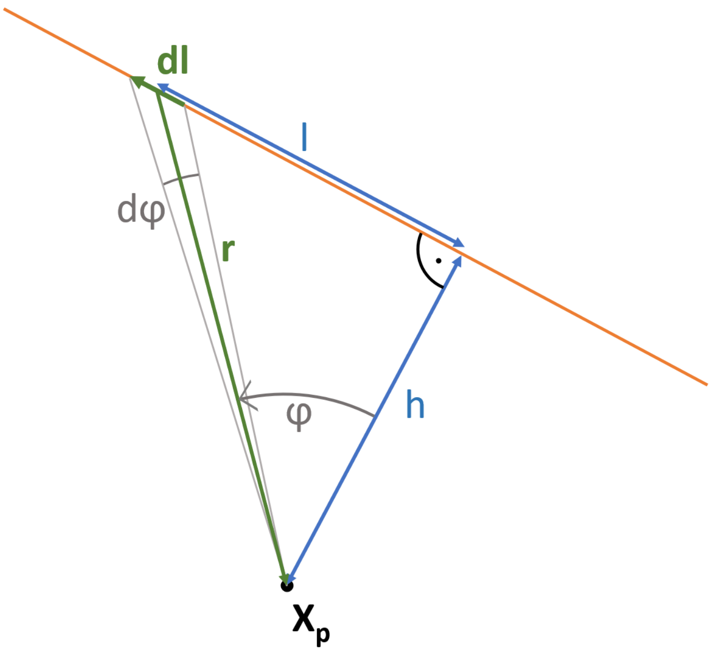

Figure 2: A vortex element on a straight vortex line

First, the equation for the induction of an infinite straight-line vortex will be derived. Hence, the focus lies on the integral

$$\boldsymbol{v_i} \propto \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \tag{2}$$

In total, there are 2 unknowns currently in eq. (2): the length $|\boldsymbol{dl}|$ and the vector $\boldsymbol{r}$, from which $|\boldsymbol{r}|^3$ can be obtained. The direction of the vortex has to be known. Therefore, $\boldsymbol{dl} = |\boldsymbol{dl}|\, \boldsymbol{e_v}$, with the unit vector in the direction of the vortex $\boldsymbol{e_v}$, is known as soon as dl is known.

First, let us examine $|\boldsymbol{dl}|$. Assuming (for now, derivation will follow later) a known h, the right triangle of fig. 2 brings

$$l = h \tan(\varphi) \tag{3}$$

The same logic can be used for $|\boldsymbol{dl}|$. The tip of $\boldsymbol{dl}$ is at the angle $\varphi + d\varphi/2$, while the root of $\boldsymbol{dl}$ is at the angle $\varphi - d\varphi/2$. The length from the line h to the tip of $\boldsymbol{dl}$ minus the length from the line h to the root of $\boldsymbol{dl}$, which is exactly $|\boldsymbol{dl}|$, becomes

$$|\boldsymbol{dl}| = h\left(\tan\left(\varphi + \frac{d\varphi}{2}\right) - \tan\left(\varphi - \frac{d\varphi}{2}\right)\right) \tag{4}$$

Using basic trigonometric angle sum identities for $\tan()$, $\cos()$, and $\sin()$ shows

$$\tan(\alpha+\beta) = \frac{\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)} \tag{5}$$

which can now be applied to eq. (4):

$$dl = h\left(\frac{\sin(\varphi)\cos\left(\frac{d\varphi}{2}\right) + \cos(\varphi)\sin\left(\frac{d\varphi}{2}\right)}{\cos(\varphi)\cos\left(\frac{d\varphi}{2}\right) - \sin(\varphi)\sin\left(\frac{d\varphi}{2}\right)} - \frac{\sin(\varphi)\cos\left(\frac{d\varphi}{2}\right) - \cos(\varphi)\sin\left(\frac{d\varphi}{2}\right)}{\cos(\varphi)\cos\left(\frac{d\varphi}{2}\right) + \sin(\varphi)\sin\left(\frac{d\varphi}{2}\right)}\right) \tag{6}$$

The simplifications $\cos\left(\frac{d\varphi}{2}\right) = 1$ and $\sin(d\varphi/2) = d\varphi/2$ then result in

$$dl = h\left(\frac{\sin(\varphi) + \cos(\varphi)\frac{d\varphi}{2}}{\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}} - \frac{\sin(\varphi) - \cos(\varphi)\frac{d\varphi}{2}}{\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}}\right) \tag{7}$$

Bringing both summands inside the parenthesis on the same denominator yields

$$dl = h\left(\frac{\left(\sin(\varphi) + \cos(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}\right) - \left(\sin(\varphi) - \cos(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}\right)}{\left(\cos(\varphi) - \sin(\varphi)\frac{d\varphi}{2}\right)\left(\cos(\varphi) + \sin(\varphi)\frac{d\varphi}{2}\right)}\right) \tag{8}$$

The terms in eq. (8) now have to be multiplied by one another. While doing so, terms with $d\varphi^2$ can be neglected. The result will be

$$dl = h\left(\frac{\sin(\varphi)\cos(\varphi) + \sin^2(\varphi)\frac{d\varphi}{2} + \cos^2(\varphi)\frac{d\varphi}{2} - \sin(\varphi)\cos(\varphi) + \sin^2(\varphi)\frac{d\varphi}{2} + \cos^2(\varphi)\frac{d\varphi}{2}}{\cos^2(\varphi) + \sin(\varphi)\cos(\varphi)\frac{d\varphi}{2} - \sin(\varphi)\cos(\varphi)\frac{d\varphi}{2}}\right) \tag{9}$$

$$= h\left(\frac{\left(\sin^2(\varphi) + \cos^2(\varphi)\right)d\varphi}{\cos^2(\varphi)}\right) \tag{10}$$

$$dl = \frac{h}{\cos^2(\varphi)}d\varphi \tag{11}$$

With that, $\boldsymbol{dl}$ is known by $\boldsymbol{dl} = |\boldsymbol{dl}|\,\boldsymbol{e_v}$ (again with the unit vector of the direction of the vortex $\boldsymbol{e_v}$).

Now we turn onto $\boldsymbol{dl} \times \boldsymbol{r}$. First, rewrite it into

$$\boldsymbol{dl} \times \boldsymbol{r} = |\boldsymbol{dl}|\,\boldsymbol{e_v} \times \boldsymbol{r} = \frac{h}{\cos^2(\varphi)}d\varphi\,(\boldsymbol{e_v} \times \boldsymbol{r}) \tag{12}$$

Leaving us with an unknown term $\boldsymbol{e_v} \times \boldsymbol{r}$. However, its magnitude is the area of the parallelogram spanned by $\boldsymbol{e_v}$ and $\boldsymbol{r}$. This area is the same as the height h multiplied with the base length $|\boldsymbol{e_v}| = 1$. Therefore, the value of this area is always h, regardless of where the point $\boldsymbol{r}$ is based on the vortex. Furthermore, since both $\boldsymbol{r}$ and $\boldsymbol{e_v}$ always lie in the plane that we are looking at, the direction of the vector $\boldsymbol{e_v} \times \boldsymbol{r}$ is always the same: normal to the plane (in this case coming out of the plane since $\boldsymbol{x_p}$ always lies on the left of the vortex when following $\boldsymbol{dl}$). In summary, we've just shown that the resulting vector from $\boldsymbol{e_v} \times \boldsymbol{r}$ has a constant magnitude h and a constant direction:

$$\boldsymbol{e_v} \times \boldsymbol{r} = h\boldsymbol{e_i} \tag{13}$$

The unit vector $\boldsymbol{e_i}$ is the direction of the induced velocities at point $\boldsymbol{x_p}$ and is currently unknown to us but just like h, we will find expressions for that later.

The last thing from eq. (2) we have not yet looked at is $|\boldsymbol{r}|^3$. From the right triangle, it quickly follows that

$$r = \frac{h}{\cos(\varphi)} \tag{14}$$

and with that the integral becomes

$$\boldsymbol{dl} \times \boldsymbol{r} = df\boldsymbol{e_v} \times \boldsymbol{r} = \frac{h}{\cos^2(\varphi)} d\varphi h\boldsymbol{e_i} = \frac{h^2}{\cos^2(\varphi)} \boldsymbol{e_i} d\varphi \tag{15}$$

$$r^3 = \left(\frac{h}{\cos(\varphi)}\right)^3 \tag{16}$$

$$\boldsymbol{v_i} \propto \int\limits_V \frac{\boldsymbol{dl} \times \boldsymbol{r}}{r^3} = \int\limits_{\varphi_V} \frac{\cos(\varphi)}{h} d\varphi \boldsymbol{e_i} \tag{17}$$

With φ_V being all angles φ belonging to the vortex. Equation (17) can be readily integrated, noting that neither h nor $\boldsymbol{e_i}$ are dependent on φ.

$$\boldsymbol{v_i} \propto \frac{1}{h} \sin(\varphi)\Big|_{\varphi_V} \boldsymbol{e_i} = \frac{1}{h}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right) \boldsymbol{e_i} = \frac{2}{h}\boldsymbol{e_i} \tag{18}$$

For an infinite vortex line, φ_V spans from $\varphi_{Vs} = -\pi/2$ to $\varphi_{Ve} = pi/2$ with. The last two unknowns for the induced velocities of the infinite straight-line vortex are h and $\boldsymbol{e_i}$. Those are now derived.

As stated and used before, h stands in direct connection to the area of the parallelogram spanned by the vector $\boldsymbol{r}$ and $\boldsymbol{dl}$. Since $\boldsymbol{r}$ is any vector pointing from the vortex core to the control point[1], we can define an arbitrary point $\boldsymbol{x_{vs}}$ on the vortex core (which has to be known) and get

$$\boldsymbol{r_s} = \boldsymbol{x_p} - \boldsymbol{x_{vs}} \tag{19}$$

And instead of $\boldsymbol{dl}$ we can use any vector $\boldsymbol{r_v}$ along the vortex core which points in the same direction as the vorticity vector (if it was the opposite direction, the induced velocity vector $\boldsymbol{v_i}$ would be negative of its true values). Hence, we take a second point $\boldsymbol{x_{ve}}$ on the vortex core and get

$$\boldsymbol{r_v} = \boldsymbol{x_{ve}} - \boldsymbol{x_{vs}}\,, \quad \boldsymbol{r_v} = |\boldsymbol{r_v}|\,\boldsymbol{e_v} \tag{20}$$

With the unit vector of the direction of the vortex $\boldsymbol{e_v}$ and the length $|\boldsymbol{r_v}|$ that $\boldsymbol{r_v}$ is pointing along the vortex. The area A of the parallelogram spanned by $\boldsymbol{r_s}$ and $\boldsymbol{r_v}$ ($\boldsymbol{r_v}$ represents the base of the parallelogram) can now be calculated in two different ways

$$A = |\boldsymbol{r_s} \times \boldsymbol{r_v}| \tag{21}$$

$$A = h\,|\boldsymbol{r_v}| \tag{22}$$

Combining both equations yields

$$h = \frac{|\boldsymbol{r_s} \times \boldsymbol{r_v}|}{|\boldsymbol{r_v}|} \tag{23}$$

[1] Remember that we showed that $\boldsymbol{r}$ does not influence the area A.

Again, h is the normal distance from the vortex core to the control point. Hence, it is very valuable in figuring out whether the control point lies directly on the extension of the vortex core or very close to it, where a solid body rotation might be a better assumption than an irrotational vortex. Now, $\boldsymbol{e_i}$ (the direction of the induced velocities) is straightforward. Equation (13) already shows the necessary equation. Instead of $\boldsymbol{r}$ we can use $\boldsymbol{r_s}$ because $\boldsymbol{r}$ has no influence on the result of the cross product as long as it is pointing from the vortex core to the induction point. Recall the paragraph above eq. (13) for that. In the end, we get

$$\boldsymbol{e_v} \times \boldsymbol{r_s} = h\boldsymbol{e_i} \quad \Big| \cdot |\boldsymbol{r_v}| \tag{24}$$

$$|\boldsymbol{r_v}|\, \boldsymbol{e_v} \times \boldsymbol{r} = |\boldsymbol{r_v}|\, h\boldsymbol{e_i} \tag{25}$$

$$\boldsymbol{r_v} \times \boldsymbol{r_s} = |\boldsymbol{r_v}|\, h\boldsymbol{e_i} \tag{26}$$

$$\boldsymbol{e_i} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_v}|\, h} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_v}| \frac{|\boldsymbol{r_s} \times \boldsymbol{r_v}|}{|\boldsymbol{r_v}|}} \tag{27}$$

$$\boldsymbol{e_i} = \frac{\boldsymbol{r_v} \times \boldsymbol{r_s}}{|\boldsymbol{r_s} \times \boldsymbol{r_v}|} \tag{28}$$

Where eq. (27) used eq. (23) and with that, everything is known for the induced velocities of the infinite straight-line vortex.

4 Finite straight-line vortex

So far, we have not done unnecessary work in terms of finding an expression for the induction of the finite straight-line vortex. The only changes now are concerning φ_V from eq. (18). For that, consider fig. 3. The solid orange line shows the finite vortex, which is bound by the start of the

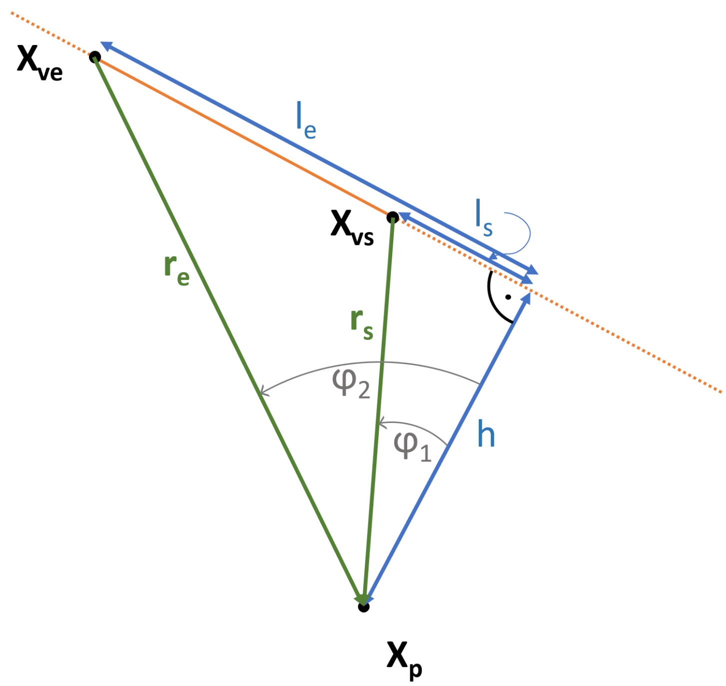

Figure 3: Sketch of a finite straight-line vortex

vortex $\boldsymbol{x_{vs}}$ and the end of the vortex $\boldsymbol{x_{ve}}$. These names are not by coincidence the same as above but show that we can use the expressions for $\boldsymbol{e_i}$ and h from the infinite vortex case. Furthermore, we have a normal distance l_s from the h line to the start of the vortex and a normal distance l_e

to the end of the vortex. Lastly, there are the vectors $\boldsymbol{r_s}$ and $\boldsymbol{r_e}$ pointing from the start and end of the vortex to the induction point, respectively. The vortex vector $\boldsymbol{r_v}$ pointing from $\boldsymbol{x_{vs}}$ to $\boldsymbol{x_{ve}}$ is not drawn due to clarity reasons. Now, we want to know the angles φ_1 and φ_2 (which we already knew for the infinite vortex case).

There are many ways of calculating the φ's since we already know two sides and one angle in the right triangles of fig. 3, rendering the triangles fully determined already. However, one idea is to find equations for which the φ's are in $\sin()$ functions, as we could then use that expression directly in eq. (18). Looking at the right triangles we find

$$\sin(\varphi_1) = \frac{l_s}{|\boldsymbol{r_s}|} \tag{29}$$

$$\sin(\varphi_2) = \frac{l_e}{|\boldsymbol{r_e}|} \tag{30}$$

Which leaves l_s and l_e open for calculation. Again, trigonometric functions or Pythagoras' theorem could be applied, but there exists a computationally easier solution. This solution is solely based on multiplication and division, not on evaluating square roots or trigonometric functions. The length l_s is the distance of $\boldsymbol{r_s}$ pointing parallel to the vortex. With the unit vector $\boldsymbol{e_v}$ pointing in the direction of the vortex, we would have

$$l_s = -\boldsymbol{r_s}\boldsymbol{e_v} \tag{31}$$

$$l_e = -\boldsymbol{r_e}\boldsymbol{e_v} \tag{32}$$

With the minus because the vectors $\boldsymbol{r_s}$ and $\boldsymbol{r_e}$ are pointing in the opposite direction of $\boldsymbol{e_v}$ (when viewing the direction of $\boldsymbol{e_v}$), but l_s is defined positive for the case of opposite direction. We can multiply both sides with the length of the vortex $|\boldsymbol{r_v}|$ to get rid of the unit vector $\boldsymbol{e_v}$

$$l_s\,|\boldsymbol{r_v}| = -\boldsymbol{r_s}\boldsymbol{e_v}\,|\boldsymbol{r_v}| = -\boldsymbol{r_s}\boldsymbol{r_v} \tag{33}$$

$$l_e\,|\boldsymbol{r_v}| = -\boldsymbol{r_e}\boldsymbol{e_v}\,|\boldsymbol{r_v}| = -\boldsymbol{r_e}\boldsymbol{r_v} \tag{34}$$

$$\tag{35}$$

Which can be inserted into eqs. (29) and (30) to get

$$\sin(\varphi_1) = -\frac{\boldsymbol{r_s}\boldsymbol{r_v}}{|\boldsymbol{r_s}|\,|\boldsymbol{r_v}|} \tag{36}$$

$$\sin(\varphi_2) = -\frac{\boldsymbol{r_e}\boldsymbol{r_v}}{|\boldsymbol{r_e}|\,|\boldsymbol{r_v}|} \tag{37}$$

And with that the induced velocities of the finite straight-line vortex are derived:

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi}\int\limits_V \frac{\boldsymbol{dl}\times\boldsymbol{r}}{r^3} \tag{38}$$

$$= \frac{\Gamma}{4\pi h}\sin(\varphi)\Big|_{\varphi_1}^{\varphi_2}\boldsymbol{e_i} \tag{39}$$

$$= \frac{\Gamma}{4\pi h}\left(\sin(\varphi_2) - \sin(\varphi_1)\right)\boldsymbol{e_i} \tag{40}$$

$$= \frac{\Gamma}{4\pi h}\left(-\frac{\boldsymbol{r_e}\boldsymbol{r_v}}{|\boldsymbol{r_e}|\,|\boldsymbol{r_v}|} + \frac{\boldsymbol{r_s}\boldsymbol{r_v}}{|\boldsymbol{r_s}|\,|\boldsymbol{r_v}|}\right)\frac{\boldsymbol{r_v}\times\boldsymbol{r_s}}{|\boldsymbol{r_s}\times\boldsymbol{r_v}|} \tag{41}$$

$$\boldsymbol{v_i} = \frac{\Gamma}{4\pi h\,|\boldsymbol{r_v}|}\left(\boldsymbol{r_v}\left(\frac{\boldsymbol{r_s}}{|\boldsymbol{r_s}|} - \frac{\boldsymbol{r_e}}{|\boldsymbol{r_e}|}\right)\right)\frac{\boldsymbol{r_v}\times\boldsymbol{r_s}}{|\boldsymbol{r_s}\times\boldsymbol{r_v}|} \tag{42}$$

5 Example python code

The code can also be found on GitHub.

```python
import numpy as np

def vortex_induction_factor(vortex_start: np.ndarray,
                            vortex_end: np.ndarray,
                            induction_point: np.ndarray) -> np.ndarray:
    """
    This function calculates the induction at a point 'induction_point' from a straight vortex line between the
    two points 'vortex_start' and 'vortex_end' for a unity circulation. The returned value is a vector of induced
    velocities.
    :param vortex_start: numpy array of size (3,)
    :param vortex_end: numpy array of size (3,)
    :param induction_point: numpy array of size (3,)
    :return:
    """
    r_s = induction_point-vortex_start  # vector from induction point to the start of the vortex
    r_e = induction_point-vortex_end  # vector from the induction point to the end of the vortex
    r_v = vortex_end-vortex_start  # vector representing the vortex

    l_s = np.linalg.norm(r_s)  # distance between the induction point and the start of the vortex
    l_e = np.linalg.norm(r_e)  # distance between the induction point and the end of the vortex
    l_v = np.linalg.norm(r_v)  # length of the vortex

    h = np.linalg.norm(np.cross(r_v, r_s))/l_v  # shortest (normal) distance between the control point and an
    # infinite extension of the vortex filament
    if h <= 1e-10:  # the control point lies too close normal to the vortex line
        # todo handle control points that lie very close to the vortex core
        return np.zeros(3)  # for now assume no induction
    e_i = np.cross(r_v, r_s)/(h*l_v)  # unit vector of the direction of induced velocity
    return e_i/(4*np.pi*h*l_v)*(np.dot(r_v, (r_s/l_s-r_e/l_e)))
```