Time-Dependent Two-Nanoparticle Network

Orthodox Coulomb-Blockade Treatment with an AC Source Drive

0 Regime of Validity & Road-Map ★

1. **Orthodox picture.** Sequential, *incoherent* single-electron tunnelling; no cotunnelling.
2. **Deep-blockade limit**

$$eU_0,\ k_BT \ll E_C \equiv \frac{e^2}{2C_\Sigma}, \qquad \omega_0 \ll \frac{1}{R_{ij}C_\Sigma}, \tag{1}$$

   so islands behave as open-circuit capacitors during the RF cycle; Section **??** then gives closed-form harmonic rates.

3. **Beyond the blockade.** Violating (**??**) (large bias, high T, or fast drive) allows real current to flow *during* the cycle. The same rate formula still applies, but island potentials $\phi_i(t)$ must be obtained self-consistently using the Floquet master-equation framework (Section **??**).

1 System Layout

Two metallic nanoparticles (**NP 1**, **NP 2**) are connected in series between a driven source **S** and a grounded drain **D**, atop a static back-gate **G**. Direct tunnelling to the gate is forbidden.

Node	Potential	Excess charge
Source (S)	$U_S(t) = U_0 \cos \omega_0 t$	–
NP 1	$\phi_1(t)$	$Q_1(t)$
NP 2	$\phi_2(t)$	$Q_2(t)$
Drain (D)	$U_D = 0$	–
Gate (G)	U_G (static)	–

Every neighbouring pair carries a geometric capacitance C_{ij} and a tunnel resistance R_{ij}.

2 Electrostatics

Approximate self- and mutual capacitances for metallic spheres (radii r_i, centre spacing d):

$$C_i \simeq K_s r_i, \qquad C_{ij} \simeq K_m \frac{r_i r_j}{r_i + r_j + d}, \tag{2}$$

with geometry factors K_s, K_m.

Charge–potential relation for the two floating islands:

$$\boldsymbol{Q} = \mathbf{C}\,\boldsymbol{\phi}, \qquad \mathbf{C} = \begin{pmatrix} C_{11} & -C_{12} \\ -C_{12} & C_{22} \end{pmatrix}, \tag{3}$$

where $C_{11} = C_{S1} + C_{12} + C_1$ and $C_{22} = C_{2D} + C_{12} + C_2$. The inverse reads

$$\mathbf{C}^{-1} = \frac{1}{\Delta} \begin{pmatrix} C_{22} & C_{12} \\ C_{12} & C_{11} \end{pmatrix}, \qquad \Delta = C_{11}C_{22} - C_{12}^2. \tag{4}$$

Thus $E_C \sim e^2/(2\,\mathrm{tr}\,\mathbf{C})$.

3 Free-Energy Cost for One Tunnelling Event

$$\boxed{\Delta F_{i\to j} = e(\phi_i - \phi_j) + \frac{e^2}{2}\bigl(C^{-1}_{ii} + C^{-1}_{jj} - 2C^{-1}_{ij}\bigr)}. \tag{5}$$

For island–electrode transfer (infinite reservoir capacitance):

$$\Delta F_{i\to E} = e(\phi_i - U_E) + \frac{e^2}{2}C^{-1}_{ii}. \tag{6}$$

4 Exact Stochastic Dynamics

With configuration probability $P(\boldsymbol{n}, t)$ ($\boldsymbol{n} = (n_1, n_2)$) the master equation is

$$\frac{\mathrm{d}P(\boldsymbol{n}, t)}{\mathrm{d}t} = \sum_{\boldsymbol{m}\neq\boldsymbol{n}} \bigl[\Gamma_{\boldsymbol{m}\to\boldsymbol{n}} P(\boldsymbol{m}, t) - \Gamma_{\boldsymbol{n}\to\boldsymbol{m}} P(\boldsymbol{n}, t)\bigr]. \tag{7}$$

Orthodox rate:

$$\boxed{\Gamma_{ij}(\Delta F) = \frac{\Delta F}{e^2 R_{ij}} \frac{1}{1 - \mathrm{e}^{-\Delta F/k_B T}}.} \tag{8}$$

5 From Master Equation to Kirchhoff's Law

Define $\langle Q_i\rangle = e\langle n_i\rangle$. Taking the first moment of (**??**) and regrouping gives the ensemble-averaged tunnel current

$$I_{T,ij} = e\sum_{\boldsymbol{n}} \bigl[\Gamma_{j\to i} - \Gamma_{i\to j}\bigr] P(\boldsymbol{n}, t). \tag{9}$$

Because $\langle Q_i\rangle = \sum_k C_{ik}\langle\phi_i - \phi_k\rangle$, we arrive at

$$\boxed{\sum_k C_{ik}\dot{\langle}\phi_i - \phi_k\rangle + \sum_j I_{T,ij}(t) = 0.} \tag{10}$$

When may KCL be treated as continuous?

Criterion	Physical meaning	Consequence
Many hops per window Δt	Shot noise $\sim 1/\sqrt{N}$ small	Currents look smooth; (**??**) reliable.
$\Delta t \gg R_{ij}C_\Sigma$ (or $k_BT, eU_0 \gg \hbar\omega$)	Fast fluctuations averaged	Potentials treatable as continuous.
Linear/weakly nonlinear regime	Higher moments negligible	KCL with *mean* currents closes equations.

For single-electron pumps or ns resolution, solve the full stochastic dynamics; KCL then holds trajectory-by-trajectory with Dirac-δ spikes.

6 Circuit-Averaged KCL Equations

Applying (**??**) to node 1 yields

$$C_{S1}\dot{(}U_S - \phi_1) + C_{12}\dot{(}\phi_2 - \phi_1) + C_1\dot{(}-\phi_1) + I_{T,S1} + I_{T,12} = 0, \tag{11}$$

with an analogous equation for node 2.

7 Deep-Blockade Analytic Solution

Assuming (**??**), tunnelling currents vanish in (**??**) and potentials lock to a single cosine:

$$\mathbf{C}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}C_{S1}U_0\cos\omega_0 t\\ 0\end{pmatrix} \implies \quad \phi_i(t) = \alpha_i U_0 \cos\omega_0 t, \tag{12}$$

with

$$\alpha_1 = \frac{C_{S1}C_{22}}{\Delta}, \qquad \alpha_2 = \frac{C_{S1}C_{12}}{\Delta}. \tag{13}$$

Define $\Delta F_{ij}(t) = A_{ij} + B_{ij}\cos\omega_0 t$ where $A_{ij} \sim E_C$ and $B_{ij} = e(\alpha_i - \alpha_j)U_0$. Using $1/(1-e^{-x}) = \sum_{\ell=0}^{\infty} e^{-\ell x}$ and $e^{-z\cos\theta} = \sum_{m=-\infty}^{\infty}(-1)^m I_m(z)e^{im\theta}$, one finds

$$\boxed{\Gamma_{ij}^{(n)} = -\frac{1}{e^2R_{ij}}\sum_{\ell=0}^{\infty} e^{-\ell A_{ij}/k_BT}\Big[A_{ij}(-1)^n I_n(\ell\beta_{ij}) + \tfrac{B_{ij}}{2}\big((-1)^{n-1}I_{n-1} - (-1)^{n+1}I_{n+1}\big)\Big]}, \tag{14}$$

with $\beta_{ij} = B_{ij}/k_BT$.

Physical implications

- **Photon-assisted tunnelling**: ℓ counts absorbed/emitted quanta; I_n distributes them into harmonics.
- **Odd/even rule**: Symmetric drive with zero DC offset suppresses even n.
- **Sweet spot**: Richest spectrum for $E_C/k_BT \sim eU_0/k_BT \sim 1$.
- **Large-island limit**: If $r_2 \gg r_1$, only S–1 and 1–2 junctions contribute non-zero B_{ij}, mapping to a single-electron transistor.

8 Beyond the Blockade: Floquet Master Equation

When (**??**) is violated, write each periodic quantity as a Fourier series, e.g. $\phi_i(t) = \sum_m \phi_i^{(m)} e^{im\omega_0 t}$. Substituting into KCL and the master equation yields an infinite set of linear equations for $\phi_i^{(m)}$ and $P_{\boldsymbol{n}}^{(m)}$. Numerical truncation reproduces (**??**) in the blockade limit and remains valid outside it.

9 References

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2. H. Grabert and M. H. Devoret (eds.), *Single Charge Tunneling* (Plenum, 1992).

3. J. Tien and J. Gordon, Phys. Rev. **129**, 647 (1963).