EBA3500 Fall 2021
Exercises 1: Matrices and Python

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This is not a course in mathematics, but statistics and machine learning builds heavily on mathematics. Linear algebra is especially important. I do not expect you prove the results rigourosly. But I expect you to try to *convince yourself*. You should feel reasonably confident that you understand what's going on. To get to this point, do not stare blankly on the paper! Try to write down concrete example matrices, see what happens. Please use Python too!

Mathematical concepts can only be understood throuh exploration and experimentation. Even John von Neumann (widely regarded as the greatest genius of all time – I'm not exaggerating!) reportedly once said "Young man, in mathematics you don't understand things. You just get used to them."

1. Working with multiplication.

   (a) Verify that $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 3 \\ 2 & 1 & 3 \\ 4 & 4 & 3 \end{bmatrix}$ using the definition of matrix multiplication from the lectures.

   (b) Calculate $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

   (c) When a, b are numbers, it's always true that $ab = ba$. Is it always the case that $AB = BA$? (*Hint:* Look at the two previous exercises.)

   (d) Let $\mathbf{O}$ be the $m \times n$ matrix consisting only of 0s and A be an $m \times n$ matrix. What is $A\mathbf{O}$? For instance, calculate

   $$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad A\mathbf{O} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (e) Remember that the ith unit vector $\mathbf{e}_i$ is the vector of only 0s with 1 in the ith position. Calculate

   $$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{e}_1, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{e}_2, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix} \mathbf{e}_3.$$

   (*Hint:* Remember that you may use Python!)

2. Working with the dot product.

   (a) Calculate the dot product between $x = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix}$. Use the definition of matrix multiplication to calculate $x^T y$.

   (b) Convince yourself that $x \cdot y = x^T y$ when x, y are vectors in $\mathbb{R}^n$. (*Hint:* Remember that vectors are column vectors! Use the definition of matrix multiplication.)

   (c) Calculate

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   (d) Calculate $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ using the definition of matrix multiplication. (*Hint:* Do the transpose first!)

   (e) Let X be an $n \times m$ matrix and y a vector in $\mathbb{R}^n$. Convince yourself that $X^T y = \begin{bmatrix} x_1 \cdot y \\ x_2 \cdot y \\ \vdots \\ x_m \cdot y \end{bmatrix}$, where x_j is the jth column of X. This corresponds to the common definition of matrix multiplication in textbooks. (*Hint:* Use the definition of matrix multiplication.)

3. Let A be a matrix and $\circ$ the element-wise product.

   (a) Is there a matrix so that $A \circ J = A = J \circ A$ for all A? If so, what is it? (*Hint:* What happens if A is a real number? Remember that you can experiment with Python!)

   (b) Let A be a matrix. Is there always a matrix B so that $A \circ B = J$? If not, what conditions must A satisify for this to be the case? Identify B using the notation b_{ij}. (*Hint:* What happens if A is a real number?)

   (c) Let's call A element-wise invertible if $A \circ B = J$ for some B and invertible if $AC = I$ for some C. Are all element-wise invertible matrices invertible? How about the other way around? (*Hint:* Consider I, the identity matrix under matrix multiplication, and J, the identity matrix under element-wise multiplication. Again, try to experiment with Python.)

4. Working with transposes.

   (a) Verify the three transposition facts in Python using the matrices

$$A = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 7 & -2 \\ -1 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

(b) When A is an $n \times n$ matrix and x a vector in $\mathbb{R}^n$, the function $x^T Ax$ is called a *quadratic form*. Write out $x^T Ax$ for the matrix A above and $x = (x_1, x_2, x_3)$.

5. Working with inverse matrices.

   (a) Does $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ have an inverse? (*Hint:* A matrix has an inverse if and only if $Ax = b$ has a unique solution x for every b. Also, try to use Python.)

   (b) What is "the inverse of the inverse", $(A^{-1})^{-1}$? (*Hint*: You can use Python to calculate the inverse of the inverse for a couple of example matrices. In addition, remember that he inverse is *unique*. Finally, what is $1/(1/a)$ when a is a number?)