60615A : Decision Analysis
Session 5

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Example : House insurance

- A client calls to receive an estimate of the annual premiums he would have to pay for this insurance, he lives 10 km from downtown, his land is 5 000 squared feet and he doesn't possess a pool.
- Generate 1000 possible scenarios of the profits/losses realized by the insurance company after the first year if the parties agreed that the client would pay an annual insurance premium of 2000$.
- Is this a good deal for the insurance company ?
- Is this a good deal for the client ?
- See Excel file.

Bayesian inference

- We wish to model $P(Z)$ using $\{z_1, z_2, ..., z_M\}$
- Hypothesis followed by the Bayesian approach :
  1. Z follows a parametric form : $f(z;\theta)$
  2. Even before looking at our data, we have a subjective belief of the value of θ : i.e. $f(\theta)$
  3. We know that $f(\{z_1, z_2, ..., z_M\}|\theta) = \prod_{i=1}^M f(z_i;\theta)$
- Implications :
  1. Before seeing the data, $P(Z \in A) = \int P(Z \in A|\theta)f(\theta)d\theta = \int \int_A f(z;\theta)f(\theta)dzd\theta$
  2. After seeing the data, apply Bayes' theorem to determine $P(Z \in A|\mathcal{O})$, where $\mathcal{O} = \{z_1, z_2, ..., z_M\}$

$$f(\theta|\mathcal{O}) = \frac{f(\mathcal{O}|\theta)f(\theta)}{\int f(\mathcal{O}|\theta)f(\theta)d\theta} \qquad P(Z \in A|\mathcal{O}) = \int \int_A f(z;\theta)f(\theta|\mathcal{O})dzd\theta$$

where we exploit that Z is independent of $\mathcal{O}$ if θ is known

- Note : $f(\theta|\mathcal{O})$ is our belief of θ after studying $\mathcal{O}$

The conjugate prior of a distribution

- In general, it is difficult to compute $f(\theta|\mathcal{O})$ because of the integral

$$\int f(\mathcal{O}|\theta)f(\theta)d\theta = \int \left(\prod_{i=1}^{M} f(z_i;\theta)\right) f(\theta)d\theta$$

- If $f(\theta)$ is the conjugate prior for $f(z;\theta)$, then $f(\theta|\mathcal{O})$ takes the same parametric form as $f(\theta)$

Distribution	Conjugate prior	Parameters update
Bernoulli	Beta(α,β)	$\alpha' = \alpha + \sum_i z_i\ ,\ \beta' = \beta + \sum_i (1 - z_i)$
Poisson	Gamma(k,θ)	$k' = k + \sum_i z_i\ ,\ \theta' = \theta/(M\theta + 1)$
Exponential	Gamma(k,θ)	$k' = k + M\ ,\ \theta' = \theta/(1 + \theta \sum_i z_i)$
...	...	...

Objective of a Monte Carlo simulation

- In a problem with uncertainty, the value of our objective must be considered as a random variable.
  - Example : generated profit, project's success, magnitude of environmental impact, etc.
- Monte Carlo simulation allows one to study the stochastic nature of this random variable.
  - Estimation of the expected value : expected profit, expected impact, ...
  - Probability distribution : probability of success, distribution function, variance, percentiles, ...
- A Monte Carlo simulation enables the comparison of different alternatives.
- It does not directly enable the optimization of a decision.

Monte Carlo simulation procedure

1. Design the influence diagram.
2. Describe the influence relations associated to each box.
3. Choose the alternatives to evaluate.
   - Choose the function that maps influence variables to actions.
4. Randomly draw M scenarios of realizations of the random variables.
   - Start with the random variables that are not influenced and generate a realization according to their distribution.
   - For each variable that are only influenced by already computed variables, generate a realization according to the conditional distribution.
   - Repeat until a realization has been generated for all variables.
   - Evaluate the objective for this scenario of realization.
5. Analyze some statistics that are based on the empirical distribution obtained for the objective.
6. Estimate the level of confidence of each estimate to find out if more scenarios should be used.

Generating a discrete variable

- We want to simulate a random variable taking the values $\{z_1, z_2, ..., z_N\}$ with respective probabilities $\{p_1, p_2, ..., p_N\}$.
- Method : generate U uniformly on $[0, 1]$ and draw Z according to the table :

If $U \in [a, b]$	then $Z =$
$[\ 0\ ,\ p_1\ ]$	z_1
$]\ p_1\ ,\ p_1 + p_2\ ]$	z_2
$]\ p_1 + p_2\ ,\ p_1 + p_2 + p_3\ ]$	z_3
...	...
$]\ p_1 + ... + p_{n-1}\ ,\ 1\ ]$	z_n

Generating a continuous random variable

Inverse transform method (see Excel file) :

- Assumption : the cumulative distribution function $F : \mathbb{R} \to [0, 1]$ is continuous and strictly increasing
- Method : if U is a uniformly distributed random variable over $[0, 1]$, then $V = F^{-1}(U)$ is distributed according to F
- Proof (AI) :

$$F_V(v) \quad = \quad P(F^{-1}(U) \leq v) = P(U \leq F(v)) = F(v)$$

- Example : exponential distribution,
  - $F(z) = (1 - \exp(-\lambda z))\mathbb{1}\{z \geq 0\}$
  - $V = F^{-1}(U) = -(1/\lambda)\ln(1 - U)$
- Example in Excel : «=norminv(rand() ;0 ;1)» generates a random value distributed according to the normal distribution (see Excel file)