60615A : Decision Analysis
Monte Carlo Simulation Part I

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Outline

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1. Statistical analysis
2. Evaluate the estimation error

Empirical distribution

- Definition : the empirical distribution is obtained by giving an equal probability to each values in a scenario set, called a sample.
- The empirical distribution based on the sample $Z_1, Z_2, ..., Z_M$ is

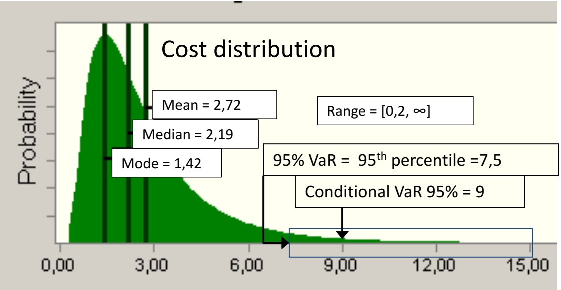
$$F_Z^M(z) = \frac{1}{M}\sum_{i=1}^{M} \mathbb{1}\{Z_i \leq z\}$$

  In fact, $F_Z^M(z)$ is the proportion of time we observe that Z_i is smaller or equal to z in the sample.
- The analysis of a Monte-Carlo simulation is done assuming that the empirical distribution captures the likelihood of all possible realizations.
- Example : see nuclear plant case study (Excel file)

Relevant statistics

There exists several measures useful to quantify risk :

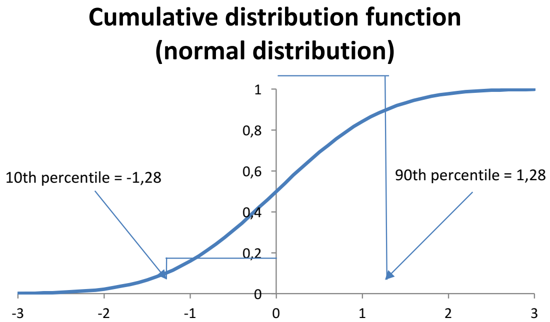
- Expected value
- Standard deviation
- Range
- Probability of reaching a target
- Percentile
- Value-at-risk
- Conditional value-at-risk
- Expected shortfall



Percentile / Value-at-risk / Conditional value-at-risk

- Definition : the n-th percentile is the smallest value below which we are assured that the random variable has more than $n\%$ chances of occurring.
  - If $F_Z(\cdot)$ is continuous (and strictly increasing), the n-th percentile is the value z for which $F_Z(z) = n/100$
- Definition : the value-at-risk for a level of confidence of α is the value of losses for which we believe we have a $\alpha \times 100\%$ chances of doing better.
  - If $F_Z(\cdot)$ is continuous and Z is a cost, $VaR_\alpha(Z)$ is the value z for which $F_Z(z) = \alpha$ (for a profit it is $F_Z(-z) = 1 - \alpha$)
- Definition : the conditional value-at-risk for a level of confidence of α is the expected value of losses conditional to obtaining a worst value than $VaR_\alpha(Z)$.
  - If Z is a cost, we can compute it with : $E[Z|Z \geq VaR_\alpha(Z)] = E[Z \cdot \mathbb{1}\{Z \geq VaR_\alpha(Z)\}]/(1-\alpha)$

Example



- 10-th percentile = -1,28 ; 90-th percentile = 1,28
- VaR-90% = 1,28 (wheter Z is a profit or an expense)
- CVaR-90% $= E[Z|Z \geq 1,28] \approx 1.75$ (if $Z =$ expense), $-E[Z|Z \leq -1,28] \approx 1.75$ (if Z is a profit)

Stochastic dominance

- Definition : the alternative x_A dominates stochastically the alternative x_B if for every level of performance, the alternative x_A has a higher chance of surpassing this level than the alternative x_B.
- Mathematically, if $g(x, Z) = \text{profit}$:

$$P(g(x_A, Z) \geq \alpha) \geq P(g(x_B, Z) \geq \alpha)\,,\ \forall\, \alpha \in \mathbb{R}$$
$$P(g(x_A, Z) \leq \alpha) \leq P(g(x_B, Z) \leq \alpha)\,,\ \forall\, \alpha \in \mathbb{R}$$

- Consequence of interest : every person that maximizes expected utility will agree on the fact that x_A dominates x_B.
- Can be verified by comparing the distribution functions.

Deterministic dominance

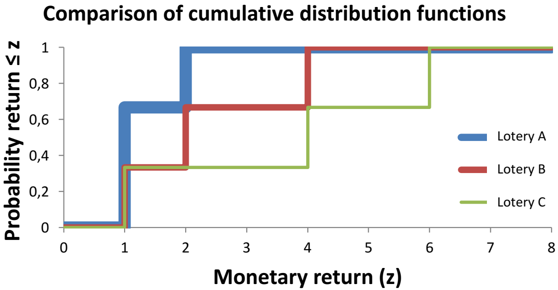
- Definition : the alternative x_A dominates deterministically the alternative x_B if we are guaranteed that the outcomes of x_A will be preferred to the outcomes of x_B under all possible realizations of the uncertain parameters.
- Mathematically, if $g(x, Z) = \text{profit}$:

$$P(g(x_A, Z) \geq g(x_B, Z)) = 1$$

- Sufficient condition : x_A dominates x_B if the most negative outcomes of the alternative x_A are preferred to the most positive outcomes of the alternative x_B.
- Can't always be verified by comparing the distribution functions.
- Determ. dominance implies stoch. dominance but not the reverse.

Example of Dominance

- Consider the 3 lotteries related to the roll of a dice :
  - Lottery A : (1-2) win 1$, (3-4) win 1$, (5-6) win 2$
  - Lottery B : (1-2) win 1$, (3-4) win 2$, (5-6) win 4$
  - Lottery C : (1-2) win 6$, (3-4) win 4$, (5-6) win 1$



- «A is determ. dominated by B but not by C» , «A & B are stoch. dominated by C»

Outline

Estimation of the expected value

- Let $Z_1, Z_2, ..., Z_M$ be the simulated values of the uncertain parameters of the problem.
- In fact, the values $g(x, Z_1), g(x, Z_2), ..., g(x, Z_M)$ are independent and identically distributed random variables since each is the result of a Monte-Carlo simulation.
- The Monte-Carlo estimator approximates $E[g(x, Z)]$ with

$$\hat{\theta}_M(x, Z) = \frac{1}{M}\sum_{i=1}^{M} g(x, Z_i)$$

- Note that the estimated value is a random variable : i.e., the values estimated by two analysts will usually not be the same.

Estimation of the expected value

The Monte-Carlo estimator is :

- Without bias : $E[\hat{\theta}_M(x,Z)] = E[g(x,Z)]$
- Convergent : $Var[\hat{\theta}_M(x,Z)] = Var[g(x,Z)]/M \rightarrow 0$ when $M \rightarrow \infty$
  I.e., the strong law of large numbers guarantees that $\hat{\theta}_M$ converges almost surely to $E[g(x,Z)]$.

$$P(\lim_{M\rightarrow\infty} |\hat{\theta}_M - E[g(x,Z)]| = 0) = 1$$

- Asymptotically normal : by the central limit theorem :

$$\frac{\hat{\theta}_M - E[g(x,Z)]}{\sigma/\sqrt{M}} \rightarrow \mathcal{N}(0,1) \text{ when } M \rightarrow \infty$$

where $\sigma^2 = Var[g(x,Z)]$

Confidence interval for the expected value (Excel file)

- Let $\hat{\theta}_M$ be a Monte-Carlo estimator of $E[g(x, Z)]$.
- Let $\hat{\sigma}^2_M$ be the unbiased MC estimator of $Var[g(x, Z)]$.

$$\hat{\sigma}^2_M = \frac{1}{M-1}\sum_{i=1}^{M}(g(x, Z_i) - \hat{\theta}_M)^2$$

- When the size of M is sufficiently large,

$$\hat{\theta}_M \approx \mathcal{N}(E[g(x, Z)], \hat{\sigma}^2_M/M)$$

- We can therefore estimate a confidence interval of level $1 - \alpha$

$$E[g(x, Z)] \in \left[\hat{\theta}_M - \Phi^{-1}(1 - \frac{\alpha}{2})\frac{\hat{\sigma}_M}{\sqrt{M}}, \hat{\theta}_M + \Phi^{-1}(1 - \frac{\alpha}{2})\frac{\hat{\sigma}_M}{\sqrt{M}}\right]$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution $\mathcal{N}(0, 1)$.

MC Quantile Estimator

- Definition : for every $\alpha \in (0,1)$, the MC estimator $\hat{z}_\alpha^M$ of the α-quantile of the distribution F_X is the α-quantile of the empirical distribution $F_Z^M(z)$.
- In practice :
  - Considering that $Z_{(1)}, Z_{(2)}, \ldots, Z_{(M)}$ is the ordered version of the sample $Z_1, Z_2, \ldots, Z_M$
  - We can measure
$$\hat{z}_\alpha^M = Z_{(\lceil M\alpha \rceil)}$$
where $\lceil x \rceil$ is the smallest superior integer to x.
- Confidence interval of level $1-\beta$:

$$Z_{(\lfloor M(\alpha-\Delta_\beta)\rfloor)} \le z_\alpha \le Z_{(\lceil M(\alpha+\Delta_\beta)\rceil+1)}$$

where $\Delta_\beta = \Phi^{-1}(1-\frac{\beta}{2})\sqrt{\alpha(1-\alpha)}/\sqrt{M}$

Confidence interval - Example

- Suppose the size of the sample is 10000, what are the indices of the ordered sample list that allow us to determine a 95% confidence interval for the 10th percentile ?
  Hint : $\Phi^{-1}(97.5\%) \approxeq 1.96$

- The margin of error on the index of this percentile is $\Delta = 1.96 \times \sqrt{0.1(1-0.1)/10000} = 0.0059$
- The minimum index is therefore $\lfloor 10000 \times (0.1 - 0.0059) \rfloor = 941$
- The maximum index is therefore $\lceil 10000 \times (0.1 + 0.0059) \rceil + 1 = 1060$