60615A : Decision Analysis
Monte Carlo Simulation Part I

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Outline

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Objective of a Monte Carlo simulation

- In a problem with uncertainty, the value of our objective must be considered as a random variable.
  - Example : generated profit, project's success, magnitude of environmental impact, etc.
- Monte Carlo simulation allows one to study the stochastic nature of this random variable.
  - Estimation of the expected value : expected profit, expected impact, ...
  - Probability distribution : probability of success, distribution function, variance, percentiles, ...
- A Monte Carlo simulation enables the comparison of different alternatives.
- It does not directly enable the optimization of a decision.

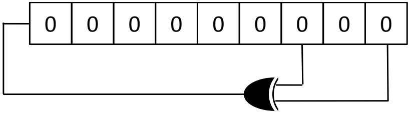
Monte Carlo simulation procedure

1. Design the influence diagram.
2. Describe the influence relations associated to each box.
3. Choose the alternatives to evaluate.
   - Choose the function that maps influence variables to actions.
4. Randomly draw M scenarios of realizations of the random variables.
   - Start with the random variables that are not influenced and generate a realization according to their distribution.
   - For each variable that are only influenced by already computed variables, generate a realization according to the conditional distribution.
   - Repeat until a realization has been generated for all variables.
   - Evaluate the objective for this scenario of realization.
5. Analyze some statistics that are based on the empirical distribution obtained for the objective.
6. Estimate the level of confidence of each estimate to find out if more scenarios should be used.

Outline

Generation of Bernoulli events ($p = 0.5$)

- Some devices really simulates this event through a physical phenomena (e.g. thermal noise).
- Our computers simulate it using a deterministic process.
- This is the difference between random and pseudo-random.
- Linear Feedback Shift Register (see Excel file) :
  - This system generates a sequence of 0's and 1's, such that we cannot distinguish the sequence from a random sequence (with 50% chance that the next value is 1).



Uniform distribution on the interval $[0, 1]$

«How to generate a uniform value between 0 and 1 using a quarter (25 cent coint) ?»

Iterative Process (see Excel file) :

- Divide the interval in two sub-intervals of the same size.
- Execute a fair coin toss : if tail, keep the left interval, else the right.
- After k tosses, the size of the interval is $1/2^k$.
- Stop when the resulting interval is small enough.
- Keep as a random value the average value on this interval.

Generating a discrete variable I

- We want to simulate a random variable taking the values $\{z_1, z_2, ..., z_N\}$ with respective probabilities $\{p_1, p_2, ..., p_N\}$.
- Method : generate U uniformly on $[0, 1]$ and draw Z according to the table :

If $U \in [a, b]$	then $Z =$
$[\ 0\ ,\ p_1\ ]$	z_1
$]\ p_1\ ,\ p_1 + p_2\ ]$	z_2
$]\ p_1 + p_2\ ,\ p_1 + p_2 + p_3\ ]$	z_3
...	...
$]\ p_1 + ... + p_{n-1}\ ,\ 1\ ]$	z_n

Generation of a discrete variable II (Additional Information)

- We want to simulate a random variable taking the values $\{z_1, z_2, ..., z_N\}$ with respective probabilities $\{p_1, p_2, ..., p_N\}$.
- Theorem : if U is uniform on $[0, 1]$, then

$$V = \sum_{i=1}^{N} z_i \mathbb{1}\left\{\sum_{j=1}^{i-1} p_j \leq U < \sum_{j=1}^{i} p_j\right\}$$

follows the described discrete distribution.

- Proof (AI) :

$$P(V = z_i) = P\left(\sum_{j=1}^{i-1} p_j \leq U \leq \sum_{j=1}^{i} p_i\right) = F_U\left(\sum_{j=1}^{i} p_j\right) - F_U\left(\sum_{j=1}^{i-1} p_j\right)$$

$$= \sum_{j=1}^{i} p_j - \sum_{j=1}^{i-1} p_j = p_i.$$

Generating a continuous random variable

Inverse transform method (see Excel file) :

- Assumption : the cumulative distribution function $F : \mathbb{R} \to [0,1]$ is continuous and strictly increasing
- Method : if U is a uniformly distributed random variable over $[0,1]$, then $V = F^{-1}(U)$ is distributed according to F
- Proof (AI) :

$$F_V(v) \quad = \quad P(F^{-1}(U) \leq v) = P(U \leq F(v)) = F(v)$$

- Example : exponential distribution,
  - $F(z) = (1 - \exp(-\lambda z))\mathbb{1}\{z \geq 0\}$
  - $V = F^{-1}(U) = -(1/\lambda)\ln(1 - U)$
- Example in Excel : «=norminv(rand() ;0 ;1)» generates a random value distributed according to the normal distribution (see Excel file)

Generating a random vector

- We want to generate a scenario of realizations for a vector of random variables for which the joint probability function is $P_Z(z_1, z_2, z_3, ..., z_n)$.
- Compute the marginal and conditional probability functions.

$$P_1(Z_1 = y) = \sum_{z_2} \sum_{z_3} \cdots \sum_{z_n} P_Z(y, z_2, z_3, ..., z_n)$$

$$P_2(Z_2 = y | z_1) = \sum_{z_3} \sum_{z_4} \cdots \sum_{z_n} P(z_1, y, z_3, ..., z_n) \Big/ P_Z(Z_1 = z_1)$$

...

$$P_n(Z_n = y | z_1, ..., z_{n-1}) = P_Z(z_1, ..., z_{n-1}, y) / P_Z(Z_1 = z_1, ..., Z_{n-1} = z_{n-1}$$

- Method : if $(V_1, V_2, ..., V_n)$ is a set of random variables generated as above, then V is distributed following P_Z.

$$V_1 \sim P_1(Z_1) \quad V_2 \sim P_2(Z_2 | V_1) \quad ... \quad V_n \sim P_n(Z_n | V_1, V_2, ..., V_{n-1})$$

Rejection method - Theory (AI)

- Hypothesis : f_Z & f_V are density functions on $\mathbb{R}^n$
  1. We know how to simulate V.
  2. There exist a constant c such as $f_Z(z) \leq cf_V(z)$, $\forall\, z \in \mathbb{R}^n$
- Method : consider a sequence of pairs $(U_1, V_1), (U_2, V_2), \ldots,$ where each U_i is independent and uniformly distributed on $[0, 1]$ and each V_i is independent and generated following f_V. If i^* is the 1[st] index i such that $cU_i \leq f_Z(V_i)/f_V(V_i)$, then V_{i^*} is distributed according to f_Z.
- Proof when $n = 1$ (AI) :

$$P(V_{i^*} \leq z) = P\left(V \leq z \middle| cU \leq \frac{f_Z(V)}{f_V(V)}\right) = \frac{P(V \leq z \,\&\, cU \leq f_Z(V)/f_V(V))}{P(cU \leq f_Z(V)/f_V(V))}$$

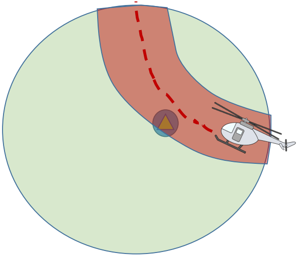
$$\propto \int_{-\infty}^{z} P\left(cU \leq \frac{f_Z(y)}{f_V(y)}\right) f_V(y)dy = \int_{-\infty}^{z} (1/c)\frac{f_Z(y)}{f_V(y)} f_V(y)dy$$

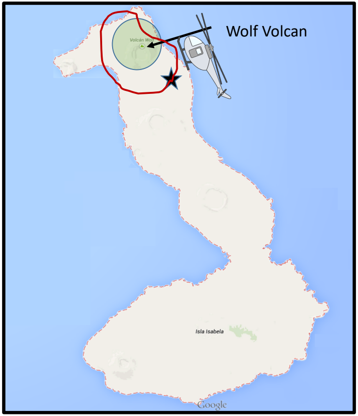
$$= (1/c)F_Z(z) \Rightarrow P(V_{i^*} \leq z) = P(Z \leq z)$$

- If $c >> 1$, we can wait a long time for V_{i^*} $(E[i^*] = c)$

Example of random generation I

What are your chances of seeing a Galapagos iguana during a helicopter tour of the Wolf volcan?





Example of random generation II

Generate a point uniformly in a disc of radius 1

$$f_Z(z_1, z_2) = (1/\pi)\mathbb{1}\{z_1^2 + z_2^2 \leq 1\}$$

- Consider V uniform on the square $[-1, 1] \times [-1, 1]$

$$f_V(v_1, v_2) = (1/4)\mathbb{1}\{-1 \leq v_1 \leq 1 \;\&\; -1 \leq v_2 \leq 1\}$$

- We can generate V by simulating $V_1 \sim U(-1, 1)$ and $V_2 \sim U(-1, 1)$ and assemble $V = (V_1, V_2)$
- Let's use the rejection method (see Excel file) :
  - We choose c :

$$c = \inf\{c | f_Z \leq c f_V\} = \inf\{c | (1/\pi \leq c/4\} = 4/\pi$$

  - We accept (U, V_1, V_2) if

$$U \leq \frac{\pi}{4} \frac{(1/\pi)\mathbb{1}\{V_1^2 + V_2^2 \leq 1\}}{(1/4)\mathbb{1}\{-1 \leq V_1 \leq 1, -1 \leq V_2 \leq 1\}} = \mathbb{1}\{V_1^2 + V_2^2 \leq 1\}$$

I.e, if $V_1^2 + V_2^2 \leq 1$, we say $Z_1 = V_1$ and $Z_2 = V_2$