MATH 60604A Statistical Modelling

Chapter 3 Exercises

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Question 1

The Gamma distribution is parametrized in terms of α and β, with $\alpha > 0$ and $\beta > 0$. For $X \sim \text{Gamma}(\alpha, \beta)$, the density is defined for $x > 0$ as follows

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$$

where, for positive integer values of α, $\Gamma(\alpha) = (\alpha - 1)! = (\alpha - 1) \times (\alpha - 2) \times \cdots \times 2 \times 1$.

We will focus on the case where $\alpha = 2$ and β is unknown. For a random sample $X_1, \ldots, X_n \stackrel{iid}{\sim} \text{Gamma}(2, \beta)$,

a) Write an expression for the likelihood function $L(\beta)$.

b) Write an expression for the log-likelihood function $LL(\beta)$.

c) Derive an expression for the maximum likelihood estimator for the parameter β in terms of a random sample $X_1, \ldots, X_n \stackrel{iid}{\sim}$ Gamma$(2, \beta)$.

d) Suppose the following random sample is observed from the Gamma$(2, \beta)$ distribution:

$$x_1 = 15, x_2 = 7, x_3 = 12, x_4 = 7, x_5 = 22, x_6 = 25, x_7 = 1, x_8 = 12, x_9 = 2, x_{10} = 6$$

Based on this random sample, calculate the MLE $\hat{\beta}_{mle}$.

Question 2

Consider the Pareto Type II (or Lomax) distribution, with density given by

$$f(y) = \frac{\alpha\theta^\alpha}{(y + \theta)^{\alpha+1}}, \qquad \text{for } y > 0$$

Suppose it is known that $\theta = 1$. For a random sample $Y_1, \ldots, Y_n$ from the Pareto Type II distribution (with $\theta = 1$),

a) Write an expression for the likelihood function. Show your work and be sure to simplify the expression.

b) Write an expression for the log-likelihood function. Show your work and be sure to simplify the expression.

c) What is the maximum likelihood estimator for α? Show your work.