# MATH 60604A Statistical Modelling

## Chapter 3 Solutions

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### Question 1

**a) Write an expression for the likelihood function $L(\beta)$.**

For an independent random sample, we have:

$$
\begin{aligned}
L(\beta) &= \prod_{i=1}^{n} f(x_i) \\
&= \prod_{i=1}^{n} \frac{\beta^{\alpha}}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i} \\
&= \frac{\beta^{\alpha n}}{\Gamma(\alpha)^n} \left\{ \prod_{i=1}^{n} x_i^{\alpha-1} \right\} \exp\left\{ -\beta \sum_{i=1}^{n} x_i \right\}
\end{aligned}
$$

Note that we can replace $\alpha$ with 2 throughout the above expression (note $\Gamma(2) = 1! = 1$), which then simplifies to

$$
L(\beta) = \beta^{2n} \left\{ \prod_{i=1}^{n} x_i \right\} \exp\left\{ -\beta \sum_{i=1}^{n} x_i \right\}
$$

**b) Write an expression for the log-likelihood function $LL(\beta)$.**

$$
LL(\beta) = \alpha n \ln(\beta) - n \ln\{\Gamma(\alpha)\} + (\alpha - 1) \sum_{i=1}^{n} \ln(x_i) - \beta \sum_{i=1}^{n} x_i
$$

Note that we can replace $\alpha$ with 2 throughout the above expression, which then simplifies to:

$$
LL(\beta) = 2n \ln(\beta) + \sum_{i=1}^{n} \ln(x_i) - \beta \sum_{i=1}^{n} x_i
$$

**c) Derive an expression for the maximum likelihood estimator for the parameter $\beta$ in terms of a random sample $X_1, \ldots, X_n \overset{iid}{\sim}$ Gamma$(2, \beta)$.**

To find the MLE, we take the partial derivative with respect to $\beta$ and set equal to zero:

$$\frac{\partial}{\partial \beta} LL(\beta) = \frac{2n}{\beta} - \sum_{i=1}^{n} x_i = 0$$
$$\Leftrightarrow \frac{2n}{\beta} = \sum_{i=1}^{n} x_i$$
$$\Rightarrow \beta = \frac{2n}{\sum_{i=1}^{n} x_i}$$

Thus, the MLE is

$$\hat{\beta}_{mle} = \frac{2n}{\sum_{i=1}^{n} x_i} = 2/\bar{X}$$

We should check that this is indeed a maximum,

$$\frac{\partial^2}{\partial \beta^2} LL(\beta) = -\frac{2n}{\beta^2} < 0$$

**d) Suppose the following random sample is observed from the Gamma$(2, \beta)$ distribution:**

$$x_1 = 15, x_2 = 7, x_3 = 12, x_4 = 7, x_5 = 22, x_6 = 25, x_7 = 1, x_8 = 12, x_9 = 2, x_{10} = 6$$

**Based on this random sample, calculate the MLE $\hat{\beta}_{mle}$.**

From this data, $\bar{x} = \frac{1}{10} \sum_{i=1}^{10} x_i = 109/10 = 10.9$ and thus the MLE is $\hat{\beta}_{mle} = 2/10.9 = 0.1834862$.

## Question 2

**a) Write an expression for the likelihood function. Show your work and be sure to simplify the expression.**

For a random sample $Y_1, \ldots, Y_n$, the likelihood function is

$$L(\alpha) = \prod_{i=1}^{n} \left( \frac{\alpha}{(Y_i + 1)^{\alpha+1}} \right)$$
$$= \alpha^n \prod_{i=1}^{n} (Y_i + 1)^{-(\alpha+1)}$$

**b) Write an expression for the log-likelihood function. Show your work and be sure to simplify the expression.**

The log-likelihood is

$$LL(\alpha) = n\ln(\alpha) - (\alpha+1)\sum_{i=1}^{n}\ln(Y_i+1)$$

**c) What is the maximum likelihood estimator for $\alpha$? Show your work.**

$$\begin{aligned}\frac{\partial}{\partial\alpha}\,LL(\alpha) &= \frac{n}{\alpha} - \sum_{i=1}^{n}\ln(Y_i+1)\\ &= 0\\ \Leftrightarrow \frac{n}{\alpha} &= \sum_{i=1}^{n}\ln(Y_i+1)\\ \Rightarrow \alpha &= \frac{n}{\sum_{i=1}^{n}\ln(Y_i+1)}\end{aligned}$$

Thus, the MLE is $\hat{\alpha} = n/\sum_{i=1}^{n}\ln(Y_i+1)$.

Note: we should check that

$$\frac{\partial^2}{\partial\alpha^2}\,LL(\alpha) = \frac{-n}{\alpha^2} < 0$$

So that we know we have a maximum.