Synthetic turbulence

1 Motivation

Standard numerical methods, such as finite volume schemes, have proven useful in studying turbulence. However, these numerical simulations tend to be computationally expensive, making it difficult to simulate turbulence under a wide variety of physical conditions, e.g., different sound speeds.

The aim of this project is to devise a method that can be used to construct synthetic turbulence. In order for the method to be considered successful, it should require only a limited amount of computational resources. This would be especially useful when increasing the resolution of the system being simulated.

Furthermore, the synthetic turbulence should contain all the statistical properties of turbulence, as predicted by theory and observed in traditional fluid simulations. It is easy to generate a set of random-phased waves with a given power-spectrum in Fourier space, with the amplitude distribution dependent on the direction of a uniform magnetic field. However, theory and simulations show that the wave amplitudes depends on the direction of the local mean magnetic field. The method for synthetic turbulence should thus be able to take the direction of the local magnetic field into account.

2 Specifying scalar field

The scalar field is constructed in k-space using the expression

$$\tilde{\psi}(k_x, k_y, k_z) = E^{1/2} \left(\cos\phi + i \sin\phi \right), \tag{1}$$

where $\phi = [0, 2\pi]$ is a random phase. In three dimensions, the amplitude E of the waves are defined by the GS95 scaling

$$E = k_\perp^{-10/3} \exp\left(-\frac{k_\parallel}{k_\perp^{2/3}} \right). \tag{2}$$

The parallel direction $k_\parallel$ is defined along k_x, while the perpendicular direction $k_\perp$ is defined in the $k_y - k_z$ plane. An inverse Fourier transform is then used to obtain the scalar field ψ in real space.

3 Deformation method

This method has been found to work well at large scales.

Figure 1 is an illustration of the idea behind the method. A scalar field (*left*) is initially specified, as described in Section 2, along with a magnetic field (*centre*, note that this figure shows the magnetic field lines). The scalar field should be deformed in such a manner that it follows the magnetic field lines, as shown in the figure on the right.

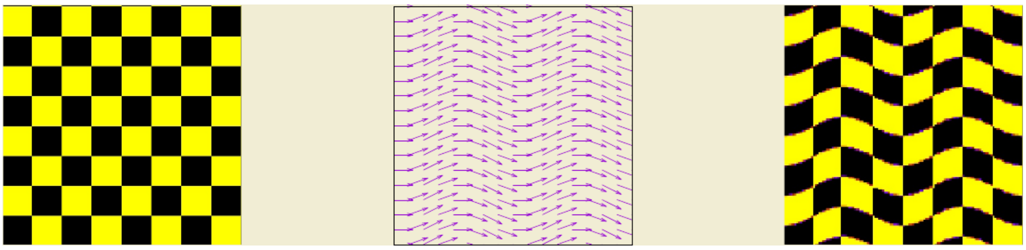

Figure 1: An illustration of the displacement method. Given an initial scalar field (*left*) and a magnetic field (*centre*), the scalar field should be deformed to follow the magnetic field (*right*).

The deformation is achieved by calculating the deviation (or displacement) of the given magnetic field from a uniform magnetic field. This deviation is calculated at every point in space, and once this is known, this deviation is used to deform the scalar field.

This deformation procedure is done as follows (at this point it is assumed that the magnetic field is specified):

3.1 Filter magnetic field

A set of magnetic field distributions $\mathbf{B^{(i)}}(\mathbf{r})$, with $i = n, n-1, \ldots, 0$, are built by transforming the magnetic field components to Fourier space and using

$$\mathbf{\tilde{B}^{(i)}}(\mathbf{r}) = \langle \mathbf{\tilde{B}^{(i+1)}}(\mathbf{r}) \rangle_{k>k_i}, \tag{3}$$

where $\langle \cdot \rangle_{k>k_i}$ filters out all spatial frequencies $k > k_i$. This means that when transforming back to real space $\mathbf{B^{(n)}}(\mathbf{r}) = \mathbf{B}(\mathbf{r})$ and $\mathbf{B^{(0)}}(\mathbf{r}) = \mathbf{B_0}$, where $\mathbf{B_0}$ is the average magnetic field.

Currently the value of k_i is defined as

$$k_i = \sqrt{2} \cdot 2^{n-i}, \tag{4}$$

where

$$n_x = n_y = n_z = 2^n. \tag{5}$$

Here n_x is the number of grid points along the x-direction, etc, and n is a free parameter.

The important thing to note is that every subsequent magnetic field distribution $\mathbf{B^{(i)}}(\mathbf{r})$ increasingly resembles a uniform magnetic field.

3.2 Calculate $\delta\xi^{(i)}$

Guided by the MHD induction equation

$$\frac{\delta B}{\delta t} = \nabla \times (\mathbf{V} \times \mathbf{B}), \tag{6}$$

we make an ansatz and choose

$$\mathbf{B}^{(i+1)} - \mathbf{B}^{(i)} = \nabla \times \left(\delta\xi^{(i)} \times \mathbf{B}^{(i)}\right), \tag{7}$$

where $\delta\xi^{(i)}$ is defined as a spatial displacement.

Transforming to k-space, we obtain

$$\tilde{\mathbf{B}}^{(i+1)} - \tilde{\mathbf{B}}^{(i)} = i\mathbf{k} \times \left(\delta\tilde{\xi}^{(i)} \times \tilde{\mathbf{B}}^{(i)}\right). \tag{8}$$

Multiplying both sides by $i\mathbf{k}$ and simplifying, we obtain

$$\left(\tilde{\mathbf{B}}^{(i+1)} - \tilde{\mathbf{B}}^{(i)}\right) \times i\mathbf{k} = i^2k^2\left(\delta\tilde{\xi}^{(i)} \times \tilde{\mathbf{B}}^{(i)}\right) - \left[\left(\delta\tilde{\xi}^{(i)} \times \tilde{\mathbf{B}}^{(i)}\right) \cdot i\mathbf{k}\right] i\mathbf{k}. \tag{9}$$

In practice it was found that simplifying the above equation by defining

$$\delta\tilde{\xi}_*^{(i)} \equiv \delta\tilde{\xi}^{(i)} \times \tilde{\mathbf{B}}^{(i)}, \tag{10}$$

and requiring that

$$\delta\tilde{\xi}_*^{(i)} \cdot i\mathbf{k} = 0 \tag{11}$$

leads to better results. One therefore only needs to solve

$$\delta\tilde{\xi}_*^{(i)} = -\frac{1}{k^2}\left[\left(\tilde{\mathbf{B}}^{(i+1)} - \tilde{\mathbf{B}}^{(i)}\right) \times i\mathbf{k}\right]. \tag{12}$$

Once $\delta\tilde{\xi}_*^{(i)}$ has been calculated, we transform back to real space to obtain $\delta\xi_*^{(i)}$.

3.3 Calculate displacement field

Making a second ansatz, a displacement field $\delta\mathbf{R}^{(i)}(\mathbf{r})$ is calculated from the expression

$$\delta\xi_*^{(i)} = \delta\mathbf{R}^{(i)} \times \mathbf{B}^{(i)}, \tag{13}$$

where it is assumed that

$$\delta\mathbf{R}^{(i)} \cdot \mathbf{B}^{(i)} = 0. \tag{14}$$

Every point in space can now be associated with a total displacement

$$\Delta\mathbf{R}(\mathbf{r}) = \sum_{i=0}^{i=n} \delta\mathbf{R}^{(i)}(\mathbf{r}) \tag{15}$$

If $S(\mathbf{r})$ is the original scalar field, the deformed scalar field $S'(\mathbf{r})$ is obtained by moving every part of the original scalar field to the new coordinates $\mathbf{r} + \Delta\mathbf{R}(\mathbf{r})$, i.e.,

$$S'(\mathbf{r}') = S(\mathbf{r} + \Delta\mathbf{R}). \tag{16}$$

In the numerical implementation of the technique this step requires interpolation, and care should thus be taken.

4 Squares method

This method has been found to work well at small scales.

The method starts by specifying a magnetic field, similar to e.g., the field shown in the centre panel of Figure 1. On large scales the local magnetic field is, to a good approximation, given by the average magnetic field. A scalar field, as described in Section 2, is constructed for all $k < k_1$, where k_1 is a user-specified value. Here the parallel direction is defined along the global x-axis

Next, the initial domain is divided into a number of sub-domains, and the average magnetic field in every sub-domain is calculated. The average magnetic field, and consequently the parallel direction, will vary between the different sub-domains. A scalar field for $k_1 \leq k < k_2$ is constructed in every sub-domain, taking into account the specific directions of the average magnetic field in every sub-domain.

The above process is repeated, i.e., the sub-domains are divided, and new scalar fields are added using wave numbers $k_2 \leq k < k_3$. This process is repeated until the Nyquist frequency has been reached.

There is no prescription on choosing the values $k_1, k_2, \ldots$, and one unfortunately has to rely on some trial-and-error.