X Marks the Spot:

Unlocking the Treasure of Spatial-X Models

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Abstract

In recent years, political scientists have made extensive use of spatial econometric models to test a wide range of theories. In a review of spatial papers, we find that a majority of these studies use the spatial autoregressive (SAR) model. Although this is a powerful method that reveals inferences about diffusion processes, it is also highly restrictive and makes assumptions that often are not appropriate given the expressed theories. We contend that spatial-X (SLX) models are a better reflection of typical theories about spatial processes. Our simulations demonstrate that SLX models consistently retrieve the direct and indirect effects of covariates when the true data-generating process reflects other spatial processes. SAR models, on the other hand, tend to find phantom higher-order effects that are not present in the data. We further demonstrate how SLX models reveal heterogeneity in patterns of spatial dependence in countries' defense burdens that SAR models cannot discover.

Keywords: spatial theories, spatial models, spatial-x, SLX

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Political scientists have increasingly made use of spatial econometric models. The appeal of these models is that they relax rigid assumptions about the independence of observations across space. While the move toward these techniques is encouraging for the building of more realistic models of politics, there is frequently a substantial disjuncture between theoretical propositions, what is actually being tested, and how results are interpreted. This has particularly been the case when researchers rely only on the spatial autoregressive (henceforth SAR) model to test their theories.

The SAR model captures contemporaneous interdependence in outcomes—how the value of the dependent variable in one unit, y_i, affects the value of the dependent variable in another unit, y_j. Although the SAR model is the most popular spatial model among political scientists, it is a highly restrictive model that researchers should use with caution. We argue that, depending on the theory being tested and the pattern of spatial effects, the spatial-X (henceforth SLX) model may be preferable to the SAR. This is the case because the SLX model allows the spatial processes to influence the outcome through one or more independent variables. Examples of this include trash in a neighbor's yard affecting the value of your home and lax gun regulations in nearby states producing negative externalities (Knight 2013). Since there is no implicit endogeneity, both the estimation and the interpretation of SLX models are considerably easier than with SAR models.

Ideally, selecting the appropriate spatial econometric model should be driven by one's theory about the presence of spatial dependence in the outcomes, observables, and/or unobservables (Cook, Hays, and Franzese 2015). After selecting a theoretically-grounded model, appropriate specification tests (such as a Lagrange multiplier test) should further buttress those choices (Darmofal 2015). In the absence of an appropriately specific theory, one at-first obvious approach would be to borrow a strategy from time series analyses (Hendry 1995): start with a general model that includes all three types of spatial dependence, and then gradually pare down the model by testing restrictions (Vega and Elhorst 2015). A full interpretation of the various total, indirect and direct effects and specification tests would then complete the process (LeSage and Pace 2009; Whitten, Williams, and Wimpy n.d.). In practice, however, the estimation and interpretation of a general model that includes spatial relationships from all three sources is very difficult and thus it is rarely done.

So what do political science researchers do? In our review of the literature, we found that the overwhelming majority of spatial publications in the discipline start and end with the SAR model. This is the case despite the fact that the underlying theories, as expressed by the authors, often do not match the assumptions that the SAR model implicitly imposes. We also find that researchers rarely interpret the variety of quantities of interest available in the SAR models that they estimate.

In this paper we make a case for an approach to spatial model building that takes advantage of the flexibility and relative simplicity of the SLX model. In this approach, the theoretical implications of different model specifications are quite transparent and the assumptions imposed by the SAR model can easily be turned into testable propositions. This approach simplifies the preliminary stages of model specification, and minimizes inferential errors while maximizing the ease of interpretation. We offer these recommendations knowing full well that they are no substitute for careful theorizing and modeling of the data-generating process (DGP). At the same time, we think that this approach offers a better alternative to the current one that dominates the use of spatial econometrics in political science. By doing so, we hope to illuminate a path forward and provide a practical midway point between current unsatisfactory practice and the ideal modeling approaches advocated by Cook, Hays, and Franzese (2015) and Vega and Elhorst (2015).

In the sections that follow, we begin with a brief overview of SAR and SLX models and point out how they differ in expectations about endogeneity, feedback, and higher-order effects. We demonstrate that SLX is more flexible in terms of producing an empirical test that closely matches underlying theory. We then identify a troubling pattern from our survey of the use of spatial econometric models: theories predicting spillovers among only first-order neighbors are often tested by SAR models that impose higher-order (and feedback) effects. In a series of Monte Carlo experiments we explore which type of model is more robust to errors in expectations about the DGP; more specifically, what happens when the true model is an SLX, but we estimate an SAR and vice versa. These experiments show that SAR models perform poorly in terms of identifying the correct direction of spatial dependence, and this problem worsens as the degree of spatial heterogeneity increases. SLX models, however, do an adequate job of characterizing spatial effects of processes typically encountered in political science. We then provide an illustration of countries' defense burdens which shows how SLX models can reveal interesting patterns of spatial heterogeneity—both in terms of *how* the countries are connected and *to what extent*—that SAR models cannot. Most notably, we show that instability (in the form of interstate war) spills over into neighbors' defense burdens, and that these effects are above and beyond what one might attribute to positive spatial dependence from a SAR model. We conclude with a discussion of the implications of our findings for future research along with some potential paths forward for model selection.

Modeling Spatial Dependence in Politics

It has been a little more than a decade since political scientists wrote the first papers about the potential for spatial econometric models of political phenomena (Beck, Gleditsch, and Beardsley 2006; Franzese and Hays 2007; Ward and Gleditsch 2008). In the wake of these path-breaking works, there has been a rapid increase in the employment of spatial econometric models by political scientists. Scholars have made theoretical arguments about how spatial relationships help to determine policies (Gray 1973; Simmons and Elkins 2004; Neumayer, Plümper, and Epifanio 2014), conflict (Buhaug and Gleditsch 2008; Garcia and Wimpy 2016), party competition (Williams and Whitten 2015; Bohmelt et al. 2016), terrorism (Midlarsky, Crenshaw, and Yoshida 1980; Neumayer and Plümper 2010*a*) and many other outcomes.

This section provides an overview of two of the most popular spatial econometric models: SAR and SLX models. Spatial dependence occurs in both models, whether it is in the outcomes (SAR) or in the observables (SLX). The researcher specifies the manner in which all the observations are connected to each other via an $N \times N$ weights matrix ($\mathbf{W}$) in both models, so it is worthwhile spending some time on this critical task. For ease of presentation, we explore the basic mechanics of each model in the context of a simple contiguity or first-order weights matrix.[1] In Figure 1 we show a toy example of how these types of matrices are constructed. On the far left side of this figure, we show a map of six units with arrows depicting first-order neighbors. For now, only consider the first-order contiguity weights matrix, $\mathbf{W}$, where each cell identifying a pair of neighboring units contains a one and all other cells contain zeros. Since being a neighbor is symmetrical, this matrix is symmetrical. And since a unit cannot be a neighbor of itself, the main diagonal contains all zeros. As we will see below, these features of $\mathbf{W}$ have meaningful consequences for substantive inferences.

[Figure 1 about here.]

The Spatial Autoregressive (SAR) Model

The equation for a basic SAR model is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \rho\mathbf{W}\mathbf{y} + \boldsymbol{\epsilon} \tag{1}$$

and the reduced equation (isolating $\mathbf{y}$ on the left side) is

$$\mathbf{y} = (\mathbf{I}_n - \rho\mathbf{W})^{-1}\mathbf{X}\boldsymbol{\beta} + (\mathbf{I}_n - \rho\mathbf{W})^{-1}\boldsymbol{\epsilon}. \tag{2}$$

[1] In addition to making the presentation more clear, this is the most popular type of weights matrix used in political science research. In our survey of the literature (described below), we found that 72.3% of the models reported used a weights matrix specified based on geography. Regardless, the same types of patterns and problems discussed in this paper persist in SAR models regardless of the specification of $\mathbf{W}$.

From the infinite series expansion of the spatial multiplier,

$$(\mathbf{I}_n - \rho\mathbf{W})^{-1} = (\mathbf{I}_n + \rho\mathbf{W} + \rho^2\mathbf{W}^2 + \rho^3\mathbf{W}^3 + \ldots) \tag{3}$$

we can see that higher order and feedback effects, or *global effects*, are present in any and all SAR models. Consider the second weights matrix depicted in Figure 1 ($\mathbf{W}^2$), which has two meaningful features. First, unlike $\mathbf{W}$, the main diagonal elements—which convey the relationship of a unit with itself—do not equal zero. These values reflect feedback effects where the effect of a unit on its neighbors comes back to affect the unit itself.

Second, we can see that $\mathbf{W^2}$ contains non-zero values for all cells containing second-order neighbors.[2] The result is that the effects extend beyond first- and second-order neighbors (because of $\mathbf{W}$ raised to increasingly higher values in Equation 3), and they occur simultaneously at time t. We can thus label them as "global effects."

While interpreting the coefficients is a reasonable place to start, it is a bad place to stop in the interpretation of an SAR model. Given all of the different higher order and feedback terms implicit in $\mathbf{W}^2$ and the higher order terms in Equation 3, a general interpretation of the estimated ρ masks substantial variation in the estimated effects across units. For instance, if we want to infer the effect of a single x on y, then one approach is to examine the partial derivatives matrix (LeSage and Pace 2009; Whitten, Williams, and Wimpy n.d.):

$$\left[\frac{\partial \mathrm{E}(y)}{\partial x_1} \cdots \frac{\partial \mathrm{E}(y)}{\partial x_N}\right] = (\mathbf{I} - \hat{\rho}\mathbf{W})^{-1}\hat{\beta} \tag{4}$$

where the resulting $N \times N$ matrix (N is the total number of observations) contains both the impacts of x_i on y_i, or direct effects (along the diagonal), and the impacts of x_i on y_j, or indirect effects (along the off-diagonal).

[2] The value in these cells reflects the number of paths through which each pair of units are second-order neighbors. So, for example, $\text{cell}_{AE} = \text{cell}_{EA} = 1$ because A and E are only connected through B, but $\text{cell}_{BC} = \text{cell}_{CB} = 2$ because B and C are connected through both A and D.

The Spatial-X (SLX) Model

In contrast to the SAR model, the SLX model offers a framework in which researchers may choose whether to model local or global spatial relationships. In addition, the SLX model is relatively easier to estimate and interpret. In practice, the SLX model has been most often discussed as a modeling strategy for exogenous spatial effects from a direct neighbor. This effort has primarily been driven by the work of LeSage and Pace (2009) in which they suggest that the SLX model works best in the case of "externalities," or "local spillovers". Spillovers would appear to accurately characterize many political science phenomena; something happens to a neighbor that affects the outcome of interest.

The equation for a basic SLX model can we written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{Z}\boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{5}$$

where $\mathbf{Z}$ is the matrix of variables expected to exert spatial influence on $\mathbf{y}$ through a theoretically specified $\mathbf{W}$ matrix that connects observations to each other through a vector of spatial parameters $\boldsymbol{\theta}$.[3]

If we start with simple SLX model with a single independent variable, x_1, such that $\mathbf{X} = \mathbf{Z} = x_1$, we calculate the effect of x_1 on y as

$$\left[\frac{\partial \mathrm{E}(y)}{\partial x_1}\right] = \hat{\beta}_1 + \hat{\theta}_1\mathbf{W}. \tag{6}$$

In contrast to the infinite series expansion in the SAR spatial multiplier, the indirect effect ($\hat{\theta}_1\mathbf{W}$)—or "neighbor effect"—is only present at the first-order. In other words, with the specification presented in Equation 5, the effect of x_1 is limited to only local effects; the effect does not continue to second-order neighbors, and there are no feedback effects. However, as

[3]We use $\mathbf{Z}$ to reflect the possibility that $\mathbf{Z}$ and $\mathbf{X}$ can have different contents.

we discuss below, if we want to specify higher order and feedback effects, we can incorporate them into an SLX model.

Without realizing it, scholars often use SLX models by including spatially-weighted independent variables. In fact, any model that controls for the sum or average of neighbors' attributes reflects an SLX specification (Drolc, Gandrud, and Williams 2019). The most common example incorporates a temporally-lagged spatial lag (TLSL), which tests a common argument in theories of policy diffusion that the diffusion process—or the influence of neighbors' covariates more generally—occurs with a temporal lag. This variant of the SLX equation can be written as:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{W}\mathbf{y}_{t-1}\theta + \boldsymbol{\epsilon} \tag{7}$$

While the estimation is straight-forward, the presence of the TLSL adds a temporal dimension to the quantities of interest.[4]

In the next section, we provide some guidelines as to when to use one model over the other by highlighting each model's flexibility and ability to derive correct causal inferences.

Theory, Specification, and Estimation in SAR and SLX Models

The general motivating force behind spatial econometric models is to test theories about how y_i is a function of some aspect of unit j. If a researcher's expectations are that the

[4]A common misconception is that this is "an alternative specification of the spatial autoregressive model" (Beck, Gleditsch, and Beardsley 2006, 40). However, the TLSL forces the causality to go in a single direction (i.e., from y_{t-1} to y_t), which eliminates the defining feature of SAR models: spatial endogeneity. Neumayer and Plümper (2010*b*, 158) note that a model with a temporal lag "is not strictly speaking a spatial lag model". Feedback for i from x_{t-1} can occur by first influencing i's neighbors at time t, which then influences j's neighbors (including i) at time $t+1$. In the first section of our Supplemental Materials document, we provide some more discussion of the features of the TLSL model and why it is more like an SLX model than a SAR model.

spatial relationships are between y_i and y_j, and that these relationships are global (including feedback from y_i to y_j and back to y_i) and occur immediately, then the SAR model is clearly the appropriate model to choose. On the other hand, if a researcher's expectations are that the spatial relationships in their study are between y_i and some variable z_j, and these relationships are local in nature, then the SLX model is clearly the appropriate model to choose. But what about a researcher whose expectations are less sharp? Under these circumstances, we echo the suggestion by Vega and Elhorst (2015) that the SLX model is a better place to start than the SAR. The advantages of the SLX model are four-fold: greater flexibility in specifying lower-order (local) versus higher-order (global) effects, the ability to relax and test common factor restrictions, the ease of estimation and interpretation, and more flexible and realistic specifications of temporal processes. We will explore each of these in turn.

First, the SLX model offers a more flexible approach to modeling lower-order versus higher-order spatial effects than the SAR model. As we outlined above, the SAR model implicitly imposes a number of strong assumptions that may not be immediately obvious to scholars, and that often run contrary to their theoretical expectations about the spatial processes at work. In most incarnations of the SAR model, there is no way to limit spatial effects to only local first-order effects.[5] Although the SAR model certainly captures *local* effects, by construction it also finds higher-order effects. The infinite series expansion (Equation 3) reveals that all observations are eventually impacted by any change in x; this is the case even if there is no theoretical reason to expect that 9th-order neighbors would be influenced (albeit in a small fashion) by a change.[6] The SAR model offers no flexibility in terms of how many orders of neighbors are impacted. Feedback effects start in the second order (e.g., $\mathbf{W^2}$ in Figure 1). Though one may theorize that the total effect of a change in x_j for observation i arises only from first-order local effects, the SAR model forces there to

[5] In the case of SAR models using directed dyadic data, one can limit the effects to the first-order by distinguishing between the *sources* and *targets* of stimulus (e.g., Neumayer and Plümper (2010*b*, 152–154)).

[6] The one exception to this would be for a unit that has no connectivity with any other. In Figure 1 such a unit would appear as an island with values equal to zero for the relevant column and row in $\mathbf{W}$.

be higher-order and feedback effects, whether this accurately reflects the true spatial process or not. The consequence is that SAR models will risk finding a *global* process even if one is not present.

In an SLX model, expectations about local first-order processes can be easily tested through the specification of a model like that in Equation 5. But can SLX, on the other hand, perform adequately when the true spatial process includes higher-order effects? The answer is "yes." By altering the model specification, scholars can incorporate higher-order effects into their models of spatial processes. For example, imagine that we have an expectation of higher-order effects that stop at the 3rd-order. For ease of exposition, we limit our model to a single independent variable $\mathbf{x}$. In this case, we would estimate an SLX model specified as

$$\mathbf{y} = \beta\mathbf{x} + \theta_1\mathbf{W}\mathbf{x} + \theta_2\mathbf{W^2}\mathbf{x} + \theta_3\mathbf{W^3}\mathbf{x} + \boldsymbol{\epsilon}.^7 \tag{8}$$

The total estimated effect of x on y would be

$$\left[\frac{\partial \mathrm{E}(y)}{\partial x_1} \cdots \frac{\partial \mathrm{E}(y)}{\partial x_N}\right] = \hat{\beta} + \hat{\theta}_1\mathbf{W} + \hat{\theta}_2\mathbf{W}^2 + \hat{\theta}_3\mathbf{W}^3 \tag{9}$$

where $\hat{\beta}$ is the estimated direct (or zero-order) effect of x, $\hat{\theta}_1$ is the estimated first-order indirect effect, $\hat{\theta}_2$ is the estimated second-order indirect effect, and $\hat{\theta}_3$ is the estimated third-order indirect effect. It is worth noting that restrictions on the order of effects can be tested, so one strategy is to start with an n-order specification and then use hypothesis tests to pare down the model.[8]

With the model specification shown in Equation 8, the SLX model mimics the SAR model in its estimation of feedback effects. However, there might be situations where one expects higher-order neighbor effects but does not expect that the spatial process will have feedback

[7] In this specification, $\mathbf{X} = \mathbf{x}$, a single vector containing the values for independent variable x and $\mathbf{Z} = (\mathbf{x} \quad \mathbf{x} \quad \mathbf{x})$.

[8] One word of caution is that if there are few neighboring observations in the weights matrix, then higher-order representations of $\mathbf{W}$ can exacerbate problems of multicollinearity and inflate standard errors.

effects. In these cases, one can exchange the terms with $\mathbf{W}$ squared, cubed, and raised to higher powers for n-order contiguity matrices such as $\mathbf{W}_{2^{nd}}$ in Figure 1 (as we demonstrate in the *Supplemental Materials* document). One can also test whether the spatial process exhibits feedback loops by comparing the model fit (through information criterion such as AIC and BIC) of an SLX specification with traditional squared and higher order $\mathbf{W}$ terms versus that from one with nth-order contiguity matrices.

The second relative advantage of the SLX model is the ability to relax and test common factor restrictions. In the case of an SAR model, there is an imposed assumption that all of global spatial dependence works through a single spatial parameter, ρ. For example, consider estimating an SAR model where we have two independent variables, x_1 and x_2 in $\mathbf{X}$. Since there is only one estimated ρ in the model, the infinite series expansion of the spatial multiplier (found in Equation 3) is identical and the only difference in the estimated spatial effects comes from different coefficient estimates in $\hat{\boldsymbol{\beta}}$. In other words, the strength of spatial dependence, and the manner in which higher-order effects reverberate throughout the system, is identical for both x_1 and x_2 by construction. Moreover, one spatial autocorrelation coefficient (ρ) is used to represent the declining spatial effects across all orders of neighbors, albeit at an order of magnitude smaller at each additional order of neighbors (i.e., ρ, ρ^2, ρ^3, and so on). Rather than this common factor restriction being imposed by the model (in the case of SAR), in the context of an SLX model it becomes a testable proposition; we can test the accuracy of these restrictions and modify the specification accordingly.[9]

It is not outside the realm of possibility to consider political science applications in which spatial relationships may vary across independent variables. In such cases of spatial heterogeneity, the SLX model offers a far more straight-forward and accurate approach by allowing scholars to estimate different spatial parameters for the independent variables. Unlike the SAR model, the SLX model also provides the flexibility of spatially lagging only

[9]For example, in Moore and Shellman (2007)'s analysis on refugees' destination, they estimate different parameters for the spatial contiguity effects of a number of independent variables.

those variables from a neighboring unit that are theorized to impact y_i via some form of spatial dependence. In other words, $\mathbf{X}$ and $\mathbf{Z}$ do not have to be the same. Some covariates might only have direct effects (i.e., only be in $\mathbf{X}$), and others might only have indirect effects (i.e., only be in $\mathbf{Z}$). Furthermore, the $\mathbf{W}$s can vary across the different z variables (if different interconnectivities are theorized to exist), and this model allows for spatial heterogeneity (e.g., when $\theta_1 \neq \theta_2$). As opposed to the complicated multiple SAR model, estimation for an SLX model with different interconnectivities is rather straightforward.

The third advantage of SLX models to applied researchers is ease of estimation and interpretation. Perhaps the most notable argument for the SLX model comes from the Gibbons and Overman (2012) critique of the general enterprise of spatial econometrics. These authors argue that analysts using the SAR model end up with models that are weakly-identified due to the hurdles involved in estimation of these endogenous models. The authors further point out that the model being estimated rarely fits the expressed theory. As such, they make an argument for the far-simpler, exogenous SLX model.

SLX models provide much more meaningful coefficients in terms of allowing scholars to quickly and correctly make inferences about the spatial processes at work. While the SAR model produces a *global* coefficient of average spatial dependence (ρ), getting substantive impacts and estimates of uncertainty from other coefficients of interest is notably difficult. Moreover, the ρ does not easily distinguish between local and higher-order connectivity, thus making it hard to make substantive inferences about the degree of spatial dependence from one neighbor to another. Given the complexity of calculating quantities of interest from SAR models, it is perhaps understandable that few scholars move beyond simply interpretations of the sign and significance of ρ.

In contrast, the SLX model is easier to interpret. The biggest difference in interpretation is that in an SLX model, the respective $\hat{\beta}$ can be interpreted as is; $\hat{\beta}$ is the estimated change in y_i for a one-unit increase in x_i. On the other hand, in an SAR model, the $\hat{\beta}$ represents the estimated "direct effect" of x_i on y_i, which, due to spatial dependence in the outcomes, we never actually observe because they are merely the pre-spatial effects that have yet to be filtered through the spatial dependence. This is complicated by the fact that the effect of x_i on y_i also depends on the strength of feedback. The spatial coefficients from an SLX model, the $\hat{\theta}$ terms, are interpreted as the estimated impact of a one-unit increase in the relevant z variable on the outcome variable in a neighboring unit.[10] In comparison, this effect is difficult, if not impossible, to ascertain from merely examining the $\hat{\rho}$ value from an SAR model.

The fourth advantage of SLX models is that they offer more flexible and realistic specifications of temporal processes. Up until now, we have ignored possible temporal variations in our data other than to note that by construction SAR models impose that all effects of y_j on y_i and feedback effects from y_i back on itself happen immediately. If our theory is that the effect of y_j on y_i and any associated feedback effects take time to happen, we can test this through an SLX model by including $\mathbf{y}_{t-1}$ as a term in our specification of $\mathbf{Z}$ (see Equation 7).

In the next section, we explore patterns in how political scientists have selected and interpreted spatial econometric models.

Spatial Dependence in Practice

To provide a systematic overview of how researchers in political science have used spatial econometric models, we reviewed every work published through the end of 2015 that cited Beck, Gleditsch, and Beardsley (2006) and/or Franzese and Hays (2007)—two highly in-

[10]As we mentioned above, we are limiting the present discussion to cases where $\mathbf{W}$ is a row-standardized contiguity matrix. With more complicated $\mathbf{W}$ where the weights matrix contains values other than 0 and 1, the estimated effect of a one-unit increase in x_j on y_i is $\hat{\theta}$ times w_{ij}.

fluential early papers promoting the use of spatial models.[11] We coded the use of these techniques in the main empirical model that was reported.

[Table 1 about here.]

Table 1 shows a taxonomy of the use of spatial econometric models by political scientists in terms of the type of spatial theory expressed and the model estimated. There are three patterns worth noting in this table. First, although SAR models recover global effects—or those where all non-isolate observations are influenced by changes in one observation—only 8.9% (5 of 94 studies) expressed theories that are global in nature. In those rare cases when scholars explicitly state global theories, on the other hand, they tend to appropriately test those theories with a model based on dependence in the outcomes (i.e., SAR or a more complex model that includes an SAR process). Second, when political science researchers have expressed a theory that is either local or not specific in terms of the types of spatial expectations, almost half the time (46.1%) they have estimated an SAR model which imposes global spatial effects.

Third, only 10.6% (10 out of 94 studies) of the publications we reviewed feature more than one pattern of spatial dependence. There is also generally little discussion of the process—if any—used by scholars to pare down more complex models to simpler models. Altogether, this is indicative of scholars beginning with a specific model in mind (usually the SAR), rather than a general-to-specfic approach. We return to this point in our discussion of paths forward below.

Among those papers in our survey that estimated an SAR model, the overwhelming majority offered a theory of indirect effects—the effect of x_i on y_j—that were local in nature rather than global. In fact, only 8.9% (4 out of 45 studies) of those that estimated an SAR model explicitly expressed a global theory. Essentially this means that scholars theorize that

[11] This initial search yielded 155 publications. We narrowed this down to 94 studies that reported results from at least one spatial econometric model.

explanatory variables may have an effect on first-order neighbors (local) but are silent about potential impacts on higher-order neighbors (global) (Elhorst 2014). This latter finding is striking because the SAR model imposes global relationships across all spatial units. In contrast, spatial relationships in SLX models can be specified as either local or global.

Another common pattern emerges in our in-depth exploration: scholars often estimate a series of SAR models, each with a different specification of the weights matrix (e.g., Flores 2011; Gassebner, Gaston, and Lamla 2011; Goldsmith 2007; Obinger and Schmitt 2011).[12] Recall that the SAR model imposes a common factor restriction that all of the spatial dependence operates through one parameter, ρ, connected through a properly specified $\mathbf{W}$. An SLX model, on the other hand, easily allows different variables to influence the outcome through multiple spatial dependence paths; the final model can then be pared down based on traditional model selection criteria. The examples referenced above highlight at least three problems that arise from estimating separate SAR models: first, it severely limits whether authors can determine which model best approximates the data-generating process. Second, if multiple weighting schemes are operating simultaneously and are correlated, then any model including only one will be biased. Third, even if only one weights matrix correctly specifies the spatial process, if explanatory variables operate through the weights matrix in different ways, then the SAR model will not be able to disentangle that spatial heterogeneity.[13]

In the next section we present the results from two sets of Monte Carlo experiments in which we explore how SLX and SAR models perform under different data generating processes (DGPs).

[12] These examples do not represent a comprehensive list. Instead, we identify them as representative examples of instances where the inferences might be different with an SLX model, either due to methodological differences or because the SLX model represents a more appropriate way of testing the theoretical expectations.

[13] This is not to say that there are no good examples of scholars carefully selecting the appropriate empirical model and then painstakingly exploring the substantive meaning of those models; Freeman and Quinn (2012: 67), for example, reveal how income inequality and financial integration influence democracy through multiple channels of spatial dependence.

Experiments

In this section we provide two sets of Monte Carlo experiments to explore the robustness of both models to errors in expectations about the DGP. In the first set of experiments, we expect that SLX models will perform admirably in deriving inferences about the effects of explanatory variables, even though the true process is generated by an SAR. One of the advantages of SLX models that we described above is that they can mimic the higher-order effects of SAR models in situations that scholars are likely to encounter in practice. The opposite, however, is unlikely to be true. In the second set of experiments we expect that SAR models are too inflexible to effectively deal with spatial heterogeneity that often accompanies an SLX DGP.

SAR Data-Generating Process

In the first set of experiments we generate data using the reduced-form equation for the SAR model found in Equation 2, with matrix $\mathbf{X}$ containing a single variable drawn from a uniform distribution, $x \in [-10, 10]$, and where $\beta = 1$. $\mathbf{W}$ is an $N \times N$ symmetric row-standardized contiguity weights matrix, where each element below the diagonal is randomly drawn from a Bernoulli distribution. We simulated 1000 data sets at each of nine different scenarios defined by the strength of the spatial autocorrelation, $\rho \in \{-0.8, 0.8\}$.[14] In each section of Table 2, we display our findings from estimations using a different SLX specification.[15]

The first SLX model that we estimate includes one spatially-lagged independent variable ($\mathbf{Wx}$) and is thus specified as

[14] It is worth noting, however, that values of ρ with an absolute value close to 1 are exceedingly rare in political science research.

[15] In Table 1 of our appendix we present the same results for an SAR estimation. Not surprisingly, the SAR model does an excellent job of recovering the true DGP.

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta + \boldsymbol{\epsilon}. \tag{10}$$

The βs are not directly comparable across SAR and SLX specifications; instead, with the use of the partial derivatives matrix one can easily compare the average direct, indirect and total effects across models (LeSage and Pace 2009). Additionally, since these effects can be partitioned into n-order effects, we assess the performance of the model at different orders. Table 2 shows how often the estimated SLX model specification's 95% confidence intervals include the true effects given the DGP characterized in Equation 2. Each section of the table provides the recovery rates for a different SLX model specification (more on the others below), and each column represents a different strength of spatial lag coefficient (ρ in Equation 2).

[Table 2 about here.]

There are two clear patterns in the top section of Table 2. First, the simple SLX model recovers the true average direct effect at an acceptably high rate (nearly 95%) for all values of ρ except 0.8.[16] The recovery rates for the zero-order direct effect (characterized by the β_{SLX}), on the other hand, dip below 95% at ρ values lower than -0.4 and higher than 0.4. This means that β_{SLX} is capturing the true average direct effect at common values of ρ, but it is not an accurate reflection of the zero-order direct effect (β_{SAR}). The second clear pattern is that the inclusion of one $\mathbf{W}x$ term is enough to recover the true first-order indirect effects nearly 95% of the time for all $\rho \in [-0.6, 0.6]$, but overall this model specification does a poor job of capturing the true total indirect effects. These two patterns are a consequence of an overly-simplistic SLX model specification given the true DGP. The lack of higher-order $\mathbf{W}$ matrices means that there cannot be second-, third-, or higher-order indirect effects and that there can be no feedback effects. This is why the simple SLX in the top section of

[16]It is worth noting that negative values are quite rare and that we have never seen such a high value for ρ in a political science application.

Table 2 has a recovery rate for all higher-order direct and indirect effects that is 0% at all values of ρ (represented by dashes).

These results from the experiments displayed in the top section of Table 2 should certainly warrant caution from scholars. To what extent can we eliminate some of these concerns by modifying the SLX model specification? In the second set of experiments, we add a first-order weights matrix squared ($\mathbf{W}^2$) and estimate a second SLX coefficient (θ_2) with a model specified as

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W}^{\mathbf{2}}\mathbf{x}\theta_2 + \boldsymbol{\epsilon}. \tag{11}$$

Recall that $\mathbf{W}$ matrices raised to higher-orders allow for spatial effects that cycle through neighbors of neighbors, and back to the originator (see the first-order squared matrix in Figure 1). This second model specification, therefore, should help address the poor performance of the first model in estimating the average indirect effects created in the DGP.

From the second section in Table 2, we can see that the addition of the squared weights matrix improves the performance of the model in all of the weak spots detailed above. First, the zero-order direct effects are now recovered at 95% for nearly all the values of ρ, in addition to the second-order direct effects (feedback effects), which results in even more accurate estimates of the average direct effect. Second, the ability to model second-order indirect effects not only allows one to gain accurate inferences about those effects (nearly 95% recovery rates at all values), but also about the average indirect effects overall. Instead of poor performance at values outside of $\rho \in [-0.2, 0.2]$, the average indirect effects are well-recovered at a much broader range of values ($\rho \in [-0.6, 0.4]$). Furthermore, in these experiments the improvement in model performance does not come at a cost of multicollinearity; on average the highest variance inflation factor (VIF) is a little more than 2.[17]

[17]Of course, scholars must make their own decisions as to whether the increased multicollinearity in their model is worth being able to derive more accurate inferences about spatial processes.

As we discussed earlier, the infinite series expansion of the spatial multiplier in the SAR model (see Equation 3) means that the indirect effects increase forever, but at a declining rate. To mimic this sort of process with an SLX model, one would have to include a large number of higher-order weights matrices. The utility of this approach, however, would be limited as the size of the indirect effects from higher-order effects is quite small, and this would likely exacerbate any issues of multicollinearity. The next step, then, is to see if including a cubed weights matrix effectively recovers the direct and indirect effects by estimating a model specified as

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W^2}\mathbf{x}\theta_2 + \mathbf{W^3}\mathbf{x}\theta_3 + \boldsymbol{\epsilon}. \tag{12}$$

The recovery rates presented in the bottom section of Table 2 reveal the untapped promise of the SLX model even in those processes ripe with spatial diffusion. In all but the most extreme cases of positive spatial autocorrelation (i.e., when $\rho > 0.6$), the SLX model with both squared and cubed $\mathbf{W}$ terms recovers the average direct effect, average indirect effect, and by implication, the average total effect. Thus, in the vast majority of cases that scholars observe in practice, one could estimate an SLX model and make the same substantive inferences regarding the impact of a variable on the observation, its neighbors, and the other observations. The only drawback is higher multicollinearity (the average VIF score is now above 5), which leads to somewhat inflated standard errors. This is certainly not a reason to avoid estimating the SLX model, and is a similar decision to those made by scholars choosing whether to include lower-order terms in interactive models.[18]

In the next section we explore the opposite scenario; where we have an SLX DGP, but we estimate an SAR model.

[18]In the *Supplemental Materials* document, we also present the performance of an SLX model with a matrix specified as $\mathbf{W_{2nd}}$ in addition to $\mathbf{W}$.

SLX Data-Generating Process

Our next set of experiments assesses the performance of SAR and SLX models when the spatial processes operate through dependence in the observables and there are no higher-order spatial effects. In other words, how do the models perform when the data are generated in a manner consistent with a SLX model? As before, we choose to compare the performance of SLX and SAR models based on whether they can recover the true average direct and indirect effects, calculated via the partial derivatives approach.

We expect that SLX models—not surprisingly—will perform well.[19] Our expectations for SAR models, on the other hand, are mixed. We expect the SAR to do poorly in terms of recovering the true indirect effects because it will find evidence of higher-order and feedback effects in the data that we know do not exist. We also expect that situations of spatial heterogeneity—when the explanatory variables operate through different spatial processes—will be particularly problematic for recovering the correct estimates of direct and indirect effects. We generate data for our first set of SLX DGP Monte Carlo experiments as

$$\mathbf{y} = \mathbf{x_1}\beta_1 + \mathbf{x_2}\beta_2 + \mathbf{W}\mathbf{x_1}\theta_1 + \mathbf{W}\mathbf{x_2}\theta_2 + \boldsymbol{\epsilon}, \tag{13}$$

where x_1 and x_2 are drawn from uniform distributions, $x_1 \in [-10, 10]$ and $x_2 \in [-5, 5]$, and β_1 and β_2 are set to 1. $\mathbf{W}$ is an $N \times N$ symmetric row standardized contiguity weights matrix, where each element below the main diagonal is randomly drawn from a Bernoulli distribution. In order to assess how these models perform under varying degrees of spatial heterogeneity, we conduct 1000 simulations of nine scenarios where we vary the magnitude of the spatial coefficients (θ_1 and θ_2).

The first step is to determine whether the SAR can recover the true total effects of x_1 on $\mathbf{y}$. Figure 2 shows histograms for 1000 estimated average total effects (with 95% confidence

[19]Results from these estimates are shown in Table 2 of our *Supplemental Materials* document.

intervals via the percentile method depicted with vertical dashed lines), as we vary the values of θ_1 and θ_2. More specifically, the histograms are arranged to represent variation on two critical elements of these models. Moving vertically depicts varying levels of *spatial effects*, ranging from weak (bottom row) to moderate (middle row) to strong (top row). Moving horizontally depicts varying levels of *spatial heterogeneity*, ranging from none (left column) to moderate (middle column) and substantial (right column). In each experiment, the true total effect is the sum of the direct effect ($\beta_1 = 1$) and the indirect effect (θ_1) for x_1, or 0.8, 1.4 and 0.2 (going from the bottom row to the top row). If the 95% confidence intervals (dashed vertical lines) for the calculated total effects from the SAR include the true total effects from the DGP (solid vertical lines), then we can conclude that estimating the SAR model when the true DGP is an SLX will recover the substantive effects of x_1.

[Figure 2 about here.]

Figure 2 shows that both spatial effect size and spatial heterogeneity have meaningful influences on the degree of bias for the SAR under these SLX DGPs. In nearly all of the scenarios, the true total effect appears far away from the bulk of the estimated total effects from the simulations. As one goes from no spatial heterogeneity (first column) to moderate and substantial heterogeneity (third column), the true total effect is less likely to be located near the center of the 95% confidence interval. Thus, when there are multiple patterns of spatial dependence in one model, estimating an SAR model (with one $\mathbf{W}$) is likely to provide incorrect inferences about x_1.

Moving up the rows (from weak to strong spatial effect size), the consequences for inferences become even more severe; while the SAR confidence intervals include the true total effect under weak and moderate (with the true total effect falling near the edges of the intervals for substantial heterogeneity) spatial effects, the true total effect is much smaller than those predicted by the SAR model under strong spatial effects. This should cause scholars to be cautious about using SAR models when either they expect strong spatial dependence or when they expect multiple spatial processes are at work.

While Figure 2 provides clear evidence of overestimation of the total effects, it does not identify whether the source of overestimation is related to the direct effects or indirect effects. Our expectation was that it would be caused by the latter, since the estimation of the direct effect is rather straight-forward in both types of models. Indeed, across all scenarios, the SAR model is able to recover the true average direct effects in at least 95% of the simulations.

Another source of bias in the SAR's calculation of indirect effects is the common factor restriction that forces all of the independent variables to influence the dependent variable via the same spatial multiplier, whether this makes sense or not. We can imagine circumstances where there are different causal processes (or at least different functional forms) operating across the explanatory variables. Figure 3 confirms our suspicions about the dangers of the common factor restriction in circumstances of substantial heterogeneity. The 95% confidence interval only includes the true indirect effect for x_2 (in this case, θ_2) in five of the nine scenarios. More troubling is the fact that under substantial spatial heterogeneity (last column), scholars would conclude that the indirect effects of x_2 are negative (positive), when in fact the true spatial pattern is positive (negative) spatial dependence. Because there are both positive and negative spatial patterns present in the DGP that the SAR model cannot disentangle, there is a risk that scholars will falsely conclude that a spatial pattern exists that is completely opposite of the true pattern.

[Figure 3 about here.]

We suspect that the inability of the SAR model to accurately predict indirect effects in some cases is due to the "phantom" higher-order and feedback effects found by the SAR model. Recall that our experiments are structured so that there are no higher-order effects; by only including one θ for each variable, the indirect effects are limited to being first-order effects (i.e., from first-order neighbors of i). As a result of the SAR model's infinite expansion of the spatial multiplier and the fact that weights matrices are treated as polynomials rather than higher-order contiguity matrices, the total effects of x on y will be distributed across orders of neighbors. Thus, the SAR model may find higher-order effects, whether they are there or not.

Figure 4 shows the average estimated second- (top row) and third-order (bottom row) indirect effects for x_1 from the SAR model. The three columns reflect three scenarios of weak, moderate and strong spatial effects (under no spatial heterogeneity), and the solid vertical line shows the true higher-order effects (0). The histograms in Figure 4 demonstrate that the SAR model will often incorrectly discover higher-order (and feedback) effects when they are not actually present (with the lone exception being the third-order effects under weak spatial effects). Indeed, the size of these phantom effects increases with the strength of spatial dependence. The issue is two-fold. First, the DGP is strictly exogenous, so there is no reason for unit i's neighbors to simultaneously influence unit i. Second, there are no higher-order effects within the DGP. The SAR model thus produces inferences about feedback and global effects that are not warranted by the true DGP. It is clear that there is a substantial inferential penalty for estimating an SAR model on an SLX DGP.

[Figure 4 about here.]

Application

If scholars believe that strategic responses occupy a central place in almost any explanation of military expenditures (e.g., Richardson 1960; Palmer 1990; Powell 1999), then spatial econometric models are appropriate. The advancement of spatial econometric models means that scholars can model the precise manners in which choices (such as military expenditures) or conditions (such as instability) spill over into neighboring countries (see, for instance, Shin and Ward 1999; Flores 2011; Plümper and Neumayer 2015). Of course, theory might suggest that defense burdens in i directly influence defense burdens in i's neighbors through a single global autocorrelation coefficient (SAR); or, defense burdens in i influence defense burdens in i's neighbors through multiple patterns of connectivities and/or with a temporal delay (SLX); or, defense burdens in i are unrelated to defense burdens in i's neighbors (non-spatial OLS). Though theory can guide this decision, the true process driving military spending is unknown. We argue that a reasonable starting place given this inherent uncertainty is the SLX model since it provides a more flexible specification for estimating spatial heterogeneity and because it avoids some of the problematic assumptions hidden in SAR models.

To provide an applied example of the modeling choices that we discussed and simulated above, we collected data from 1953–2008 in 193 countries, both developed and developing, democratic and authoritarian. We measured defense spending in a manner similar to other scholars (Whitten and Williams 2011; Phillips 2015) as a country's defense burden, or military expenditures as a percentage of GDP. We use the measure developed by Phillips (2015), who uses expenditure data from both the Stockholm International Peace Research Institute and the Correlates of War national material capabilities data (Singer and Small 1972) divided by GDP data from the Penn World Table Version 6.3 (Heston, Summers, and Aten 2009). Both theories of budgeting and previous research tell us that defense burdens are highly autoregressive; the budget in one year is highly dependent on the previous year, plus responses to short-term fluctuations.[20] As a result, the defense burden variable is non-stationary in most countries.[21] This poses a substantial problem for inferences since it increases the risk of spurious results (Granger and Newbold 1974). Since defense burdens are first-order integrated, taking the first difference makes the series stationary. To further control for the

[20] This is a general feature of budgeting processes in organizations. As Ostrom and Marra (1986, 822) note, "given the complexity of the behavior under examination and the cognitive limitations of decision makers, it is highly unlikely that the budget decisions of any of the organizations are rebuilt from zero each year".

[21] We estimate separate augmented Dickey-Fuller tests for each country with a long enough time series to get stable test statistics. In the original *defense burden* variable we can reject the null hypothesis of a unit root at the 90% confidence level in only 29% of the countries. We reject the null hypothesis in 99% of the countries for the $\Delta defense\ burden_{t-1}$ series.

dynamics of defense burden, we include the level in the previous year ($defense\ burden_{t-1}$) and the previous change ($\Delta defense\ burden_{t-1}$).[22]

For the purposes of comparing different ways of estimating spatial econometric models, we begin with a simple model of defense burden. We include *total population (logged)* to account for larger countries having larger defense burdens. We also take into account the impact of domestic and international instability by including a dichotomous variable representing the presence of a civil war that is an armed conflict resulting in at least 25 battle-related deaths in a year (Allansson, Melander, and Themner 2017), and interstate war (coded 1 if the hostility level of the militarized interstate dispute reaches 5). We expect that both interstate wars and civil wars will have a meaningful impact on defense burdens. A large portion of the time period under examination occurs in the Cold War, where military spending is driven by alliance commitments and actions of the two superpowers (for a review of this literature, see Goldsmith (2003, 557–559)). We thus believe that the degree of responsiveness to the superpowers' changes in spending will be conditioned by the presence of an alliance with the US (Plümper and Neumayer 2015). To take this into account, we include two interaction variables (and their lower-order terms) made up of *alliance with US* (coded 1 if the state has an alliance with the US according to the Correlates of War), Δ*US military expenditures*, and Δ*USSR/Russian military expenditures*. Finally, Plümper and Neumayer (2010) demonstrate that unmodeled spatial and temporal shocks can falsely suggest spatial dependence. To guard against this possibility, we include *trend* (which is coded 1 for 1950 and increases each year), a dichotomous variable for *1992*[23], and regional variables. Though there are a variety of ways of specifying the connections between countries (for some alternatives, see below), for

[22] There is a considerable debate about the casual use of lagged dependent variables to control for temporal dependence (see Wilkins (2018) for an overview of this debate).

[23] In a model with annual fixed effects, 1992 is the only year that is statistically different from the others.

the simple preliminary analysis we use a binary, un-row-standardized[24] contiguity[25] weights matrix.

We first compare the results of the non-spatial OLS model (first column) to the SAR model (second column) in Table 3. The results clearly show that ignoring spatial dependence forces OLS to overestimate the immediate impacts of the covariates; almost all of the OLS coefficients are larger (either more negative or more positive) than their SAR counterparts (Ward and Gleditsch 2008: 68-69). Most notably, the non-spatial OLS would lead scholars to falsely conclude that there is no spatial dependence in states' defense burdens. The SAR model corrects this mistake. As expected, our estimate of the global spatial autocorrelation coefficient, ρ, is statistically significant and positive, indicating that the defense burdens of contiguous states are positively correlated. In other words, we have found evidence that increases to one state's defense burden *simultaneously* induce increases in their neighbors' defense burdens (see also Flores 2011). The result is that covariates, such as civil and interstate wars, influence defense burdens directly (via the βs) and indirectly through spatial diffusion (via ρ).[26]

[Table 3 about here.]

However, as we discussed above, the various assumptions underpinning SAR models may not be consistent with how these processes actually work. Most notably, it is possible that to the extent that spillovers occur, they occur with a delay. Indeed, due to incrementalism in budgeting, defense outlays are likely to be highly path dependent and thus responsive to other countries' outlays with a temporal lag. In other words, if states are responsive

[24]As Neumayer and Plümper (2016) note, the decision to row-standardize the weights matrix should follow closely from one's theory. In the case of defense burdens, we believe that all countries are not equally influenced by their neighbors' burdens. Instead, the degree to which one state is influenced is based partly on the number of neighbors.

[25]Only those states that are contiguous by land or river are coded as neighbors, according to the Correlates of War project.

[26]In the *Supplemental Materials* we move beyond the coefficients to provide a more in-depth exploration of total, direct and indirect effects.

to the spending patterns by other states, responses will occur one-year later rather than simultaneously. Based on this clear distinction, there is ample justification for estimating an SLX model with temporal lags rather than with an endogenous SAR model where spatial effects are instantaneous.

In the case of defense spending, we believe that the SLX model offers the added benefit of allowing one to properly model spatial heterogeneity. More specifically, as outlined above, the SLX model offers a great deal of flexibility in specifying the particular spatial patterns (if any) that are theorized to govern different variables. This heterogeneity can be accomplished by determining which other countries are important spatially (via the $\mathbf{W}$) and by allowing the strength (and sign) of the spatial clustering to vary by variable. This flexibility is particularly helpful when it comes to modeling conditional patterns of spatial dependence, and varying degrees of higher-order effects. We think this is more appropriate than estimating one global spatial autocorrelation coefficient (ρ) that dictates the same spatial process for all the variables.

To demonstrate the flexibility in modeling various spatial patterns, we add the SLX variables derived from three weights matrices (contiguity, alliance commitments and defense pacts) to reflect the following expectations:

- We expect that civil wars in neighboring states will have an impact on states' defense burdens, but there are possibly conflicting reasons for the impact (Phillips 2014). To account for these spatial patterns, we include $\textit{contiguity} \times \textit{civil war}_{t-1}$.
- States are likely to increase their defense burdens in response to interstate wars that either increase their own instability or demand that they fulfill alliance commitments. To account for these expectations, we include $\textit{contiguity} \times \textit{interstate war}_{t-1}$ and $\textit{alliance} \times \textit{interstate war}_{t-1}$.
- Temporally-lagged spatial lag (TLSL) variables measure the influence of neighbors' defense burdens in the previous year. We believe that states will respond positively

to prior military spending patterns by both contiguous states and those states with which they share a defense pact. We include *contiguity*$\times$*defense burden*$_{t-1}$ and *defense pact*$\times$*defense burden*$_{t-1}$.

Results from the SLX model (Model 2 in Table 3) show that a flexible estimation technique (such as SLX) is necessary to derive accurate inferences about spatial patterns. Recall that the SAR model is limited to one specification of the neighbors' connections (via $\mathbf{W}$) and one global spatial autocorrelation parameter (ρ).[27] The SLX model shows that both of these constraints are questionable at best. It reveals that countries' defense burdens are connected in more than one way; a single covariate (such as *defense burden*$_{t-1}$) can exhibit spatial dependence through multiple weights matrices (in this case, contiguity and defense pacts), and the nature of this connectivity may differ across covariates (i.e., *interstate war*$_{t-1}$ and *defense burden*$_{t-1}$).

The second constraint of the SAR model—that the single ρ parameter accurately reflects the patterns of spatial dependence—is also potentially damaging. The coefficients for the spatial estimates (θs) are statistically different (at the 95% confidence level) in some cases, which means that, for example, the effects of contiguity are more negative for civil wars than interstate wars, and interstate war can influence defense burdens differently based on either contiguity or alliance patterns.[28] Since the coefficients for the two SLX variables for *defense burden*$_{t-1}$ are in the opposite direction as the direct effect, it points to a different spillover story than the one from the SAR model. In both models, increases in i's *defense burden* at $t-1$ lead to decreases at time t for i ($\beta = -0.14$ in the SLX model and $\beta = -0.12$ in the SAR model). In the SLX model, an increase in i's defense burden at $t-1$ positively spills over in the defense burdens of i's neighbors at t ($\theta_1 = 0.006$ and $\theta_2 = 0.003$); in the SAR

[27] This is the same obstacle that Flores (2011) faces. He makes a number of statements suggesting that countries' responses to their neighbors' defense burdens are conditioned by circumstances such as alliance commitments and neighborhood size. These patterns of conditional spatial dependence are quite reasonable, but Flores (2011) is unable to make those inferences from SAR models with one global autocorrelation parameter.

[28] In the former case, F-tests suggest that we can reject the null of equal coefficients at the 99% confidence level; in the latter case, we can reject the null of equal coefficients at the 95% confidence level.

model, an increase in i's defense burden at $t-1$ decreases its defense burden at t, which leads to decreases in i's neighbors due to positive spatial dependence ($\rho = 0.07$). The two models therefore give contrasting explanations of the influence of *defense burden* in neighbors depending on whether y_j is influenced by x_i (in the SLX), or y_i (in the SAR). The fact that the direct and indirect coefficients are different signs in the SLX model implies that the SAR model (with its global autocorrelation coefficient) cannot accurately reflect the spatial patterns present in $defense\ burden_{t-1}$.

This exploration of defense spending highlights a significant advantage of SLX models related to spatial heterogeneity. In this case, states' defense burdens are spatially dependent in more than one way, and the strength of spatial dependence varies across connection type and covariate. Both of these—combined with the ability to model higher-order and conditional spatial effects[29]—suggest that SLX models offer the key to unlocking a treasure trove of spatial inferences.

Paths Forward

From our discussion and findings above it is clear that we are advocates of the simplicity offered by the SLX approach. Our goal, however, is not to suggest that the SLX is a superior model for all spatial econometric enterprises. Rather, we advocate for this approach based on several important realities when it comes to the adoption of spatial models in political science. First, many applied researchers are not exposed to spatial models in a way that is complementary to their training (i.e., building from basic linear models). Likewise, many researchers interested in spatial models likely find the current offerings of software tools insurmountable to getting started.[30] Without more accessibility widespread adoption is less

[29] In the *Supplemental Materials* we first reveal that the extent to which civil wars drive military spending in nearby states depends on the particular region, and then demonstrate how to incorporate higher-order effects in the SLX model.

[30] It is worth pointing out that this is changing with more recent releases of Stata and continual updating of the available R packages. On the other hand, estimation and interpretation are still far less straightforward

likely. How then, should applied researchers proceed when choosing the spatial model that best fits their theory?

A variety of typologies exist for categorizing spatial processes (e.g., Darmofal (2015); Vega and Elhorst (2015)), but a particularly elegant typology is demonstrated by Cook, Hays, and Franzese (2015). The authors note that spatial dependence can arise from spatial clustering in the outcomes (i.e., $\rho \mathbf{W} y$), spatial clustering in the observables (i.e., $\mathbf{WZ}\boldsymbol{\theta}$), or spatial clustering in the unobservables (i.e., $\lambda \mathbf{W} \mu$).These typologies demonstrate, among other things, that there are more than just two models to consider on the spatial econometric menu. Without repeating the full listing, several of these models have particular relevance for our discussion in this paper—especially when it comes to model choice. The so-called Spatial Durbin model (SDM) is in some ways a middle ground between an SLX and SAR because it allows for modeling spatial dependence both between outcomes (e.g., y_j, y_i) and between predictors and outcomes (i.e., x_j, y_i) (Elhorst 2014: 9):

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \rho \mathbf{W}\mathbf{y} + \mathbf{WZ}\boldsymbol{\theta} + \boldsymbol{\epsilon} \tag{14}$$

Cook, Hays, and Franzese (2015) suggest starting with a more general spatial model and then testing the restrictions to pare down the model of unnecessary spatial components. Since it is difficult to recover estimates of the general nesting spatial (GNS) model (i.e., one with estimates of all three spatial parameters) due to weak identification, Cook, Hays, and Franzese (2015) suggest two strategies depending on the goal of the project. If there is concern about accurately characterizing the spatial process, then a spatial autocorrelation model is appropriate; if there is concern about getting accurate estimates of the effects of covariates, then the spatial Durbin model is appropriate. Our approach is similar in that we advocate thinking carefully about theory and whether there are any expectations for higher-order effects, specifying the theoretically-motivated model, and then testing the accuracy

than competing approaches of dealing with spatial heterogeneity—many of which aim to treat it is as nuisance rather than substance.

of those model specifications. If any differences between our approaches arise, it is the point of departure—which we advocate in most political science applications is more closely approximated by the SLX model. This is especially the case in the context of the SAR being the only relatable alternative.

Our advice for the applied researcher is to first follow theory and apply the most analogous spatial model. If for example, the goal is to account for all spatial dependence then the SDM and/or other alternatives (e.g., spatial error models) that capture and eliminate spatial noise in the unobservables may be more useful. It is often the case, however, that one's theory does not provide clear guidance and available diagnostics are uncertain. Which of the two prominent models is less prone to inferential errors as a result of misspecifying the type of dependence? In the *Supplemental Materials* document we provide evidence that the SLX model is more robust to misspecifying the source of the spatial dependence. Additional Monte Carlo analysis based on a DGP with dependence in the error terms (one that is consistent with a spatial error model) shows that simple SLX specifications correctly recover true average effects of variables at expected rates. SAR models, on the other hand, have unsatisfactory recovery rates (i.e., much lower than 95%) for first-order indirect effects, and as a result, the total effects. The SAR is only really appropriate for those theories that specify global dependence in the outcomes among all units—something that we demonstrate is relatively rare in political science. When in doubt, political scientists are well-served to start with an SLX.

Conclusion

The increased use of spatial models in political science is a welcome trend that involves the relaxation of the incredible assumption that there are no contagion or spillover effects across our observations. However, as our review of this literature demonstrates, there is often a substantial disjuncture between expressed theories of spatial relationships and the models that political scientists estimate. This disjuncture has been particularly prevalent in projects that use the SAR model. We have shown that the SLX model is a good starting point for spatial econometrics in political science because of its flexibility in model specification, ease in estimation, and simplicity in interpretation.

Building on previous works, we provide a comprehensive comparison of the limits of both the SAR and SLX models using simulations. Our Monte Carlo analyses indicate that, even if the true DGP is SAR, the SLX performs quite well at detecting spatial relationships at typically-observed levels. The same cannot be said of the SAR when the true DGP is SLX. This is especially the case if there are even relatively small amounts of heterogeneity in terms of the spatial effects across independent variables in the DGP. Our suggested approach places the power of specification in the researcher's hands and the ability to turn unrealistically imposed assumptions from SAR models into testable propositions.

Our findings in this study lead us to a set of four considerations that researchers should keep in mind when employing spatial models to test their theories. First, researchers should carefully consider the nature of spatial dependence in their theories and choose a model that best reflects their expectations of how these processes work. Second, unless their theories are about global dependence among outcomes, they should start with a simple SLX model and work from there towards more complex models and specifications as appropriate. Third, when they suspect that the spatial diffusion processes are heterogeneous, researchers should employ the SLX with varied specifications of matrices based on these expectations. And fourth, researchers should move beyond a cursory discussion of the direction and statistical significance of the spatial parameters to an interpretation of all the estimated effects (i.e., direct, indirect and total).

Although we have critiqued the choice of the SAR model for many political science applications, we do not begrudge its use under the proper circumstances—theorized endogenous and global dependence among outcomes. In this paper, we aimed to move the spatial revolution in political science to a place where researchers are more easily able to implement, interpret, and understand the applications. The SLX model fills this void by providing a more flexible mapping onto most political science theoretical expectations and by presenting fewer challenges to specification and interpretation.

References

Allansson, Marie, Erik Melander, and Lotta Themner. 2017. "Organized Violence, 1989–2016." *Journal of Peace Research* 54 (4): 574–587.

Beck, Nathaniel, Kristian Skrede Gleditsch, and Kyle Beardsley. 2006. "Space is More Than Geography: Using Spatial Econometrics in the Study of Political Economy." *International Studies Quarterly* 50 (1): 27–44.

Bohmelt, Tobias, Lawrence Ezrow, Roni Lehrer, and Hugh Ward. 2016. "Party Policy Diffusion." *American Political Science Review* 110 (2): 397–410.

Buhaug, Halvard, and Kristian Skrede Gleditsch. 2008. "Contagion or Confusion? Why Conflicts Cluster in Space." *International Studies Quarterly* 52 (2): 215–233.

Cook, Scott J., Jude C. Hays, and Robert J. Franzese. 2015. Model Specification and Spatial Interdependence. In *APSA Conference Paper*.
URL: *http://www.sas.rochester.edu/psc/polmeth/papers/Cook_Hays_Franzese.pdf*

Darmofal, David. 2015. *Spatial Analysis for the Social Sciences.* Cambridge University Press.

Drolc, Cody, Christopher Gandrud, and Laron K. Williams. 2019. "Taking Time (and Space) Seriously: How Scholars Falsely Infer Policy Diffusion from Model Misspecification." *Policy Studies Journal.* Forthcoming.

Elhorst, J Paul. 2014. "Linear Spatial Dependence Models for Cross-Section Data." In *Spatial Econometrics.* Springer pp. 5–36.

Flores, Alejandro Quiroz. 2011. "Alliances as Contiguity in Spatial Models of Military Expenditures." *Conflict Management and Peace Science* 28 (4): 402–418.

Franzese, Robert J., Jr., and Jude C. Hays. 2007. "Spatial Econometric Models of Cross-Sectional Interdependence in Political Science Panel and Time-Series-Cross-Section Data." *Political Analysis* 15 (2): 140–164.

Garcia, Blake E., and Cameron Wimpy. 2016. "Does Information Lead to Emulation? Spatial Dependence in Anti-Government Violence." *Political Science Research and Methods* 4 (1): 27–46.

Gassebner, Martin, Noel Gaston, and Michael J Lamla. 2011. "The Inverse Domino Effect: Are Economic Reforms Contagious?" *International Economic Review* 52 (1): 183–200.

Gibbons, Stephen, and Henry G. Overman. 2012. "Mostly Pointless Spatial Econometrics." *Journal of Regional Science* 52 (2): 172–191.

Goldsmith, Benjamin E. 2003. "Bearing the Defense Burden, 1886-1989: Why Spend More?" *Journal of Conflict Resolution* 47 (5): 551–573.

Goldsmith, Benjamin E. 2007. "Arms Racing in 'Space': Spatial Modelling of Military Spending around the World." *Australian Journal of Political Science* 42 (3): 419–440.

Granger, Clive W.J., and Paul Newbold. 1974. "Spurious Regressions in Econometrics." *Journal of Econometrics* 2 (2): 111–120.

Gray, Virginia. 1973. "Innovation in the States: A Diffusion Study." *American Political Science Review* 67 (4): 1174–1185.

Hendry, David F. 1995. *Dynamic econometrics*. Oxford University Press on Demand.

Heston, Alan, Robert Summers, and Bettina Aten. 2009. Penn World Table Version 6.3. Technical report Center for International Comparisons of Production, Income and Prices at the University of Pennsylvania.
URL: *http://datacentre2.chass.utoronto.ca/pwt/*

Knight, Brian. 2013. "State Gun Policy and Cross-State Externalities: Evidence from Crime Gun Tracing." *American Economic Journal: Economic Policy* 5 (4): 200–229.

LeSage, James P., and R. Kelley Pace. 2009. *Introduction to Spatial Econometrics*. Boca Raton: Chapman & Hall/CRC.

Midlarsky, Manus I., Martha Crenshaw, and Fumihiko Yoshida. 1980. "Why Violence Spreads: The Contagion of International Terrorism." *International Studies Quarterly* 24 (2): 262–298.

Moore, Will H., and Stephen M. Shellman. 2007. "Whither Will They Go? A Global Study of Refugees' Destinations, 1965–1995." *International Studies Quarterly* 51 (4): 811–834.

Neumayer, Eric, and Thomas Plümper. 2010*a*. "Galton's Problem and Contagion in International Terrorism along Civilizational Lines." *Conflict Management and Peace Science* 27 (4): 308–325.

Neumayer, Eric, and Thomas Plümper. 2010*b*. "Spatial Effects in Dyadic Data." *International Organization* 64 (1): 145–166.

Neumayer, Eric, and Thomas Plümper. 2016. "W." *Political Science Research and Methods* 4 (1): 175–193.

Neumayer, Eric, Thomas Plümper, and Mariaelisa Epifanio. 2014. "The "Peer-Effect" in Counterterrorist Policies." *International Organization* 68 (1): 211–234.

Obinger, Herbert, and Carina Schmitt. 2011. "Guns and Butter?: Regime Competition and the Welfare State during the Cold War." *World Politics* 63 (2): 246–270.

Ostrom, Charles W., and Robin F. Marra. 1986. "Us Defense Spending and the Soviet Estimate." *American Political Science Review* 80 (3): 819–842.

Palmer, Glenn. 1990. "Alliance Politics and Issue Areas: Determinants of Defense Spending." *American Journal of Political Science* 34: 190–211.

Phillips, Brian J. 2015. "Civil War, Spillover and Neighbors' Military Spending." *Conflict Management and Peace Science* 32 (4): 425–442.

Plümper, T., and E. Neumayer. 2010. "Model Specification in the Analysis of Spatial Dependence." *European Journal of Political Research* 49 (3): 418–442.

Plümper, Thomas, and Eric Neumayer. 2015. "Free-Riding in Alliances Testing: An Old Theory with a New Method." *Conflict Management and Peace Science* 32 (3): 247–268.

Powell, Robert. 1999. *In the Shadow of Power: States and Strategies in International Politics*. Princeton University Press.

Richardson, Lewis F. 1960. *Arms and Insecurity*. Pittsburg, PA: Boxwood.

Shin, Michael, and Michael D. Ward. 1999. "Lost in Space: Political Geography and the Defense-Growth Trade-off." *Journal of Conflict Resolution* 43 (6): 793–817.

Simmons, Beth A., and Zachary Elkins. 2004. "The Globalization of Liberalization: Policy Diffusion in the International Political Economy." *American Political Science Review* 98 (1): 171–189.

Singer, J. David, and Melvin Small. 1972. *The Wages of War, 1816-1965: A Statistical Handbook*. New York: John Wiley & Sons.

Vega, Solmaria Halleck, and J. Paul Elhorst. 2015. "The SLX Model." *Journal of Regional Science* 55 (3): 339–363.

Ward, Michael D., and Kristian Skrede Gleditsch. 2008. *Spatial Regression Models*. Thousand Oaks: Sage.

Whitten, Guy D., and Laron K. Williams. 2011. "Buttery Guns and Welfare Hawks: The Politics of Defense Spending in Advanced Industrial Democracies." *American Journal of Political Science* 55 (1): 117–134.

Whitten, Guy D., Laron K. Williams, and Cameron Wimpy. n.d. "Interpretation: The Final *Spatial* Frontier." *Political Science Research and Methods* Forthcoming.

Wilkins, Arjun S. 2018. "To Lag or Not to Lag?: Re-evaluating the Use of Lagged Dependent Variables in Regression Analysis." *Political Science Research and Methods* 6 (2): 393–411.

Williams, Laron K., and Guy D. Whitten. 2015. "Don't Stand So Close to Me: Spatial Contagion Effects and Party Competition." *American Journal of Political Science* 59 (2): 309–325.

Table 1: Applications of Spatial Econometrics in Political Science

	Spatial Theory Type	
Model Used	Global	Local
SAR	80.0%	46.1%
	(4)	(41)
SLX	0.0%	38.2%
	(0)	(34)
Complex	20.0%	10.1%
	(1)	(9)
Other	0.0%	5.6%
	(0)	(5)

Note: This table summarizes the 94 studies citing Beck, Gleditsch and Beardsley (2006) and/or Franzese and Hays (2007) that reported results from at least one spatial econometric model. Cell entries are column percentages on top of raw totals in parentheses.

Table 2: Recovery Rates for Various SLX Model Specifications: SAR Data-Generating Process

	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
Simple SLX model: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta + \boldsymbol{\epsilon}$									
Direct: Total	94.9	94.4	94.8	93.7	95.5	93.5	95.0	94.0	88.9
0 Order	77.9	91.9	95.1	93.5	95.5	93.4	94.8	90.7	81.0
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	15.1	49.9	85.2	93.7	95.2	93.5	40.3	0	0
1st Order	91.4	93.8	95.5	93.1	95.2	94.5	94.4	93.8	93.1
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with a squared term: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W^2}\mathbf{x}\theta_2 + \boldsymbol{\epsilon}$									
Direct: Total	95.0	94.0	95.1	93.8	95.4	94.0	95.2	94.4	91.0
0 Order	94.2	95.1	95.2	94.2	93.6	94.8	94.8	94.9	95.8
2nd Order	93.2	93.9	94.7	94.6	94.0	94.7	93.6	93.4	93.8
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	93.0	94.7	94.8	94.7	93.9	94.5	93.3	83.7	2.6
1st Order	93.0	94.9	95.9	94.0	94.9	94.9	94.4	93.2	92.6
2nd Order	93.2	93.9	94.7	94.6	94.0	94.7	93.6	93.4	93.8
SLX model with squared and cubed terms: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W^2}\mathbf{x}\theta_2 + \mathbf{W^3}\mathbf{x}\theta_3 + \boldsymbol{\epsilon}$									
Direct: Total	95.0	94.2	94.6	94.3	96.1	94.4	94.5	93.7	93.4
0 Order	94.2	95.1	95.3	94.1	93.9	94.9	94.3	94.9	95.9
2nd Order	93.9	94.0	94.8	95.2	94.2	95.2	93.8	93.2	93.3
3rd Order	94.2	93.6	94.7	93.5	95.0	94.4	95.0	93.9	93.4
Indirect: Total	95.5	95.1	95.6	94.4	95.2	94.5	93.9	92.6	87.0
1st Order	94.3	94.7	95.1	93.9	94.9	93.7	93.9	95.2	94.4
2nd Order	93.9	94.0	94.8	95.2	94.2	95.2	93.8	93.2	93.3
3rd Order	94.2	93.6	94.7	93.5	95.0	94.4	95.0	93.9	93.4

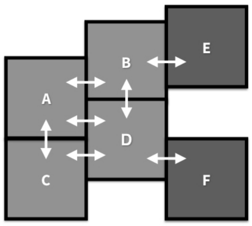
Table 3: Non-Spatial OLS, SAR and SLX Models of Neighborhood Effects on Defense Burdens

	OLS Model 1	SAR Model 2	SLX Model 3
Spatial Estimates (ρ and θ)			
ρ		0.07***	
		(0.004)	
Contiguity×Civil War$_{t-1}$			-0.11*
			(0.07)
Contiguity×Interstate War$_{t-1}$			0.18***
			(0.07)
Ally×Interstate War$_{t-1}$			-0.006
			(0.03)
Contiguity×Defense Burden$_{t-1}$			0.006***
			(0.002)
Defense Pact×Defense Burden$_{t-1}$			0.003***
			(0.001)
Non-Spatial Estimates (β)			
Civil War$_{t-1}$	-0.47***	-0.45***	-0.39***
	(0.17)	(0.16)	(0.15)
Interstate War$_{t-1}$	0.46***	0.45***	0.37**
	(0.16)	(0.16)	(0.16)
Total Population (Logged)$_{t-1}$	0.03	0.03	0.02
	(0.02)	(0.02)	(0.02)
Alliance with US	-0.21**	-0.22**	-0.15*
	(0.10)	(0.09)	(0.08)
ΔUS Defense Burden	-0.04	-0.03	-0.05
	(0.07)	(0.07)	(0.08)
ΔUSSR/Russia Defense Burden	0.006	-0.01	-0.05***
	(0.02)	(0.02)	(0.02)
US Ally×Δ US Defense Burden	0.14	0.11	0.21
	(0.12)	(0.11)	(0.13)
US Ally×Δ USSR Defense Burden	0.005	0.01	0.02
	(0.03)	(0.03)	(0.02)
Annual Trend	-0.007***	-0.006***	-0.004**
	(0.002)	(0.002)	(0.002)
1992	1.13***	0.74***	-1.26***
	(0.28)	(0.27)	(0.21)
Defense Burden$_{t-1}$	-0.12***	-0.12***	-0.14***
	(0.006)	(0.01)	(0.006)
ΔDefense Burden$_{t-1}$	-0.14***	-0.13***	-0.14***
	(0.01)	(0.02)	(0.01)
Constant	0.46**	0.41*	0.22
	(0.22)	(0.21)	(0.17)
N	6,328	6,328	7,266

Note: Models include regional fixed effects. The SAR model excludes isolates.
* p-value < 0.1; ** p-value < 0.05; *** p-value < 0.01

Figure 1: Spatial Arrangement with Associated Squared ($\mathbf{W}^2$) and Second-Order ($\mathbf{W}_{2^{nd}}$) Weights Matrices

Map



W

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	0	1	1	0
C	1	0	0	1	0	0
D	1	1	1	0	0	1
E	0	1	0	0	0	0
F	0	0	0	1	0	0

First-Order Neighbors

$\mathbf{W}^2$

	A	B	C	D	E	F
A	3	1	1	2	1	1
B	1	3	2	1	0	1
C	1	2	2	1	0	1
D	2	2	1	4	1	0
E	1	0	0	1	1	0
F	1	1	1	0	0	1

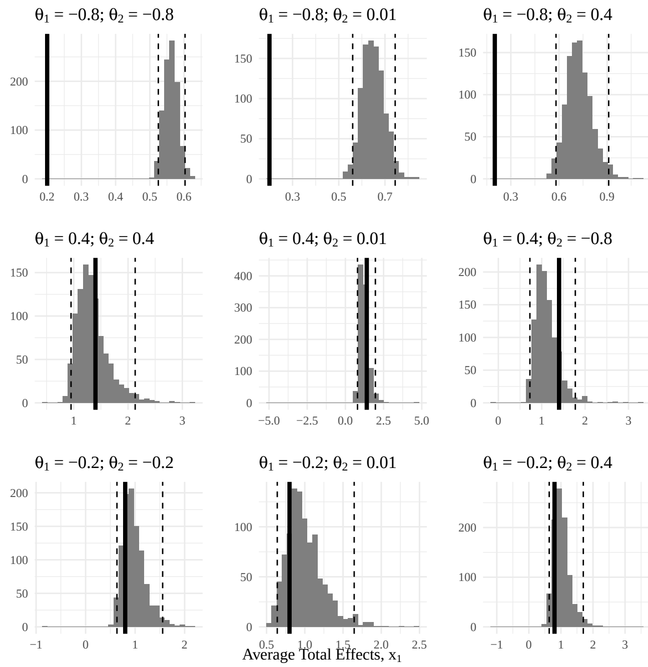
First-Order Neighbors (Squared)

$\mathbf{W}_{2^{nd}}$

	A	B	C	D	E	F
A	0	0	0	0	1	1
B	0	0	1	0	0	1
C	0	1	0	0	0	1
D	0	0	0	0	1	0
E	1	0	0	1	0	0
F	1	1	1	0	0	0

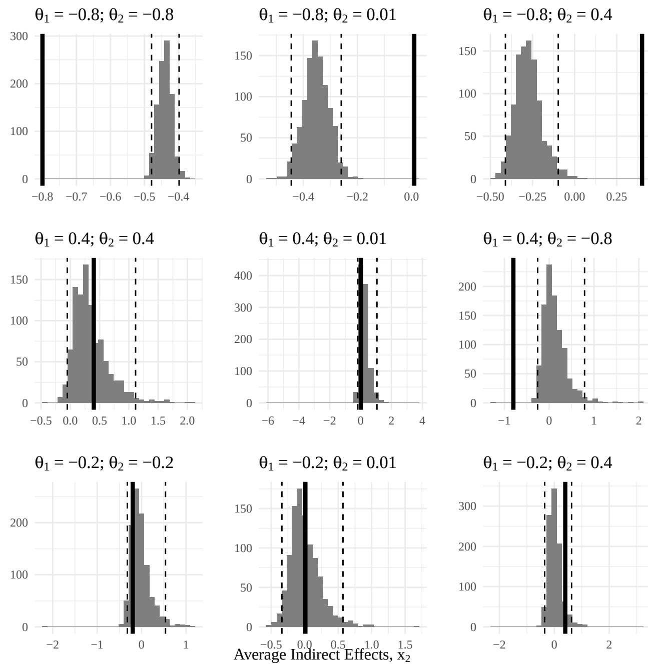
Second-Order Neighbors

Figure 2: Recovery of the True Total Effects of x_1 in the SAR Model across Strength of Spatial Autocorrelation and Spatial Heterogeneity



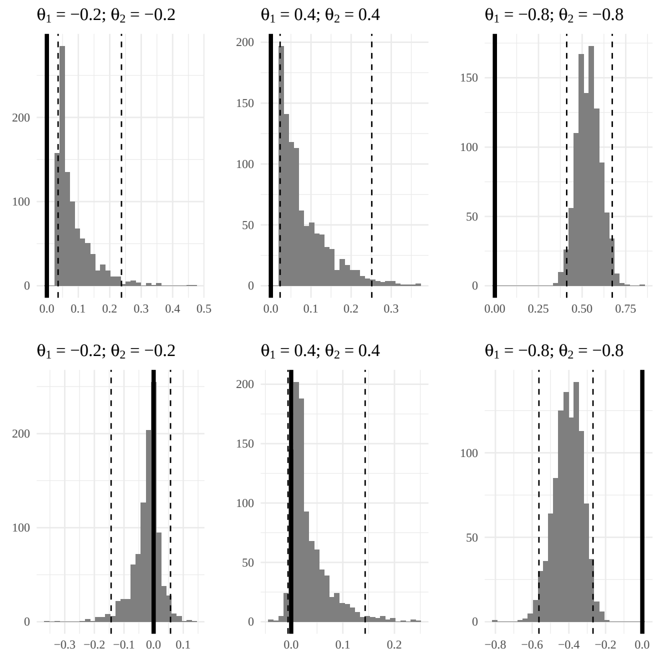
Note: θ values refer to those from the SLX Monte Carlo simulations. Vertical lines represent 95% confidence intervals (dashed) and the true total effect of x_1 (solid).

Figure 3: Recovery of the True Indirect Effects of x_2 in the SAR Model across Strength of Spatial Autocorrelation and Spatial Heterogeneity



Note: θ values refer to those from the SLX Monte Carlo simulations. Vertical lines represent 95% confidence intervals (dashed) and the true indirect effect for x_2 (solid).

Figure 4: SAR Model Indicates the Presence of Higher-Order Indirect Effects for x_1 When They Are Not There



Note: θ values refer to those from the SLX Monte Carlo simulations. Vertical lines represent 95% confidence intervals (dashed) and the true higher-order effects for x_1 (solid).

X Marks the Spot:
Unlocking the Treasure of Spatial-X Models
Supplemental Materials

Cameron Wimpy Laron K. Williams Guy D. Whitten

Time lag spatial lag models

In our paper we describe a model with a temporally lagged spatial lag model as being a variant of an SLX model rather than a SAR model. This is because such a model does not have one of the defining characteristics of SAR models, endogenous regressors. As Lesage and Pace (2009: 192) write, such a model "relies on past period dependent variables and contains no simultaneous spatial interaction." In this section we provide some more details about models with temporally lagged spatial lag model terms and how the results from them are interpreted. To do this, we follow a set of notational and presentational conventions used in Chapter 4 of Elhorst (2014). In that chapter, Elhorst provides a taxonomy of a set of models that are dynamic in space and time.

Elhorst breaks the effects estimated by such models into four different categories: short-term direct effects, short-term indirect effects, long-term direct effects, and long-term indirect effects. The distinction between short-term and long-term effects is a common feature of time series models while, as we discuss in our paper, the distinction between direct and indirect effects is a common feature of spatial models.

For our purposes, we will consider a set of models beginning with one that is not dynamic in either time or space and then a select set of models that are dynamic in only one dimension before discussing the temporally lagged spatial lag model which is dynamic in both dimensions. Across all of these models we will assume a common panel data structure with observations that vary across units and over time. If we follow Elhorst's notational convention of using only temporal subscripts, a simple regression model without any spatial or temporal dynamics would be written as

$$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t \tag{1}$$

which we call a "non-spatial static model."[1] Because this model has no temporal dynamics, all effects estimated from it will be short-term and because this model has no spatial dynamics,

[1] Following this notational convention is convenient because it allows the temporal dimension for each term to be indentified by the subscripts and the spatial dimension of each term to be identified by the presence or absence of the connectivity matrix ($\mathbf{W}$). $\mathbf{y}_t$ and $\mathbf{X}_t$ contain the N observations for each unit at

all effects from it will be direct. Thus for a one unit increase in a particular independent variable, $\mathbf{x}_k$, the effect will simply be an immediate (short-term direct) increase of β_k.

If we add temporal dynamics to Equation 1 in the form of a lagged dependent variable, our model becomes

$$\mathbf{y}_t = \mathbf{y}_{t-1}\phi + \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t \tag{2}$$

which we will call a "non-spatial dynamic model." For a one unit increase in a particular independent variable, $\mathbf{x}_k$, there will now be both a short-term effect of β_k and a long-term effect of $\beta_k(1-\phi)^{-1}$. From a spatial perspective, both of these effects are direct because they are caused by changes in the value of the independent variable in one unit on the value of the dependent variable for that same unit.

If we add spatial dynamics to Equation 1 in the form of a spatially lagged independent variable, our model becomes

$$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \mathbf{W}\mathbf{Z}_t\boldsymbol{\theta} + \boldsymbol{\epsilon}_t \tag{3}$$

which we will call a "temporally static SLX" model.[2] As the name implies, the effects from such a model will all be short-term. For a one unit increase in a particular independent variable, $\mathbf{x}_k$, there will be a short-term direct effect of β_k. This model, of course, also has indirect effects are come from the $\mathbf{W}\mathbf{Z}_t\boldsymbol{\theta}$. Thus, for instance, the effects of a global increase in a spatially-specified independent variable, $\mathbf{z}_k$, would be $\mathbf{W}\theta_k$.

If we add spatial dynamics to Equation 1 in the form of a spatially lagged dependent variable, our model becomes

$$\mathbf{y}_t = \rho\mathbf{W}\mathbf{y}_t + \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t \tag{4}$$

which we will call a "temporally static SAR" model. As with the temporally static SLX, the effects from such a model will all be short-term. The direct effect from a SAR model is $[(\mathbf{I}_N - \rho\mathbf{W})^{-1}\boldsymbol{\beta}_k\mathbf{I}_N]^{\bar{d}}$, where following Elhorst's notation, $\bar{d}$ is a calculation of the mean diagonal element of a matrix.[3] And the indirect effects from such a model are $[(\mathbf{I}_N - \rho\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\overline{rsum}}$, where $\overline{rsum}$ is a calculation of the mean row sum of the non diagonal elements of a matrix.

Turning to the model of interest, we write the temporally lagged spatial lag model as

$$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \mathbf{W}\mathbf{y}_{t-1}\theta + \boldsymbol{\epsilon}_t \tag{5}$$

time t so that $\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t$ expands into $\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \vdots \\ \mathbf{y}_T\end{bmatrix} = \begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \vdots \\ \mathbf{X}_T\end{bmatrix} + \begin{bmatrix}\boldsymbol{\epsilon}_1\\ \boldsymbol{\epsilon}_2\\ \vdots \\ \boldsymbol{\epsilon}_T\end{bmatrix}$ and each of the vectors and matrices in this expansion expands further into N items. E.g., each of the $\mathbf{y}_t = \begin{bmatrix}y_{1t}\\ y_{2t}\\ \vdots \\ y_{Nt}\end{bmatrix}$.

[2]As discussed in the paper, we specify the independent variables associated with spatial-X effects as $\mathbf{Z}$ to emphasize the point that the variables in $\mathbf{X}$ and $\mathbf{Z}$ do not have to be the same. Elhorst, and many others, specify both matrices identically as $\mathbf{X}$ or, as in Elhorst's Chapter 4 discussion of different combinations of models in time and space, $\mathbf{X}_t$.

[3]The components of the short-term direct effects for the SAR model are often separated into the pre-spatial direct effect, β_k, and then the spatially-filtered direct effect, $[(\mathbf{I}_N - \rho\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\bar{d}}$.

using θ instead of ρ to emphasize that this model is essentially an SLX model. But, because this model is both temporally and spatially dynamic, it will have a combination of both short-term and long-term effects as well as direct and indirect effects. In essence, the θ term in Equation 5 plays the role of an individual θ_k term inside $\boldsymbol{\theta}$ in Equation 3 with the addititonal complication that the temporal impact of this term works in a fashion along the lines of the ϕ term in Equation 2. Thus, the temporally lagged spatial lag model has unfiltered short-term direct effects, β_k, just like those of the non-spatial dynamic model and the temporally static SLX, but it has no short-term indirect effects. This is the case *precisely* because the spatially lagged component of the model, $\mathbf{W}\mathbf{y}_{t-1}\theta$, is also temporally lagged. To better understand how this term works, we can write Equation 5 back one time period as

$$\mathbf{y}_{t-1} = \mathbf{X}_{t-1}\boldsymbol{\beta} + \mathbf{W}\mathbf{y}_{t-2}\theta + \boldsymbol{\epsilon}_{t-1} \tag{6}$$

and then substitute the right-hand side of Equation 6 into Equation 5,

$$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \mathbf{W}(\mathbf{X}_{t-1}\boldsymbol{\beta} + \mathbf{W}\mathbf{y}_{t-2}\theta + \boldsymbol{\epsilon}_{t-1})\theta + \boldsymbol{\epsilon}_t \tag{7}$$

which, if the data being modeled are temporally stationary, meaning $|\theta| < 1$, will lead to decreasing effects as we move more temporally distant from any change in lagged values of $\mathbf{y}_t$, $\mathbf{X}_t$, or $\boldsymbol{\epsilon}_t$. These longterm effects consist of own unit, or direct, effects of $[(\mathbf{I}_N - \theta\mathbf{W})^{-1}\beta_{1k}\mathbf{I}_N]^{\bar{d}}$ and indirect effects of $[(\mathbf{I}_N - \theta\mathbf{W})^{-1}\beta_{1k}\mathbf{I}_N]^{\overline{rsum}}$.

In Table 1 we provide the specifications of all 5 of the models that we discuss in this section and in Table 2 we provide a listing of the four different effects from each model. If we look across the entries for each model in Table 2, we can see that the temporally lagged spatial lag model is very different from the temporally static SAR. It does not have the defining characteristic of short-term endogenous effects. Instead, what it has is a combination of long-term direct effects and long-term indirect effects which combine elements of the non-spatial dynamic model and the temporally static SLX. And as we note in the paper, the SAR spatial multiplier matrix, $(\mathbf{I}_N - \rho\mathbf{W})^{-1}$ incorporates immediate feedback through terms like $\mathbf{W}^2$ and $\mathbf{W}^3$. In contrast, the temporal decay multiplier of the temporally lagged spatial lag model, $(\mathbf{I}_N - \theta\mathbf{W})^{-1}$, is simply a spatially weighted geometric lag (aka a Koyck lag) function which, as long as stationarity conditions are met, means that temporally more distant changes have smaller effects.

Additional Experiments

SAR Model Performance for SAR DGP

In this section we detail how the "correct" models perform for SAR and SLX data-generating processes, respectively. In both cases, the recovery rates of the coefficients are, as expected, close to 95%. We provide these results in Tables 3 and 4.

Table 1: Different specifications of temporal and spatially lagged models

Model Name	Specification
Non-spatial static model	$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t$
Non-spatial dynamic model	$\mathbf{y}_t = \mathbf{y}_{t-1}\phi + \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t$
Temporally static SLX	$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \mathbf{W}\mathbf{Z}_t\boldsymbol{\theta} + \boldsymbol{\epsilon}_t$
Temporally static SAR	$\mathbf{y}_t = \rho\mathbf{W}\mathbf{y}_t + \mathbf{X}_t\boldsymbol{\beta} + \boldsymbol{\epsilon}_t$
Temporally lagged spatial lag model	$\mathbf{y}_t = \mathbf{X}_t\boldsymbol{\beta} + \mathbf{W}\mathbf{y}_{t-1}\theta + \boldsymbol{\epsilon}_t$

Notes:

Second-Order Neighbor Model Performance for SAR DGP

One possibility that we explored in the paper addressed the particular functional form of the higher-order weights matrices. A specification mirroring the SAR by squaring $\mathbf{W}$ (seen in Equation 9 in the manuscript) produces non-zero values along the diagonal of the partial derivatives matrix, which means that there are feedback effects, and higher-order effects more generally. If deemed unnecessary by theory, the functional form of the weights matrix can be modified so that it expressly prohibit feedback effects. This specification would identify higher-orders of contiguity (see the second-order weights matrix, $\mathbf{W_{2nd}}$ in Figure 1). The third set of experiments evaluates how well a model specified as

$$\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W_{2nd}}\mathbf{x}\theta_2 + \boldsymbol{\epsilon} \quad (8)$$

deals with an SAR DGP where feedback effects are present. As we can see from the third section in Table 5, which is the same as Table 2 in the manuscript with this additional set of results, this change in the specification from $\mathbf{W^2}$ to $\mathbf{W_{2nd}}$ results in a serious reduction in the recovery rates of the zero-order direct effect, which is now larger, on average, because it has to account for the spatial effects that would otherwise be modeled as feedback effects. Furthermore, the recovery rates for the second-order direct effects are, by construction, 0%.[4] We would only advocate this type of model specification if two conditions are met: first, the theory is quite clear about the impossibility of feedback effects, and second, these feedback effects are shown to be zero in robustness checks.

Spatial Error Model (SEM)

A popular alternative to the SAR and SLX models is the spatial error model (SEM).[5] Instead of the spatial dependence arising in the outcomes (as in y_i influencing y_j, and vice versa)

[4]Indeed, if we include second- and third-order contiguity weights matrices, we are able to model some of the higher order effects but still none of the feedback effects.

[5]The description of the SEM draws heavily from Ward and Gleditsch (2008: 65-67).

Table 2: Effects from different specifications of temporal and spatially lagged models

Model Name	Short-term direct effects	Short-term indirect effects	Long-term direct effects	Long-term indirect effects
Non-spatial static model	β_k	none	none	none
Non-spatial dynamic model	β_k	none	$\beta_k(1-\phi)^{-1}$	none
Temporally static SLX	β_k	$\mathbf{W}\theta_k$	none	none
Temporally static SAR	$[(\mathbf{I}_N-\rho\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\bar{d}}$	$[(\mathbf{I}_N-\rho\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\overline{rsum}}$	none	none
Temporally lagged spatial lag model	β_k	none	$[(\mathbf{I}_N-\theta\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\bar{d}}$	$[(\mathbf{I}_N-\theta\mathbf{W})^{-1}\beta_k\mathbf{I}_N]^{\overline{rsum}}$

Notes: In this table we follow some useful notational conventions used by Elhorst (2014) where $\bar{d}$ is a calculation of the mean diagonal element of a matrix and $\overline{rsum}$ is a calculation of the mean row sum of the non-diagonal elements of a matrix.

Table 3: Recovery Rates for the SAR Model Specification: SAR Data-Generating Process

	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
SAR model: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \rho\mathbf{W}\mathbf{y} + \boldsymbol{\epsilon}$									
β	94.2	94.5	94.7	93.3	95.6	93.6	95.6	94.8	95.1
ρ	93.5	95.7	95.1	93.9	94.0	94.6	95.1	94.2	95.4

or in the observables (as in x_i influencing y_j), the SEM models spatial dependence in the unobservables, or errors:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \lambda\mathbf{W}\mu + \boldsymbol{\epsilon} \tag{9}$$

where the overall error is decomposed into ϵ, "a spatially uncorrelated error term that satisfied the normal regression assumption, and [μ], which is a term indicating the spatial component of the error term" (Ward and Gleditsch 2008: 65-66). If λ is 0, then there is no spatial dependence in the errors and an OLS can be safely estimated; if λ is not 0, then "we have a pattern of spatial dependence between the errors for connected observations" (Ward and Gleditsch 2008: 66). This poses no complications for generating quantities of interest, as "the differences in the independent variables in i do not have effects on outcomes in observations connected to i" (Ward and Gleditsch 2008: 67). Essentially, this means that variables will only have a direct effect (i.e., x_i on y_i) and no indirect effects (such as x_i on y_j), feedback or otherwise.[6]

We echo Beck, Gleditsch and Beardsley's (2006: 30) conclusion that the SEM is not appropriate in most political science applications. This is because a variable can have an impact on neighboring observations if omitted (and thus part of the error term), but not if it is included. Consider the example of economic growth in an SEM model:

> [...] remember that the "errors" are just the variables that we either chose not to measure, or could not measure. In particular, they are errors from the perspective of the analyst, not the perspective of policy makers in the country. Thus, if Germany grew more quickly because of some variable not included in the specification, that growth would affect all other countries. But if Germany grew more quickly because it had a left government, and if that variable were included in the specification, then this extra German growth would have no impact on the growth in other countries (30).

It remains to be seen how SLX and SAR models perform when the true data-generating process features spatial dependence in the unobservables, which is typically consistent with

[6] Darmofal (2015: 107-108) succinctly states that "because the spatial multiplier in a spatial error model pertains only to the errors, substantive covariates do not vary in their equilibrium effects based on the spatial locations of the observations in a spatial error formulation."

Table 4: Recovery Rates for the SLX Model Specification: SLX Data-Generating Process

	$\theta_1 = -0.8$; $\theta_2 = -0.8$	$\theta_1 = -0.8$; $\theta_2 = 0.01$	$\theta_1 = -0.8$; $\theta_2 = 0.4$	$\theta_1 = 0.4$; $\theta_2 = 0.4$	$\theta_1 = 0.4$; $\theta_2 = 0.01$	$\theta_1 = 0.4$; $\theta_2 = -0.8$	$\theta_1 = -0.2$; $\theta_2 = -0.2$	$\theta_1 = -0.2$; $\theta_2 = 0.01$	$\theta_1 = -0.2$; $\theta_2 = 0.4$
SLX model: $\mathbf{y} = \mathbf{x_1}\beta_1 + \mathbf{x_2}\beta_2 + \mathbf{W}\mathbf{x_1}\theta_1 + \mathbf{W}\mathbf{x_2}\theta_2 + \boldsymbol{\epsilon}$									
Direct									
β_1	95.0	96.1	95.5	94.7	94.7	95.0	94.7	95.3	94.4
β_2	93.7	94.9	95.5	95.7	95.0	94.3	95.5	95.0	95.0
Indirect									
θ_1	95.0	95.1	95.4	96.5	94.5	94.6	95.2	93.7	94.1
θ_2	95.3	94.9	94.5	95.0	95.1	94.7	94.6	94.8	95.2

Table 5: Recovery Rates for Various SLX Model Specifications: SAR Data-Generating Process

	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
Simple SLX model: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta + \boldsymbol{\epsilon}$									
Direct: Total	94.9	94.4	94.8	93.7	95.5	93.5	95.0	94.0	88.9
0 Order	77.9	91.9	95.1	93.5	95.5	93.4	94.8	90.7	81.0
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	15.1	49.9	85.2	93.7	95.2	93.5	40.3	0	0
1st Order	91.4	93.8	95.5	93.1	95.2	94.5	94.4	93.8	93.1
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with a squared term: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W^2}\mathbf{x}\theta_2 + \boldsymbol{\epsilon}$									
Direct: Total	95.0	94.0	95.1	93.8	95.4	94.0	95.2	94.4	91.0
0 Order	94.2	95.1	95.2	94.2	93.6	94.8	94.8	94.9	95.8
2nd Order	93.2	93.9	94.7	94.6	94.0	94.7	93.6	93.4	93.8
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	93.0	94.7	94.8	94.7	93.9	94.5	93.3	83.7	2.6
1st Order	93.0	94.9	95.9	94.0	94.9	94.9	94.4	93.2	92.6
2nd Order	93.2	93.9	94.7	94.6	94.0	94.7	93.6	93.4	93.8
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with a second-order term: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W_{2nd}}\mathbf{x}\theta_2 + \boldsymbol{\epsilon}$									
Direct: Total	95.5	93.6	95.0	93.9	96.3	93.3	95.7	93.8	91.0
0 Order	76.1	89.9	95.1	93.7	96.3	93.8	95.3	89.5	77.9
2nd Order	0	0	0	0	0	0	0	0	0
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	95.3	94.0	95.1	95.8	95.0	95.4	94.5	90.6	36.0
1st Order	96.5	95.4	95.9	94.2	95.2	95.6	95.0	91.7	88.4
2nd Order	95.0	94.5	94.7	95.8	94.5	95.3	94.1	94.6	93.6
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with squared and cubed terms: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W^2}\mathbf{x}\theta_2 + \mathbf{W^3}\mathbf{x}\theta_3 + \boldsymbol{\epsilon}$									
Direct: Total	95.0	94.2	94.6	94.3	96.1	94.4	94.5	93.7	93.4
0 Order	94.2	95.1	95.3	94.1	93.9	94.9	94.3	94.9	95.9
2nd Order	93.9	94.0	94.8	95.2	94.2	95.2	93.8	93.2	93.3
3rd Order	94.2	93.6	94.7	93.5	95.0	94.4	95.0	93.9	93.4
Indirect: Total	95.5	95.1	95.6	94.4	95.2	94.5	93.9	92.6	87.0
1st Order	94.3	94.7	95.1	93.9	94.9	93.7	93.9	95.2	94.4
2nd Order	93.9	94.0	94.8	95.2	94.2	95.2	93.8	93.2	93.3
3rd Order	94.2	93.6	94.7	93.5	95.0	94.4	95.0	93.9	93.4

an SEM model. In other words, if one is uncertain about the true spatial process at work and diagnostics are uncertain, how dangerous is it to begin with a SLX or SAR model if there is spatial dependence in the unobservables? LeSage and Pace (2009: 157-159) point out that the result is "unbiased but inefficient coefficient estimates" and "inference regarding dispersion of the explanatory variables based on the asymptotic variance-covariance matrix for the SAR model will be misleading, since error dependence is ignored when constructing the variance-covariance matrix".

To explore this possibility we simulate data based on the following equation (Darmofal 2015: 102):

$$\mathbf{y} = \mathbf{x}\beta + \boldsymbol{\epsilon}, \quad \epsilon = \lambda\mathbf{W}\epsilon + \xi \tag{10}$$

with matrix $\mathbf{X}$ containing a single variable drawn from a uniform distribution, $x \in [-10, 10]$, and where $\beta = 1$. $\mathbf{W}$ is an $N \times N$ symmetric row-standardized contiguity weights matrix, where each element below the diagonal is randomly drawn from a Bernoulli distribution. We simulated 1000 data sets at each of nine different scenarios defined by the strength of the spatial error dependence, $\lambda \in \{-0.8, 0.8\}$.[7] As with the experiments presented in the manuscript, we focus on the recovery rates for direct and indirect effects of x since those reflect both the coefficients and their uncertainty. Table 5 shows the results of the experiments.

Two clear patterns emerge from Table 6. The first pattern is that the SAR model does a poor job of recovering the first-order indirect effect (which is actually 0) in the SEM DGP for models of λ that are lower than lower than 0 or larger than 0.2. In those case, the SAR model finds false evidence that x_i influences y_j through the outcomes (i.e., by influencing y_i). Since the SAR model finds "phantom" first-order indirect effects that do not actually exist, it also does a poor job in recovering the true average total effects (not shown). The second pattern is that the various specifications of the SLX model—ranging from a simple model with only first-order indirect effects to one with first- through third-order effects—recovers estimates of the direct and indirect effects that are consistently close to 95% for all values of λ. The SLX is more flexible in this respect because the θs are effectively 0, which rules out higher-order effects. For these two reasons, it appears as though the SLX model is more robust to incorrectly specifying the spatial dependence when it actually occurs in the unobservables. If one is only concerned about generating meaningful inferences for the explanatory variables, then this type of misspecification is not problematic; of course, if one is interested in modeling the actual spatial process, then the SLX would be unable to show that the spatial dependence actually exists in the unobservables.

[7] It is worth noting that values of λ close to the absolute value of 1 are exceedingly rare in practice.

Table 6: Recovery Rates for SAR and Various SLX Model Specifications: SEM Data-Generating Process

	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8
SAR model: $\mathbf{y} = \mathbf{x}\beta + \rho\mathbf{W}\mathbf{y} + \boldsymbol{\epsilon}$									
Direct: Total	94.6	94.3	94.5	93.4	95.3	93.8	95.1	94.7	93.4
0 Order	94.7	94.4	94.3	93.8	95.6	93.6	95.3	94.7	93.5
2nd Order	98.0	99.4	99.8	99.8	100	99.9	99.9	99.6	99.0
3rd Order	100	100	100	100	100	100	100	100	100
Indirect: Total	59.9	72.1	83.8	90.1	93.9	97.7	97.3	96.2	93.1
1st Order	66.9	77.6	87.3	91.2	94.0	94.3	90.0	82.6	73.1
2nd Order	98.0	99.4	99.8	99.8	100	99.9	99.9	99.6	99
3rd Order	100	100	100	100	100	100	100	100	100
Simple SLX model: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta + \boldsymbol{\epsilon}$									
Direct: Total	94.9	94.8	94.9	93.8	95.5	93.5	95.3	94.9	93.9
0 Order	94.9	94.8	94.9	93.8	95.5	93.5	95.3	94.9	93.9
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	94.0	95.5	95.8	93.1	95.2	94.5	94.8	94.3	95.6
1st Order	94.0	95.5	95.8	93.1	95.2	94.5	94.8	94.3	95.6
2nd Order	–	–	–	–	–	–	–	–	–
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with a squared term: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W}^2\mathbf{x}\theta_2 + \boldsymbol{\epsilon}$									
Direct: Total	95.0	94.1	95.0	93.8	95.4	94.0	95.2	95.1	94.0
0 Order	94.2	95.1	95.2	94.2	93.6	94.7	94.7	95.0	95.3
2nd Order	94.4	94.0	94.9	94.6	94.0	94.7	93.8	93.9	94.8
3rd Order	–	–	–	–	–	–	–	–	–
Indirect: Total	96.2	95.1	95.0	94.7	93.9	94.4	93.6	92.2	93.3
1st Order	94.8	95.5	96.0	94.0	94.9	94.9	94.8	93.9	95.6
2nd Order	94.4	94.0	94.9	94.6	94.0	94.7	93.8	93.9	94.8
3rd Order	–	–	–	–	–	–	–	–	–
SLX model with squared and cubed terms: $\mathbf{y} = \mathbf{x}\beta + \mathbf{W}\mathbf{x}\theta_1 + \mathbf{W}^2\mathbf{x}\theta_2 + \mathbf{W}^3\mathbf{x}\theta_3 + \boldsymbol{\epsilon}$									
Direct: Total	94.8	94.2	94.6	94.3	96.1	94.5	94.8	94.5	95.2
0 Order	94.2	95.0	95.3	94.1	93.9	94.8	94.3	94.9	95.4
2nd Order	94.8	94.0	95.0	95.2	94.2	95.3	93.9	93.3	94.3
3rd Order	94.6	93.8	94.8	93.6	95.0	94.6	95.3	94.3	94.4
Indirect: Total	95.8	95.5	95.8	94.5	95.2	94.5	94.0	93.2	93.4
1st Order	94.7	94.8	95.1	93.9	94.9	93.7	93.9	95.2	94.3
2nd Order	94.8	94.0	95.0	95.2	94.2	95.3	93.9	93.3	94.3
3rd Order	94.6	93.8	94.8	93.6	95.0	94.6	95.3	94.3	94.4

Application

In this section we explore substantive effects from the defense spending application in the manuscript, as well as some models to show how deftly SLX handles conditional spatial dependence.

As shown in the SAR model of Table 7, both civil and interstate wars influence defense burdens. Of course, the coefficients themselves are only the estimated direct effects of those covariates on defense burden. To understand the total effect of the covariates, it is important to utilize the partial derivatives approach. In the manuscript we showed that the total impact of the covariates on the defense burden depend on the coefficient (β), the size of the change in x (in the case of civil and interstate wars, the difference between a value of 0 and 1), the global spatial autocorrelation coefficient (ρ), and each state's distribution of neighbors ($\mathbf{W}$). Each state potentially has a different configuration of neighbors, which leads to different indirect effects for each.

To simplify matters, we examine the average direct, indirect and total effects in Table 8. From this table we can see that the estimated average total effect of a civil war at time $t-1$ in a neighboring state on the defense burden of the focal state at time t is a reduction in military spending as a percentage of GDP of -0.62%. The effect of a civil war in a state results in a reduction in that state's defense burden, on average, of -0.46%. Note that this effect is slightly larger than the coefficient for *civil* war_{t-1}; the difference is the result of feedback effects, or the effects of civil war in state i influencing its neighbor j, which feeds back to affect state i. The average indirect effect—or, the average effect of a civil war in the focal state on other states is -0.17%. These effects demonstrate that civil wars can meaningfully impact states' defense burdens, and over a quarter of the overall effect spills over into neighboring states. The effects of *interstate* war_{t-1} are similar in magnitude and distribution between direct and indirect effects, except for being positive. On average, experiencing an interstate war at time $t-1$ increases that state's defense burden by 0.46%, and spills over to increase the defense burdens of neighboring states by 0.17%.

As we demonstrated with our Monte Carlo experiments, the consequences of model choices can range from understating the overall effects, to making the opposite inference regarding indirect effects. Recall that in the case of civil and interstate wars, the SLX variables were consistent with the pattern of positive spatial dependence in the SAR model (consistent with the positive ρ). When we compare the various effects of the SLX to the SAR model for both of these variables (see Table 2), we see that the SLX produces average total and indirect effects that are much larger (almost two and three times larger in the case of *interstate* war_{t-1}), and smaller direct effects. Since the SLX model does not force the spatial autocorrelation to be represented by one parameter, the indirect effects are free to vary in size based on the particular covariate. In the case of *defense* $burden_{t-1}$, the average total effects are much smaller in the SLX model because the coefficients for the two SLX variables are signed in the opposite direction from the *defense* $burden_{t-1}$ variable. In the SAR model, contrary to expectations, increasing one's defense burden by 1% is estimated to decrease contiguous neighbors' defense burdens by -0.04%; in the SLX model, this same

Table 7: Non-Spatial OLS, SAR and SLX Models of Neighborhood Effects on Defense Burdens

	OLS Model 1	SAR Model 2	SLX Model 3
Spatial Estimates (ρ and θ)			
ρ		0.07***	
		(0.004)	
Contiguity×Civil War$_{t-1}$			-0.11*
			(0.07)
Contiguity×Interstate War$_{t-1}$			0.18***
			(0.07)
Ally×Interstate War$_{t-1}$			-0.006
			(0.03)
Contiguity×Defense Burden$_{t-1}$			0.006***
			(0.002)
Defense Pact×Defense Burden$_{t-1}$			0.003***
			(0.001)
Non-Spatial Estimates (β)			
Civil War$_{t-1}$	-0.47***	-0.45***	-0.39***
	(0.17)	(0.16)	(0.15)
Interstate War$_{t-1}$	0.46***	0.45***	0.37**
	(0.16)	(0.16)	(0.16)
Total Population (Logged)$_{t-1}$	0.03	0.03	0.02
	(0.02)	(0.02)	(0.02)
Alliance with US	-0.21**	-0.22**	-0.15*
	(0.10)	(0.09)	(0.08)
ΔUS Defense Burden	-0.04	-0.03	-0.05
	(0.07)	(0.07)	(0.08)
ΔUSSR/Russia Defense Burden	0.006	-0.01	-0.05***
	(0.02)	(0.02)	(0.02)
US Ally×Δ US Defense Burden	0.14	0.11	0.21
	(0.12)	(0.11)	(0.13)
US Ally×Δ USSR Defense Burden	0.005	0.01	0.02
	(0.03)	(0.03)	(0.02)
Annual Trend	-0.007***	-0.006***	-0.004**
	(0.002)	(0.002)	(0.002)
1992	1.13***	0.74***	-1.26***
	(0.28)	(0.27)	(0.21)
Defense Burden$_{t-1}$	-0.12***	-0.12***	-0.14***
	(0.006)	(0.01)	(0.006)
ΔDefense Burden$_{t-1}$	-0.14***	-0.13***	-0.14***
	(0.01)	(0.02)	(0.01)
Constant	0.46**	0.41*	0.22
	(0.22)	(0.21)	(0.17)
N	6,328	6,328	7,266

Note: Models include regional fixed effects. The SAR model excludes isolates.
* p-value < 0.1; ** p-value < 0.05; *** p-value < 0.01

Table 8: Average Direct, Indirect and Total Effects of Covariates on Defense Burden

Variables	**W Specification**	**Avg. Effects**	**SAR**	**SLX**			
			M2	M3	M4	M5	M6
Civil War$_{t-1}$	Contiguity	Direct	-0.46	-0.39			
		Indirect	-0.17	-0.36			
		Total	-0.62	-0.74			
Civil War$_{t-1}$	Region (Europe)	Indirect			-4.63		
	Region (Middle East)	Indirect			-6.32		
	Region (Africa)	Indirect			2.04		
	Region (Asia)	Indirect			-1.37		
	Region (Americas)	Indirect			-1.54		
Interstate War$_{t-1}$	Contiguity	Direct	0.46				
		Indirect	0.17				
		Total	0.63				
	Contiguity + Alliance	Direct		0.37			
		Indirect		0.51			
		Total		0.88			
	Contiguity ($\mathbf{W}$)	Indirect					0.51
	Contiguity ($\mathbf{W^2}$)	Direct					0.05
		Indirect					0.19
		Total					0.24
	Contiguity ($\mathbf{W^3}$)	Direct					0.01
		Indirect					0.11
		Total					0.12
	Contiguity ($\sum_{n=1}^{3} \mathbf{W^n}$)	Direct					0.30
		Indirect					0.80
		Total					1.10
Interstate War$_{t-1}$	Region (Europe)	Indirect				-1.43	
	Region (Middle East)	Indirect				2.16	
	Region (Africa)	Indirect				0.21	
	Region (Asia)	Indirect				-0.84	
	Region (Americas)	Indirect				0.44	
Defense Burden$_{t-1}$	Contiguity	Direct	-0.12				
		Indirect	-0.04				
		Total	-0.16				
	Contiguity + Defense	Direct		-0.14			
		Indirect		0.05			
		Total		-0.09			

Note: SAR Model 1 uses a binary, un-row-standardized contiguity weights matrix.

change is estimated to increase contiguous neighbors' defense burdens by 0.02%. The latter estimation technique is more flexible and provides more realistic inferences.

Another advantage in the flexibility of the SLX is the ability to properly model conditional patterns of spatial dependence. In the first SLX model (Model 2 in Table 7), we demonstrated that the spatial effects of *defense burden*$_{t-1}$ were conditioned by patterns of neighbors via contiguity and defense pact. To explore this further, let us examine how the flexibility of SLX models allows us to easily estimate region-specific SLX variables.[8] Due to security agreements, colonial histories, regional organizations, and other characteristics (see **?**, 429–430 for a summary), covariates that might spillover in one region are contained in another. In Models 4 and 5 (Table 9) we estimate separate region-specific parameters for both variables (*civil war*$_{t-1}$ and *interstate war*$_{t-1}$). By doing so, we can show how the effects of wars depend on the regions in which they occur.

The results in Table 9 (and the effects depicted in Table 2) demonstrate that the non-conditional SLX model (Model 3) clouded a great deal of region-specific heterogeneity in the spatial patterns. The resulting average indirect effect of *civil war*$_{t-1}$ in Model 3 was slightly negative (-0.36). This value represents a rough average of the indirect effects across regions, and obscures the fact that in one of the regions (Africa) civil wars in one's region actually increases states' defense burdens. This is also the case in *interstate war*$_{t-1}$, as both Europe and Asia/Oceania respond in the opposite manner as the other regions to interstate wars in the region. These inferences—while relatively easy to derive in the SLX setting—would be prohibitively difficult, if not impossible, with an SAR model.

One set of circumstances where the SAR model is generally more appropriate than the SLX is in the case of higher-order effects beyond the first-order. If these effects are expected to occur simultaneously, then it is generally correct to estimate an SAR. If, however, the higher-order effects are based on spatial clustering in the observables, then the SLX can be modified to estimate higher-order effects. In Model 6 (Table 9) we add two higher-order effects to represent the possibility that changes in defense burdens by second- and third-order contiguous neighbors might have an effect on the focal state. Adding the two higher-order effects SLX variables has the benefit of allowing for feedback effects (as shown in the direct effects for second- and third-order contiguity in Table 2), but has the downside of increasing multicollinearity (both higher-order effects have variance inflation factors above 11). In this case, all three SLX variables are positive, and it is clear that the positive effects decline considerably at each additional order of contiguity. F-tests suggest that we cannot reject the null hypothesis that the two higher-order coefficients are jointly equal to 0, which means that the specification with first-order contiguity is sufficient.

While the F-tests suggest that the specification with the first-order contiguity is sufficient,[9] it is instructive to graphically explore how the $\mathbf{W}$ (i.e., the distribution of contiguous neighbors in this model) influences the size of the indirect effects. In Figure 1 we depict the average total effects of *interstate war*$_{t-1}$ for all European countries in 2007 (except for Russia

[8]For this example, we use the Correlates of War's regional classification (see Stinnett et al. 2002 for a description).

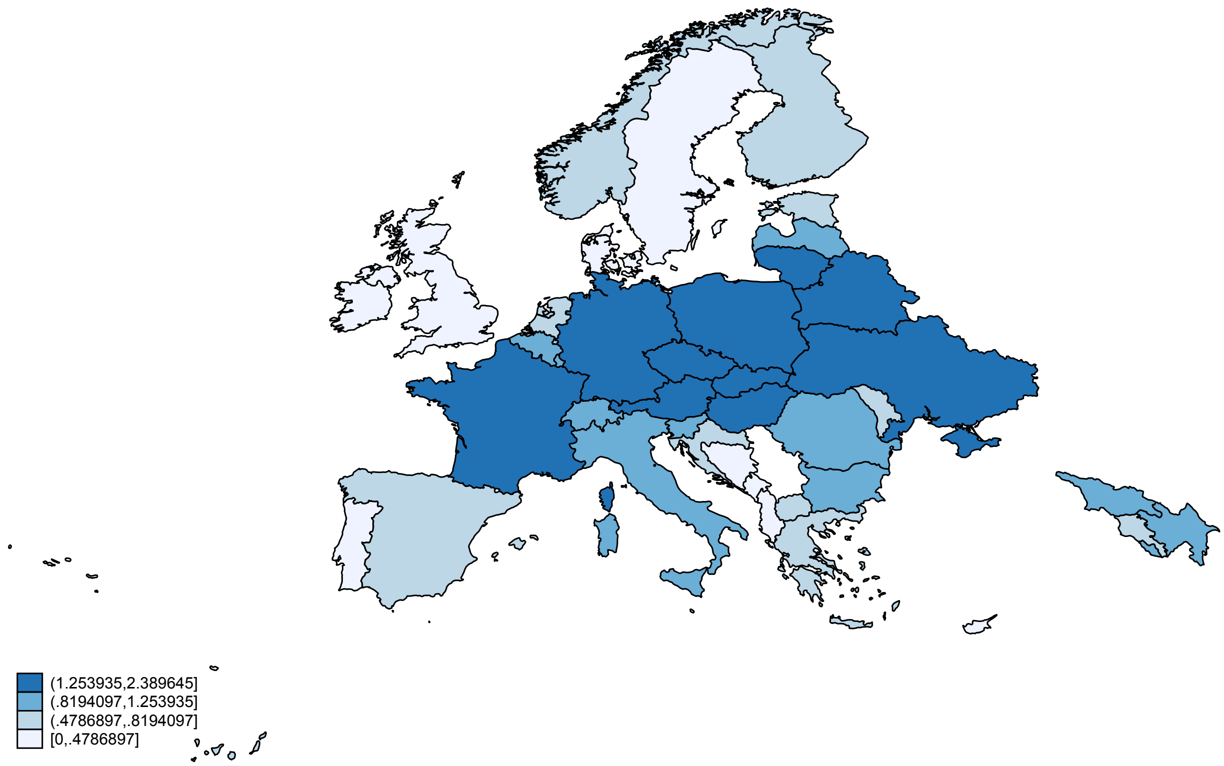
[9]F-tests suggest that we cannot reject the null hypothesis that the two higher-order coefficients are jointly equal to 0.

Table 9: Conditional SLX Models of Neighborhood Effects on Defense Burdens

	Model 4	Model 5	Model 6
Spatial Estimates (θ)			
Contiguity×Interstate War$_{t-1}$			0.16
			(0.12)
Contiguity^2×Interstate War$_{t-1}$			0.02
			(0.05)
Contiguity^3×Interstate War$_{t-1}$			0.002
			(0.01)
Region×Civil War$_{t-1}$	-0.16	-0.008	
	(0.10)	(0.03)	
Region (Middle East)×Civil War$_{t-1}$	-0.06		
	(0.13)		
Region (Africa)×Civil War$_{t-1}$	0.23**		
	(0.11)		
Region (Asia)×Civil War$_{t-1}$	0.11		
	(0.12)		
Region (Americas)×Civil War$_{t-1}$	0.21*		
	(0.12)		
Region×Interstate War$_{t-1}$	-0.001	-0.05	
	(0.02)	(0.05)	
Region (Middle East)×Interstate War$_{t-1}$		0.12*	
		(0.07)	
Region (Africa)×Interstate War$_{t-1}$		0.06	
		(0.05)	
Region (Asia)×Interstate War$_{t-1}$		0.02	
		(0.06)	
Region (Americas)×Interstate War$_{t-1}$		0.06	
		(0.10)	
Non-Spatial Estimates (β)			
Civil War$_{t-1}$	-0.41***	-0.39***	-0.42***
	(0.15)	(0.15)	(0.15)
Interstate War$_{t-1}$	0.44***	0.44***	0.24
	(0.15)	(0.15)	(0.21)
Total Population (Logged)$_{t-1}$	0.04**	0.04**	0.03*
	(0.02)	(0.02)	0.02)
Annual Trend	-0.003*	-0.004**	-0.004***
	(0.002)	(0.002)	0.002)
1992	-1.00***	-1.04***	-1.05***
	(0.20)	(0.20)	(0.20)
Defense Burden$_{t-1}$	-0.12***	-0.12***	-0.12***
	(0.006)	(0.006)	(0.006)
ΔDefense Burden$_{t-1}$	-0.15***	-0.15***	-0.15***
	(0.01)	(0.01)	(0.01)
Constant	0.22	0.24	0.23
	(0.17)	(0.17)	(0.17)
N	7,266	7,266	7,266

Note: Models include regional fixed effects.

Figure 1: Total Effects of *Interstate* War_{t-1} across European States in 2007 (Model 6)



Note: The values represent the total effects—including direct and indirect effects—for each European state (i.e., the total in each row of the partial derivatives matrix), given an interstate war in every European state in 2007. Russia, which is omitted for graphical purposes, has the largest total effects of any European state (2.74).

for illustration purposes). There is a great deal of variation in the size of the effects, and that variation is largely consistent with the historical record detailing countries' responses to interstate wars. For example, the countries that have the largest total indirect effects are also those with the most contiguous neighbors (Russia, Germany, Poland, Ukraine, and Austria).[10] On the other hand, countries like Malta, Cyprus, United Kingdom and Ireland have the smallest total effects.

[10]This is a function of our choice to not row-standardize the weights matrix.