Project 2 - AST3220

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Analytical problems and methods

We know that $n_i^{(0)}$ denotes a particle's number density in thermal and chemical equilibrium. In this case we assume that the Standard model particles p_3 and p_4 are always in thermal and chemical equilibrium meaning that $n_3 = n_3^{(0)}$ and $n_4 = n_4^{(0)}$. From this the right-hand side of the Boltzmann equation gives:

$$a^{-3}\frac{d(n_1a^3)}{dt} = -n_1^{(0)}n_2^{(0)}\langle\sigma v\rangle\left[\frac{n_1n_2}{n_1^{(0)}n_2^{(0)}} - \frac{n_3^{(0)}n_4^{(0)}}{n_3^{(0)}n_4^{(0)}}\right] \tag{1}$$

$$= -\langle\sigma v\rangle\left[n_1n_2 - n_1^{(0)}n_2^{(0)}\right] \tag{2}$$

Taking the time derivative of both the number density of particle 1 and the scale factor gives:

$$a^{-3}\left(a^3\frac{dn_1}{dt} + n_1\frac{da^3}{dt}\right) = \frac{dn_1}{dt} + a^{-3}n_1\frac{d}{dt}(a\cdot a\cdot a) \tag{3}$$

$$= \frac{dn_1}{dt} + a^{-3}n_1((\dot{a}a + a\dot{a}) + aa\dot{a}) \tag{4}$$

$$= \frac{dn_1}{dt} + 3n_1\frac{\dot{a}}{a} \tag{5}$$

We remember the definition of the Hubble parameter as $H(t) = \dot{a}/a$. Inserting this give then the simplified version of the Boltzmann equation as:

$$\frac{dn_1}{dt} + 3Hn_1 = -\langle\sigma v\rangle\left[n_1n_2 - n_1^{(0)}n_2^{(0)}\right] \tag{6}$$

This equation is also valid for the antiparticles, that is for n_2 instead of n_1.

A chemical potential is a process of released or absorbed energy due to change of particle number in thermodynamics. If we assume the chemical potential to be zero, then the reaction rates between the annihilation and the production of the WIMP particles must be equal. As we assume chemical potential to be zero implies that there is an equally amount of a particle and its anti-particle. We get $n_\chi = n_1 + n_2 = 2n_1$.

Now we will introduce two new terms as we want to rewrite the Boltzmann equation on a dimensionless form. The first term is a new time variable:

$$x = \frac{m_\chi}{T} \tag{7}$$

where m_χ is the mass of the WIMP and T is the temperature. The other term is a quantity defined as:

$$Y = \frac{n_\chi}{s} \tag{8}$$

where n_χ is the number density of a particle and its antiparticle and s is the entropy density of the universe. We know from (Elgarøy 2020, eq. 4.31) an derived expression for the entropy density. We will in this project assume that $\hbar = c = G = k_B = 1$ leaving us with only:

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \tag{9}$$

where g_{*s} is the effective number of relativistic degrees of freedom contributing to the entropy density and is found, by including both bosons and fermions, to be 106.75 (Baumann 2012, p. 48). We have found that the Boltzmann equation may we written as Equation 6. By implementing Equation 8 in the Boltzmann equation and also $n_\chi = n_1 + n_2 = 2n_1$, we get:

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\langle \sigma v \rangle \left[n_\chi^2 - \left(n_\chi^{(0)} \right)^2 \right] \tag{10}$$

$$\Rightarrow s\frac{dY}{dt} + Y\frac{ds}{dt} + 3HYs = -\langle \sigma v \rangle s^2 \left[Y^2 - Y_{\text{eq}}^2 \right] \tag{11}$$

$$\Rightarrow s\frac{dY}{dt} + Y\left(\frac{ds}{dt} + 3Hs \right) = -\langle \sigma v \rangle s^2 \left[Y^2 - Y_{\text{eq}}^2 \right] \tag{12}$$

If we assume a conservation of entropy, we have that $sa^3 = C$ where C is just a constant. Taking the time derivative of the entropy density results then in:

$$\frac{ds}{dt} = \frac{d}{dt}\left(\frac{C}{a^3} \right) = -3C\frac{\dot{a}}{a^4} = -3s\frac{\dot{a}}{a} = -3Hs \tag{13}$$

Implementing this in the the Boltzmann equation above results in:

$$\frac{dY}{dt} = -s \langle \sigma v \rangle \left[Y^2 - Y_{\text{eq}}^2 \right] \tag{14}$$

Now we may extend the terms of dY/dt to the following:

$$\frac{dY}{dt} = \frac{dY}{dx}\frac{dx}{dT}\frac{dT}{ds}\frac{ds}{dt} = \frac{dY}{dx}\frac{dx}{dT}\left(\frac{ds}{dT} \right)^{-1}\frac{ds}{dt} \tag{15}$$

We have expressions for all Y, x and s where we already know the left-hand side of the equation above. For the sake of keeping the calculations simple, we will compute each of the remaining differential terms one by one starting with the x-term:

$$\frac{dx}{dT} = \frac{d}{dT}\left[\frac{m_\chi}{T}\right] = -\frac{m_\chi}{T^2} \tag{16}$$

$$\frac{ds}{dT} = \frac{d}{dT}\left[\frac{2\pi^2}{45}g_{*\mathrm{s}}T^3\right] = \frac{6\pi^2}{45}g_{*\mathrm{s}}T^2 \tag{17}$$

Implementing this in Equation 15 gives:

$$\frac{dY}{dt} = \frac{dY}{dx}\left(-\frac{m_\chi}{T^2}\right)\left(\frac{6\pi^2}{45}g_{*\mathrm{s}}T^2\right)^{-1}\dot{s} \tag{18}$$

$$= -\frac{dY}{dx}x\frac{45}{6\pi^2 g_{*\mathrm{s}}T^3}\dot{s} \tag{19}$$

$$= -\frac{dY}{dx}\frac{x}{3}\frac{\dot{s}}{s} \tag{20}$$

We have an expression for $\dot{s}$ as we found from Equation 13:

$$= \frac{dY}{dx}\frac{x}{\cancel{3}}\frac{\cancel{3}\cancel{s}H}{\cancel{s}} = \frac{dY}{dx}xH \tag{21}$$

Flipping the equation leaving dY/dx on the left side gives:

$$\frac{dY}{dx} = \frac{1}{xH}\frac{dY}{dt} \tag{22}$$

$$= -\frac{s\langle\sigma v\rangle}{xH}\left[Y^2 - Y_{\mathrm{eq}}^2\right] \tag{23}$$

where we now have the Hubble parameter $H = H(x)$ expressed as a function of the new time variable x.

Now that we have shorten the Boltzmann equation, we want to find an expression for the $s/H(x)$ term. We already have a expression of the entropy density as shown in Equation 9 and now need an expression for the Hubble parameter. We assume that the decoupling process happens in the radiation dominating era and that the energy density of relativistic particles is the only factor that is involved in the first Friedmann equation. With this, the Hubble parameter using our units becomes:

$$H^2 = \frac{8\pi}{3}\rho_{\mathrm{r}} \tag{24}$$

where ρ_r is the density in the radiation dominating era, the ultrarelativistic particles. Now we get:

$$\frac{s}{H(x)} = \frac{2\pi^2 g_{*\mathrm{s}}T^3}{45}\sqrt{\frac{3}{8\pi}}\frac{1}{\sqrt{\rho_{\mathrm{r}}}} \tag{25}$$

In the early universe we assumed that only ultrarelativistic particles are present. With the units we have defined and introducing (Elgarøy 2020, eq. 4.21), we get the expression:

$$\frac{s}{H(x)} = \frac{2\pi^{\cancel{2}} g_{*s} T^{\cancel{3}}}{45} \sqrt{\frac{3}{8\pi}} \sqrt{\frac{30}{\cancel{\pi^2} g_* \cancel{T^4}}} \tag{26}$$

$$= \frac{2\pi\sqrt{90}}{45} \frac{g_{*s}}{\sqrt{g_*}} \frac{1}{\sqrt{8\pi}} T \tag{27}$$

Inserting for T from Equation 7 gives:

$$\frac{s}{H(x)} = \frac{2\pi\sqrt{90}}{45} \frac{g_{*s}}{\sqrt{g_*}} \frac{m_\chi}{\sqrt{8\pi}} \frac{1}{x} \tag{28}$$

where g_* is the effective number of relativistic degrees of freedom. Usually $g_{*s} \neq g_*$ but in the early universe, the difference is small and of little significance (Elgarøy 2020, p. 71). Before beginning with the numerical implementation, we should introduce a new parameter to make the Boltzmann equation cleaner. That is:

$$\lambda = \frac{2\pi\sqrt{90}}{45} \frac{g_{*s}}{\sqrt{g_*}} \frac{m_\chi}{\sqrt{8\pi}} \langle \sigma v \rangle \tag{29}$$

Including this new parameter reduces the Boltzmann equation even further:

$$\frac{dY}{dx} = -\frac{\lambda}{x^2} \left[Y^2 - Y_{\text{eq}}^2 \right] \tag{30}$$

Numerical problems and methods

We have found an expression for the Boltztmann equaion i.e. Equation 30 that will be solved numerically. First we must introduce some initial conditions. The decoupling process will be studied within $x \in [1, 10^3]$ and we assume the mass of the WIMP to be $m_\chi = 10^3$ GeV. As Y will vary over several orders of magnitude, it would be wise to implement a new variable $W = \ln \lambda Y$ and $\lambda Y = y$. The reason is to prevent occurrence of numerical errors. The adjusted Boltzmann equation is then:

$$\frac{dW}{dx} = \frac{d}{dx}(\ln \lambda Y) \tag{31}$$

$$= \frac{d}{dY}(\ln \lambda Y) \frac{dY}{dx} \tag{32}$$

Insert Equation 30:

$$= \frac{1}{Y} \left(-\frac{\lambda}{x^2} \left[Y^2 - Y_{\text{eq}}^2 \right] \right) \tag{33}$$

$$= \frac{\lambda}{x^2}\left[Y_{\text{eq}}^2 Y^{-1} - Y\right] \tag{34}$$

$$= \frac{\cancel{\lambda}}{x^2}\left[\left(\frac{e^{W_{\text{eq}}}}{\cancel{\lambda}}\right)^2\left(\frac{e^{W}}{\cancel{\lambda}}\right)^{-1} - \frac{e^{W}}{\cancel{\lambda}}\right] \tag{35}$$

$$= \frac{1}{x^2}\left[e^{2W_{\text{eq}}-W} - e^{W}\right] \tag{36}$$

This will be the final Boltzmann equation to be solved numerically. Note that we need to find the equilibrium value W_{eq}. We may find this by using the definition of y_{eq} as following:

$$y_{\text{eq}} = 9.35 \times 10^9 \frac{g}{2}\sqrt{\frac{100}{g_*}}\left(\frac{m_\chi}{1000 \quad \text{GeV}}\right)\left(\frac{\langle\sigma v\rangle}{10^{-10} \quad \text{GeV}^{-2}}\right)x^{3/2}e^{-x} \tag{37}$$

g is the internal degrees of freedom of the dark matter particles. This equation will be compared with the Boltzmann differential equation that is to be solved. We assume the WIMP to be fermions that has spin $S = 1/2$. The degrees of freedom of the WIMP is then calculated as follows:

$$g = 2S + 1 = 1 + 1 = 2 \tag{38}$$

The equilibrium state is assumed to happen in the early stage of the Universe, meaning that we will set $x = 1$ in Equation 37. The relation between W and y is then used to find W_{eq}. We notice then that the only parameters in our model is the mass of the WIMP, m_χ, and the thermally averaged cross section $\langle\sigma v\rangle$. We will firstly consider three cases for the cross section, that is $\langle\sigma v\rangle_1 = 10^{-9}$ GeV, $\langle\sigma v\rangle_2 = 10^{-10}$ GeV and $\langle\sigma v\rangle_3 = 10^{-11}$ GeV.

Python is the language tool used in this project and our first implementation is to create a function that calculates y_{eq} based on the initial condition. W_{eq} is then calculated as W_0 as we need an initial condition to solve the differential equation. From the library package SciPy, we will use a integration tool called `solve_ivp`. The solver requires a callable function that is to be integrated, a tuple that contains the integration limits, an initial condition that is W_0 in this case and also an integration method. We will define the callable function as `dW_dx` which returns the right-hand side of Equation 36. The endpoints of the time variable x enters as the integration limits. The integration method used is the Radau method as it is an implicit method. The problem we are solving are stiff so using a explicit method requires more steps which would be inconvenient. We are solving for the adjusted variable W, so for comparing with the thermal and chemical equilibrium function y_{eq}, we need to exponentiate the results of the differential equation.

Next is to find the dark matter abundance based on the results from the differential equation and compare it to the observational value of the abundance. The dark matter abundance today is quantified as following:

$$\Omega_{\mathrm{dm},0}h^2 = 1.69\frac{x_f}{20}\sqrt{\frac{100}{g_*}}\left(\frac{10^{-10} \quad \mathrm{GeV}^{-2}}{\langle\sigma v\rangle}\right) \tag{39}$$

where x_f is the time of decoupling, or in this case when y becomes less than its initial value y_0. We will first use the initially three chosen values of the cross section $\langle\sigma v\rangle$ and evaluate the results with the observational abundance which seems to be $[\Omega_{\mathrm{dm},0}]_{\mathrm{observed}} \approx 0.12$.

Next is to evaluate the dark matter abundance over an interval of the cross section, that is to evaluate the abundance in a range of $\langle\sigma v\rangle \in [10^{-14}, 10^{-7}]$ in units of GeV^{-2}. We want to find at what value the cross section is consistent with the observed value of the abundance. An error of ± 0.05 should be included as the observed value may contain small errors.

Results

Figure 1 shows how the function y evolves for different values of $\langle\sigma v\rangle$ as the time variable x increases. We notice that for all three cases of $\langle\sigma v\rangle$, y is exponentially large and decrease very fast at the beginning until about $x = 50$ as it behaves more constant. The results of the numerical integration seems to converge towards an arbitrary value. In contrast with the y_{eq} function where all three cases goes to zero, the three cases of thermally average cross section becomes approximately $y_1 = 24$, $y_2 = 22$ and $y_3 = 19$ for $\langle\sigma v\rangle_1$, $\langle\sigma v\rangle_2$ and $\langle\sigma v\rangle_3$ respectively.

Using equation Equation 39 with $\langle\sigma v\rangle_1 = 10^{-9}$ GeV^{-2}, $\langle\sigma v\rangle_2 = 10^{-10}$ GeV^{-2} and $\langle\sigma v\rangle_3 = 10^{-11}$ GeV^{-2} results in the dark matter abundances $(\Omega_{\mathrm{dm},0}h^2)_1 = 0.05$, $(\Omega_{\mathrm{dm},0}h^2)_2 = 0.49$ and $(\Omega_{\mathrm{dm},0}h^2)_3 = 4.94$ respectively. We notice that $\langle\sigma v\rangle_1 = 10^{-9}$ is the thermally averaged cross section that results in a value closest to $(\Omega_{\mathrm{dm},0}h^2)_{\mathrm{observed}}$. Figure 2 shows at what value of $\langle\sigma v\rangle$ that corresponds to the observable value of the dark matter abundance. We see that the thermally average cross section should lie between $2.9 \cdot 10^{-10}$ GeV^{-2} and $7 \cdot 10^{-10}$ GeV^{-2} to match the observable value within reason.

Discussion

During this project we have studied the WIMP through a thermally and chemical equilibrium process by taking advantage of the Boltzmann equation. As a result based on the assumptions we have made during this project, we have estimated what the thermally average cross section should be when comparing with the observed dark matter abundance.

The code seems to produce an overflow in the Boltzmann equation as the parameter $W \to \infty$ for small $\langle\sigma v\rangle$. This leads to an numerical error as the values gets too extreme and also of no interest. We may therefore neglect the error as it does not give any result close to the observational value.

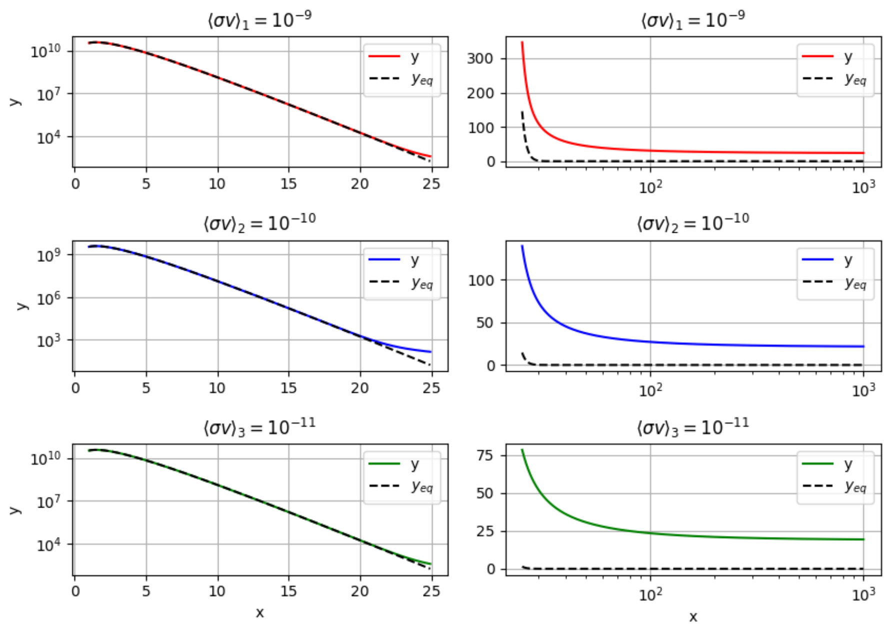

Figure 1: The resulting plots from integrating the Boltzmann equation. The plots for each thermally average cross section is split into two plots where the left-most plots shows the results in the interval $x \in [1, 25]$ and the right-most plot shows in the interval $x \in [25, 10^3]$.

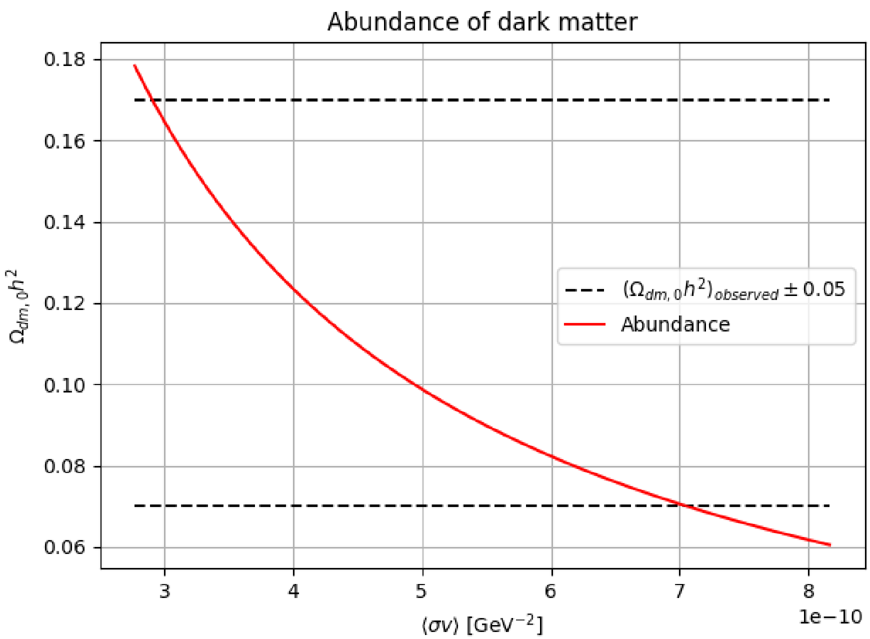

Figure 2: The dark matter abundance. A visualization of the dark matter abundance as a function of the thermally average cross section. The plot shows only the numerical part that lies around the observable abundance. The observable dark matter abundance is approximately 0.12.

References

Elgarøy, Øystein (2020). "AST3220 – Cosmology I". In: URL: https://www.uio.no/studier/emner/matnat/astro/AST3220/v20/undervisningsmateriale/lectures_2019.pdf.

Baumann, Daniel (2012). "Cosmology". In: *Part III Mathematical Tripos*.