AST3220, spring 2020: Project 3

Extraordinarily annoying repetition of the sermon from project 1 and 2

This project consists of a set of tasks, some analytical, some numerical. You should structure your answers as a report with an introduction, methods, results, discussion and conclusion. It is important that you explain how you think, just writing down a bunch of equations with no explanations will not give you a maximum score. I recommend that you write the report using LaTeX. Posting handwritten lecture notes and solutions to problems is a privilege that belongs to the lecturer alone. Your figures should have a clear layout with proper axis labels and units, and with a caption explaining what the figure shows. The figures should be referenced in the main text. You are also required to hand in your source code in a form that can be easily compiled.

Inflation without approximation

In the lectures and in the problems we have studied inflation analytically with the slow-roll approximation. If we forego nice, closed expressions we can however, solve the full equations numerically, and this is what you will do in this project.

I remind you that the Planck energy, Planck mass, and Planck length are defined by, respectively

$$E_{\mathrm{P}}^2 = \frac{\hbar c^5}{G}, \;\; m_{\mathrm{P}}^2 = \frac{\hbar c}{G}, \;\; l_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{1}$$

Assuming spatial flatness and that the scalar field dominates the energy density, the equations governing the evolution of the scalar field and the scale factor are

$$\begin{aligned} \ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) &= 0 & (2) \\ H^2 &= \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right]. & (3) \end{aligned}$$

Before solving these equations numerically it is useful to rewrite them in terms of dimensionless quantities. First, define

$$H_i^2 \equiv \frac{8\pi G}{3c^2} V(\phi_i), \tag{4}$$

where ϕ_i is the initial value of the field, and then introduce the variables

$$\tau = H_i t \tag{5}$$

$$h = \frac{H}{H_i} \tag{6}$$

$$\psi = \frac{\phi}{E_{\mathrm{P}}} \tag{7}$$

$$v = \frac{\hbar c^3}{H_i^2 E_{\mathrm{P}}^2} V \tag{8}$$

a) Check that these variables are dimensionless.

b) Show that equations (2) and (3) can be rewritten as

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{9}$$

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{10}$$

We need to think about the initial conditions. It is convenient to shift the origin of the time coordinate so we can start at $\tau = 0$. The condition for h is trivial: $h(0) = H(0)/H_i = 1$. For $\psi = \phi/E_{\mathrm{P}}$ we should choose a value that makes sure that we get inflation, and that means that the slow-roll conditions should be fulfilled. But since the equation for the scalar field is a second-order equation, we also need an initial value for $d\psi/d\tau$.

c) Give an argument for why

$$\left(\frac{d\psi}{d\tau}\right)_{\tau=0} = -\frac{1}{3}\left(\frac{dv}{d\psi}\right)_{\psi=\psi_i}, \tag{11}$$

where ψ_i is the inital value of ψ.

We are now ready to look at specific models. Let's try

$$V(\phi) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2, \tag{12}$$

from the example starting on page 115 in the lecture notes. Let $mc^2 = 0.01 E_{\mathrm{P}}$.

d) Use the slow-roll conditions to choose an appropriate initial value for the field.

e) Solve equations (9) and (10) numerically and plot the results. Based on the lectures, how would you expect ψ to behave? Does the numerical solution conform with your expectation?

f) In the same plot, plot the slow-roll solution from the lecture notes. When does it start to deviate significantly from the exact, numerical solution?

g) Using the result for h, plot $\ln[a(\tau)/a_i]$, where $a_i = a(\tau = 0)$. (Hint: Start with the definition $H = \dot{a}/a$ and integrate.) Estimate how many e-foldings we get and compare with the slow-roll result.

h) Show that in terms of the dimensionless variables

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{13}$$

i) What would you expect the ratio in equation (13) to be in the slow-roll regime? And in the oscillating phase? Plot the numerical result and compare.