Project 3 - AST3220

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Method

Analytical problems and methods

We will work with two equations that govern the evolution of the scalar field and the scale factor. This is with the assumption of spatial flatness and the domination of the scalar field over energy density. The two equations are:

$$\ddot{\phi} + 3H\dot{\phi} + \hbar c^3 V'(\phi) = 0 \tag{1}$$

The second equation is the resulting first of the Friedmann equation:

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right] \tag{2}$$

where ϕ is the scalar field, H is the Hubble parameter, $\hbar$ is Planck constant, c is the speed of light, V is the potential energy and G is the gravitational constant. Another important equation that we will use is the initial Hubble parameter which is given as following:

$$H_i^2 \equiv \frac{8\pi G}{3c^2} V(\phi_i) \tag{3}$$

where ϕ_i is the initial value of the field.

First we need to figure out the units of the various parameters. The Hubble parameter has the units of $[H] = [\dot{a}/a] = 1/\mathrm{s}$ since $\dot{a}$ is the time derivative of the scale factor. The units of τ becomes:

$$[\tau] = [H_i t] = \frac{1}{\cancel{s}} \cdot \cancel{s} = 1 \tag{4}$$

We remember that $H_i = H(\phi_i)$ is just the initial value of the Hubble parameter meaning that they have the same units. This leaves us with:

$$[h] = \left[\frac{H}{H_i}\right] = 1 \tag{5}$$

The unit of the scalar field is energy. Since E_P is the Planck energy, these units in theses two terms will cancel each other out leaving us with:

$$[\psi] = \left[\frac{\phi}{E_{\mathrm{P}}}\right] = 1 \tag{6}$$

Next for the units of the reduced potential, we know the definition of the Planck energy. If we insert the definition, the potential is reduced to:

$$v = \frac{\hbar c^3}{H_i^2 E_P^2} V = \frac{1}{H_i^2} \frac{G}{c^2} V \tag{7}$$

If we insert Equation 3, we get:

$$[v] = \left[\frac{3}{8\pi} \frac{V}{V(\phi_i)}\right] = 1 \tag{8}$$

We have then found that the variables τ, h, ψ and v are all dimensionless.

First we will use the definition of Equation 6 and flip it so that ϕ stays isolated on either sides of the equation sign and insert it onto Equation 1. Also, Equation 4 will be used to replace the time derivation:

$$\frac{d^2}{d(\tau/H_i)^2} \psi E_P + 3H \frac{d}{d(\tau/H_i)} \psi E_P + \hbar c^3 V'(\phi) = 0 \tag{9}$$

Both E_P and H_i are constant at all time so that:

$$H_i^2 E_P \frac{d^2\psi}{d\tau^2} + 3H H_i E_P \frac{d\psi}{d\tau} + \hbar c^3 \frac{dV}{d\phi} = 0 \tag{10}$$

Dividing by $H_i^2 E_{\mathrm{P}}$ and insert Equation 6 for ϕ in the third term gives:

$$\frac{d^2\psi}{d\tau^2} + 3\frac{H}{H_i}\frac{d\psi}{d\tau} + \frac{\hbar c^3}{H_i^2 E_{\mathrm{P}}} \frac{dV}{d(\psi E_{\mathrm{P}})} = 0 \tag{11}$$

In the second term, we remember the define variable from Equation 5 and in the third term we may use Equation 8 which leaves us with:

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{dv}{d\psi} = 0 \tag{12}$$

Which is what we wanted to end up with. Next we want to rewrite Equation 2 by using the dimensionless variables as well. First we start by dividing the equation by H_i^2 so that the left term becomes h^2. Next we will here also use Equation 4 and Equation 6 on the time derivative of ϕ:

$$h^2 = \frac{8\pi G}{3c^2 H_i^2} \left[\frac{H_i^2 E_{\mathrm{P}}^2}{2\hbar c^3} \left(\frac{d\psi}{d\tau}\right)^2 + V(\phi)\right] \tag{13}$$

$$\Rightarrow h^2 = \frac{8\pi G}{3c^2 \cancel{H_i^2}} \frac{\cancel{H_i^2} E_{\mathrm{P}}^2}{\hbar c^3} \left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + \frac{\hbar c^3}{H_i^2 E_{\mathrm{P}}^2} V(\phi)\right] \tag{14}$$

$$\Rightarrow h^2 = \frac{8\pi}{3}\frac{GE_\mathrm{P}^2}{\hbar c^5}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{15}$$

where we remember the definition of the Planck energy as $E_\mathrm{P}^2 = \hbar c^5/G$ which simplifies the equation above further into:

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v(\psi)\right] \tag{16}$$

Now we have shown that Equation 1 and Equation 2 can be rewritten into Equation 12 and Equation 16 respectively.

Equation 12 is recognized as an harmonic oscillator with damping as we have an term that is dependent on the single time derivative of the field. We assume the scalar field to be initially at large scale, meaning that at $t = 0$ we have $\phi_i >> 0$. The scalar field will initially starts to "roll-down" the potential hill towards the bottom, or towards $V = 0$. If it rolls sufficiently slow, hence the slow-roll condition, the potential is essentially constant i.e. the acceleration of the field is zero and therefore the first term at the left side on Equation 12 is just zero. We remember the definition of $h = H/H_i$ where H_i is the initial value of the Hubble parameter. The value of h is just 1 since we are looking at the initial phase of the inflation, meaning that we are left with just:

$$\left(\frac{d\psi}{d\tau}\right)_{\tau=0} = -\frac{1}{3}\left(\frac{dv}{d\psi}\right)_{\psi=\psi_i} \tag{17}$$

We can find the initial value of the scalar field by introducing an equation that tells us something about the growth of the scale factor, namely the number of e-foldings. If the slow roll condition are met, the number of e-foldings can be found by (Elgarøy 2020, eq. 6.10) as:

$$N(t) = \frac{8\pi}{E_\mathrm{P}}\int_{\phi_\mathrm{end}}^{\phi}\frac{V}{V'}d\phi \tag{18}$$

where ϕ_end represents the field at the end of the inflation and is found from the criterion $\epsilon(\phi_\mathrm{end}) = 1$. We will set $\phi = \phi_i$ in the upper integral limit so that we integrate throughout the period of inflation. From (Elgarøy 2020, p. 117) we have that approximately 60 e-foldings is necessary for the inflation to be useful. With the potential energy $V(\phi) = m^2c^4\phi^2/2(\hbar c)^3$, we get:

$$60 = \frac{8\pi}{E_\mathrm{P}^2}\int_{\phi_\mathrm{end}}^{\phi_i}\frac{\cancel{m^2c^4}\phi^{\cancel{2}}}{2\cancel{(\hbar c)^3}}\frac{4\cancel{(\hbar c)^3}}{\cancel{m^2c^4}\cancel{\phi}}d\phi \tag{19}$$

$$\Rightarrow \int_{\phi_\mathrm{end}}^{\phi_i}\phi d\phi = 15\frac{E_\mathrm{P}^2}{\pi} \tag{20}$$

$$\Rightarrow \phi_i^2 = \phi_\mathrm{end}^2 + 30\frac{E_\mathrm{P}^2}{\pi} \tag{21}$$

We need to find ϕ_{end} by the criterion mentioned earlier. The definition of ϵ is:

$$\epsilon = \frac{E_{\text{P}}^2}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{22}$$

where $\epsilon << 1$ for the slow-roll condition to be met. So ϕ_{end} becomes:

$$\epsilon_{\text{end}} = \frac{E_{\text{P}}^2}{16\pi}\left(\frac{2}{\phi_{\text{end}}}\right)^2 = 1 \tag{23}$$

$$\Rightarrow \phi_{\text{end}} = \frac{E_{\text{P}}}{\sqrt{4\pi}} \tag{24}$$

Inserting this into Equation 21 gives:

$$\phi_i^2 = \frac{E_{\text{P}}^2}{4\pi} + 30\frac{E_{\text{P}}^2}{\pi} \tag{25}$$

$$\Rightarrow \phi_i = \sqrt{\frac{121E_{\text{P}}^2}{4\pi}} = \frac{11}{2\sqrt{\pi}}E_{\text{P}} \tag{26}$$

We have found an initial value for the scalar field by assuming the slow-roll condition is met and with an approximation of the e-foldings.

Numerical problems and methods

We want to solve Equation 12 and Equation 16 numerically. We recognise the first equation as an oscillating spring with damping. If we flip this equation so that we get the double derivative on the left side, we have:

$$\frac{d^2\psi}{d\tau^2} = -3h\frac{d\psi}{d\tau} - \frac{dv}{d\psi} \tag{27}$$

This is the equation that will be calculated through every iterations. We have found the initial value $(d\psi/d\tau)_i$ and ψ_i which mean that we may calculate the initial double derivative:

$$\left(\frac{d^2\psi}{d\tau^2}\right)_i = -3h\left(\frac{d\psi}{d\tau}\right)_i - \left(\frac{dv}{d\psi}\right)_i \tag{28}$$

If we remember Equation 17, we notice that $(d^2\psi/d\tau^2)_i = 0$. We will then estimate the single derivative as:

$$\frac{d\psi'}{d\tau} = \frac{d^2\psi}{d\tau^2} = \psi'' \tag{29}$$

discretising gives:

$$\frac{\psi_i' - \psi_{i-1}'}{\Delta\tau} = \psi_{i-1}'' \tag{30}$$

$$\Rightarrow \psi_i' = \psi_{i-1}' + \psi_{i-1}'' \Delta\tau \tag{31}$$

The next goes for the numerical approach of ψ:

$$\frac{d\psi}{d\tau} = \psi' \tag{32}$$

We will discretise the last equation and remember that we have just calculated ψ_i' so that we may implement the Euler-Cromer method:

$$\frac{\psi_i - \psi_{i-1}}{\Delta\tau} = \psi_i' \tag{33}$$

$$\Rightarrow \psi_i = \psi_{i-1} + \psi_i' \Delta\tau \tag{34}$$

From Equation 27 we must also find $dv/d\psi$. We know the definition of v from Equation 7 and ψ from Equation 6 and since we are given the potential $V = E_P^2\phi^2/2 \cdot 10^4(\hbar c)^3$, we get:

$$\frac{dv}{d\psi} = \frac{d}{d\psi}\left(\frac{\cancel{\hbar c^3}}{H_i^2 \cancel{E_P^2}} \frac{1}{2 \cdot 10^4} \frac{\cancel{E_P^2}}{(\hbar \cancel{c})^{\cancel{3}2}} E_P^2 \psi^2\right) \tag{35}$$

$$= \frac{d}{d\psi}\left(\frac{E_P^2}{2 \cdot 10^4 \cdot \hbar^2 H_i^2}\psi^2\right) = \frac{E_P^2}{10^4 \cdot \hbar^2 H_i^2}\psi \tag{36}$$

This is implemented in the code as a callable function as it only needs ψ as input. Next is to find a from Equation 27. This is calculated by taking the square-root of Equation 16 since we have already found $d\psi/d\tau$ and have an expression for $v(\psi)$.

In the same plot as above, we will include the analytical solution of the scalar field found from (Elgarøy 2020, p. 116):

$$\phi(t) = \phi_i - \frac{mc^2 E_P}{\hbar\sqrt{12\pi}}t \tag{37}$$

We will let $mc^2 = 0.01E_P$ and we also need the field to become dimensionless before we compare the numerical and analytical solution with eachother. This is done by calculating the scalar field ϕ with respect to an appropriate interval of t and then use Equation 6 to get $\psi_{\text{analytical}}$.

Next we will try to estimate how many e-foldings we get from our results. From (Elgarøy 2020, p. 108), we have that the e-foldings is given by:

$$e\text{-foldings} = \ln\left(\frac{a(t_f)}{a_(t_i)}\right) \tag{38}$$

where $a(t_i)$ is the initial scale factor and $a(t_f)$ is the final scale factor. We want to use the result of h, calculated from solving the numerical approach on the scalar field, to plot the ratio between the scale factors $a(\tau)/a_i$. We need to start with the definition of the Hubble parameter:

$$H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{d\tau}{dt}\frac{da}{d\tau} = \frac{H_i}{a}\frac{da}{d\tau} \tag{39}$$

Where we have used Equation 4 to find H_i. We further get:

$$\frac{H}{H_i} = h = \frac{1}{a}\frac{da}{d\tau} \tag{40}$$

$$\Rightarrow \int_0^\tau h d\tau = \int_{a_i}^{a(\tau)} \frac{1}{a} da \tag{41}$$

$$\Rightarrow \int_0^\tau h d\tau = \ln\left(\frac{a(\tau)}{a_i}\right) \tag{42}$$

The integral on the left side is the term to calculate with the values of h already calculated. We will use an numerical approach by summing over every products $h_i \cdot d\tau$ towards an arbitrary value τ and plot it.

Since we are dealing with a scalar field that is homogeneous, i.e. ϕ is only time dependent, the energy density of the field can be derived from (Elgarøy 2020, eq. 6.1) as:

$$\rho_\phi c^2 = \frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 + V(\phi) \tag{43}$$

and the pressure from (Elgarøy 2020, eq. 6.2) as:

$$p_\phi = \frac{1}{2\hbar c^3}\left(\frac{d\phi}{dt}\right)^2 - V(\phi) \tag{44}$$

We want to implement the dimensionless variables in both of these equations starting of with Equation 43:

$$\rho_\phi c^2 = \frac{1}{2\hbar c^3}\left(\frac{dE_P\psi}{d\tau/H_i}\right)^2 + \frac{H_i^2 E_P^2}{\hbar c^3} v \tag{45}$$

$$= \frac{H_i^2 E_P^2}{\hbar c^3}\left(\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v\right) \tag{46}$$

Since the difference in Equation 44 is a minus sign, we get:

$$p_\phi = \frac{H_i^2 E_P^2}{\hbar c^3}\left(\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v\right) \tag{47}$$

The ratio between the pressure and the energy field then becomes:

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 - v}{\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + v} \tag{48}$$

The ratio will be plotted so that we may see what happens to the energy during the slow-roll phase and in the oscillating phase.

Results

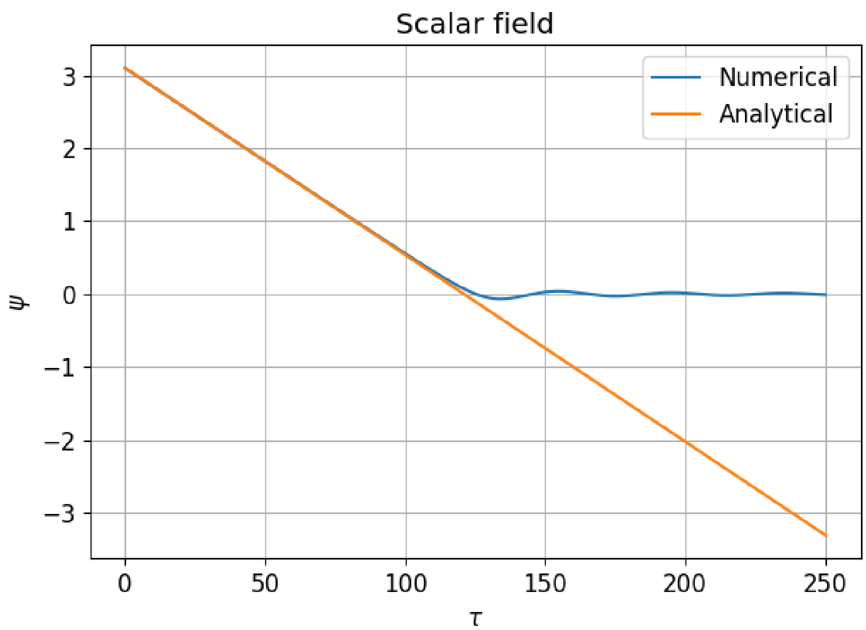

Figure 1: A numerical and analytical result on the scalar field with respect to time. Notice that the scalar field ψ and time τ are both dimensionless.

We notice from Figure 1 that the numerical and analytical results of the scalar fields behaves quite the same in the slow-roll phase. We see from the analytical Equation 37 that the inflation stays linearly at all time which coincides with the numerical solution up to a point. The numerical approach start oscillating at $\psi = 0$ and then the amplitude gets smaller and smaller. This happens when the slow-roll starts fading at around $\tau \approx 100$.

Figure 2 shows how the number of e-foldings starts to converge as the slow-roll condition ends with respect to τ. It looks to be converging towards 63 after some time during the inflation.

We notice from Figure 3 how the ratio between the energy densities and the pressure evolves with time. The ratio is almost constant during the slow-roll regime before it eventually starts to oscillate with constant frequency.

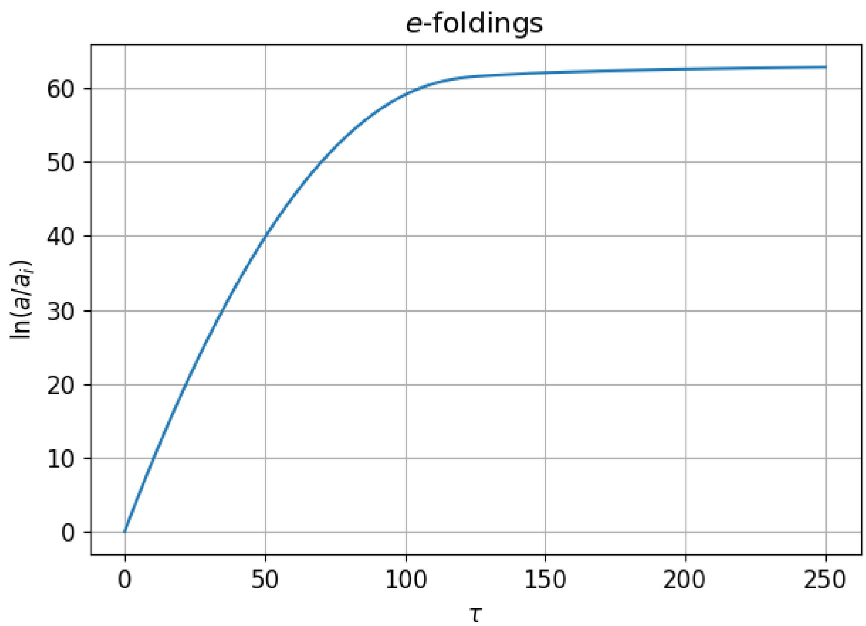

Figure 2: The evolution of e-foldings with respect to τ.

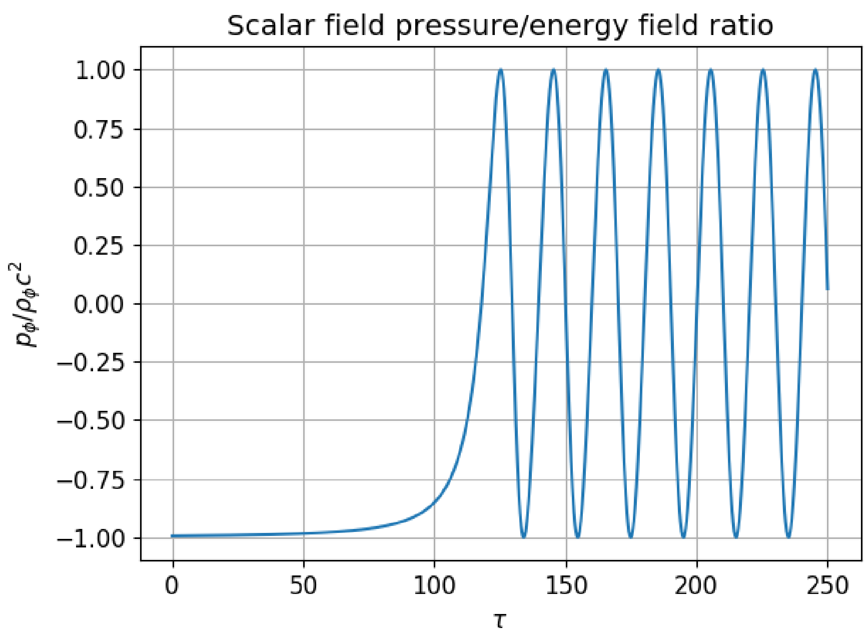

Figure 3: The ratio between the pressure and energy density as function of τ.

Conclusion

We remember that Equation 37 is just the solution for the slow-roll regime, and not for what happens after. This means that the numerical and analytical solution match very well until the field reaches zero and starts oscillating.

We found that based on the numerical calculations of the scale factor $a(\tau)$, that the e-foldings must be approximately 63 as we have stated that the e-foldings should be around 60 for the inflation to be useful. The numerical solution fits good with the statements made above. There are still some deviation from the value used to find ϕ_i as may be a result of numerical errors or the method used are not sufficient enough.

Since we are dealing with a slow-roll condition, the expansion velocity of the Universe is basically constant and therefore the potential should be constant as there is no loss or gain in energy. We should then expect Equation 48 to be constant throughout the slow-roll phase. The potential will eventually reach zero and start oscillating. Since the slow-roll condition is still present, the amplitude of the oscillation should decrease with time. As we see from Equation 48, amplitude should be normalize at all time and that is confirmed by Figure 3. We observe that the oscillation will be present for the rest of time.

References

Elgarøy, Øystein (2020). "AST3220 – Cosmology I". In: URL: https://www.uio.no/studier/emner/matnat/astro/AST3220/v20/undervisningsmateriale/lectures_2019.pdf.