Project 1 - AST3220

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Analytical problems and methods

Gauss' law states that if there exist an electrically charged point Q in a three dimensional space, we may enclose it with a sphere with a surface area S. The charged point Q will create a electric field surrounding it. The electric field caused by Q will traverse the sphere such that the total electric flux on the area is Q/ϵ_0 where ϵ_0 is a constant. The flux is also found to be $\phi_E = \oiint_S \mathbf{E}d\mathbf{A} = E \cdot S$ where E is the electric field. We know the expression of a closed sphere surface as $S = 4\pi r^2$ where r is the radius from the charged point. The electric field is then given as:

$$E = \frac{\phi_E}{S} = \frac{Q}{\epsilon_0 4\pi}\frac{1}{r^2} \tag{1}$$

or

$$\mathbf{E} = \frac{Q}{\epsilon_0 4\pi}\frac{|\mathbf{r}|}{r^2} \tag{2}$$

where $|\mathbf{r}|$ is just the direction of $\mathbf{r}$. This shows us that the electric field does fall with $1/r^2$. Note that the electric field may be either attractive or repulsive depending on the charge of Q. The same yields for a gravitational field but with a point mass m instead of a charge and the gravitational constant G. The expression goes like:

$$g = Gm\frac{1}{r^2} \tag{3}$$

or

$$\mathbf{g} = -Gm\frac{|\mathbf{r}|}{r^2} \tag{4}$$

The reason for the negative sign is because gravity is found to be attractive at all time.

We now consider the differential equation:

$$\nabla\phi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)\phi = s(\mathbf{r}, t) \tag{5}$$

where $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. We want to solve the equation for $s(\mathbf{r}, t) = 0$ and we will do so by separation of variable. Redefines $\phi(\mathbf{r}, t) = h(\mathbf{r}) \cdot g(t)$ will give:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \mathbf{r}^2}\right) h(\mathbf{r}) g(t) = 0 \tag{6}$$

$$h(\mathbf{r}) \frac{\partial^2}{\partial t^2} g(t) = g(t) \frac{\partial^2}{\partial \mathbf{r}^2} h(\mathbf{r}) \tag{7}$$

$$\frac{1}{g(t)} \frac{\partial^2}{\partial t^2} g(t) = \frac{1}{h(\mathbf{r})} \frac{\partial^2}{\partial \mathbf{r}^2} h(\mathbf{r}) \tag{8}$$

Each side has each of is own variable dependency which means that each term must be equal to a constant that we will set to be $-K^2$. Now we want to solve the left term first. We will try with $g(t) = C_1 e^{-i\omega t}$:

$$\frac{1}{g(t)} \frac{\partial^2}{\partial t^2} g(t) = \frac{1}{g(t)} \frac{\partial^2}{\partial t^2} C_1 e^{-i\omega t} = \frac{1}{\cancel{C_1 e^{-i\omega t}}} (-\omega^2) \cancel{C_1 e^{-i\omega t}} = -\omega^2 \tag{9}$$

so $\omega = K$ where ω is the frequency. Now we want to solve the right term on Equation 8. We remember the definition of $\mathbf{r} = (x, y, z)$ meaning that $h(\mathbf{r})$ must be taken the derivative of with respect to x, y and z. So now trying $h(\mathbf{r}) = C_2 e^{i\mathbf{k}\cdot(x,y,z)}$. $\mathbf{k}$ is related to the wave vector, the propagating direction of a wave. We get:

$$\frac{1}{h(\mathbf{r})} \frac{\partial^2}{\partial \mathbf{r}^2} h(\mathbf{r}) \tag{10}$$

$$= \frac{1}{h(\mathbf{r})} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) C_2 e^{i\mathbf{k}\cdot(x,y,z)} \tag{11}$$

$$= \frac{C_2}{h(\mathbf{r})} \left(e^{i(k_y y + k_z z)} \frac{\partial^2}{\partial \mathbf{x}^2} e^{ik_x x} + e^{i(k_x x + k_z z)} \frac{\partial^2}{\partial \mathbf{y}^2} e^{ik_y y} + e^{i(k_x x + k_y y)} \frac{\partial^2}{\partial \mathbf{z}^2} e^{ik_z z}\right) \tag{12}$$

$$= \frac{\cancel{C_2}}{\cancel{C_2 e^{i\mathbf{k}\cdot(x,y,z)}}} (-k_x^2 - k_y^2 - k_z^2) \cancel{e^{i\mathbf{k}\cdot(x,y,z)}} = -k^2 \tag{13}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ so $k = K$. This means that the ω and k obeys the dispersion relation $\omega = k$. We now have a solution of $\phi(\mathbf{r}, t)$ which is:

$$\phi(\mathbf{r}, t) = h(\mathbf{r}) \cdot g(t) = C_2 e^{i\mathbf{k}\cdot\mathbf{r}} \cdot C_1 e^{-i\omega t} = C e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t} \tag{14}$$

where $C = C_1 + C_2$. If the right boundary conditions are met, we may find it that $C = 1/(2\pi)^{3/2}$ that gives the fully solution:

$$\phi(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t} \tag{15}$$

Q.E.D.

The definition of a Fourier transform is given as:

$$F(k) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{16}$$

where k is the spatial frequency of a wave also called the wavenumber. The inverse Fourier transform is given as:

$$f(x) = \mathcal{F}[F(k)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{-ikx} dk \tag{17}$$

where $1/2\pi$ is the normalization factor. Now we are assuming a static point source which implies that ϕ is independent of time t such that $-\nabla^2\phi = s(\mathbf{r}) = \lambda\delta(\mathbf{r})$. Fourier transforming this has the solution $k^2\Phi(\mathbf{k}) = \lambda \Rightarrow \Phi(\mathbf{k}) = \lambda/k^2$. We remember that $\mathbf{r}$ is spatial i.e. in three dimensions meaning that we get a k-component in all three dimensions. If we inverse transform this and take into account for the dimensions, we get:

$$\phi(\mathbf{r}) = \mathcal{F}^{-1}[\Phi(\mathbf{k})] = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \Phi(\mathbf{k}) e^{-i(k_x,k_y,k_z)\cdot(x,y,z)} dxdydz \tag{18}$$

$$= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\lambda}{k^2} e^{-i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k} \tag{19}$$

$$= \frac{\lambda}{(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{e^{-i\mathbf{k}\cdot\mathbf{r}}}{k^2} d\mathbf{k} \tag{20}$$

It would be wise to convert to spherical coordinates with the new parameters $x = k\sin\phi\cos\theta$, $y = k\sin\phi\sin\theta$ and $z = k\cos\phi$ where $\phi \in [0,\pi]$, $\theta \in [0, 2\pi]$ and $k \in [0,\infty)$. $\mathbf{k}$ has a direction radially outwards from the source. Therefore we should integrate over this parameter instead of $\mathbf{r}$. The dot product of two vecto is defined as $\mathbf{k}\cdot\mathbf{r} = kr\cos\phi$ where ϕ is the angle between the two vectors. With the Jacobi determinant $J = k^2\sin\phi$, we get:

$$= \frac{\lambda}{(2\pi)^3} \iiint \frac{e^{-ikr\cos\phi}}{\cancel{k^2}} \cancel{k^2} \sin\phi \; dk d\theta d\phi \tag{21}$$

$$= \frac{2\pi\lambda}{(2\pi)^3} \iint e^{-ikr\cos\phi} \sin\phi \; d\phi dk \tag{22}$$

By substituting $u = \cos\phi$ gives $d\phi = -du/\sin\phi$. The new limits are $u(0) = 1$ and $u(\pi) = -1$ so we are integrating from 1 to -1. By flipping the limits will add a minus-sign in the integral. We may therefore instead use $d\phi = du/\sin\phi$. We then get:

$$= \frac{\lambda}{(2\pi)^2} \iint e^{-ikru} \cancel{\sin\phi} \frac{du}{\cancel{\sin\phi}} dk \tag{23}$$

$$= \frac{\lambda}{(2\pi)^2} \iint e^{-ikru}\ dudk \tag{24}$$

$$= \frac{\lambda}{(2\pi)^2} \iint (\cos kru - i \sin kru)\ dudr \tag{25}$$

So with a little help from a integral calculator with the limits $u \in [-1, 1]$, we get:

$$= \frac{\lambda}{(2\pi)^2} \int \frac{2}{kr} \sin kr\ dk \tag{26}$$

Using the fact that $\int_0^\infty \sin x/x dx = \pi/2$ so that we obtain:

$$= \frac{\lambda}{2\pi^{\cancel{2}} r} \frac{\cancel{\pi}}{2} \tag{27}$$

$$= \frac{\lambda}{4\pi r} \tag{28}$$

Q.E.D.

The hierarchy problem describes the huge discrepancy between expectation and reality. For instance if we assume that the constants c, G and $\hbar$ are the only parameters in theory of fundamental nature, the Planck energy is given as $E_P = \sqrt{\hbar c^5/G} = 10^{28}$ eV. The Planck energy is then a characteristic energy scale in fundamental theory of nature. Based on the standard model, the electromagnetic and the weak force are unified to an important energy scale called the electroweak scale E_{EW}. The standard model suggest in a fundamental theory that the electroweak scale should be of the same order as the Planck scale. In reality, $E_{EW} \approx 10^{13}$ eV which is a factor difference of 10^{15} from the Planck scale. A Hierarchy problem like this is what we will look deeper into through this project.

The convolution theorem states that:

$$\mathcal{F}[f(w)h(w)] = \mathcal{F}[f(w)] * \mathcal{F}[h(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(p)H(w-p)dp \tag{29}$$

The normalization factor $1/2\pi$ is fine to be included in the Fourier transform instead of the inverse Fourier transform. In this case $h(w) = \delta(w)$ where $\delta(w = 0) = \infty$ and $\delta(w \neq 0) = 0$. The Fourier transform of Dirac-delta function is $\mathcal{F}[\delta(w)] = 1$ which implies that $H(w-p) = 1$. This results in:

$$\mathcal{F}[g(w)\delta(w)] = \int_{-\infty}^{\infty} \frac{dp}{2\pi} G(p) \tag{30}$$

for $w = 0$.

The fact that we have a static point-source implies a time independent field ϕ. And so by Fourier transforming (Brown, Mathur, and Verostek 2018, eq. 44) we get:

$$\mathcal{F}\left[\left(\cancel{\frac{\partial^2}{\partial t^2}} - \nabla^2 - \frac{\partial^2}{\partial w^2}\right)\phi + l\delta(w)\left(\cancel{\frac{\partial^2}{\partial t^2}} - \nabla^2\right)\phi\right] \tag{31}$$

$$= \mathcal{F}\left[\left(-\nabla^2 - \frac{\partial^2}{\partial w^2}\right)\phi\right] + \mathcal{F}\left[-l\delta(w)\nabla^2\phi\right] \tag{32}$$

$$= \mathcal{F}[-\nabla^2\phi] + \mathcal{F}\left[-\frac{\partial^2}{\partial w^2}\phi\right] + \mathcal{F}\left[-l\delta(w)\nabla^2\phi\right] \tag{33}$$

$$= k^2\Phi(\mathbf{k}) + p^2\Phi(p) + l\mathcal{F}\left[-\delta(w)\nabla^2\phi\right] \tag{34}$$

where k is the spatial frequency (wavenumber) of ϕ and p is the frequency in w-dimension of ϕ. We recognise the last term as $\mathcal{F}[-\delta(w)\nabla^2\phi(x,y,z,w)] = k^2F(\mathbf{k})$ which leave us with:

$$= k^2\Phi(\mathbf{k}) + p^2\Phi(p) + lk^2F(\mathbf{k}) = \lambda \tag{35}$$

$$\Rightarrow (k^2 + p^2)\Phi(\mathbf{k}, p) + lk^2F(\mathbf{k}) = \lambda \tag{36}$$

$$\Rightarrow \Phi(\mathbf{k}, p) + \frac{lk^2}{k^2 + p^2}F(\mathbf{k}) = \frac{\lambda}{k^2 + p^2} \tag{37}$$

Q.E.D.

Now we want to solve for $F(\mathbf{k})$. We will start by integrating Equation 37 with respect with p on both sides. Also we will divide every term with 2π:

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\Phi(\mathbf{k}, p) + \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{lk^2}{k^2 + p^2}F(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{\lambda}{k^2 + p^2} \tag{38}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dp}{2\pi}\Phi(\mathbf{k}, p) + \frac{lk^2}{2\pi}F(\mathbf{k})\int_{-\infty}^{\infty} \frac{dp}{k^2 + p^2} = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} \frac{dp}{k^2 + p^2} \tag{39}$$

From (Brown, Mathur, and Verostek 2018, eq. C3), we know that:

$$F(\mathbf{k}) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\Phi(\mathbf{k}, p) \tag{40}$$

We will implement this in Equation 39 which then results in:

$$\Rightarrow F(\mathbf{k}) = \left(\frac{\lambda}{2\pi} - \frac{lk^2}{2\pi}F(\mathbf{k})\right)\int_{-\infty}^{\infty} \frac{dp}{k^2 + p^2} \tag{41}$$

$$\Rightarrow F(\mathbf{k}) = \left(\frac{\lambda}{2\pi} - \frac{lk^2}{2\pi}F(\mathbf{k})\right)\frac{1}{k}\left[\arctan\frac{p}{k}\right]_{-\infty}^{\infty} \tag{42}$$

$$\Rightarrow F(\mathbf{k}) = \frac{1}{2\cancel{\pi}k}\left(\lambda - lk^2F(\mathbf{k})\right)\cancel{\pi} \tag{43}$$

$$\Rightarrow F(\mathbf{k}) + \frac{lkF(\mathbf{k})}{2} = \frac{\lambda}{2k} \tag{44}$$

$$\Rightarrow F(\mathbf{k}) \left(1 + \frac{lk}{2}\right) = \frac{\lambda}{2k} \tag{45}$$

$$\Rightarrow F(\mathbf{k}) = \frac{\lambda}{2k} \frac{1}{\left(1 + \frac{lk}{2}\right)} \tag{46}$$

We notice that $F(\mathbf{k})$ only depends on the parameters k and l where l is a parameter in the DGP model.

Inverse Fourier transforming $\Phi(\mathbf{k}, p)$ using Equation 17 will give us:

$$\phi(\mathbf{r}, w) = \mathcal{F}^{-1}[\Phi(\mathbf{k}, p)] = \int_{-\infty}^{\infty} \frac{d\mathbf{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp}{2\pi} \Phi(\mathbf{k}, p) e^{-ipw} e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{47}$$

We recognise the inner integral as Equation 40 which gives us:

$$\phi(\mathbf{r}, w) = \mathcal{F}^{-1}[\Phi(\mathbf{k}, p)] = \int_{-\infty}^{\infty} \frac{d\mathbf{k}}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-ipw} \tag{48}$$

Now $w = 0$ will reduce the integral into:

$$\phi(\mathbf{r}, 0) = \mathcal{F}^{-1}[\Phi(\mathbf{k}, 0)] = \int_{-\infty}^{\infty} \frac{d\mathbf{k}}{(2\pi)^3} F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{49}$$

We notice that to find $\phi(\mathbf{r}, 0)$, we need only the term $F(\mathbf{k})$.

We will solve Equation 49 by changing to spherical coordinates as we did in Equation 21. This means that $d\mathbf{k} = k^2 \sin\phi \; dk d\theta d\phi$ and $\mathbf{k} \cdot \mathbf{r} = kr\cos\phi$. The domains are as before; $k \in [0, \infty)$, $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$:

$$\phi(\mathbf{r}, 0) = \frac{1}{(2\pi)^3} \iiint F(\mathbf{k}) e^{-ikr\cos\phi} k^2 \sin\phi \; dk d\theta d\phi \tag{50}$$

$$= \frac{2\pi}{(2\pi)^3} \iint F(\mathbf{k}) e^{-ikr\cos\phi} k^2 \sin\phi \; dk d\phi \tag{51}$$

Substituting $u = \cos\phi$ that gives $d\phi = -du/\sin\phi$. The new limits are $u(0) = 1$ and $u(\pi) = -1$. Implementing this and flipping the integral limits gives:

$$= \frac{1}{4\pi^2} \iint F(\mathbf{k}) e^{-ikru} k^2 \cancel{\sin\phi} \; dk \left(\frac{du}{\cancel{\sin\phi}}\right) \tag{52}$$

$$= \frac{1}{4\pi^2} \int F(\mathbf{k}) k^2 \int e^{-ikru} \; du dk \tag{53}$$

$$= \frac{1}{4\pi^2} \int F(\mathbf{k}) k^2 \int (\cos kru - i\sin kru) \; du dk \tag{54}$$

$$= \frac{1}{4\pi^2} \int F(\mathbf{k}) k^{\cancel{2}} \frac{2}{\cancel{k} r} \sin kr \; dk \tag{55}$$

$$= \frac{1}{2\pi^2 r} \int F(\mathbf{k}) k \sin kr \; dk \tag{56}$$

Inserting Equation 46 results in:

$$= \frac{1}{2\pi^2 r} \int \frac{\lambda}{2\cancel{k}} \frac{1}{\left(1 + \frac{lk}{2}\right)} \cancel{k} \sin kr \; dk \tag{57}$$

$$= \frac{\lambda}{4\pi^2 r} \int_0^\infty \frac{\sin kr}{\left(1 + \frac{lk}{2}\right)} \; dk \tag{58}$$

We then need r and l to calculate the exact integral. Since the integral goes from zero to infinity, we may neglect the r term in sine function in the integrand. When $r \ll l$, the denominator becomes relatively large. When $l/2$ is large, we may rewrite the integral from Equation 58 as following:

$$\phi(\mathbf{r}, 0) = \frac{\lambda}{4\pi^2 r} \int_0^\infty \frac{\sin k}{\left(1 + \frac{lk}{2}\right)} \; dk \tag{59}$$

$$\approx \frac{\lambda}{4\pi^{\cancel{2}} r} \frac{\cancel{\pi}/\cancel{2}}{l/\cancel{2}} = \frac{\lambda}{4\pi r l} \tag{60}$$

When $r \gg l$, $l/2$ becomes relatively small which let us rewrite Equation 58 as:

$$\phi(\mathbf{r}, 0) = \frac{\lambda}{4\pi^2 r} \int_0^\infty \frac{\sin k}{\left(1 + \frac{lk}{2}\right)} \; dk \tag{61}$$

$$\approx \frac{\lambda}{4\pi^2 r} \cdot 1 \tag{62}$$

Numerical problems and methods

We want to find the summation rule of the density parameters in the DGP model looking at the following equation:

$$\frac{H^2(z)}{H_0^2} = \left(\sqrt{\Omega_{\mathrm{m0}}(1+z)^3 + \Omega_{\mathrm{rc}}} + \sqrt{\Omega_{\mathrm{rc}}}\right)^2 + \Omega_{\mathrm{k0}}(1+z)^2 \tag{63}$$

where Ω_{m0} is the dust density parameter, Ω_{k0} is the curvature density parameter, Ω_{rc} is associated with the length scale r_c and z is the redshift. $H(z)$ is the Hubble constant and H_0 is the Hubble constant as of today. The summation rule is found by looking at Equation 63 as of today which implies $z = 0$ as z is associated with the scale factor $1 + z = a_0/a$. Setting $a = a_0$ implies the Universe as it looks today. We then get:

$$\frac{H^2(0)}{H_0^2} = \frac{H_0^2}{H_0^2} = 1 = \left(\sqrt{\Omega_{\mathrm{m0}} + \Omega_{\mathrm{rc}}} + \sqrt{\Omega_{\mathrm{rc}}}\right)^2 + \Omega_{\mathrm{k0}} \tag{64}$$

Before heading on to program a DGP model, we will start of with the ΛCDM model. Before finding the luminosity distance, we must implement the Friedmann equation from (Brown, Mathur, and Verostek 2018, eq. 3.37). Our models will be consist of no radiation so that $\Omega_{r0} = 0$:

$$\frac{H^2(z)}{H_0^2} = \Omega_{\mathrm{m0}}(1+z)^3 + \Omega_{\mathrm{k0}}(1+z)^2 + \Omega_{\Lambda 0} \tag{65}$$

$\Omega_{\Lambda 0}$ is the density parameter of the cosmological constant. The Friedmann equation is a dynamic equation relating to the expanding Universe. With **??**, we may find the luminosity distance from (Brown, Mathur, and Verostek 2018, eq. 3.60) which goes like:

$$d_{\mathrm{L}} = \frac{c(1+z)}{H_0\sqrt{|\Omega_{\mathrm{k0}}|}}\mathcal{S}_k\left(\sqrt{|\Omega_{\mathrm{k0}}|}\int_0^z \frac{dz'}{H(z')/H_0}\right) \tag{66}$$

We notice that the right-hand side of the Friedmann equation should allways stay positive so that we do not get complex values within the integral. This is implemented as a test in the program. $\mathcal{S}_k$ is defined as follows:

$$\mathcal{S}_{\mathrm{k}}(x) = \begin{cases} \sin x, & k = 1 \\ x, & k = 0 \\ \sinh x, & k = -1 \end{cases} \tag{67}$$

It is sufficient to just consider whether Ω_{k0} is greater, less or equal to zero. This is based on the correlation between k and Ω_{k0} where k $\propto -\Omega_{\mathrm{k0}}$. Since the integration in Equation 66 is solved numerically, the infinitesimal dz' should be small enough to avoid integration error. It is set to 10^{-3} which is of approximately an order of thousand from the biggest value of z. The units of the luminosity distance will be of c/H_0 and as a function of $z \in (0, 2)$.

We want to find two limiting cases where we may find the luminosity distance analytically. We will do this first for the ΛCDM model and then for the DGP model. First we are solving for the case where $\Omega_{\mathrm{k0}} \to 0$ which implies that $k \to 0$ and also $\Omega_{\mathrm{m0}} = 0$. This is also known as the de Sitter model, a Universe dominating with the cosmological constant parameter. This seemed to be the case in the early Universe. From Equation 67, we notice that $\sin x \approx x$ and $\sinh x \approx x$ as $x \to 0$. We are then left with:

$$\lim_{|\Omega_{\mathrm{k0}}| \to 0} d_{\mathrm{L}} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{H(z')/H_0} \tag{68}$$

Implementing Equation 65 gives:

$$d_{\mathrm{L}} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{\Lambda 0}}} = \frac{c(1+z)}{H_0}\frac{z}{\sqrt{\Omega_{\Lambda 0}}} \tag{69}$$

Since the curvature parameter Ω_{k0} is included in this case, the sum over all density parameters is equal to one; $\sum_i \Omega_{i0} = 1$ where i goes over all density parameters including Ω_{k0}. This implementation results in:

$$\Omega_{\mathrm{m}0} + \Omega_{\mathrm{k}0} + \Omega_{\Lambda 0} = 0 + 0 + \Omega_{\Lambda 0} = 1 \tag{70}$$

With this, we get the analytical solution of Equation 69 as:

$$d_{\mathrm{L}} = (1+z)z\frac{c}{H_0} \tag{71}$$

The next case will be for $\Omega_{\mathrm{k}0} = \Omega_{\Lambda 0} = 0$ so that $\mathcal{S}(x) = x$ as previous. In this model the Universe is filled with dust and known as the Einstein-de sitter model. Using Equation 66, we get:

$$d_{\mathrm{L}} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{H(z')/H_0} \tag{72}$$

$$= \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_{\mathrm{m}0}(1+z')^3}} \tag{73}$$

Remember that $\Omega_{\mathrm{k}0} + \Omega_{\mathrm{m}0} + \Omega_{\Lambda 0} = 1$ implying that $\Omega_{\mathrm{m}0} = 1$ which gives:

$$= \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{(1+z')^{3/2}} \tag{74}$$

$$= (1+z)\left(2 - \frac{2}{\sqrt{z+1}}\right)\frac{c}{H_0} \tag{75}$$

We have found two analytical solution that we will compare with our code for the ΛCDM model. Now we want to find two solution for the luminosity distance for DGP model. Starting with $\Omega_{\mathrm{k}0} \to 0$ and $\Omega_{\mathrm{m}0} = 0$ here as well and solving Equation 63, gives:

$$\frac{H^2(z)}{H_0^2} = \left(2\sqrt{\Omega_{\mathrm{rc}}}\right)^2 = 4\Omega_{\mathrm{rc}} \tag{76}$$

Implementing this in Equation 66 gives:

$$d_{\mathrm{L}} = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{2\sqrt{\Omega_{\mathrm{rc}}}} = \frac{c(1+z)z}{2H_0\sqrt{\Omega_{\mathrm{rc}}}} \tag{77}$$

Taken account for the summation rule in Equation 64 tells us that $1 = \left(\sqrt{0 + \Omega_{\mathrm{rc}}} + \sqrt{\Omega_{\mathrm{rc}}}\right)^2 + 0 = 4\Omega_{\mathrm{rc}} \to \Omega_{\mathrm{rc}} = 1/4$. The first case of luminosity distance becomes then:

$$d_{\mathrm{L}} = (1+z)z\frac{c}{H_0} \tag{78}$$

The next case will be setting $\Omega_{\mathrm{k}0} = \Omega_{\mathrm{rc}} = 0$. This gives form Equation 64 that $1 = \left(\sqrt{\Omega_{\mathrm{m}0} + 0} + 0\right)^2 + 0 = \Omega_{\mathrm{m}0}$. The Friedmann Equation 63 gives then:

$$\frac{H^2(z)}{H_0^2} = \left(\sqrt{1 \cdot (1+z)^3 + 0} + 0\right)^2 + 0 = (1+z)^3 \tag{79}$$

Implementing this in Equation 66 gives the same integral as in Equation 73 leaving us with:

$$d_{\mathrm{L}} = (1+z)\left(2 - \frac{2}{\sqrt{z+1}}\right)\frac{c}{H_0} \tag{80}$$

We notice that for both cases, we get the same results for the ΛCDM model and the DGP model. We will use these results and compare them to our numerical calculations.

We want to find the most probable value of Ω_{m0} and $\Omega_{\Lambda 0}$ based on the data we have. If we assume that the observation will act like a Gaussian distribution and that the measurements are uncorrelated, it can be shown that the likelihood between the data and the model can be given as:

$$P(\text{data}|\text{model}) \propto \exp\left\{-\frac{1}{2}\chi^2(\vec{p})\right\} \tag{81}$$

where $\vec{p} = (\Omega_{\mathrm{m0}}, \Omega_{\Lambda 0})$ and $\chi^2(\vec{p})$ is defined as follows:

$$\chi^2(\vec{p}) = \sum_{i=1}^{N} \frac{(d_{\mathrm{L}}(z_i;\vec{p}) - d_{\mathrm{L}}^i)^2}{\sigma_i^2} \tag{82}$$

$d_{\mathrm{L}}(z_i;\vec{p})$ is the luminosity distance predicted from the model we created with the parameters Ω_{m0} and $\Omega_{\Lambda 0}$ at measured redshift z_i, d_{L}^i is the measured luminosity distance and σ_i^2 is the measurement error. We want the likelihood to be maximized to find the most probable value of the density parameters. This is equivalent to minimizing $\chi^2(\vec{p})$ which will be our focus during this part of the programming.

First part will be creating two quadratic grids for each of the density parameters Ω_{m0} and $\Omega_{\Lambda 0}$ which are transposed to each other. We may then define an additional same-sized grid corresponding to the luminosity distance that has a unique set of density parameter in each element on that grid. Both of the density parameters are sent in a another function to solve the Friedmann equation for each element in the grid for various z. Since the Friedmann equation is integrated with respect to z', we will sum over the obtained values and then take the product of the summation and the infinitesimal dz'. Next step is to consider how each element should be treated, according to Equation 67. This is done by checking every element in Ω_{k0}, whether the values are greater, equal or less than zero as mentioned above. The fact that $\Omega_{\mathrm{k0}} = 0$ supports the theory of a flat Universe. We will implement the flat Universe scenario as well in the contour plot. When this is done, each element in the d_{L}-grid is determined with the use of Equation 66. The $\chi^2(\vec{p})$ is then found by Equation 82 and stored in a separate file so that we do not need to run the code every time we want to look at the likelihood contour plot. The index corresponding to the minimum value of the $\chi^2(\vec{p})$ grid is found so that we may extract the respective density parameters Ω_{m0} and $\Omega_{\Lambda 0}$. The density parameters that relates to a best fit model is also included in that file as well as all the tested values of Ω_{m0} and $\Omega_{\Lambda 0}$.

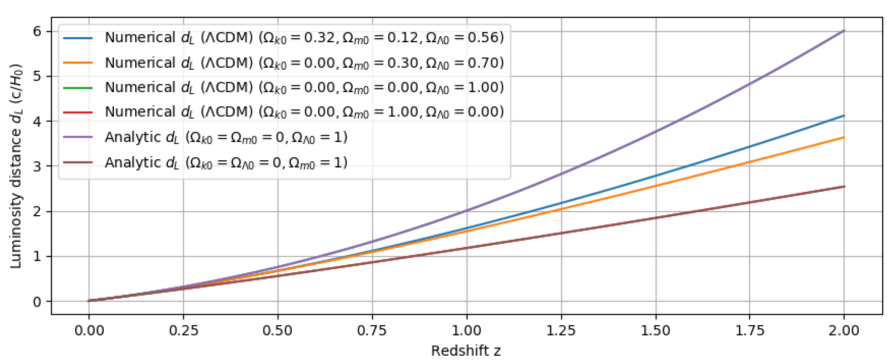
The minimum value $\chi^2_{\min}$ is found and subtracted from $\chi^2(\vec{p})$. As $\chi^2(\vec{p})$ is derived out of the Gaussian distribution, there is a 95 % chance that any of the elements that has a value below 6.17 is the true value. The value of 6.17 corresponds to 2σ off the most probable value. We will make a contour plot of this 95 % probability region to give a representation of what the values of Ω_{m0} and $\Omega_{\Lambda 0}$ seems to be.

We have found a method to estimate the most probable value of Ω_{m0} and $\Omega_{\Lambda 0}$. We will use the contour plot to choose two different cases where the parameters lies within the 95 % certainty range and compare the luminosity distance plots with the measurement. The probable parameters will be included in the plot. The point of this is to observe if the model coincide with the measurement. We will also include the de Sitter model and the Einstein-de Sitter model in the plot. The measurement data is given in units of Gpc so it is converted into units of c/H_0 as it is extracted from the file.

After we have studied the ΛCDM model, we will take a closer look at the DGP model. The main difference is how the Friedmann equation is defined. The DGP model include dark energy as a component of the Universe. Now Equation 63 is used where the same condition stands as before. The right hand side of the Friedmann equation must be positive to prevent complex values when finding the luminosity distance from Equation 66. We will find the best fit parameters of Ω_{m0} and $\Omega_{\Lambda 0}$ and see the difference in the parameters found from the ΛCDM model. Then comparing both model and study which fits the data best will be done.

Results

Figure 1: Evolution of the luminosity distance as a function of the redshift z. The green and purple line (on top of each other) corresponds to the de Sitter model where the universe is dominated by the cosmological constant density parameter, also known to be the case at the early Universe. Red and brown line (also on top of each other) corresponds to the Einstein-de Sitter model, a dust dominant Universe. Blue and orange line contains random parameter of choice.



From Figure 1 we notice that one of the limiting case, also known as the de Sitter model where $\Omega_{\mathrm{k0}} = 0$ and $\Omega_{\Lambda 0} = 1$, lies upon each other for both the analytic and numerical solution. This is also true for the other limiting case, or the Einstein-de Sitter model, where the Universe is fully covered of non-relativistic matter only. This makes us believe that the numerical and analytical solution we have done in this project coincides. All the plots passes the test making sure we are not dealing with complex values.

Figure 2: Ω_{m0} versus $\Omega_{\Lambda 0}$ contour plot for a 200×200 grid for the ΛCDM model. It shows which parameters that gives the biggest likelihood at $\Omega_{\mathrm{m0}} = 0.23$ and $\Omega_{\Lambda 0} = 0.61$. The 95 % likelihood domain is also implemented as well as the parameters that implies a flat universe as $\Omega_{\mathrm{k0}} = 0$.

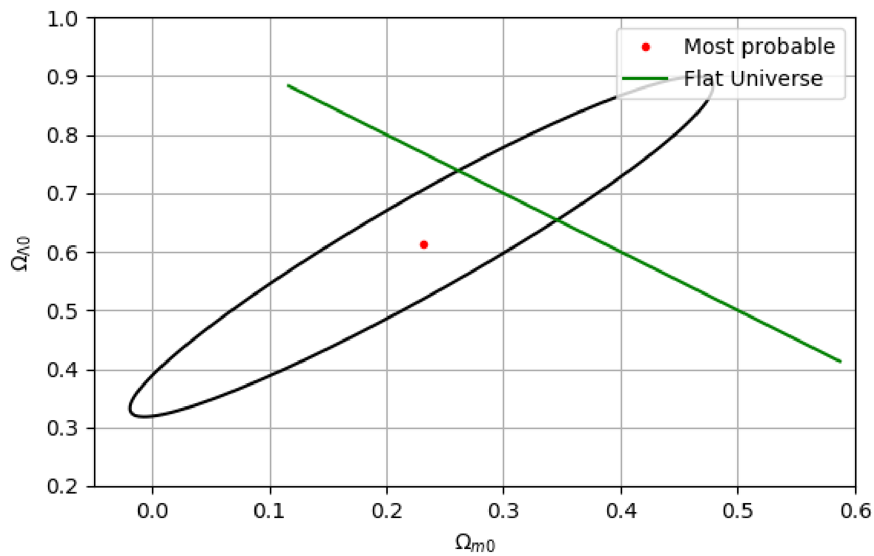
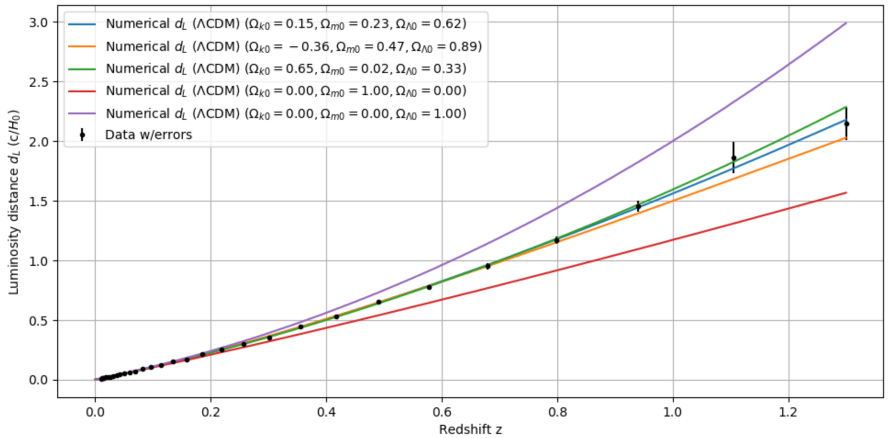


Figure 2 shows where the density parameters Ω_{m0} and $\Omega_{\Lambda 0}$ most likely are found to be within a 95 % certainty based on the data measurement. We notice that the best fit parameter are under the flat Universe line implying that $\Omega_{\mathrm{k0}} < 0$ and supports an ever expanding universe.

We notice from Figure 3 that both the de Sitter model and the Einstein-de Sitter models does not match the measured data as they follow a curve far off from the measurement errors. This quickly affirms that these models does not seems to coincide with reality based on the data used. The green and orange lines follows a model that contains parameters that lies at the 95 % likelihood domain. We notice that they are more or less just at the limits of the measured errors. The blue plot contains the parameters that fits the measurements the best.

Figure 4 shows the domain of where the measured data matches the model within a 95 % likelihood. The best fit density parameters Ω_{m0} and Ω_{rc} are marked and also just under the flat Universe curve implying a ever expanding

Figure 3: Evolution of the luminosity distance as a function of the redshift z for the ΛCDM model. The measured data is included with its errors as well as de Sitter (purple) model and Einstein-de Sitter (red) model. A blue plot containing the best fit parameters are also included. Green and orange plot are two random chosen parameters that lies at the endpoint of the 95 % likelihood domain.



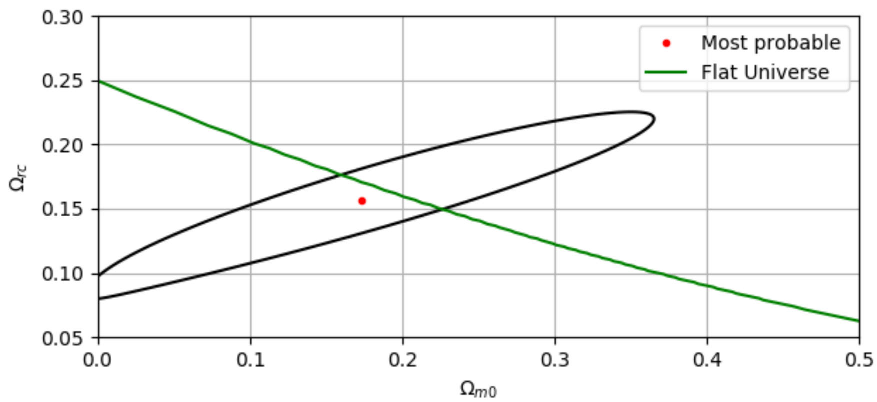
universe as $\Omega_{\mathrm{k}0} < 0$. Nevertheless, we notice that it is much closer to behave as a flat Universe. One could say that the Universe is expanding at a slower rate.

We notice that Figure 5 looks very much alike as Figure 1 meaning that they give the same results for the chosen set of density parameters. What differs is the precision of each of the random parameters from either the ΛCDM model or the DGP model.

Discussion

When plotting the luminosity distance for each of the model, we notice that they are very similar. The best fit density parameters of $\Omega_{\mathrm{m}0}$, $\Omega_{\Lambda 0}$ and $\Lambda\mathrm{rc}$ seems to give the same results as well as the two limiting cases. But we see a that the likelihood domain covers a much smaller area for the DGP model than the ΛCDM model. Because the covered area is smaller means that the precision must be greater for the DGP model. We could therefore conclude with that the DGP model fits the data best and should hold with this theory. Both of the models still gives a good representation of what builds up the Universe. The ΛCDM model should not be forgotten as this may a better representation in the future (indeed a very long way into the future perhaps).

Figure 4: Ω_{m0} versus $\Omega_{\Lambda 0}$ contour plot for a 200×200 grid of the DGP model. It shows which parameters that gives the biggest likelihood at $\Omega_{\mathrm{m0}} = 0.17$ and $\Omega_{\mathrm{rc}} = 0.16$. The 95 % likelihood domain is also implemented as well as the parameters that implies a flat universe as $\Omega_{\mathrm{k0}} = 0$.



References

Brown, Katherine, Harsh Mathur, and Mike Verostek (2018). "Exploring extra dimensions with scalar fields". In: *American Journal of Physics* 86.5, pp. 327–337. DOI: 10.1119/1.5024221. eprint: https://doi.org/10.1119/1.5024221. URL: https://doi.org/10.1119/1.5024221.

Figure 5: Evolution of the luminosity distance as a function of the redshift z for the DGP model. The measured data is included with its errors as well as the two limiting cases ($\Omega_{\mathrm{k0}} = \Omega_{\mathrm{rc}} = 0$ and $\Omega_{\mathrm{k0}} = \Omega_{\mathrm{m0}} = 0$) as numerical and analytical solution. A blue plot containing the best fit density parameters are also included. Green and orange plot are two random chosen parameters that lies at the endpoint of the 95 % likelihood domain.