Project 1 - AST3310

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February 26, 2020

Introduction

A neighbouring star is the most important source of energy for life to exist on any planet in the universe. The Sun is no exception as the Earth would be too cold if the Sun was not there to heat it up. The production of the heat at the core of a star will be the main subject in this project. More specifically modelling the energy production that occurs in the core. We will introduce various chain nuclear reaction that is the foundation for a star to radiate heat, light or other electromagnetic effects. Our focus will be of the Sun which from previous studies has a known core temperature and density. The density will be fixed at all times but it might be interesting to tweak the temperature and observe if this has any effect on the various chain reactions. The work is done numerically with the use of the object oriented language program Python.

Method

A brief introduction to chain reactions in core of a star

The first step of finding the generated energy for each branch is to calculate the reaction rate for each reaction. A reaction occurs whenever two particle collide at each other if they hold enough kinetic energy. The energy must be large enough to overcome the electrostatic potential of the atomic nucleus caused by the Coulomb repulsion. This means that the reactions must be temperature dependent. Quantum tunneling is also needed to take into consideration as the temperature itself is not enough. The temperature for two protons to collide is found to be $3 \cdot 10^{10}$ K but the Sun has a core temperature of only $15 \cdot 10^6$ K (Gudiksen 2020, p. 19). We will assume that the density and temperature of the core is radius independent. When these two factors are included, we will get the reaction rate from (Gudiksen 2020, eq. 3.23) as following:

$$r_{ik} = \frac{n_i n_k}{\rho(1+\delta_{ik})}\lambda_{ik} \tag{1}$$

The indices i and k is the reactants indicator of colliding particles. n_i and n_k are the number densities of element i and k, ρ is the mass density at the

core of the star, δ_{ik} is the Kronecker delta function and λ_{ik} is the temperature dependent part of the reaction rate in units of cubic cm per second. The reason for including δ_{ik} is because λ_{ik} is derived from integrating over both the number densities. If two identical particles collide, the derivation will integrate over two different collisions but there is in fact only one collision. The reaction rate must be divided with two to get the correct result. The values of λ_{ik} is derived and tabulated (Caughlan and Fowler 1988) and will be implemented in the code. The units in r_{ik} is given in per seconds per kilograms which tells us the frequency of a reaction per unit mass. So first of all we need to find the number density. It is found based on the amount of a substance that is present in the core of a star, its unit mass and the density in the core:

$$n_{particle} = \frac{\rho X}{A m_u} \tag{2}$$

where X is the fractional abundances of a substance, A is the nucleon number and m_u is the atomic mass unit. X is usually denoted as the amount of hydrogen that is in the star. Y is for helium and Z is for heavier metals. The atomic substances which the Sun is roughly made up of is found in Table 1. These values will be used for calculating the number densities for each reactions.

Atom	Mass fraction
X	0.7
$Y_{^3_2\mathrm{He}}$	10^{-10}
Y	0.29
$Z_{^7_3\mathrm{Li}}$	10^{-7}
$Z_{^7_4\mathrm{Be}}$	10^{-7}
$Z_{^{14}_7\mathrm{N}}$	10^{-11}

Table 1: The fractional abundance of the most common atomic species in the Sun. The Sun consist mostly of hydrogen and helium and a small amount of heavier elements.

We now have a expression for the reaction rate r_{ik} between two colliding particles, whether they are identical or different particles. So the next step is to know the fully generated energy per unit mass for each of the branches and the CNO cycle. It is given as (Gudiksen 2020, eq. 3.59):

$$\epsilon = \sum Q'_{ik} r_{ik} \tag{3}$$

The summation goes through every reaction in the nuclear chains with helium-4 as the end product. Each reaction releases some amount of energy that goes into the thermal bath and is denoted Q'_{ik}. These values may be derived by Einstein's mass-energy equivalence $E = \Delta m c^2$ where Δm is the change in mass in the reaction. The annihilation of a positron and a electron is included in the term of released energy as the energy of the produced photons contributes to the thermal bath. Each of the chain reaction has a net input of four hydrogen

atom and a net output of one helium-4 atom. The total released energy of any of the chain reaction when using atomic masses is then:

$$E = \Delta mc^2 = (4 \cdot m_{^1_1\mathrm{H}} - m_{^4_2\mathrm{He}})c^2 = 0.028758\ m_u c^2 = 26.787\ \mathrm{MeV} \qquad (4)$$

Some energy is lost due to neutrinos escaping the star, namely Q_ν. The lost neutrino energy does not contribute to the energy balance in a star. Both of the energies is tabulated and is found in Table 2 for the pp-chains and Table 3 for the CNO cycle. The CNO cycle works like a catalysts that produces hydrogen into helium-4. Carbon-12 is not produced by any of the pp-chain but through a process that is called triple-alpha process. The triple-alpha process is a set of nuclear fusions that has three alpha particles, or three helium-4 particles, and has a net result of one carbon-12 particle. The Sun is actually to cold to produce carbon-12 through this process effectively. Only when the core temperature of a star is approximately 6 $\mathrm{T}_\odot$ the production of carbon-12 is fast enough to be effective according to (Wilson 1997). The carbon that the Sun consists of must have been produced from the Sun's ancestors.

Reaction	Rate symbol $[ik]$	Q' [MeV]	Q_ν [MeV]	Branch
$^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_e$	λ_{pp}	1.177	0.265	I, II, III
$^2_1\mathrm{D} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$	λ_{pd}	5.494		I, II, III
$^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + 2\ {}^1_1\mathrm{H}$	λ_{33}	12.860		I
$^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$	λ_{34}	1.586		II, III
$^7_4\mathrm{Be} + \mathrm{e}^- \rightarrow {}^7_3\mathrm{Li} + \nu_\mathrm{e} + \gamma$	λ_{e7}	0.049	0.815	II
$^7_3\mathrm{Li} + {}^1_1\mathrm{H} \rightarrow 2\ {}^4_2\mathrm{He}$	$\lambda_{17'}$	17.346		II
$^7_4\mathrm{Be} + {}^1_1\mathrm{H} \rightarrow {}^8_5\mathrm{B} + \gamma$	λ_{17}	0.137		III
$^8_5\mathrm{B} \rightarrow {}^8_4\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e}$	λ_8	8.367	6.711	III
$^8_4\mathrm{Be} \rightarrow 2\ {}^4_2\mathrm{He}$	$\lambda_{8'}$	2.995		III

Table 2: Released energy Q' from nuclear reactions in the pp-chain. The neutrino energies Q_ν produced from the various reaction is included. Values are gathered from (Gudiksen 2020, tab. 3.2). Note that the released energy from the e7-reaction is the average released energy as Li-7 is produced in either ground state or excited state.

We may now use Equation 1, 2 and 3 to make a model of our problem. A more detailed description of how this is implemented in the code is the subject of the next subsection.

Building the model using Python

We should start of with separating constants and other temperature and density dependent variables from the main program as they are taking up space. We will create a class in a separate Python file called constants.py that needs a core

Reaction	Rate symbol $[ik]$	Q' [MeV]	Q_ν [MeV]
${}^{12}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{13}_{7}\mathrm{N} + \gamma$	λ_{p12}	1.944	
${}^{13}_{7}\mathrm{N} \rightarrow {}^{13}_{6}\mathrm{C} + \mathrm{e}^{+} + \nu_{\mathrm{e}}$	λ_{13}	1.513	0.707
${}^{13}_{6}\mathrm{C} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{14}_{7}\mathrm{N} + \gamma$	λ_{p13}	7.551	
${}^{14}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{15}_{8}\mathrm{O} + \gamma$	λ_{p14}	7.297	
${}^{15}_{8}\mathrm{O} \rightarrow {}^{15}_{7}\mathrm{N} + \mathrm{e}^{+} + \nu_{\mathrm{e}}$	λ_{15}	1.757	0.997
${}^{15}_{7}\mathrm{N} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{12}_{6}\mathrm{C} + {}^{4}_{2}\mathrm{He}$	$\lambda_{\mathrm{p15}'}$	4.966	

Table 3: Released energy Q' from nuclear reaction in the CNO chain. The neutrino energies Q_ν produced from the various reactions is included. Values are gathered from (Gudiksen 2020, tab. 3.3).

temperature and core density as input. Parameters such as mass fraction of the elements from Table 1, released energy and neutrino energy from Table 2 and Table 3 and the derived values of λ_{ik} is found in this file. Also the electron density is included in this file as it depends on the density and the mass fraction distribution in the star. We use Equation 2 to find the electron density. We assume that the elements in the core of a star is fully ionized so that electrons travels freely. Since hydrogen and helium-4 is dominating the structure of a star, heavier elements are neglected so that $Y = 1 - X$. The number density of electron is then:

$$n_{e,total} = n_H + 2n_{He} = \frac{\rho X}{m_u} + 2\frac{\rho Y}{4m_u} = \frac{\rho}{2m_u}(2X + (1 - X)) = \frac{\rho}{2m_u}(X + 1) \quad (5)$$

The electron density is needed for the electron capture by beryllium-7 in the e7-reaction. The electron capture has an upper limit if a star is colder than 10^6 K. The limit is given as $N_A\lambda_{e7} \leq 1.57 \cdot 10^{-7}/n_{e,total}$ (Gudiksen 2020, tab. 3.1) where N_A is Avogadro's constant. This is implemented in the code such that if the core temperature of the star is less than 10^6 K and if $N_A\lambda_{e7}$ exceeds the limit, the value of λ_{e7} is adjusted to a proper value.

The main program **energy_production.py** will consist of functions that represents each of the pp-branches and the CNO cycle, namely **pp_b1**, **pp_b2()**, **pp_b3()** and **CNO()**. The output from these functions corresponds to Equation 3. The released energy form each reaction Q'_{ik} is found in Table 2 and Table 3. Since both pp- and pd-reaction is included in all the pp-branches, a separate function **pp()** will be created for these two reactions to make the code as neat as possible. The reaction rate of pp-reaction is much slower than the pd-reaction. We may therefore assume that the rate of pd is instantaneous meaning that only the reaction rate of pp needs to be calculated. The released energy from both reactions is then summed. A function that calculates the reaction rate r_{ik} using Equation 1 is implemented as **r_ik(X_i, X_k, A_i, A_k, lamb)** where **X_i** and **X_k** is the mass fraction of the colliding elements and **A_i** and **A_k** are the respective nucleon number. **lamb** is λ_{ik} that corresponds to the associated collision.

We will in this model look at a moment in time meaning that we do not need to remove or add elements to the Sun. Since stars are made up of mostly hydrogen and helium-4, we assume that these elements are always disposable for the associated reactions. This is not the case for the less common elements. We need to make sure that no reaction consumes more of an element than the previous reaction are able to produce. This is done by implementing an if test that compares the reaction rate from the previous and the next step. If the next reaction rate is greater than the previous one, then the next rate should be equal the previous. These tests are implemented in each of the chain functions. There are special cases where the output element is used in more than one other reaction. This is the case for the 34-reaction. The beryllium-7 that is produced will be used in both branch 2 and branch 3 in the e7-reaction and 17-reaction respectively. If the production of beryllium-7 is slower than the consumption from e7 and 17-reaction, we can not split the distribution in half since the two next rates are probably not the same. The way to handle this problem is to find a distribution ratio:

$$P(\text{share to e7}) = \frac{r_{e7}}{r_{e7} + r_{17}} \tag{6}$$

$$P(\text{share to 17}) = \frac{r_{17}}{r_{e7} + r_{17}} \tag{7}$$

We need to precalculate the e7 and 17-reaction rate to find the ratio. If the 34-reaction is slower than the sum of e7- and 17-reaction then the next reactions should be correctly distributed by the distribution ratio just found times the previous 34-reaction:

$$r_{e7,\text{update}} = P(\text{share to e7}) \cdot r_{34} \tag{8}$$

$$r_{17,\text{update}} = P(\text{share to 17}) \cdot r_{34} \tag{9}$$

This is also the case for the pd-reaction as it produce helium-3 that goes into all three branches. But since the Sun consists of mostly hydrogen, the production of helium-3 is much greater than the consumption. Nevertheless, we will include the test as the production of helium-3 varies with temperature.

We will implement a sanity check to make sure that our code produce decent results. The sanity check compares the results produced by us with other studies of the Sun. The values to compare are the generated energy for various reactions times the core density. These are tabulated in Table 4 and getting small relative errors will only support other studies. As the reaction rate varies depending on collision of different particle, some reaction rates are much slower than others. This is as mentioned earlier the case for the pp- and pd-reaction. For branch three the 8- and 8'-reaction is neglected because of this effect. Boron-8 is unstable and will undergo a beta decay within a short period of time. Beryllium-8 will quickly decay into two alpha particles fulfilling the chain reaction. From (Lemut et al. 2006) the reaction rate of p14 is the slowest and therefore the

CNO cycle is limited by this reaction. The reactions that may be neglected is taken account for in Table 4 and also in the code.

Rate symbol $[ik]$	$r_{ik} \cdot Q_{ik} \cdot \rho$ [J m^{-3} s^{-1}]	Chain reaction
λ_{pp}	$4.04 \cdot 10^{2}$	I, II, III
λ_{33}	$8.69 \cdot 10^{-9}$	I
λ_{34}	$4.87 \cdot 10^{-5}$	II, III
λ_{e7}	$1.50 \cdot 10^{-6}$	II
$\lambda_{17'}$	$5.29 \cdot 10^{-4}$	II
λ_{17}	$1.64 \cdot 10^{-6}$	III
λ_{p14}	$9.18 \cdot 10^{-8}$	CNO

Table 4: Values are calculated using energies according to (Bahcall and Ulrich 1988) and (Caughlan and Fowler 1988). Note that the value from the pp-reaction include the released energy Q_{pp} and Q_{pd}. The same goes for the 17-reaction as it includes Q_{17}, Q_8 and Q_8' and lastly the p14-reaction includes Q_{p12}, Q_{13}, Q_{p13}, Q_{p14}, Q_{15} and Q_{p15}.

Once finished with the code and making sure that it passes the sanity checks, an evolution of the core temperature in the Sun is implemented. This is done by creating an array of the temperature at $T \in (10^4, 10^9)$. The temperature should be on a logarithmic scale to make sure that we do not miss any valuable information at low temperatures. The energy production from pp and pd will not be included in this case as it will be a dominating source of energy production at some point of temperature. All of the pp-branches would then generate almost the same amount of energy, making the plot less interesting. The reason for implementing this is to observe how the different chain reactions evolve at increasing temperature. Each of the chain reactions should be dominant at various points of temperature.

Results

When taking a closer look at Table 2, we notice that the pp/pd-reaction has to produce two deuterium before the 33-reaction is taking place. This implies that pp-one produce a net energy of 26.732 MeV whereas 1.98 % of the total energy is lost to neutrinos. As mentioned, all four chain reactions produce the same amount of energy since the input is four hydrogen atom and output is one helium-4 atom. We notice that the released energy from the chain reactions agrees with Equation 4 with an relative error of 0.2 %. When studying Table 3 as well, the percentages of lost neutrino energies is 4.04 %, 26.10 % and 6.37 % for pp-two, pp-three and CNO-cycle respectively.

The energy production at the core of the Sun from each of the chain reaction is found to be $\epsilon_{\mathrm{I}} = 0.34$ MeV/kgs, $\epsilon_{\mathrm{II}} = 22.30$ GeV/kgs, $\epsilon_{\mathrm{III}} = 1.94$ GeV/kgs and $\epsilon_{\mathrm{CNO}} = 3.54$ MeV/kgs. These values show that pp-two dominates with 92.01

% of the total energy production while pp-three contributes with only 7.98 %. Less than 0.01 % are produced from pp-one and the CNO cycle. Running the sanity check turns out to coincide with the results from Table 4 with an relative error of less than one percent separately.

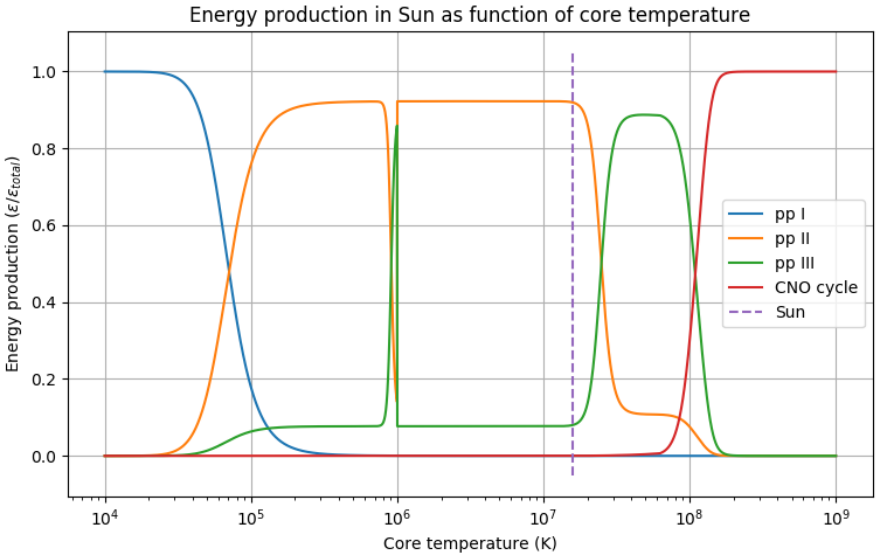

Figure 1: Evolution of the four different reaction chains in the core of the Sun as function of temperature. Each of the reactions has its dominance at different temperatures. Plot shows relative energy production.

Figure 1 shows the evolution of pp-one, pp-two, pp-three and the CNO cycle at increasing temperature in the Sun. We notice the restriction of electron capture by beryllium-7 at temperature below 10^6 K. The energy production from pp-two decreases so that pp-three is the dominating energy source just below 10^6 K. The core temperature of the Sun is marked with purple dashed line and stating that pp-two is the dominating source of energy.

Discussion

The fact that the sanity check passes makes us believe that our final results are correct. We should still be cautious with values and parameters used in this project as they may contain some kind of error, especially errors produced by numerical calculations. We have assumed the core density and temperature to be independent of radius. This should be taken into consideration for making a more realistic model of the energy production in the Sun. Still the fact that we got these results only strengthen previous studies on the energy production in the Sun.

In this project the density is set to be equal to that of the Sun. Also, the mass fraction of the various atomic species is also restricted to what builds up the Sun. Our model is therefore a special case for the Sun when it in reality exists an enormous amount of stars in the universe. The code produced has been optimized such that if one would want to model other stars that might be of interest, it can be done in an effective way. The mass fraction is defined in the **constants.py** file but may be easily changed if necessary. The code needs a core temperature and density as inputs and will then produce same sort of results as above.

We notice from Figure 1 how the different chain reactions dominates at different core temperature. The interesting part in this plot is at what temperature each of the chain reaction dominates but not necessarily on how big of a dominance. The pp-chain is the main source of energy production of the Sun until it reaches about $1.1 \cdot 10^8$ K. At this point, the CNO cycle takes over. Based on the dashed line, the model states that the pp-chain reactions is the main source of energy production in the Sun which coincides with other studies.

Conclusion

This project has given a deeper understanding of the processes in the core of stars generally but especially in the Sun. We have created a model that calculates the generated energy through four of the most common reaction processes in stars and focusing on what the Sun produces. Making sure that each individual reaction does not occur more often than it is allowed to was the greatest challenge during this project. Nevertheless, it gave a great intuition on why this must be true. We have come to an conclusion that the energy production obtained from the code seems to match other studies quite well.

References

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