Compressive Sensing with Applications to Near-Field to Far-Field Antenna Pattern Extrapolation

Semester Project

Jonas Thalmeier

Supervised by Prof. Dirk Slock, Dr. Zilu Zhao, Dr. Fangqing Xiao

20/06/2025

Table of Contents

Introduction

Motivation: Antenna Measurement Challenges

- **Near-Field (NF) to Far-Field (FF) Transformation:**
  - Necessary because direct FF measurements are often impractical (requires very large anechoic chambers).
  - NF measurements are taken close to the antenna and then transformed to FF.
  - Precise NF characterization necessary
- **Challenges:**
  - **Truncation:** Some areas might be inaccessible
  - **Measurement Grid Density:** Acquiring measurement over a dense grid is time consuming

Project Objectives and Approach

- **Main Objective:** Apply Compressive Sensing (CS) to near-field antenna measurements to enable precise FF extrapolation with relatively few measurements.
- **Approach:**
  - Understand, Implement and evaluate various Sparse Bayesian Learning (SBL) algorithms (EM, CoFEM, Tipping's FML) for antenna pattern expansion.
  - Leverage the Spherical Wave Expansion (SWE) for antenna field representation.
- **Tools Used:** Python for algorithm implementation, MATLAB for Antenna simulation.

Theoretical Foundations

Compressive Sensing Basics

- **Core Idea:** Recovering sparse signals from a small number of linear measurements.
- **Key Concepts:**
  - **Sparsity:** The signal of interest can be represented sparsely in some basis.
  - **Undersampling:** Measurements N are significantly fewer than signal dimension D ($N \ll D$).
- **Mathematical Formulation:**

$$\mathbf{t} = \Phi\mathbf{w} + \mathbf{e}$$

Where $\mathbf{t}$ is the measurement vector, Φ is the sensing matrix, $\mathbf{w}$ is the sparse signal, and $\mathbf{e}$ is noise.

Sparse Bayesian Learning (SBL)

- **Core Idea:** Inducing sparsity in weight vectors by optimizing a parameterized Gaussian prior over the signal weights.
- **Observation Model:** $\mathbf{t} = \Phi\mathbf{w} + \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.
- **Prior Distribution for w:** $p(w_i; \gamma_i) \sim \mathcal{N}(0, \gamma_i)$
- **Inference:** Hyperparameters $\boldsymbol{\gamma}$ and σ^2 are inferred e.g. via Type-II Maximum Likelihood (evidence maximization):

$$\hat{\gamma}, \sigma^2 = \arg\max p(t \mid \gamma, \sigma^2) \tag{1}$$

SBL Algorithms Implemented

EM Algorithm for SBL[1]

- **Overview:** An iterative algorithm for estimating the sparse weights $\mathbf{w}$ and hyperparameters $\boldsymbol{\gamma}$, σ^2.
- **Iterative Steps:**
  - **E-step:** Compute posterior distribution $p(\mathbf{w} \mid \mathbf{t}, \boldsymbol{\gamma}, \sigma^2)$, specifically its mean ($\boldsymbol{\mu}_w$) and covariance ($\boldsymbol{\Sigma}_w$).
  - **M-step:** Update hyperparameters $\boldsymbol{\gamma}$ and σ^2 using the computed posterior statistics.
- **Challenges:**
  - Computationally expensive due to $D \times D$ matrix inversion in each iteration: $\boldsymbol{\Sigma}_w = \left(\beta \boldsymbol{\Phi}^\top \boldsymbol{\Phi} + \mathrm{diag}(\boldsymbol{\gamma}^{-1})\right)^{-1}$.
  - Robust performance in Gaussian measurement systems, but showed instability for complex-valued (e.g., Fourier) data.

[1] Wipf and Rao, "Sparse Bayesian learning for basis selection".

CoFEM Algorithm[2]

- **Motivation:** Address the high computational cost of the EM algorithm.
- **Key Idea:** Avoids explicit computation of the full posterior covariance matrix $\mathbf{\Sigma}_w$.
- **Approach:**
  - Estimates required posterior statistics (e.g., diagonal elements of $\mathbf{\Sigma}_w$) by solving linear systems.
  - Uses techniques like Rademacher probe vectors for approximation.
- **Performance:**
  - Achieved significant runtime reduction compared to standard EM, especially for larger D.
  - For $N = 500, D = 1500$, CoFEM runtime was 98.4s vs. EM's 146.6s.
  - Reconstruction accuracy was comparable to EM for Gaussian matrices.

[2] Lin et al., "Covariance-Free Sparse Bayesian Learning".

EM vs. CoFEM: Runtime and Accuracy

Runtime Comparison:

- CoFEM is faster, especially for increasing N.
- Still, limitations for very large-scale problems.

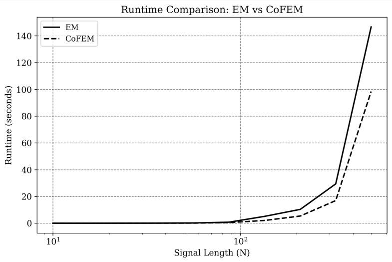


Figure 1: Runtime of EM and CoFEM for increasing N ($D = 3N$).

Accuracy (Gaussian Matrices):

- Both EM and CoFEM showed similar accuracy.
- CoFEM with unknown noise variance performed well.

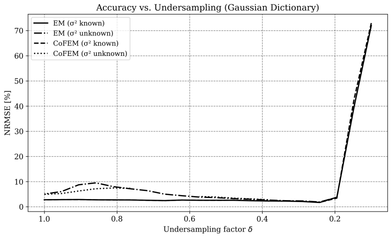


Figure 2: NRMSE vs. undersampling factor δ for EM and CoFEM (Gaussian Φ).

Stein's Unbiased Risk Estimate (SURE)

- **Purpose:** Provides an unbiased estimate of the Mean Squared Error (MSE) or "risk" of an estimator, without needing to know the true signal[3].
- **SURE Formula for $\hat{\mu}$ as a function of y (measurements):**

$$\widehat{R}(\hat{\boldsymbol{\mu}}(\mathbf{y})) = \|\mathbf{y} - \hat{\boldsymbol{\mu}}(\mathbf{y})\|_2^2 - n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n} \frac{\partial \hat{\mu}_i(\mathbf{y})}{\partial y_i}$$

- **Application in SBL:** Can be used to optimize hyperparameters (e.g., prior variances γ_i) by minimizing the SURE[4].
- **Optimal Prior Variances:** $p_i = \lfloor |\hat{x}_i(0)|^2 - \sigma^2 \rfloor_+$ (same result as Type II ML).

[3] Tibshirani and Wasserman, "Stein ' s Unbiased Risk Estimate Statistical Machine Learning , Spring 2015".

[4] Slock, "Sparse Bayesian Learning with Stein's Unbiased Risk Estimator based Hyperparameter Optimization".

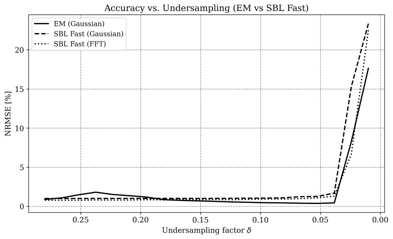
Tipping's Fast Marginal Likelihood Algorithm[5]

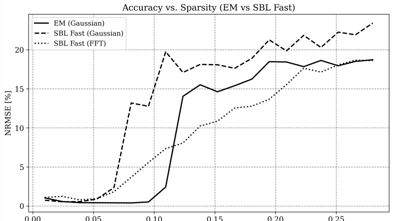
- **Connection to SURE:** While not directly using the SURE formula for optimization in its primary form, Tipping's FML effectively maximizes the marginal likelihood, which is was shown to have the same result as SURE approach.
- **Key Features:**
  - **Sequential Basis Selection and Pruning:** Iteratively adds relevant basis functions and prunes irrelevant ones.
  - **Efficient Hyperparameter Updates:** Based on closed-form solutions for Type II Maximum Likelihood.
- **Advantages:**
  - Explicit sparsity enforcement (many γ_i driven to zero).
  - Faster convergence compared to EM for many problems.
  - More robust to choice of dictionary matrix (both Gaussian and complex-valued).

[5] Tipping and Faul, "Fast Marginal Likelihood Maximisation for Sparse Bayesian Models".

Tipping's FML Performance

- **Robustness:** Consistent performance across both Gaussian and complex-valued (Fourier) systems.
- **Accuracy (Gaussian Systems):** Comparable to EM, with EM showing marginal improvements in some cases.
- **Key Finding (Complex-Valued Systems):**
  - Tipping's method significantly outperformed its own performance in Gaussian systems when applied to complex-valued dictionaries.
  - This indicates better adaptability and robustness to structured (non-i.i.d.) sensing matrices, which are common in real-world antenna problems.





Spherical Wave Expansion (SWE)

Spherical Wave Functions[6]

- **Concept:** Any electromagnetic field in a source-free region can be expressed as a superposition of spherical vector wave functions.Therefore the electric field $\mathbf{E}(r, \theta, \phi)$ can be decomposed into spherical vector wave functions $\mathbf{F}_{smn}^{(c)}$.
- **Equation:**

$$\mathbf{E}(r, \theta, \phi) = \frac{k}{\sqrt{\eta}} \sum_{s,n,m} Q_{smn}^{(3)} \mathbf{F}_{smn}^{(3)}(r, \theta, \phi)$$

  - k: wavenumber, η: wave impedance.
  - $Q_{smn}^{(3)}$: Complex expansion coefficients (sparse weights).
  - $\mathbf{F}_{smn}^{(3)}$: Spherical vector wave functions, dependent on radial distance r.

[6] Hansen, *Spherical Near-Field Antenna Measurements*.

Near-Field to Far-Field Transition

- **Far-Field Limit:** As $kr \to \infty$ (i.e., far from the antenna), the radial dependence of the spherical wave functions simplifies significantly.
- **Far-Field Functions:**

$$\mathbf{K}_{smn}(\theta, \phi) = \lim_{kr \to \infty} \left[\sqrt{4\pi} \frac{kr}{e^{ikr}} \mathbf{F}^{(3)}_{smn}(r, \theta, \phi) \right]$$

- **Significance:** The expansion coefficients $Q^{(3)}_{smn}$ determined from near-field measurements are the **same** coefficients that describe the far-field pattern. This forms the basis of NFFFT.

SBL for Determination of SWE Coefficients

SBL Framework for SWE Coefficient Estimation

- **Problem Formulation as CS:**
  - The measured electric field $\mathbf{E}(r, \theta, \phi)$ becomes the measurement vector $\mathbf{t}$.
  - The spherical wave functions $\mathbf{F}_{smn}^{(c)}$ evaluated at measurement points form the dictionary matrix Φ.
  - The unknown expansion coefficients $Q_{smn}^{(c)}$ form the sparse weight vector $\mathbf{w}$.
- **Reshaping:**
  - If N is the number of measurement points, and each point has 3 components (E-field vectors). The measurement vector $\mathbf{t}$ is flattened to dimension $3N \times 1$.
  - The dictionary matrix Φ has dimension $3N \times D$, where D is the total number of spherical wave modes considered.
  - This influences undersampling considerations

Far-Field Reconstruction Results

- **Process:**
  1. Simulate far-field data for dipole and loop antennas.
  2. Use Tipping's SBL algorithm to estimate coefficients $Q_{smn}^{(3)}$.
  3. Reconstruct the far-field pattern using the estimated coefficients.
  4. Compute relative MSE between original FF and reconstructed FF
- **Observations:** Relatively high reconstruction errors (Normalized MSE) were observed, even at $\delta = 1$ (no undersampling in N).

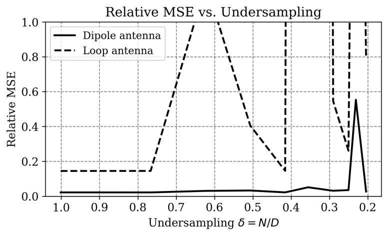


Figure 4: Relative reconstruction error (normalized MSE) as a function of the measurement-to-dimension ratio $\delta = N/D$ for dipole and loop antennas in the far-field configuration.

NF-to-FF Transformation: Process & Results

- **Process:**
  1. Simulate near-field and far-field data for dipole and loop antennas.
  2. Estimate coefficients with SBL from simulated near-field data at $r = 0.5\lambda$.
  3. Construct field from those coefficients with $\mathbf{F}_{smn}^{(c)}$ evaluated at $r = 5\lambda$
  4. Compute relative MSE between original FF and reconstructed FF
- **Results:** MSE between far-field simulation results and reconstruction from coefficients very high
- **Further investigation:** Compare coefficients computed from far-field and near-field

NF-to-FF Transformation: Discrepancies in Coefficients

- **Loop Antenna:** Dominant coefficients showed good consistency between near-field and far-field reconstructions.
- **Dipole Antenna:** Showed more significant discrepancies in the estimated coefficients.
- This indicates potential issues with the dictionary's ability to represent it sparsely, or numerical precision.

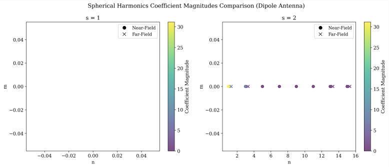


Figure 5: Comparison of wave coefficients for the dipole antenna: non-zero weights estimated by FML from near-field vs. significant coefficients from far-field.

Conclusion

Summary of Achievements

- **Theoretical Understanding:** Gained a strong understanding of SBL, different approaches for SBL algorithms and Spherical Wave Expansion.
- **Successful Algorithm Implementation:** Developed robust implementations of SBL algorithms (EM, CoFEM, FML).
- **Application:** Applied Compressive Sensing (specifically Tipping's FML) to the problem of antenna pattern extrapolation from near-field measurements.
- **Key Finding:** Tipping's FML algorithm proved to be the most robust and efficient for the antenna extrapolation problem, particularly with structured Fourier-like dictionary matrices.

Challenges Encountered

- **Numerical Instabilities:** EM algorithm struggled with complex-valued (Fourier) dictionaries.
- **Challenges with SWE via Integration:** The attempt to decompose the electromagnetic field into spherical vector wave components using the classical integration-based method was unsuccessful.
- **Unexpected High Errors:** Attempts at far-field reconstruction from NF data showed higher-than-expected errors even at full sampling, pointing to complexities beyond just undersampling.
- **Coefficient Discrepancies:** Differences in estimated coefficients for certain antenna types (e.g., dipole) between NF and FF estimations.

Future Work

- **Algorithm Optimization:** Further refine and optimize the SBL algorithms for specific antenna models and larger-scale problems.
- **Improved Basis Functions:** Investigate more accurate or specialized spherical wave basis functions to potentially improve reconstruction accuracy.
- **Real-World Validation:** Validate the framework with experimental, real-world antenna measurement data.
- **Advanced Simulations:** Transition to dedicated antenna simulation software (e.g., CST Microwave Studio) for more realistic data generation.
- **Deeper Investigation:** Conduct further research into the inherent physical assumptions and numerical implementation details of SBL for Near-Field to Far-Field Transformation.

Next Steps

- **Establish Ground Truth via Integration:** Successfully implement SWE using numerical integration to obtain accurate reference coefficients for comparison with SBL results.
- **Refine SBL Performance:** First ensure reliable reconstruction at $\delta = 1$, then gradually reduce measurements to explore the limits of undersampling robustness.
- **Sparsity Analysis of SWE Coefficients:** Study the distribution of SWE coefficients across different antenna types to better understand their sparsity structure. This can inform prior design in SBL and improve recovery performance.
- **Parameter Sensitivity Study:** Investigate how variations in key parameters affect reconstruction accuracy, including:
  - Radial measurement distance (e.g., 0.5λ vs. 1λ)
  - Measurement grid uniformity and angular resolution
  - Truncation of angular coverage
  - Noise levels and SNR
  - Antenna topology (dipole, loop, patch, etc.)

Acknowledgments

- **Supervisors:**
  - Prof. Dirk Slock
  - Dr. Zilu Zhao
  - Dr. Fangqing Xiao
- **Tools & Assistance:**
  - Python for all implementations and simulations.
  - MATLAB for antenna simulation
  - Large Language Models (LLMs) for debugging, code commenting, and text refinement.