approach against standard approximate analysis techniques indicates that the range of validity of the assumptions of the latter is rather restrictive. The numerical results for the TE polarization indicate that a Brewster angle effect occurs for a particular relation between structure period and angle. The results also show a rapid variation in the distribution of power near the Rayleigh wavelengths which is characteristic of P-type anomalies.

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Near-Field Far-Field Transformations Using Spherical-Wave Expansions

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Abstract—**Spherical-wave expansions are a well-known technique of expressing electromagnetic field data. However, most previous work has been restricted to idealized cases in which the expansion coefficients are obtained analytically. In this paper spherical-wave expansions are used as a numerical technique for expressing arbitrary fields specified by analytical, experimental, or numerical data. Numerical results on the maximum wave order needed to expand fields arising from a source of a given size are given for two practical cases, and it is found that the generally accepted wave order cutoff value corresponds to 99.9 percent or more of the power in the input pattern. Near-field patterns computed from far-field data are compared to measured data for the two cases, demonstrating the excellent numerical accuracy of the technique.**

Introduction

THERE ARE several problems in which an electromagnetic field is known (or assumed known) on one surface, and it is desired to determine the field elsewhere. A common example is when the field is specified over an antenna aperture or other "near-field" surface, and it is desired to determine the far-field pattern; the inverse of this problem is also fairly common. In a source-free region, specifying the tangential electric field everywhere on one closed surface is sufficient to determine the total field anywhere else, if certain additional conditions hold. There are two techniques for accomplishing the transformation of field data from one surface to another. 1) representing the field at an arbitrary point as an integral over the surface on which the fields are known [1], or 2) representing the field as a summation of free-space "modes" where the mode coefficients are determined by matching the fields on the surface on which the fields are known. In the latter case, the modes may be plane waves [2], cylindrical waves [3], axial waves [4], spherical waves [5], or any other set of functions which form a basis for wave solutions to Maxwell's equations. In both of these methods it is also possible to include sources, and a source-free region is assumed here only because it is a common case and eliminates unnecessary complications. The basic theory for this type of near-field transformation is well

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known. The purpose of this paper is to describe an actual implementation of spherical-wave theory for this application and to present some results of applications of the theory.

II. Determination of a Spherical-Wave Expansion

The basic properties of spherical waves have been described in a recent paper by Potter [5] and will not be repeated here. The purpose of this section is to present the essential equations for solving the following problem: an electromagnetic field $\boldsymbol{E}(\rho,\theta,\phi),\boldsymbol{H}(\rho,\theta,\phi)$ exists in a region outside a sphere (which encloses all sources) of radius $\rho_0 > 0$; given the tangential E field on the surface of a sphere of a radius $\rho_1 \geq \rho_0$, determine coefficients $a_{e,o_{mn}}$ (TE waves) and $b_{e,o_{mn}}$ (TM waves) such that for all $\rho \geq \rho_0$

$$\boldsymbol{E}(\rho,\theta,\phi) = -\sum_m \sum_n a_{e,o_{mn}}\boldsymbol{m}_{e,o_{mn}} + b_{e,o_{mn}}\boldsymbol{n}_{e,o_{mn}}$$

$$\boldsymbol{H}(\rho,\theta,\phi) = (k/j\omega\mu) \sum_m \sum_n a_{e,o_{mn}}\boldsymbol{n}_{o,e_{mn}} + b_{e,o_{mn}}\boldsymbol{m}_{o,e_{mn}}. \quad (1)$$

The functions $\boldsymbol{m}_{e,o_{mn}}$ and $\boldsymbol{n}_{e,o_{mn}}$ are the spherical-wave solutions to Maxwell's equations given by

$$\boldsymbol{m}_{e,o_{mn}} = \mp z_n(k\rho) \frac{mP_n^m(\cos\theta)}{\sin\theta} \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\imath}_\theta - z_n(k\rho) \frac{\partial}{\partial\theta} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\imath}_\phi \quad (2)$$

$$\boldsymbol{n}_{e,o_{mn}} = n(n+1)\frac{z_n(k\rho)}{k\rho} P_n^m(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\imath}_\rho + \frac{1}{k\rho}\frac{\partial}{\partial\rho}[\rho z_n(k\rho)]\frac{\partial}{\partial\theta}P_n^m(\cos\theta)\begin{matrix}\sin\\ \cos\end{matrix} m\phi\, \hat{\imath}_\theta \pm \frac{1}{k\rho}\frac{\partial}{\partial\rho}[\rho z_n(k\rho)]\frac{mP_n^m(\cos\theta)}{\sin\theta}\begin{matrix}\cos\\ \sin\end{matrix} m\phi\, \hat{\imath}_\phi$$

where $\exp(j\omega t)$ time dependence is implicitly assumed, (ρ,θ,ϕ) are spherical coordinates (Fig. 1), $k = \omega(\epsilon\mu)^{1/2} = 2\pi/\lambda$ is the propagation constant, $z_n(k\rho)$ is any solution of the spherical Bessel equation, $P_n^m(\cos\theta)$ is the associated Legendre function, and the subscripts e and o (even and odd) refer to the choice of the upper or lower sin or cos ϕ dependence.

Jones [6] has shown that any electromagnetic field can be written in this form. In the present case, since we have assumed sources of a finite extent, we need only include the solutions involving $h_n^{(2)}(k\rho)$, the spherical Hankel function of the second kind, which satisfies the radiation condition and corresponds to an outward traveling wave [7].

Several authors have obtained expressions for the coefficients of such an expansion in terms of sources or various field components [5], [6], [8], [9]. For the particular case considered here, we wish to determine the coefficients in terms of the tangential E field alone on a

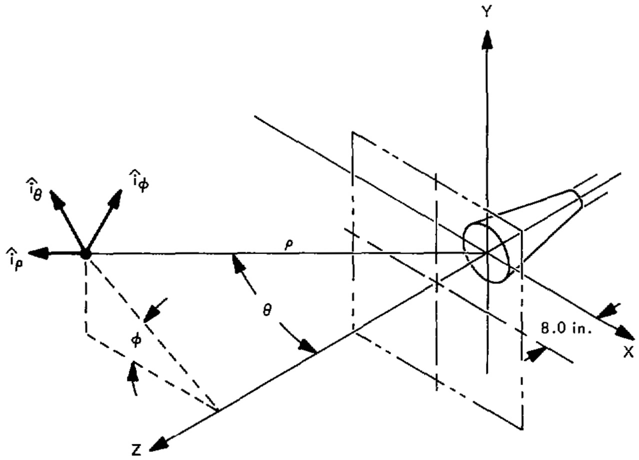

Fig. 1. Coordinate system.

sphere of radius ρ_1; the desired expressions are [10]

$$a_{e,o_{mn}} = \frac{1}{[z_n(k\rho_1)]^2}\frac{2n+1}{\pi 2n(n+1)}\frac{(n-m)!}{(n+m)!} \cdot \int_0^{2\pi}\int_0^{\pi} -\boldsymbol{m}_{e,o_{mn}}\cdot \boldsymbol{E}(\rho_1,\theta,\phi)_{\tan} \sin\theta\, d\theta\, d\phi \quad (3)$$

$$b_{e,o_{mn}} = \frac{1}{[(1/k\rho_1)(\partial/\partial\rho)[\rho_1 z_n(k\rho_1)]]^2}\frac{2n+1}{\pi 2n(n+1)}\frac{(n-m)!}{(n+m)!} \cdot \int_0^{2\pi}\int_0^{\pi} -\boldsymbol{n}_{e,o_{mn}}\cdot \boldsymbol{E}(\rho_1,\theta,\phi)_{\tan} \sin\theta\, d\theta\, d\phi$$

where $0 < \rho_0 \leq \rho_1 \leq \infty$.

Since it was assumed that all sources are contained in a sphere of radius ρ_0, spherical waves of order $n > k\rho_0$ will not contribute significantly to the field [5]. This is also shown in the results presented later. Therefore, it is also possible to obtain the coefficients by inverting a system of $2N \cong 2k\rho_0$ linear equations. That is, the truncated series is forced to equal the boundary values at $2N$ discrete points. This method has the virtue of being easily applicable to the fields being specified on an arbitrary surface. However, for the case of a spherical surface, the use of the orthogonality (3) has certain numerical advantages [10].

In conclusion, to solve the stated problem, given the tangential E field on the surface of a sphere of radius ρ_1, we can use (3) to obtain coefficients $a_{e,o_{mn}}$ and $b_{e,o_{mn}}$ which can be used via (1) to determine the total electromagnetic field everywhere outside of the sphere of radius ρ_0 which encloses the sources.

III. Applications and Results

Previous work with spherical-wave expansions has been restricted to very interesting but idealized cases in which the coefficients were obtained analytically in closed form; e.g., Potter [5] found the coefficients for a circularly symmetric optimum illumination pattern; Kennaugh and

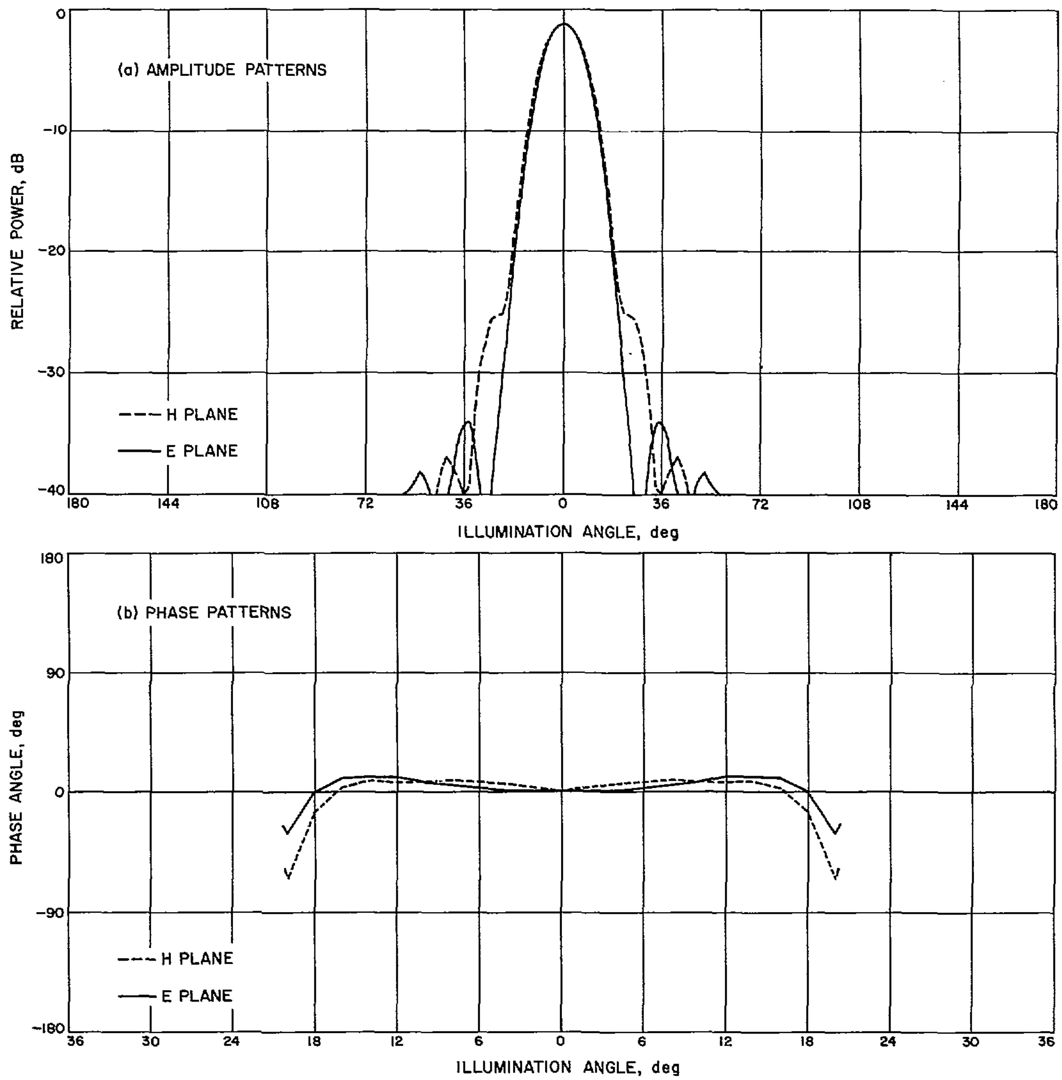

Fig. 2. Feed-horn pattern.

Ott [11] found the waves excited by a plane wave normally incident on a paraboloid; Jones [6] gives the coefficients for a plane wave and for an arbitrarily located electric dipole. However, by directly programming (3), it is possible to numerically obtain expansions for any field, whether it was originally obtained analytically, experimentally, or numerically. A program written to accomplish this has been previously described, including a complete program listing [10].

The first application of the program was the expansion of the pattern of a feed-horn such as illustrated in Fig. 1. The pattern is shown in Fig. 2. This is a far-field pattern, so the required tangential E-field data are known at $\rho_1 = \infty$. For a device with physical circular symmetry, fed by conventional waveguide linearly polarized in the y direction, it has been shown that the field must be of the form [12]

$$E(\rho,\theta,\phi) = [E_\theta(\theta) \sin \phi \, \hat{\imath}_\theta + E_\phi(\theta) \cos \phi \, \hat{\imath}_\phi] \frac{\exp(-jk\rho)}{\rho} \tag{4}$$

where $E_\theta(\theta)$ and $E_\phi(\theta)$ are simply the E- and H-plane patterns (amplitude and phase), that is the input data obtained, in this case, experimentally.

Inspection of (2) and (3) will show that, by the orthogonality properties of sines and cosines, only coefficients $a_{e,o_{mn}}$ and $b_{e,o_{mn}}$ with $m = 1$ will be present. Also, only the subscript e (even) coefficients will be present.

To determine how many coefficients are required, the empirical procedure is to 1) compute the total power in the input pattern by direct pattern integration, and 2) maintain a running total of the power in the (orthogonal) spherical waves (obtained as an explicit function of the wave coefficients) and stop when a predetermined fraction of the total power is reached. The fractional power versus the maximum wave order, for the case of the horn pattern in Fig. 2, is shown in Fig. 3. In the first attempt to expand this pattern, it was found that it was apparently impossible to get beyond 99 percent of the total power. It was subsequently determined that this was due to the fact that experimental phase data were known only for the main beam, and were simply truncated beyond this

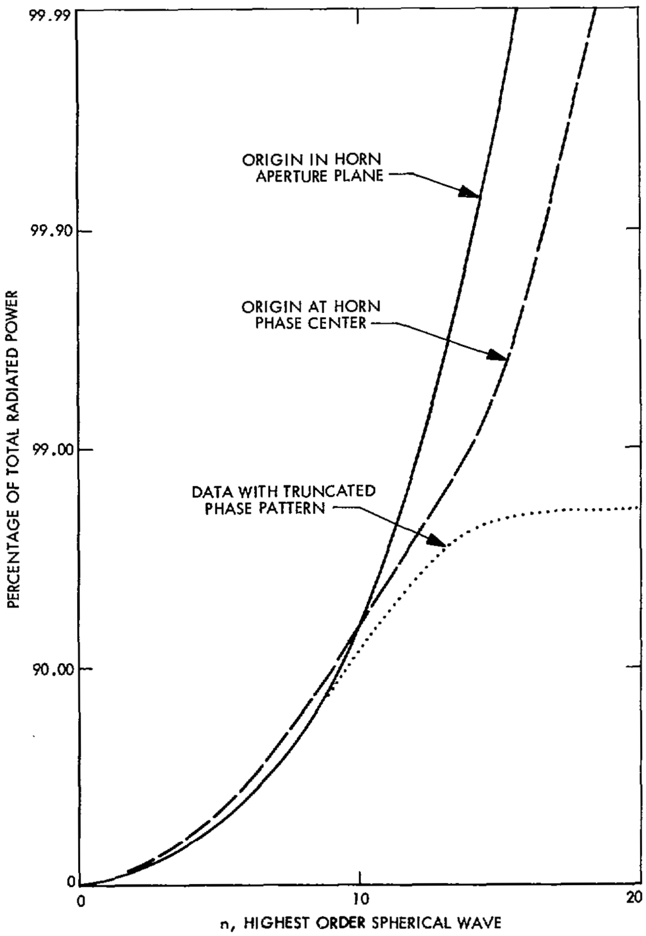

Fig. 3. Fractional power in finite spherical-wave expansion.

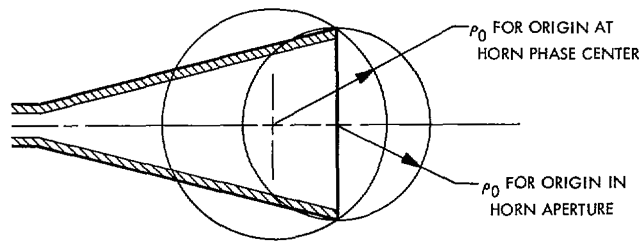

Fig. 4. Geometry of sphere enclosing aperture source.

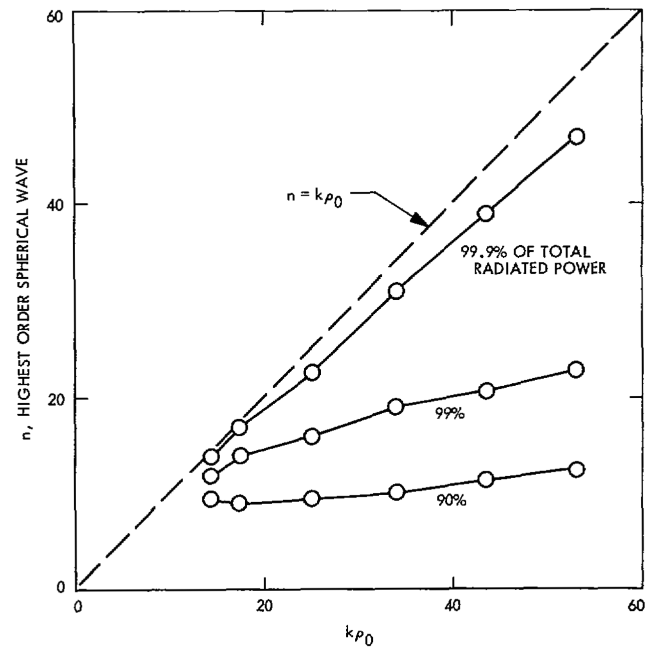

Fig. 5. Effect of enlarging $k\rho_0$ by translating coordinate system origin.

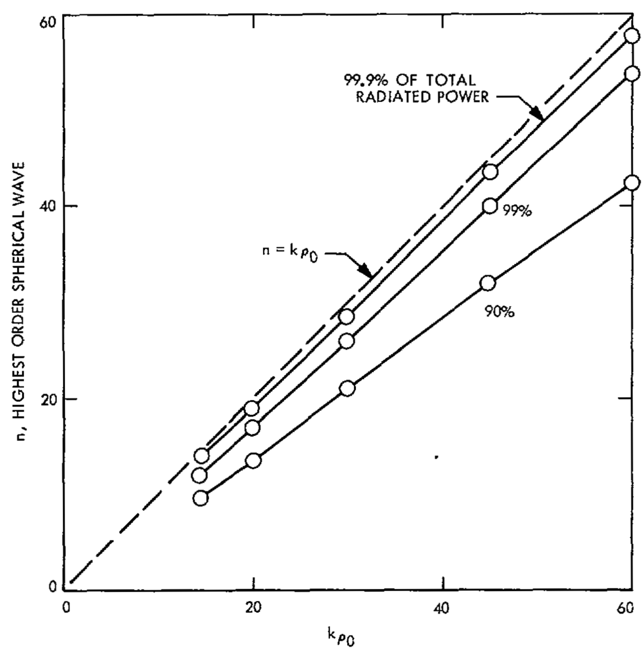

Fig. 6. Effect of enlarging $k\rho_0$ by enlarging aperture size.

point. Therefore, the program was trying to expand a physically unrealizable (discontinuous) phase pattern which was causing the difficulty. When the phase pattern was extrapolated analytically,[1] it was easy to get beyond 99.99 percent of the power, as shown in Fig. 3. When the spherical-wave expansions obtained in this manner are evaluated at $\rho = \infty$, they are indistinguishable from the input E- and H-plane amplitude and phase patterns.

A shift in the experimental phase center (or more precisely, reference center of rotation) does not affect the experimental amplitude pattern, but does produce a new experimental phase pattern. Two centers of rotation were used, providing two quite independent sets of wave coefficients. By definition, the coordinate system origin $\rho = 0$ is at the antenna center of rotation used to obtain the experimental patterns, so this corresponds to a shift in the coordinate system origin. As shown in Fig. 3, when the origin is located in the aperture plane, fewer spherical waves are required than when the origin is located at the phase center of the horn (defined such that the phase pattern is flat over the main beam). This has an elegant interpretation in terms of a wave "cutoff" phenomena; as mentioned earlier, when the sources are enclosed in a

[1] The pattern was matched over the known region with a theoretical pattern of a multimode horn, as described in [13].

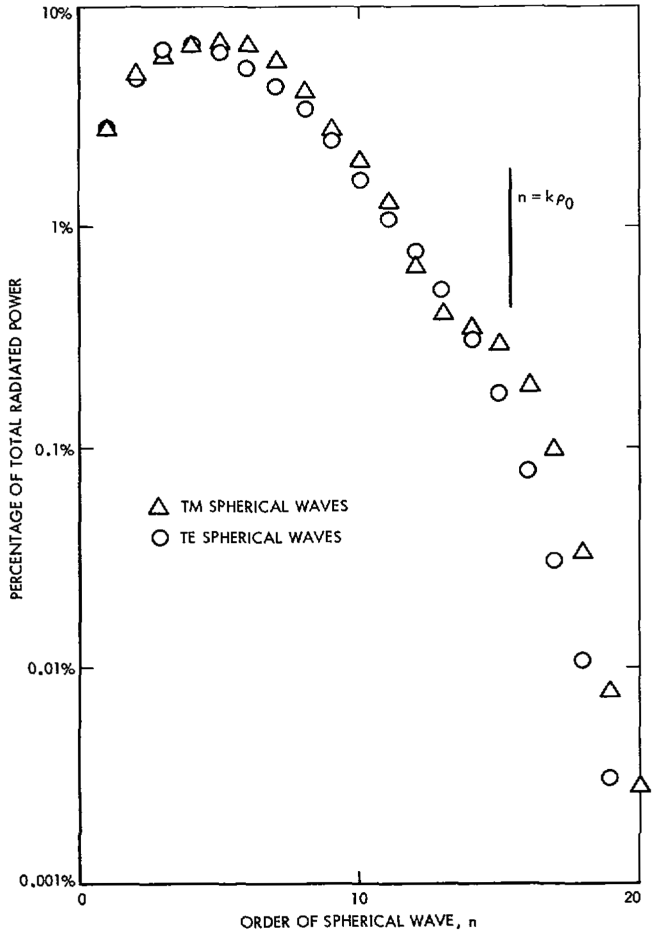

Fig. 7. Spherical-wave spectrum of feed-horn pattern.

sphere of radius ρ_0, there is a type of cutoff for waves with $n > k\rho_0$. As illustrated in Fig. 4, a bigger sphere is necessary to enclose the aperture (source) when the origin is out of the aperture plane. Data were obtained on this effect for several origin translations. For each origin, a sphere was drawn to enclose the aperture; the size of the sphere is the ordinate in Fig. 5. The abscissa shows the highest order spherical wave included in the expansion, and curves are given for several fractions of total radiated power. These results show that in this case 99.9 percent of the total radiated power is contained in spherical waves for which $n \leq k\rho_0$.

The fact that a shift in the coordinate origin changes the spherical-wave composition of a given pattern illustrates that these waves are more of a mathematical convenience than a physical reality. However, the wave expansion concept does give considerable physical insight into electromagnetic field behavior (e.g., the Fresnel region and near-field region behavior as discussed by Potter [5]) and is as valid on this basis as any other conceptual model.

Theoretical horn patterns were calculated for several aperture sizes, and these patterns were also expanded, all with respect to an origin in the aperture plane. In this case, the size of the sphere necessary to enclose the aperture simply scales with the aperture. The effect of aperture size on the required number of spherical waves is shown in Fig. 6. Again, $n = k\rho_0$ roughly corresponds to 99.9 percent of the total power. It is interesting to note that translating the phase center has a relatively minor effect on the power in the lower order waves, but enlarging the aperture has a strong effect on these waves. A detailed spherical-wave spectrum for the feed-horn pattern with the origin in the aperture plane is given in Fig. 7. This shows the fractional power for each individual wave, and again illustrates that the power drops off very rapidly beyond $n = k\rho_0$. If the E- and H-plane patterns were identical, the input pattern would be circularly symmetric, and the TE and TM wave spectrums would be identical. However, the pattern is not symmetric and this reflected in a slight difference in the wave spectrums.

IV. Experimental Verification of Near-Field Far-Field Transformations

To date there have been two experimental verifications of near-field evaluations of spherical-wave expansions obtained from far-field data. In a case previously reported [10] the input data were a computed far-field pattern of

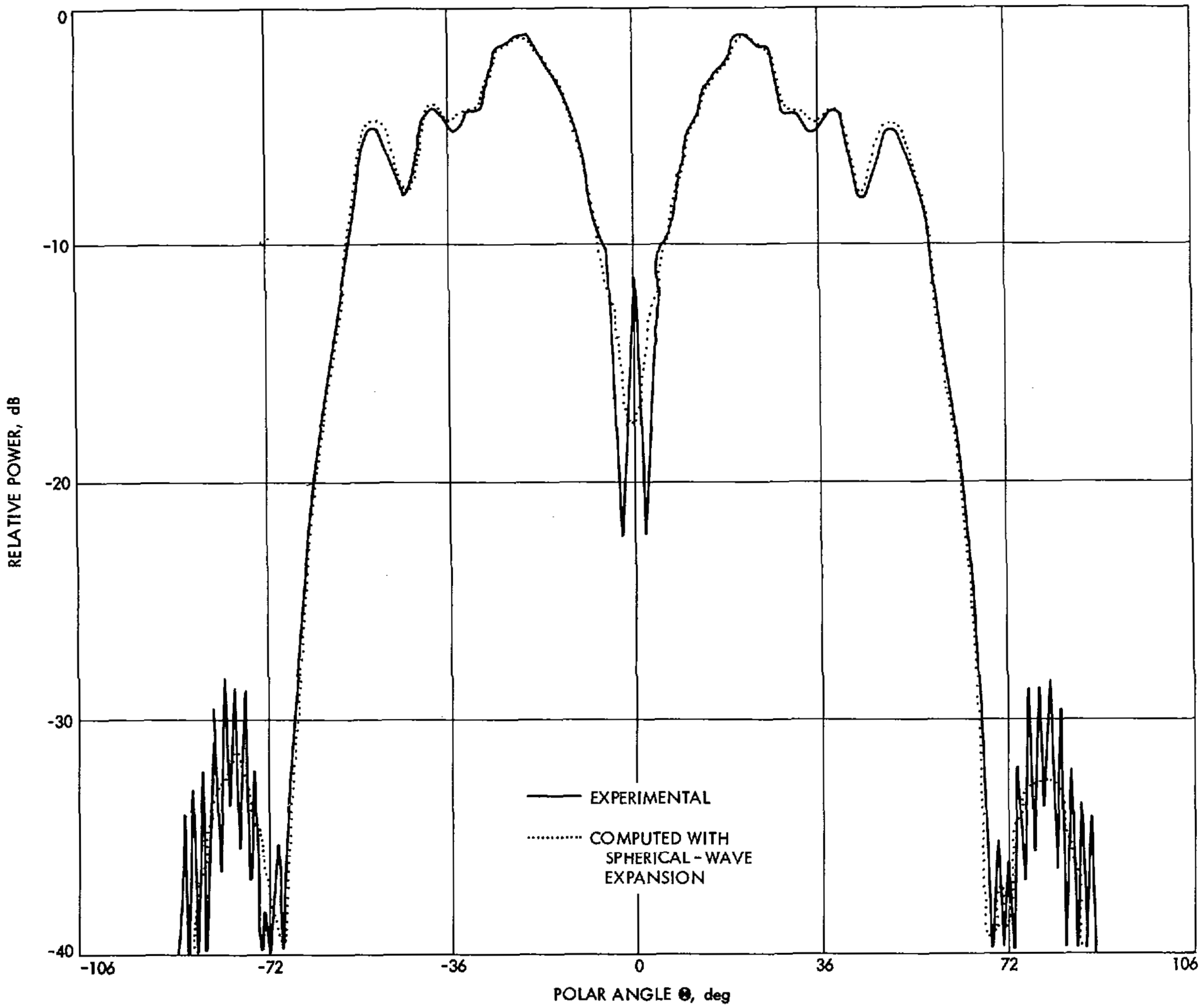

Fig. 8. Comparison of experimental and computed near-field subreflector patterns.

a subreflector of a Cassegrainian antenna system. The pattern was expanded in spherical waves and evaluated in the near-field ($0.148D^2/\lambda$ where D is the subreflector diameter). The result is compared to a pattern experimentally measured at this near-field separation in Fig. 8, demonstrating excellent agreement. (In this and the following case only E-plane or θ-component data are shown; the H-plane or ϕ-component data are virtually identical.) The wave spectrum for this pattern in shown in Fig. 9. The optimum patterns synthesized by Potter are for a Cassegrainian subreflector [5], and the spectrum of the optimum pattern is also shown in this figure (the TE and TM waves have identical spectra in this case). It is seen that the experimental pattern produced by an ordinary subreflector has roughly the correct spherical wave power spectrum. However, the relative phase of the spherical waves in the experimental pattern varies drastically, rather than being perfectly in phase as in the optimum case. Enclosing the subreflector in a sphere of radius ρ_0 leads to a value of $k\rho_0 = 73$ in this case. However, the pattern is very broad, and over 50 percent of the power is in the $n = 1$ TE and TM modes; $98\frac{1}{2}$ percent of the power is contained in the modes for $n \leq 20$, and 99.9 percent of the power for $n \leq 43$. Therefore, for this case $n \leq k\rho_0$ is a very conservative estimate of the number of spherical waves required. This case also illustrates that very complex patterns may be represented with spherical waves.

Computed near-field patterns of the feed pattern shown in Fig. 2 have been previously reported by Ludwig and Norman [13]. Recently the National Bureau of Standards measured the near-field radiation of this horn over a plane 8 in in front of the aperture, as shown approximately to scale in Fig. 1 [14]. The measured and computed data are compared in Fig. 10. Again, the agreement is excellent.

V. Conclusion

Both of the cases mentioned were motivated originally by quite real practical problems; the near-field subreflector data were used for a more accurate calculation of the performance of a Cassegrainian antenna [10]. The

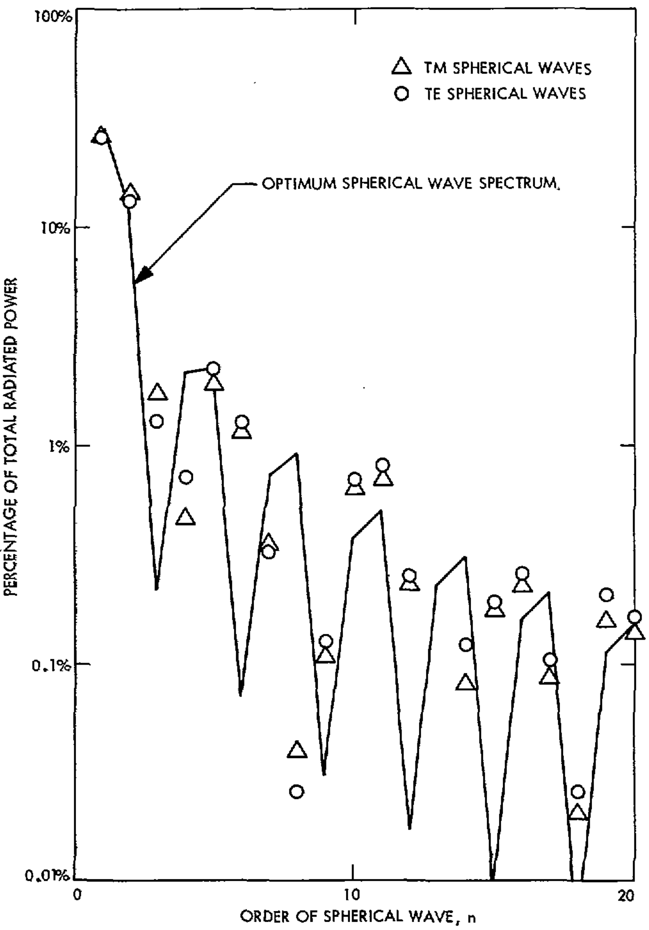

Fig. 9. Spherical-wave spectrum of subreflector pattern.

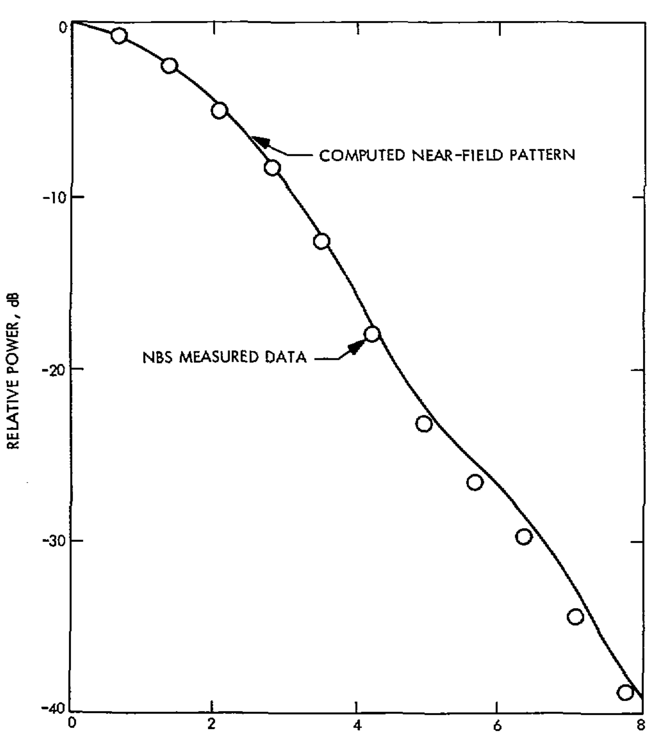

Fig. 10. Comparison of experimental and computed near-field horn patterns.

near-field horn data may be used for a very accurate evaluation of correction factors for near-field two-horn gain measurements [13], [15], [16]. These cases and other applications, such as the recent work of Jensen [17] and Hansen and Jensen [18] on the determination of far-field patterns from near-field data, illustrate that spherical-wave expansions in particular, and mode expansions in general, are a versatile and powerful analytic tool for electromagnetic problems.

Acknowledgment

The author is indebted to R. Norman of the Jet Propulsion Laboratory for the calculated near-field patterns, and to A. C. Newell of the National Bureau of Standards for the measured near-field horn patterns. P. Potter of the Jet Propulsion Laboratory stimulated the author's interest in the application of spherical-wave theory and provided valuable insights into the behavior of spherical waves.

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