Chapter 12

Multilevel model

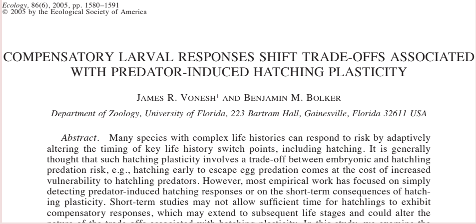
Several equivalent names:

- Multilevel models
- Hierarchical models
- Random effect models

Frog data

Data survival of hatching of reed frogs.

Figure : African reed frog, Hyperolius spinigularis



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COMPENSATORY LARVAL RESPONSES SHIFT TRADE-OFFS ASSOCIATED WITH PREDATOR-INDUCED HATCHING PLASTICITY

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Abstract. Many species with complex life histories can respond to risk by adaptively altering the timing of key life history switch points, including hatching. It is generally thought that such hatching plasticity involves a trade-off between embryonic and hatchling predation risk, e.g., hatching early to escape egg predation comes at the cost of increased vulnerability to hatchling predators. However, most empirical work has focused on simply detecting predator-induced hatching responses or on the short-term consequences of hatching plasticity. Short-term studies may not allow sufficient time for hatchlings to exhibit compensatory responses, which may extend to subsequent life stages and could alter the

Figure : Article

Frogs hatched in different tanks:

$$
\begin{aligned}
y_i &\sim Bin(n_i, p_i), \\
g(p_i) &= \alpha_i, \\
\alpha_i &\sim N(0, 10).
\end{aligned}
$$

Simple binomial model

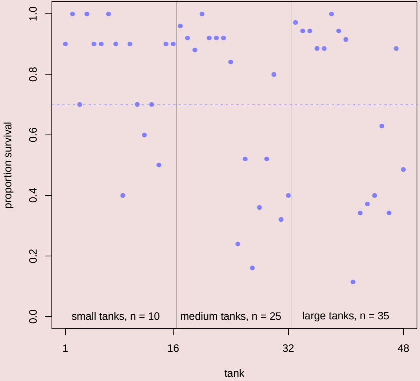


Figure : Posterior mean for each tank

DAG

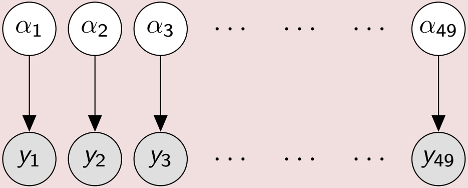
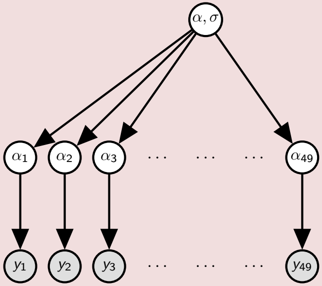


Figure : A DAG (directed acyclic graph) describing the model.

Hierarchicals model DAG



Multilevel model:

$$
\begin{aligned}
y_i &\sim Bin(n_i, p_i),\\
g(p_i) &= \alpha_i,\\
\alpha_i &\sim N(\alpha, \sigma),\\
\alpha &\sim N(0, 10),\\
\sigma &\sim HC(0, 1).
\end{aligned}
$$

Multilevel binomial model

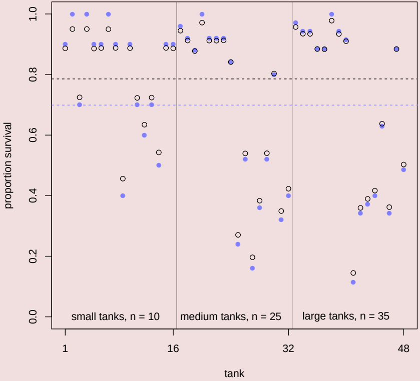


Figure : Posterior mean for each tank

Lets examine, $y_2 = 10, n_2 = 10$

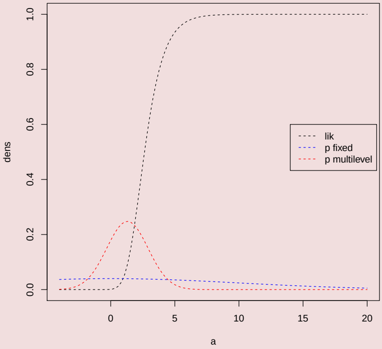


Figure : Prior + likelihood for a.

Lets examine, $y_2 = 10$, $n_2 = 10$

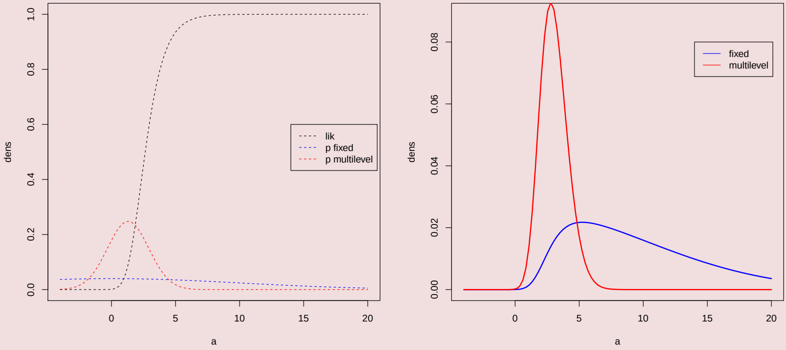


Figure : Posterior for a.

Lets examine, $y_2 = 10$, $n_2 = 10$

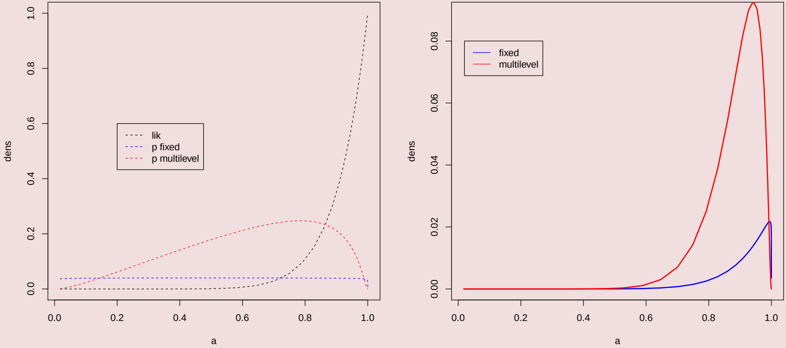
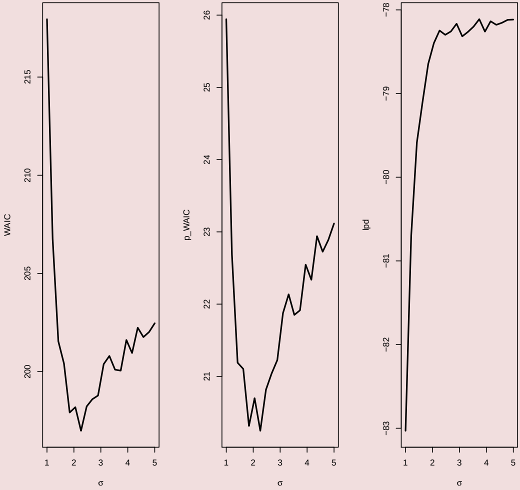


Figure : Figure on the left prior + likelihood for p. Figure on the right posterior p.

WAIC for varying σ



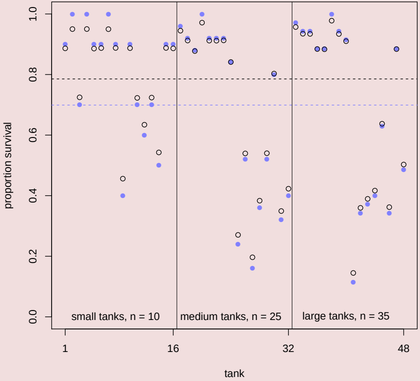


Figure : Posterior mean for each tank

Histogram of the α

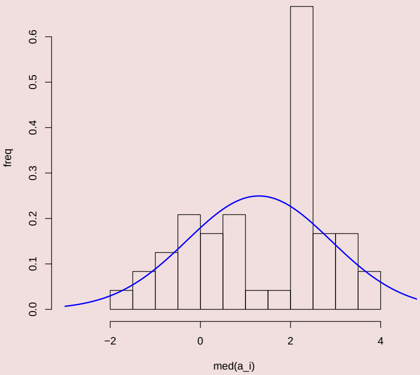


Figure : Histogram of the α_i

Multilevel mixture model:

$$
\begin{aligned}
y_i &\sim Bin(n_i, p_i),\\
g(p_i) &= \alpha_i,\\
\alpha_i &\sim \theta N(\mu_1, \sigma_1) + (1-\theta) N(\mu_2, \sigma_2),\\
\mu_1 &\sim N(0, 10),\\
\mu_2 &\sim N(0, 10),\\
\sigma_1 &\sim HC(0, 1).\\
\sigma_2 &\sim HC(0, 1).\\
\theta &\sim B(2, 2).
\end{aligned}
$$

Stan model

```
parameters{
  ordered[2] a0;
  real a[ntank];
  real<lower=0> sigma_0;
  real<lower=0> sigma_1;
  real<lower=0, upper=1> theta;
}
model{
  vector[N] mu;
  theta    ~ beta(2, 2);
  a0       ~ normal(0,10);
  sigma_0 ~ cauchy(0,1);
  sigma_1 ~ cauchy(0,1);
  for(i in 1:ntank){
   target += log_sum_exp(
     bernoulli_lpmf(1|theta) + normal_lpdf(a[i]| a0[1],sigma_0),
     bernoulli_lpmf(0|theta) + normal_lpdf(a[i]| a0[2],sigma_1)
     );
  }
  for(i in 1:N)
    mu[i] = a[tank[i]];

  y ~ binomial_logit(n, mu);
}
```

Multilevel mixture binomial model

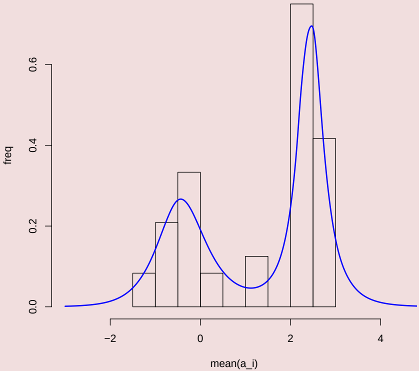


Figure : Histogram of the α_i

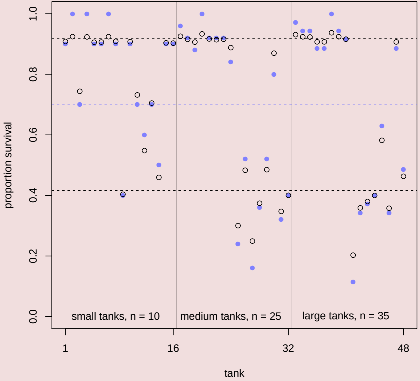
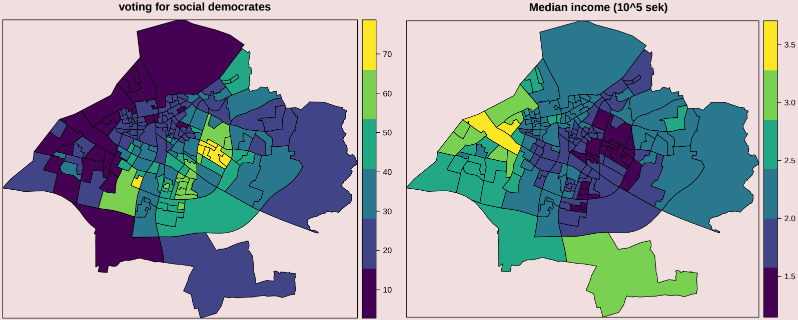


Figure : Posterior mean for each tank

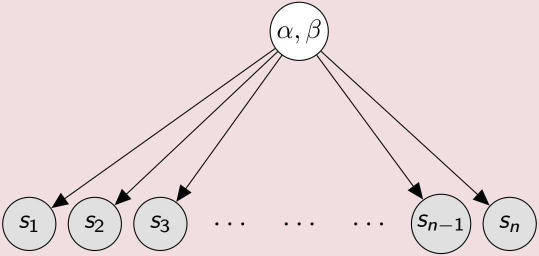
Voting in Malmö, data



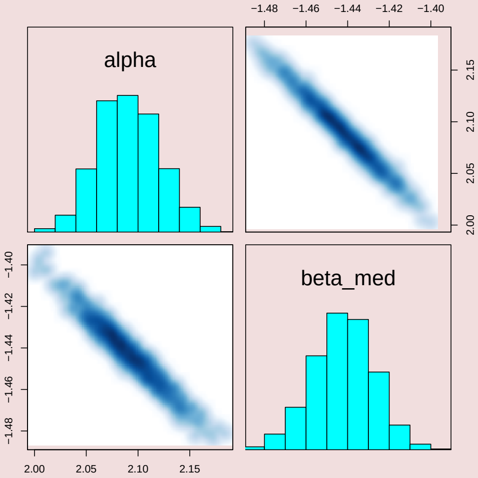
Voting in Malmö, model

$$
\begin{aligned}
s_i &\sim bin(n_i, p_i),\\
g(p_i) &= \alpha + med_i\beta,\\
\alpha &\sim N(0, 10)\\
\beta &\sim N(0, 10)
\end{aligned}
$$

Independent model DAG



Posterior parameter



Predictions

- By the model the prediction given the data is

$$\hat{Y}_i \sim Bin(n_i, p_i),$$
$$p_i \sim p(\cdot | y_1, y_2, \ldots, y_n)$$

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$$\hat{Y}_i \sim Bin(n_i, p_i),$$
$$p_i \sim p(\cdot | y_1, y_2, \ldots, y_n)$$

- The variance is:

$$V[\hat{Y} | p_i, n_i] = n_i(1 - p_i)p_i$$
$$V[\frac{\hat{Y}}{n_i} | p_i, n_i] = \frac{(1 - p_i)p_i}{n_i}$$

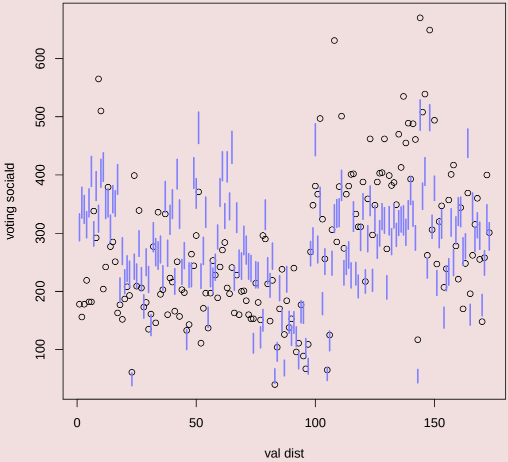

Figure : Predicition by district

Overdispertion

- Both binomial and Poisson has only one parameter.
- These models are extremely sensitivity to incorrect parameter.
- They can not adjust it variance to the data.

Solution

- This is typically solved by overdispersion model. Like Beta-binomial.

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- For each observation one adds a random non-negative parameter:

$$p(y_i|n_i) = \int Bin(y_i|n_i, p_i)h(p_i|p, \theta)p(p, \theta)dp_i dp d\theta,$$

Then one puts covariates on p not p_i.

Solution

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- For each observation one adds a random non-negative parameter:

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  Then one puts covariates on p not p_i.
- overdisperation is typically a multilevel model.

$$
\begin{aligned}
y_i &\sim Bin(n_i, p_i) \\
g(p_i) &\sim \alpha_0 + med_i \beta + Z_i \\
Z_i &\sim N(0, \sigma) \\
\alpha_0 &\sim N(0, 10) \\
\sigma &\sim HC(0, 5).
\end{aligned}
$$

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y_i &\sim Bin(n_i, p_i) \\
g(p_i) &\sim \alpha_0 + med_i \beta + Z_i \\
Z_i &\sim N(0, \sigma) \\
\alpha_0 &\sim N(0, 10) \\
\sigma &\sim HC(0, 5).
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
y_i &\sim Bin(n_i, p_i) \\
g(p_i) &\sim \alpha_i + med_i \beta \\
\alpha_i &\sim N(\alpha_0, \sigma) \\
\alpha_0 &\sim N(0, 10) \\
\sigma &\sim HC(0, 5).
\end{aligned}
$$

PI for model

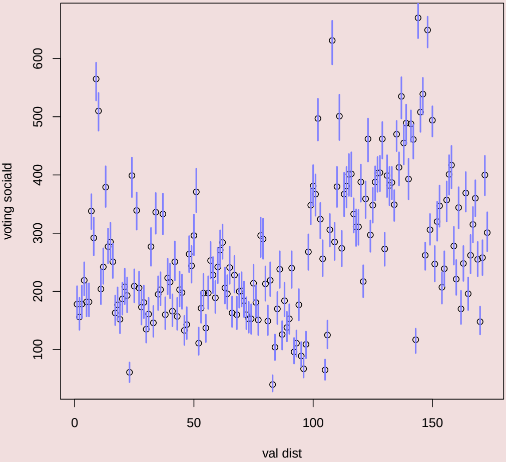


Figure : Prediction by district multilevel

PI for multilevel

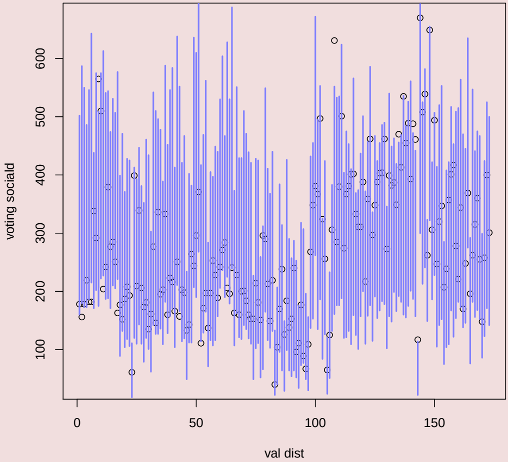


Figure : Prediction unconditional by district multilevel

Heuristically

- The variance is:

$$V[\hat{Y}|n_i] \approx n_i(1 - \hat{p}_i)\hat{p}_i + n_i^2\tilde{\sigma}$$
$$V[\frac{\hat{Y}}{n_i}|n_i] \approx \frac{(1 - \hat{p}_i)\hat{p}_i}{n_i} + \tilde{\sigma}$$

Where $\tilde{\sigma}$ is the variation from

PI for multilevel without cheating

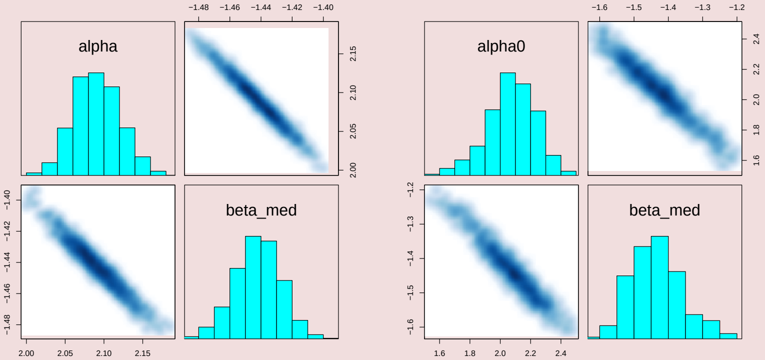


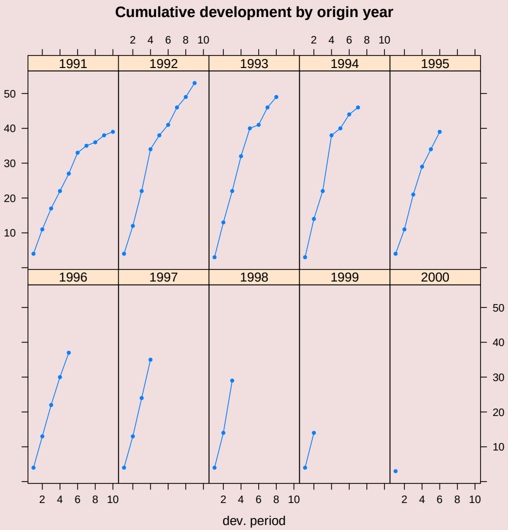
Figure : Look at parameter certainty

- Loss Developments - The goal is to estimate the ultimate cumulative loss for each accident year using historical data.

org / dev	1	2	3	4	5	6	7	8	9	10
1991	4	11	17	22	27	33	35	36	38	39
1992	4	12	22	34	38	41	46	49	53	
1993	3	13	22	32	40	41	46	49		
1994	3	14	22	38	40	44	46			
1995	4	11	21	29	34	39				
1996	4	13	22	30	37					
1997	4	13	24	35						
1998	4	14	29							
1999	4	14								
2000	3									

Table : Loss triangle. First column original year, the columns after are the development of the loss over months.

Loss triangle



Model

- A to capture this is the growth curve:

$$G(t|\omega,\theta) = (1 - e^{-(\frac{t}{\theta})^{\omega}})$$

$t-$ time after original year, $\omega, \theta-$ parameters to be fitted.

- Loss in the loss triangle is given by:

$$L_{i,j} = L_F G(12j|\omega,\theta)$$

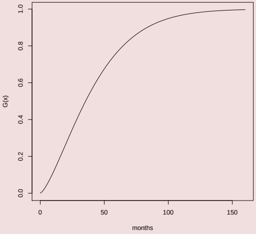


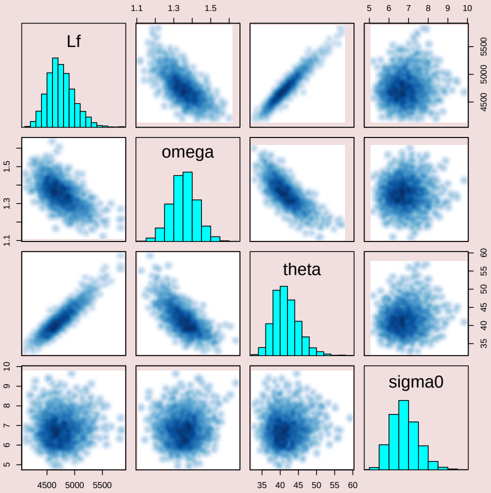
Figure : $G(t|1.4, 46)$

A first model

$$
\begin{aligned}
L_i(t) &\sim N(\mu_t, \sigma) \\
\mu_t &= L_F G(t|\omega, \theta) \\
\sigma &= \sigma_0 \sqrt{\mu_t} \\
\omega &\sim HN(1.4, 1) \\
\theta &\sim HN(46, 10) \\
\sigma_0 &\sim HC(0, 2) \\
L_F &\sim N(5000, 5000)
\end{aligned}
$$

Stan model

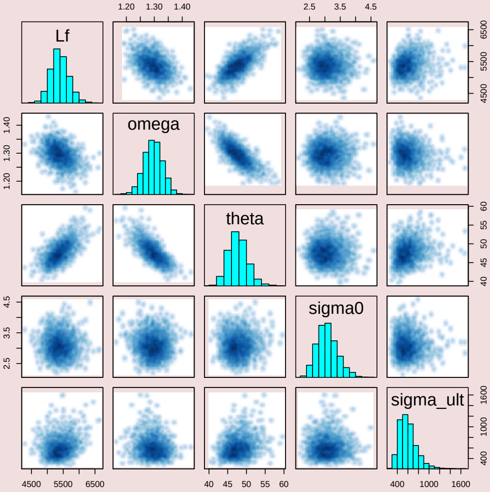
```
parameters {
   real<lower=0> theta;
   real<lower=0> omega;
   real<lower=0> Lf;
   real<lower=0> sigma0;
}
model {
   real mu[N];
   real disp_sigma[N];
   theta  ~ normal(46, 10);
   omega  ~ normal(1.4, 1);
   sigma0 ~ cauchy(0,5);
   Lf     ~ normal(5000, 5000);
   for (i in 1:N){
      mu[i] = Lf * (1-exp( -pow(dev[i] / theta, omega)));
      disp_sigma[i] = sigma0 * sqrt(mu[i]);
   }
   cum ~ normal(mu, disp_sigma);
}
```


Lf
omega
theta
sigma0
1.1
1.3
1.5
5
6
7
8
9
10
5500
5000
4500
60
55
50
45
40
35
4500
5000
5500
35
40
45
50
55
60

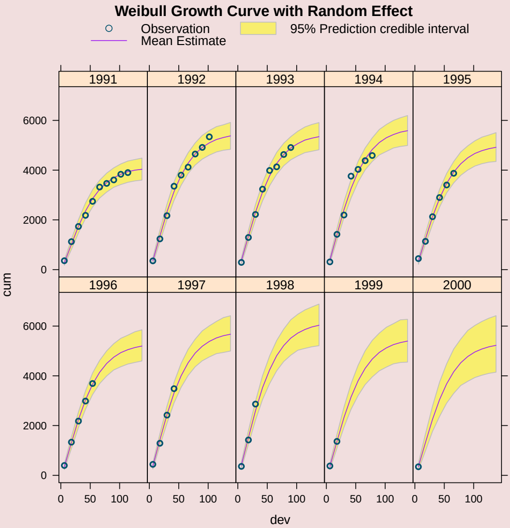
Multilevel version

$$
\begin{aligned}
L_i(t) &\sim N(\mu_{t,i}, \sigma) \\
\mu_{t,i} &= L_{F,i} G(t|\omega, \theta) \\
\sigma &= \sigma_0 \sqrt{\mu_t} \\
L_{F,i} &\sim N(L_F, \sigma_F) \\
\omega &\sim HN(1.4, 1) \\
\theta &\sim HN(46, 10) \\
\sigma_0 &\sim HC(0, 2) \\
\sigma_F &\sim HC(0, 2) \\
L_F &\sim N(5000, 5000)
\end{aligned}
$$

Result



Lf
omega
theta
sigma0
sigma_ult
1.20 1.30 1.40
2.5 3.5 4.5
4500 5500 6500
40 45 50 55 60
400 1000 1600


Weibull Growth Curve with Random Effect
Observation
Mean Estimate
95% Prediction credible interval
1991
1992
1993
1994
1995
1996
1997
1998
1999
2000
cum
dev
0
2000
4000
6000
0
50
100