14th November 2016

Prior to Posterior distribution conjugate prior

1. Assume that θ has a $B(\alpha, \beta)$ prior, and you have one binomial observation,
$$Y_1 \sim Bin(n_1, \theta),$$
   (a) what is the posterior distribution?
   (b) Suppose you make new observation, $Y_2 \sim Bin(n_2, \theta)$. What is the posterior distribution now?

2. A classical continous distribution the exponential distribution. The density of a exponential random variable, given its rate parameter θ, is
$$f(x; \theta) = \theta e^{-\theta x},$$
for $x \geq 0$. A conjugate prior for θ is the Gamma distribution
$$f(\theta; \alpha, \beta) \propto \theta^{\alpha-1} e^{-\beta\theta}.$$
Here $\theta > 0$.
   (a) If one uses an Gamma prior for θ, and observe an exponential random variable x_1 show that the posterior is a Gamma distribution; That is show that the posterior can be written on the form $\theta^{a-1}e^{-b\theta}$ for a specific value of a and b.
   (b) In R the quantile function of the gamma distribution is `qgamma(p,`α`,`β`)` and the probability function is `pgamma(y,`α`,`β`)`. Write down R code needed to compute the posterior probability of $\theta < 2$.
   (c) Suppose that you observe further independent random exponential random variables $x_2, \ldots, x_n$. Given the new data what is the posterior distribution now? HINT: Recall that if X and Y are independent then $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.
   (d) What is the posterior if we use a uniform prior, i.e. $f(\theta) \propto 1$?

3. For the likelihood of n independent normal random variables, $y_1, y_2, \ldots, y_n$, with mean μ and variance σ^2.
   (a) Explain why the joint distribution $f(y_1, y_2, \ldots, y_n; \mu, \sigma^2)$ equals $\prod_{i=1}^n f(y_i; \mu, \sigma^2)$.
   (b) Show that $f(y_1, y_2, \ldots, y_n; \mu, \sigma^2) \overset{\mu}{\propto} e^{-\sum_{i=1}^n \frac{(y_i-\mu)^2}{2\sigma^2}}$.
   (c) Show that $e^{-\sum_{i=1}^n \frac{(y_i-\mu)^2}{2\sigma^2}} \overset{\mu}{\propto} e^{-\frac{(\mu-\bar{y})^2}{2n^{-1}\sigma^2}}$.

4. For count data we have already seen the Poisson distribution. An other, more flexible, distribution for count data is the negative binomial. The

probability that negative binomial random variable Y equals k, given its probability parameter θ and size parameter z is

$$P(Y = k; \theta, z) = \frac{\Gamma(k+z)}{\Gamma(z)k!}\theta^z(1-\theta)^k,$$

where $k \in \{0, 1, 2, \ldots\}$. Here $n \in \{0, 1, 2, 3, \ldots\}$ and $\theta \in [0, 1]$. In this exerise we will assume that the parameter z is known.

(a) In R how do one compute the probability $P(Y \geq 5; \theta, z = 10)$?
HINT: In R the quantile function of the negative binomial is `qnbinom(p,z,`θ`)` and the probability function is `pnbinom(y,z,`θ`)`

(b) Assume a uniform prior for θ, i.e. $f(\theta) \propto 1$. What is the posterior distribution of theta θ given $Y_1 \sim negb(\theta, z)$? Hint the distribution is a classical distribution that we dealt with previously.

(c) After further $n - 1$ observation, $Y_2, \ldots, Y_n$ what is the posterior distribution?

(d) How would you compute a 95% posterior interval for θ in R?