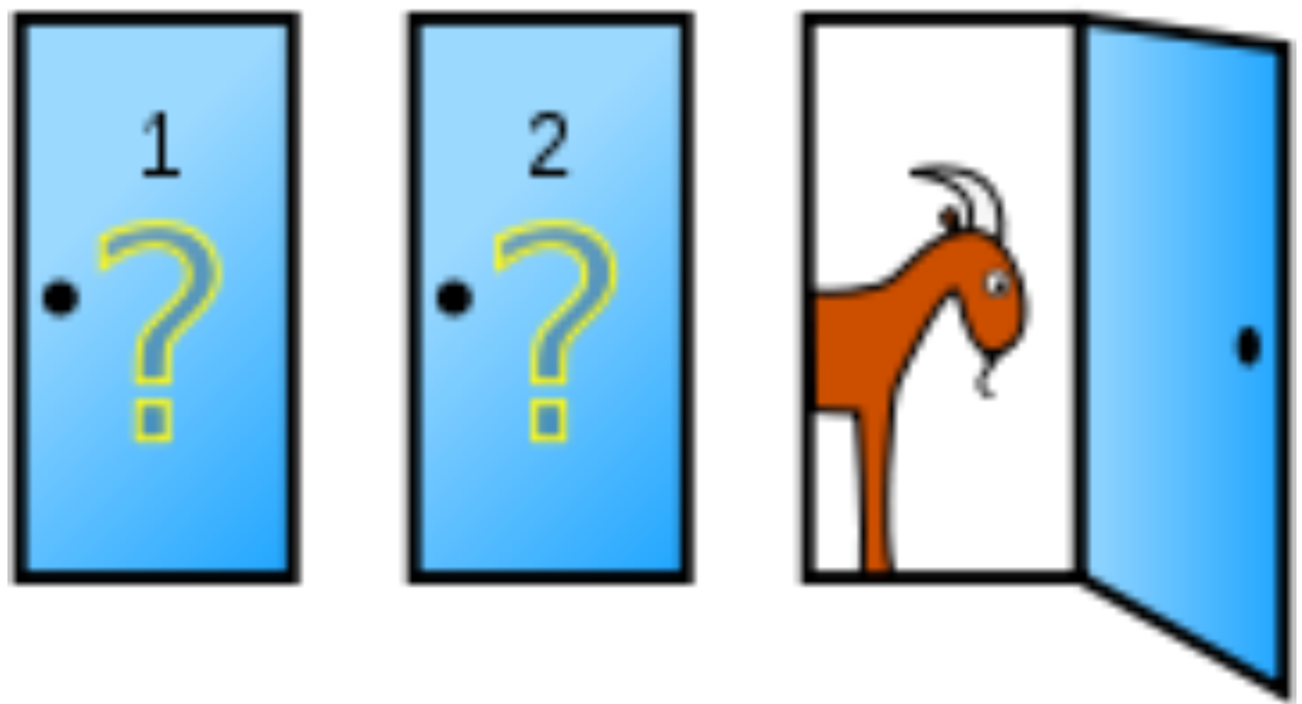
Lecture 2

Topics of the day:

1. Repetition from Last
2. Priors
3. Review of continuous distributions
4. Using R (more after lunch)

Monte Hall problem



$$A = \{\text{The car is behind the door you picked}\}$$
$$B = \{\text{The car is behind the door not chosen}\}$$
$$R = \{\text{You picked the right door right away}\}$$
$$W = R^c$$

Recall Law of total probability:

$$P(A) = P(A|R)P(R) + P(A|W)P(W)$$

$$P(B) = P(B|R)P(R) + P(B|W)P(W)$$

We will now study the coin flip example

(4.1,4.2) in book

We want to study the posterior distribution of the probability of that a coin lands on Tail.

$$P(\text{tail}) = \theta$$

If we repeat the experiment n times
the number of tails, Y, is binomially distributed:

$$P(Y|\theta, n) = \binom{n}{Y} \theta^Y (1-\theta)^{n-Y}.$$

Probability density function

(4.3.1) in book

Continuous random variable X,
with density f(x).

Probability of X taking value y is zero.

$$P(X = y) = 0$$

But probability that X is in some interval positive.

$$P(X \in [y, y + \Delta_y]) = \int_y^{y+\Delta_Y} f(x)dx$$

$$P(X \in [y, y + \Delta_y]) \approx \Delta_y f(y)$$

Important rules for density function

$$f(x) \geq 0$$

$$\int f(x)dx = 1$$

Indicator function

$$\mathbb{I}(x \in A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$$

Discrete distribution

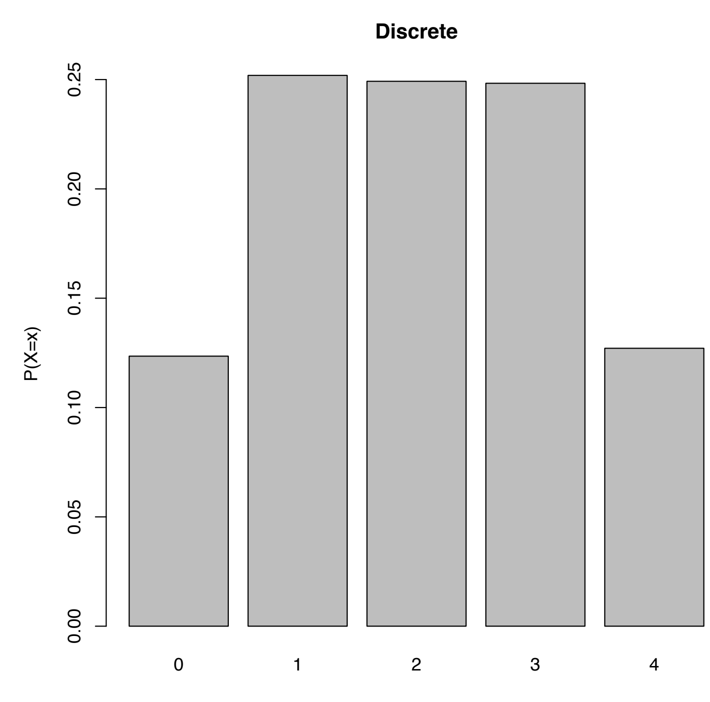
Suppose you have some data drawn from a discrete distribution P(X=k)

$$Y = (y_1, y_2, \ldots, y_N)$$

$$y_i \in \{0, 1, 2, 3, 4\}$$

$$\mathbb{I}(y_i = k) = \begin{cases} 1 & \text{if } y_i = k \\ 0 & \text{else} \end{cases}$$

$$\tilde{P}(X = k) = \frac{1}{N} \sum_{i=1} \mathbb{I}(y_i = k)$$



continous distribution

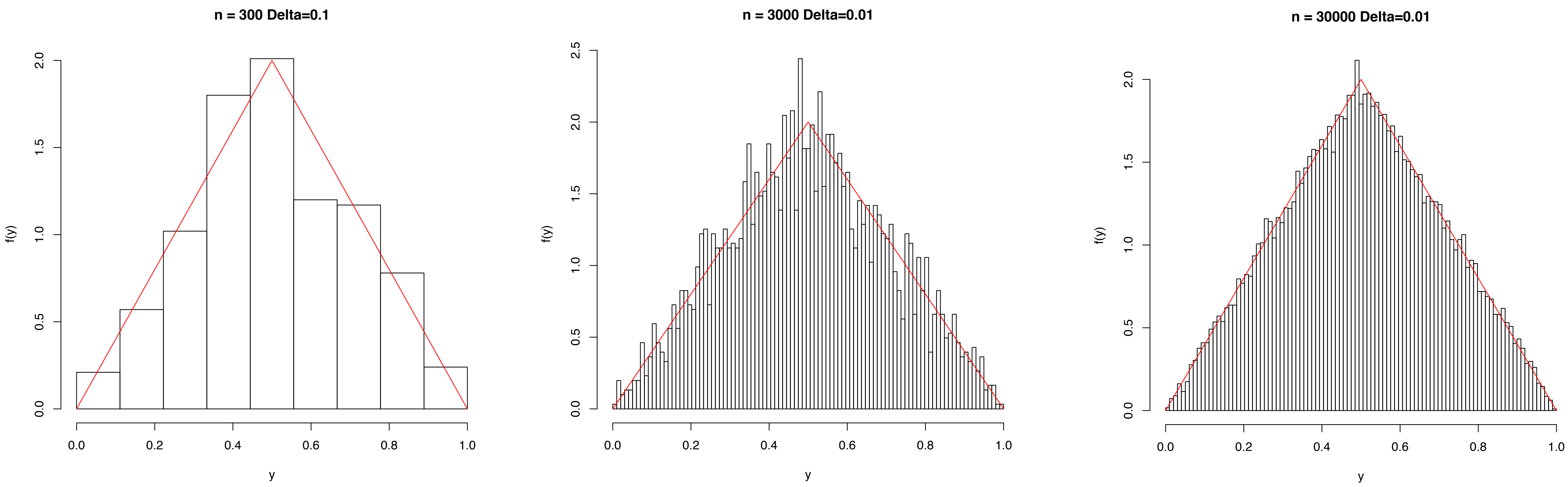
Suppose you have some data drawn from a continous distribution f(x)

$$Y = (y_1, y_2, \ldots, y_N)$$

$$\tilde{P}(X \in [x, x + \Delta]) = \frac{1}{N} \sum_{i=1} \mathbb{I}(y_i \in [x, x + \Delta])$$

example:

$$f(x) = 2 - 4|0.5 - x|$$



continuous distributions in two dimensions

For two continuous random variable X and Y, with joint density f(x, y).

$$P(\{X \in [a, a+\Delta_a]\} \cap \{Y \in [b, b+\Delta_b]\})$$
$$= \int_a^{a+\Delta_a} \int_b^{b+\Delta_b} f_{X,Y}(x, y) dx dy$$

Suppose you have some data draws from a continous distribution f(x,y)

$$[(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)]$$

$$\tilde{P}(\{X \in [a, a + \Delta_a]\} \cap \{Y \in [b, b + \Delta_b]\})$$
$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(x_i \in [a, a + \Delta_a]) \mathbb{I}(y_i \in [b, b + \Delta_b])$$

Conditional densities for continuous variables:

The conditional density of X given that Y=b,
is denoted

$$f_{X|Y}(\cdot|b)$$

A word of caution though even if X and Y are continuous variables: The random variable X|Y can be discrete.

Bayes Theorem

$$f_{Y|X}(\cdot|b) = \frac{f_{X|Y}(b|\cdot)f_Y(\cdot)}{f_X(b)}$$

Since we know that the density should integrate to one we often don't care about the denominator

$$f_{Y|X}(\cdot|b) \propto f_{X|Y}(b|\cdot)f_Y(\cdot)$$

All the rules for discrete distributions still exists

Law of total probability (marginal distributions):

$$f_X(a) = \int f_{X|Y}(a|b) f_Y(b) db$$

independence:

$$f_{X,Y}(a,b) = f_X(a) f_Y(b)$$

This is in its essence what Bayesian statistics is about:

(again)

We set a prior distribution for a parameter

$$\pi(\beta)$$

We observe data, Y, and can evaluate how likely it given a value of the parameter (likelihood),

$$f(Y|\beta)$$

We update our belief (posterior distribution), using bases theorem:

$$\pi(\beta|Y) \propto f(Y|\beta) \times \pi(\beta)$$

$$\text{Posterior} \propto \text{Likelihood} \ \times \text{Prior}$$

Bayesian coin example continuous,

(5.3.1-2 in book)

Prior:

$$\pi(\theta) \propto 0.5 - |0.5 - \theta|$$

Likelihood:

$$P(Y|\theta, n) = \binom{n}{Y} \theta^Y (1-\theta)^{n-Y}.$$

posterior:

$$\pi(\theta|Y, n) \propto \theta^Y (1-\theta)^{n-Y} \cdot (0.5 - |0.5 - \theta|)$$