Lecture 1

Bayes theorem
Conditional distributions

Topics of the day:

1. Introduction to the course
2. Introduction to Bayesian statistics
3. Bayes Theorem,
   conditional distributions

Homepage:

liveatlund.se
and
https://github.com/JonasWallin/BayesianMethods_STAE02

Examination

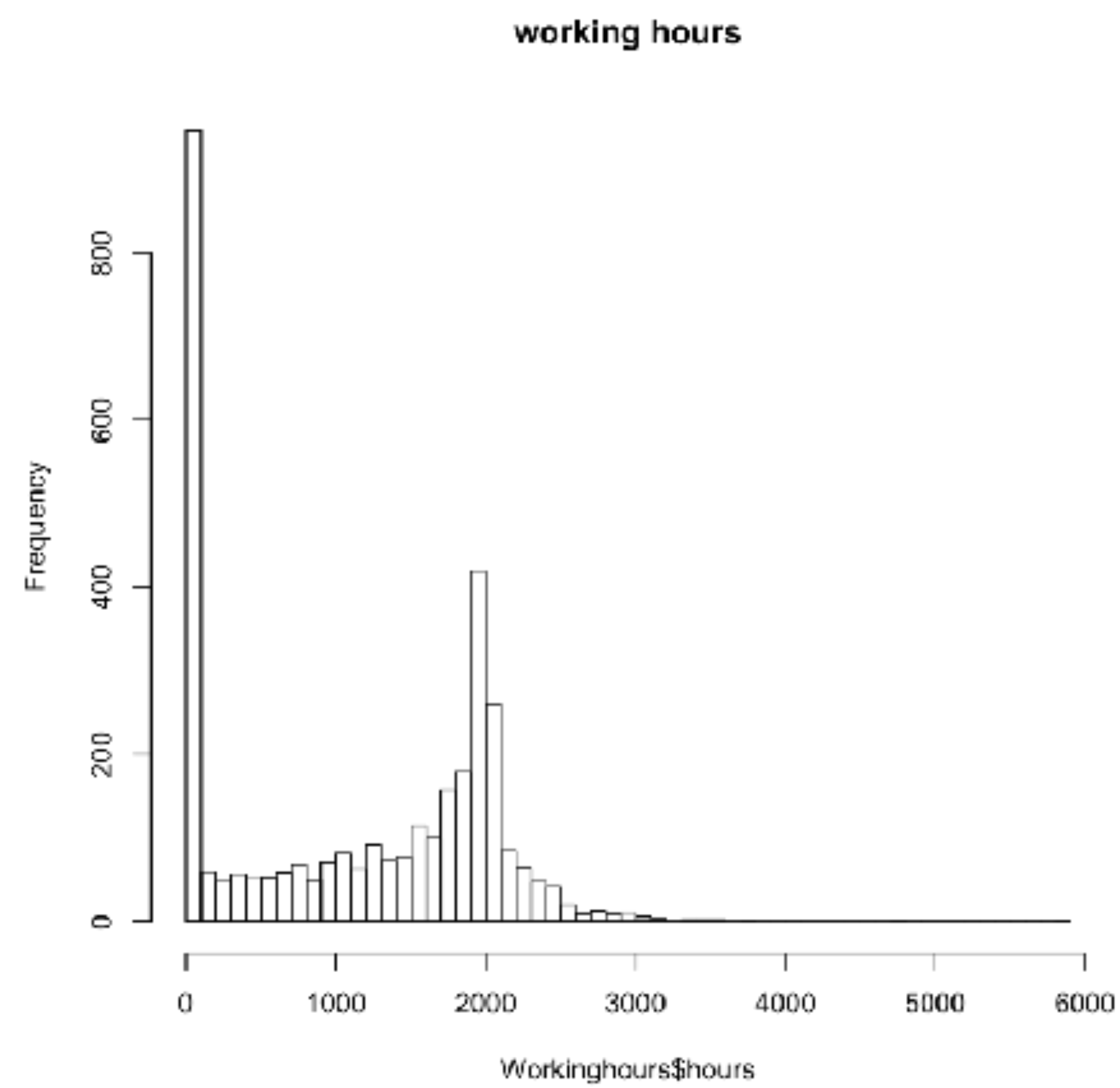
1. Small assignment each week
2. One project at the end of the course
3. Exam

Three things we want to teach you in this course:

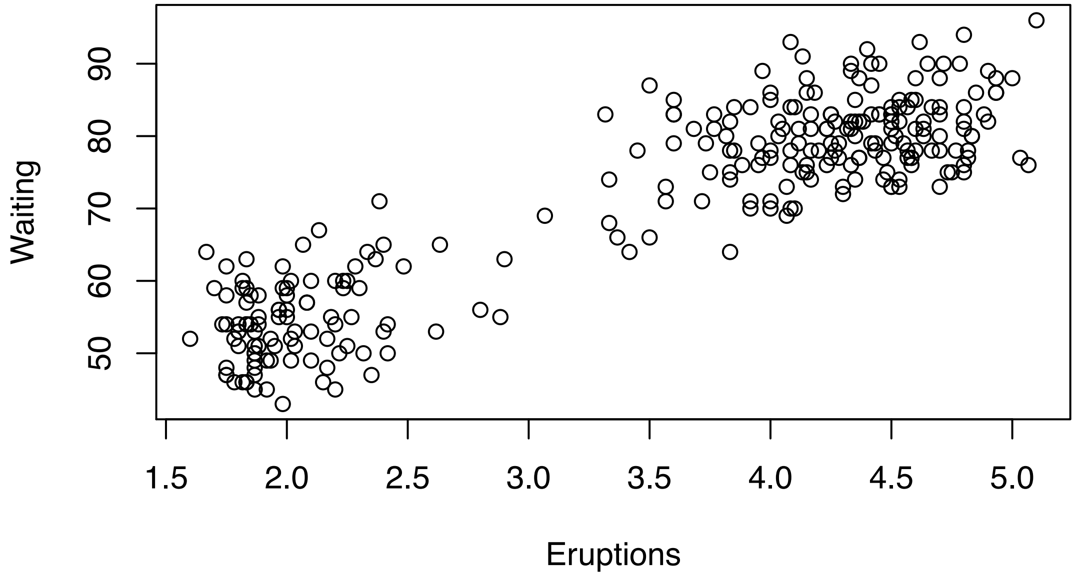
1. How to use Bayes rules.
2. Understanding about priors and how to apply it to data.
3. Tool box of statistical models and how to fit them to data.

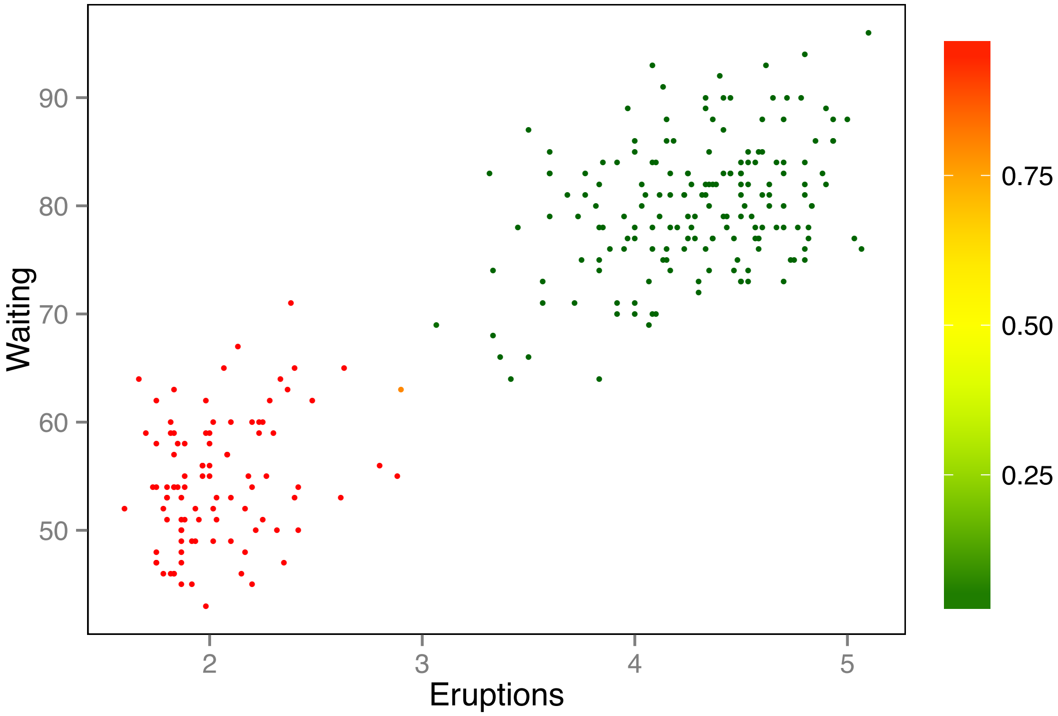
Censored Regression

$$\text{working hours} = \beta_0 + \beta_1 \text{income} + \epsilon$$

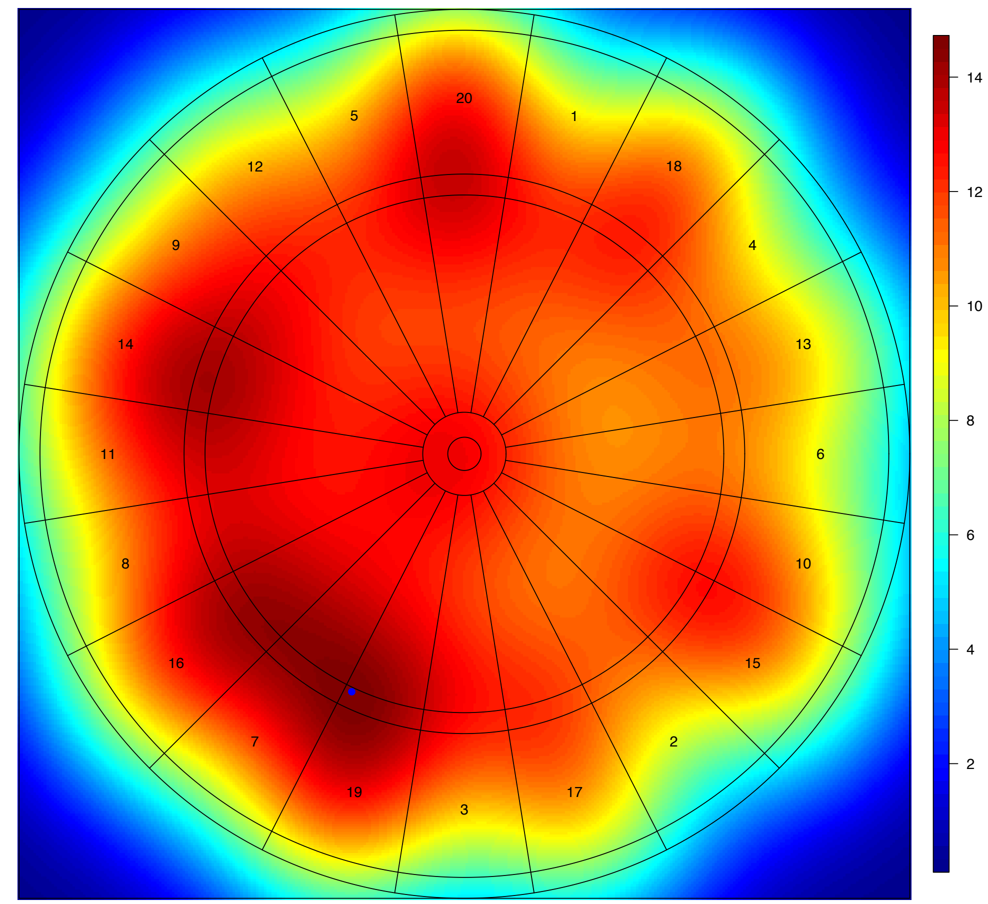


Mixture models





Using Bayesian statistics
to improve your dart game.



Stan

R

For the project later on in the course we will use R and Stan.

The essence of Bayesian statistics is:

a prior belief (prior distribution),

an observation (likelihood),

updated belief (posterior distribution)

You believe that it is one in third chance
that the ball is under either cup

Suppose you pick the first cup,
and the ball is not under.

What is the probability that it is under
cup1,cup 2 or cup 3?

Other prior:
the first cup with probability 0.69,
the second cup with probability 0.1,
and the third cup with probability 0.21.

Again you pick the first cup,
and the ball is not under.

What is the probability that it is under
cup1, cup 2 or cup 3?

Mathematical formulation of the problem:

Events:

$$A_1 = \{\text{The ball is under the first cup}\}$$
$$A_2 = \{\text{The ball is under the second cup}\}$$
$$A_3 = \{\text{The ball is under the third cup}\}$$

Prior belief:

$$P(A_1) = 0.69$$
$$P(A_2) = 0.1$$
$$P(A_3) = 0.21$$

The observations:

$$A_1^c = \{\text{the ball is not under the first cup}\}$$

The posterior distribution:

$$P(A_1|A_1^c) = ?$$

$$P(A_2|A_1^c) = ?$$

$$P(A_3|A_1^c) = ?$$

Conditional probabilities:

The probability of event A given event B

$$P(A|B)$$

Definition

$$P(B|A) = \frac{P(A,B)}{P(A)}$$

Alternative form $= \dfrac{P(A \cap B)}{P(B)}$

Both numerator means that:
P(both event A and B occurred)

Bayes Theorem

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

equation (5.1) in book

Back to the example:

First we have that probability of the complementary event is one minus the probability of the event:

$$P(A_1^c) = 1 - P(A) = 1 - 0.69 = 0.31$$

Then we use Bayes formula to calculate the posterior:

$$P(A_2|A_1^c) = \frac{P(A_1^c|A_2)P(A_2)}{P(A_1^c)} = \frac{1 \cdot 0.1}{0.31} \approx 0.322$$

$$P(A_3|A_1^c) \approx 0.678$$

We will now review some important probability laws:

The law of total probability

$$P(\cup_{i=1}^{n} B_i) = 1$$

$$P(B_i \cap B_j) = 0, \text{ for } i \neq j$$

$$P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$$

A second example

$$A = \{\text{Person have rare disease}\}$$
$$B = \{\text{Positive test}\}$$

$$\mathbb{P}(A) = 10^{-6}$$
$$\mathbb{P}(B|A) = 1.$$
$$\mathbb{P}(B|A^c) = 10^{-4}.$$

$$A = \{\text{Person have rare disease}\}$$
$$B = \{\text{Positive test}\}$$

$$\mathbb{P}(A) = 10^{-6}$$
$$\mathbb{P}(B|A) = 1.$$
$$\mathbb{P}(B|A^c) = 10^{-4}.$$

$$\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(B)} \\
&= \frac{P(B|A)P(A)}{(P(B|A)P(A) + P(B|A^c)P(A^c))} \\
&= \frac{P(B|A)P(A)}{(P(B|A)P(A) + P(B|A^c)(1 - P(A)))} \\
&= \frac{1 \cdot 10^{-6}}{1 \cdot 10^{-6} + 10^{-4} \cdot (1 - 10^{-6})} \approx \frac{1 \cdot 10^{-6}}{10^{-4}} \approx 0.01
\end{aligned}$$

Law independence:

$$P(A \cap B) = P(A)P(B)$$

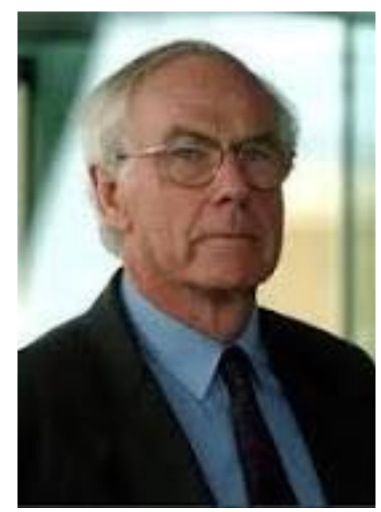
Or:

$$P(A|B) = P(A)$$

We have now looked at two toy problems. However, peoples lack of understanding of Bayes theorem and conditional probabilities can have serious consequences.

On such situation is the case of Sally Clark.

Sally Clark



Professor Sir Roy Meadow

Sally Clark was on trial for murdering two of her sons.

Her first son died a few week after his birth.
The death was ruled Sudden Infant Death Syndrome (SIDS).

A few year her second died in a similar manner.
This was viewed as highly suspicious,
and Sally and her husband was later charged with murder
of both of their Children.

During the trial Professor Sir Roy Meadow
acted as a expert witness explaining
the probability of a child dying by SIDS
(under certain factors) was about 1 in 8543.

Therefore the chance of having two cases
of SIDS in one household is approximately 1/8543^2,
which is approximately 1 in 73 millions.

This result was later viewed as strong evidence
that Sally Clark was guilty and jailed in 1999.

Two mayor errors:

First the assumption of independence

$$A_1 = \{\text{Having one child dying of SIDS}\}$$
$$A_2 = \{\text{Having two children dying of SIDS}\}$$

$$P(A_2) \neq P(A_1)^2$$

Two mayor errors:

Second error (known as Prosecutor's Fallacy):

$$I = \{\text{Person innocent}\}$$
$$E = \{\text{Evidence}\}$$

$$P(E|I) \neq P(I|E)$$

Recall Bayes:

$$P(I|E) = P(E|I)\frac{P(I)}{P(E)}$$

Sally Clark was freed in 2003,
but later died 2007 from alcohol poisoning.