Chapter 3

Inference from posterior distribution

Assumption in order:

(1) Likelihood,

(2) Parameters,

(3) Prior.

The assumption results in a Posterior.

Likelihood

- The probability of the data given the parameters. (Assumption)

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- Which parameters? The parameters you need to define a probability distribution.

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- The probability of the data given the parameters. (Assumption)
- Which parameters? The parameters you need to define a probability distribution.
- The likelihood is not a distribution for the parameters.

Example: Placenta praevia

- For the data:

$$X = \begin{cases} 1 & \text{if the child is girl,} \\ 0 & \text{if the child is boy.} \end{cases}$$

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$$X = \begin{cases} 1 & \text{if the child is girl,} \\ 0 & \text{if the child is boy.} \end{cases}$$

- The parameter is the probability of a child being a girl, p.

Example: Placenta praevia

- If you observe n_F girls of n births, the distribution given the data, is binomial:

$$\mathbb{P}(n_F|p) = \frac{n!}{n_F!(n-n_F)!}p^{n_F}(1-p)^{n-n_F}$$

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Posterior

From the prior, parameters, and likelihood, one gets posterior:

- By Bayes theorem

$$p(p|n_F) = \frac{p(n_F|p)p(p)}{p(n_F)}$$

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- Or in words:

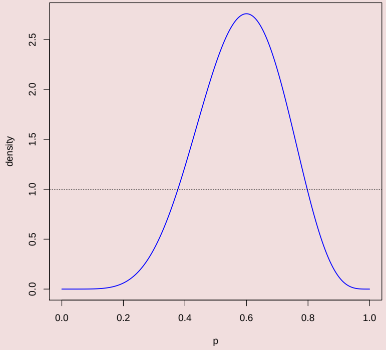
$$\text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Average Likelihood}}$$

Inference from posterior distribution

- The posterior distribution

$$p(p|n_f) \propto p^{n_f}(1-p)^{n-n_f}\mathbb{I}_{[0,1]}(p)$$

.



Inference from posterior distribution

- The posterior distribution

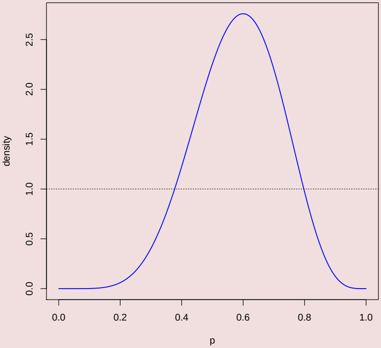
$$p(p|n_f) \propto p^{n_f}(1-p)^{n-n_f}\mathbb{I}_{[0,1]}(p)$$

.

- What is the probability that p is less then 0.5:

$$\mathbb{P}(p < 0.5|n_f) = \int_0^{0.5} p(p|n_f)dp$$

(here we would need the actual density)



Samples of the posterior

```
print(samples)
[1]  0.47474747  0.40404040  0.38383838  0.06060606
[5]  0.21212121  0.37373737  0.31313131  0.24242424  0.60606061
[10]  0.22222222  .....
```

Samples of the posterior

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[5] 0.21212121 0.37373737 0.31313131 0.24242424 0.60606061
[10] 0.22222222 .....
```

-

```
mean(samples <0.5)
mean(samples)
var(samples)
quantile(samples, c(0.05,0.95))
```

- [1] 0.2751
  [1] 0.582734
  [1] 0.01888366
  5% 95%
  0.1808081 0.6166667

Posterior distribution

What is the probability that p is less then 0.5?

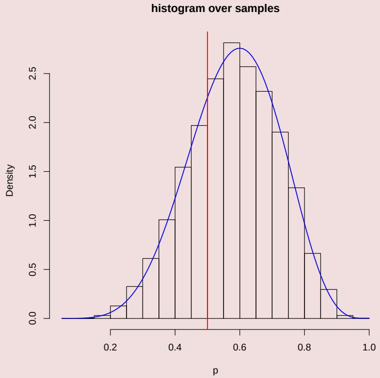
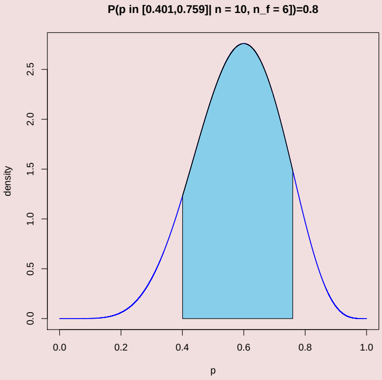
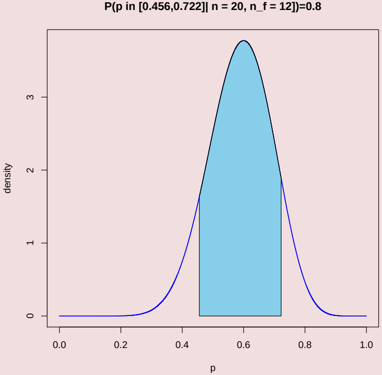


Figure: Histogram of draws from the posterior. Blue line is the posterior distribution

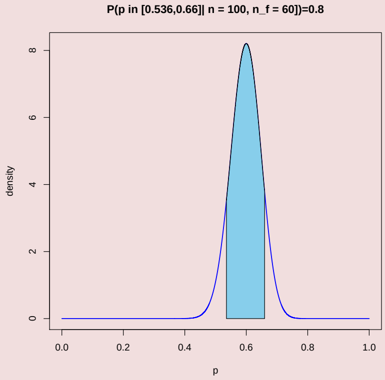
Posterior intervals



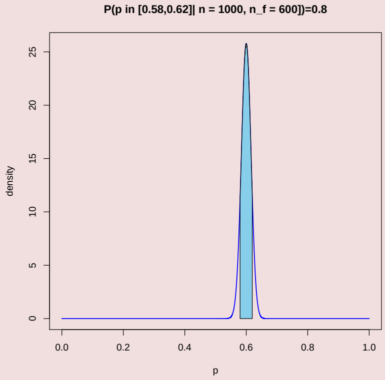
Posterior intervals



Posterior intervals



Posterior intervals



Sampling from the posterior

```
p_grid <- seq(from = 0, to  = 1, length.out = 10)
print(p_grid)
```

- ```
  [1] 0.0000000 0.1111111 0.2222222 0.3333333
  [5] 0.4444444 0.5555556 0.6666667 0.7777778
  [9] 0.8888889 1.0000000
  ```
  ```

Sampling from the posterior

- Grid approximation of the posterior:

```
p_grid <- seq(from = 0, to = 1, length.out = 10)
```

- .

```
prior <- rep(1, 10)
print(prior)
```

[1] 1 1 1 1 1 1 1 1 1 1

Grid approximation

- Grid approximation of the posterior:

```
p_grid <- seq(from = 0, to  = 1, length.out = 10)
prior  <- rep(1, 10)
```

- .

```
nf = 6
n = 10
likelihood <- dbinom(x = nf, size = n, prob = p_grid)
print(likelihood)
```

```
[1] 0.0000000000 0.0002466915 0.0092547850
[4] 0.0569018950 0.1541821742 0.2409096472
[7] 0.2276075801 0.1133711163 0.0157882546
[10] 0.0000000000
```

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```

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```
posterior <- likelihood * prior
print(posterior) # sum(posterior) = 0.8182621
```

```
[4]  0.0569018950  0.1541821742  0.2409096472
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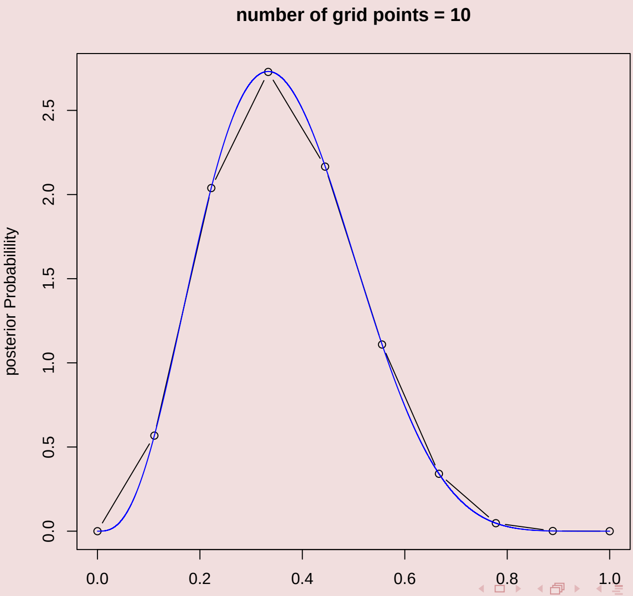
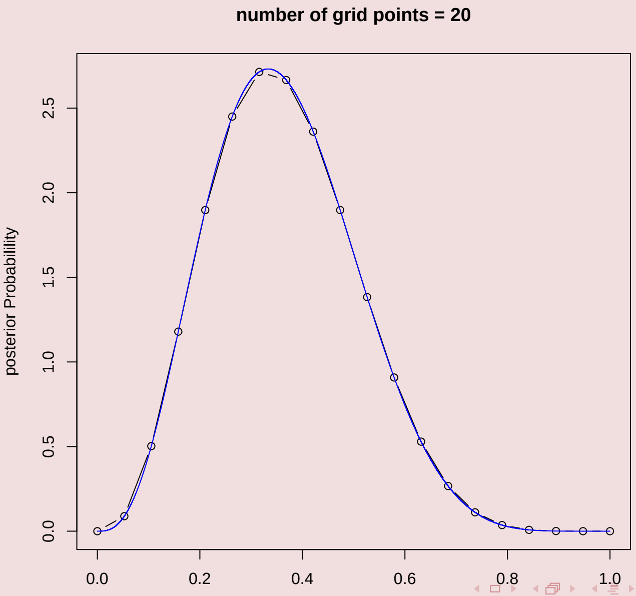
```

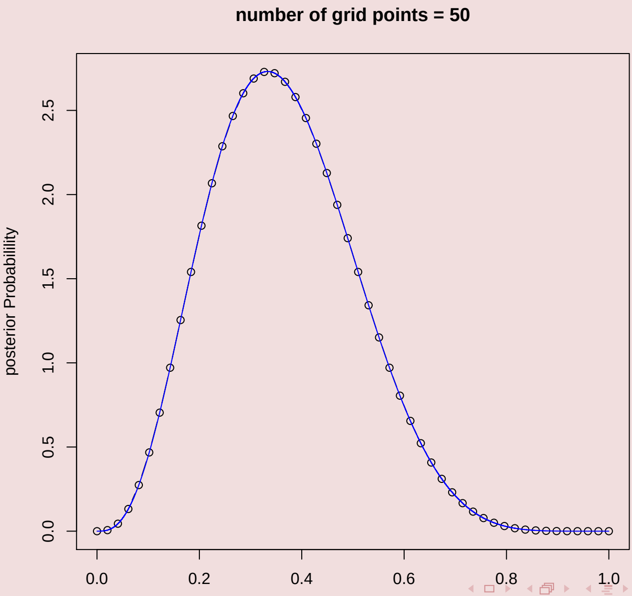
- .

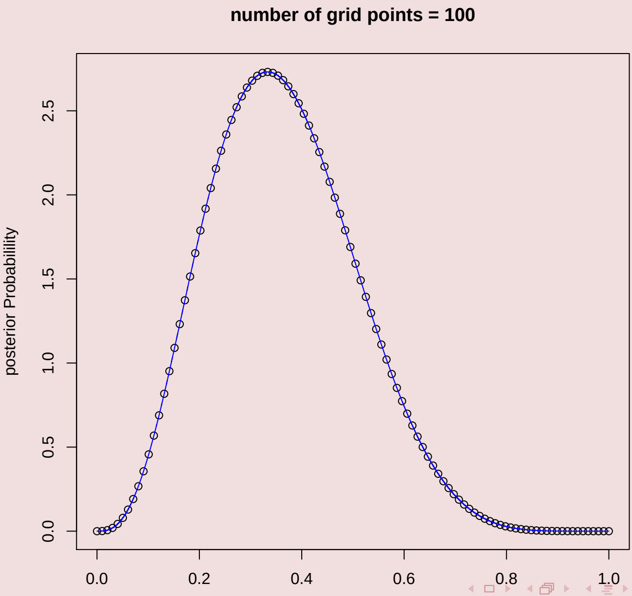
```
posterior  <- posterior / sum(posterior)
print(posterior) # sum(posterior) = 1
```

```
[4]  0.0695399334  0.1884263806  0.2944162197
[7]  0.2781597338  0.1385510953  0.0192948614
[10]  0.0000000000
```


number of grid points = 10
posterior Probabilility
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0


number of grid points = 20
posterior Probability
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0


number of grid points = 50
posterior Probability
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0


number of grid points = 100
posterior Probability
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0

Sampling from the posterior

- Grid approximation of the posterior:

```
p_grid <- seq(from = 0, to  = 1, length.out = 1000)
prior  <- rep(1, 1000)
nf = 6
n = 10
likelihood <- dbinom(x = nf, size = n, prob = p_grid)
posterior  <- likelihood * prior
posterior  <- posterior / sum(posterior)
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```

- Sampling:

```
samples <- sample(p_grid,
                  prob = posterior,
                  size = 10000,
                  replace = T)
```

Calculating $\mathbb{P}(p < 0.5)$

- $\mathbb{P}(p|n = 10, n_f = 6) = \int_0^{0.5} p(p|n = 10, n_f = 6)dp = 0.274$

Calculating $\mathbb{P}(p < 0.5)$

- $\mathbb{P}(p|n = 10, n_f = 6) = \int_0^{0.5} p(p|n = 10, n_f = 6)dp = 0.274$
- Approximation by sampling $(p_1, p_2, \ldots, p_s)$:

$$P_{est} = \frac{1}{10^4} \sum_{i=1}^{10^4} \mathbb{I}_{[0,0.5)}(p_i) = 0.281$$

In code:

```
samples <- sample(p_grid,
                  prob = posterior,
                  size = 10000,
                  replace = T)
P_est <- mean(samples < 0.5)
```

Approximation

```
samples <- sample(p_grid,
prob = posterior,
size = s,
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P_est <- mean(samples < 0.5)
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- What is the effect of the code s (the number of samples) on the approximation?

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- What is the effect of the code s (the number of samples) on the approximation?
- $s = 2$ three runs of the code

$$P^1_{est} = 0.5, P^2_{est} = 0, P^3_{est} = 0.$$

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- $s = 10$ three runs of the code

$$P^1_{est} = 0, P^2_{est} = 0.3, P^3_{est} = 0.2.$$

Approximation

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- $s = 10^5$ three runs of the code

$$P^1_{est} = 0.272, P^2_{est} = 0.274, P^3_{est} = 0.274.$$

Predictions

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- n_f^* number of girls born, n^* number of child births.

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- For a given $\hat{p}$, the distribution of n_f^* is the likelihood

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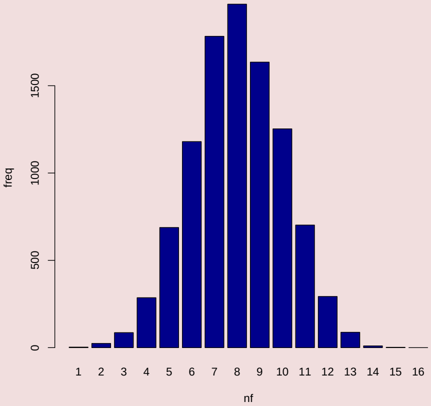
- In R:

```
p_hat     <- 0.5
nf_star   <- rbinom(n = 1000,  #number of samples
                    size = 16,
                    prob = p_hat)
```

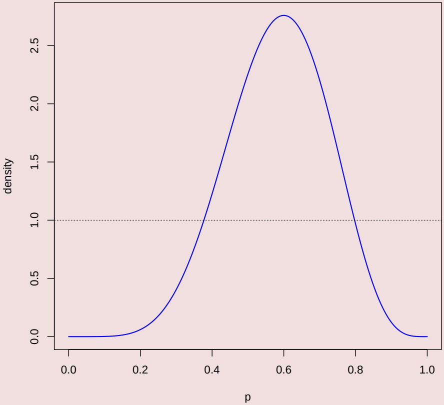
samples n_f^*

Distribution of $p(n_f^* | n = 16, \hat{p} = 0.5)$



which $\hat{p}$

Which $\hat{p}$?



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Predictions

- The answer no single $\hat{p}$!
- The predictive distribution prior to observing the data (y, n)

$$p(n_f^*|n^*) = \int p(n_f^*|n^*, p)p(p)dp.$$

- The predictive distribution after observing the data (y, n)

$$p(n_f^*|n^*, n, n_f) = \int p(n_f^*|n^*, p)p(p|n, n_f)dp.$$

Sampling from the posterior

- Prior predictive distribution

```
samples <- runif(n = 10000, min = 0, max =1)
nf_star    <- rbinom(n = 10000,
                     size = 16,
                     prob = samples)
```

Sampling from the posterior

- Prior predictive distribution

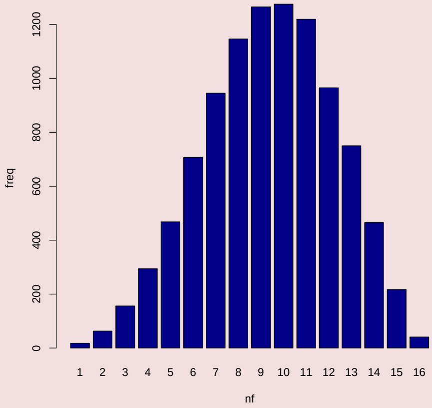
```
samples <- runif(n = 10000, min = 0, max =1)
nf_star   <- rbinom(n = 10000,
                    size = 16,
                    prob = samples)
```

- Posterior predictive distribution

```
nf = 6
n = 10
p_grid <- seq(from = 0, to  = 1, length.out = 1000)
prior  <- rep(1, 1000)
likelihood <- dbinom(x = nf, size = n, prob = p_grid)
posterior  <- likelihood * prior
posterior  <- posterior / sum(posterior)
samples <- sample(p_grid,
                  prob = posterior,
                  size = 10000,
                  replace = T)
nf_star   <- rbinom(n = length(samples),
                  size = 16,
                  prob = samples)
```

samples n_f^*

$$p(n_f^*|n^* = 16) \text{ and } p(n_f^*|n^* = 16, n = 10, n_f = 6)$$



samples n_f^*

$$p(n_f^* | n^* = 16) \text{ and } p(n_f^* | n^* = 16, n = 10, n_f = 6)$$