Submission 4.1

Jonas Wechsler

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1. P_0 : If only one student has a black card, they know that they are the only one with a black card because they do not see anybody else with a black card.
   Assume that it takes exactly k repetitions for k students with black cards.
   Consider $k+1$ students with black cards, and an individual A who has a black cards. After k turns, all the other students with black cards say nothing, so A can infer that he has a black card, and will say yes on the next turn.

2. $P_0 : 1^3 + 5 * 1 = 6, \frac{6}{6} = 1 \wedge 1 \epsilon \mathbb{N}$
   P_k : Assume $k^3 + 5k$ is divisible by 6.
   $$\begin{aligned} P_{k+1}: \quad & (k+1)^3 + 5(k+1) \\ & k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ & k^3 + 3k^2 + 8k + 6 \\ & (k^3 + 5k) + 3k(k+1) + 6 \end{aligned}$$
   A sum of numbers that are multiples of 6 is also a multiple a multiple of 6. We have already assumed that $k^3 + 5k$ is a multiple of 6 and we have already shown that 6 is a multiple of 6. If k is even, then $3k$ is a multiple of 6, so $3k(k+1)$ is also a multiple of 6. If k is odd, then $k+1$ is even and $3(k+1)$ is a multiple of 6. So, $3k(k+1)$ is a multiple of 6. Because the expression $(k^3 + 5k) + 3k(k+1) + 6$ is a multiple of 6, it is also divisible by 6.

3. P_0 : 6 can be represented with 3 2 cent coins.
   Assume n cents can be represented with 2 and 7 cent coins.
   There are 2 cases for n+1. If n+1 is even, it can be represented with 2 cent coins. If n+1 is odd, then n+1-7 is even, and can be represented with 2 cent coins and a 7 cent coin.

4. $P_0 : |1| \leq |1|$
   P_k : Assume $|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$
   $P_{k+1} : |x_1 + x_2 + ... + x_n + x_{n+1}| \leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|$
   In the case that x_{n+1} is negative, $|x_1 + ... + x_n| \geq |x_1 + ... + x_{n+1}|$, and $|x_1| + ... + |x_n| \leq |x_1| + ... + |x_{n+1}|$, meaning that $|x_1 + ... + x_{n+1}| \leq |x+1| + ... + |x_{n+1}|$. In the case that x_{n+1} is zero, $P_{k+1} = P_k$. In the case that x_{n+1} is positive, $|x_1 + ... + x_n| + |x_{n+1}| = |x_1 + ... + x_{n+1}|$, so $|x_1 + ... + x_{n+1}| \leq |x_1| + ... + |x_{n+1}|$.

5. $P_0 : (1+x)^1 \geq 1 + x$
   P_k : Assume $(1+x)^n \geq 1 + nx$
   $$\begin{aligned} P_{k+1}: \quad & (1+x)^(n+1) \geq 1 + (n+1)x \\ & (1+x)(1+x)^n \geq 1 + xn + x \\ & (1+x)(1+x)^n \geq (1+x) + nx \\ & (1+x)^n + x(1+x)^n \geq x + 1 + nx \end{aligned}$$
   $x(1+x)^n \geq x$, because $(1+x)^n \geq 1$, so if $(1+x)^n \geq 1 + nx$, then $(1+x)^n + x(1+x)^n \geq x + 1 + nx$.

6. (a) $s(k)$ is a function. $dom(s) = \mathbb{N} \geq 1$
   $codom(s) = \mathbb{R} > 0$

(b)
$$\begin{aligned} e - s(k) &= \sum_{i=0}^{\infty} \tfrac{1}{i!} - \sum_{i=0}^{k} \tfrac{1}{i!} \\ &= \sum_{i=k+1}^{\infty} \tfrac{1}{i!} \\ &= \tfrac{1}{(k+1)!} + \tfrac{1}{(k+2)(k+1)!} + \tfrac{1}{(k+3)(k+2)(k+1)!} \end{aligned}$$

where $n > 1$, $\frac{1}{k+n} < \frac{1}{k+1}$ therefore

$= \frac{1}{(k+1)!} + \frac{1}{(k+1)(k+1)!} + \frac{1}{(k+1)^2(k+1)!} >$ the previous statement.

therefore $e - s(n) < \frac{1}{(n+1)!}(\frac{1}{(n+1)} + \frac{1}{(n+1)^2} + ...)$

$\frac{1}{k!} < \frac{1}{(k+1)!}\frac{1}{(k+1)^{k+1}}$ because $k! > (k+1)!$ and $\frac{1}{(k+1)!} < 1$ and $\frac{1}{(k+1)^{k+1}}$.

(c) Because an infinite geometric series $\sum_{i=0}^{n} a * r^n$ can be written as $\frac{a}{1-r}$, the infinite geometric series in this problem can be written as.

$$\begin{aligned} e - s(k) &< \tfrac{1}{(k+1)!}\tfrac{1}{1-\frac{1}{k+1}} \\ &< \tfrac{1}{(k+1)!}\tfrac{1}{\frac{k+1-1}{k+1}} \\ &< \tfrac{1}{(k+1)!}\tfrac{1}{\frac{k}{k+1}} \\ &< \tfrac{1}{(k+1)!}\tfrac{1}{\frac{k+1}{k}} \\ &< \tfrac{1}{k*k!} \end{aligned}$$

(d) If $r = \frac{m}{n}$, $n!e = n!\frac{m}{n} = (n-1)!m$

and $n!s(n) = n!\sum_{i=0}^{n}\frac{1}{i!} = \sum_{i=0}^{n}\frac{n!}{i!}$. At this point, the denominator always cancels out with a portion of the numberator, because $n \geq i$.

(e) $e - s(n) < \frac{1}{n!*n}$ where $n \epsilon \mathbb{N}$, so $n!(e-s(n)) < \frac{1}{n}$, which means $n!(e-s(n))$ ranges from $\{\frac{1}{1}, \frac{1}{2}, ..., \frac{1}{\infty} = 0\}$, or from [0,1] and $n!(e-s(n))$ must be positive because s(n) is smaller than e and n is a natural number, therefore it ranges from [0,1]

however $n!(e-s(n)) = n! * e - n!(s(n))$ and both of those are integers if $e = \frac{m}{n}$, so $n!(e-s(n))$ must be an integer. therefore $n!(e-s(n))$ must be an integer between 0,1