Bayesian Data Analysis on Climate Data

Bayesian Computation - Spring 2019

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Why Climate Data?

Hot Topic.

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Lots of Data available.

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Bayesian Approach offers a lot of freedom.

Outline

1. Dataset
2. Models for analysis
3. Methods for approximation
4. Comparison of Models and Approximations
5. Conclusion

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Dataset - Earth Surface Temperature Data

Reference: `https://www.kaggle.com/berkeleyearth/climate-change-earth-surface-temperature-data`

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   - Land average temperature
   - Monthly data
   - Missing Data and Accuracy
2. Temperatures for all countries of the world
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   - Monthly data ⇒ convert to yearly
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2. Temperatures for all countries of the world
   - Year of first data varies
3. Other Data available

Dataset - Earth Surface Temperature Data

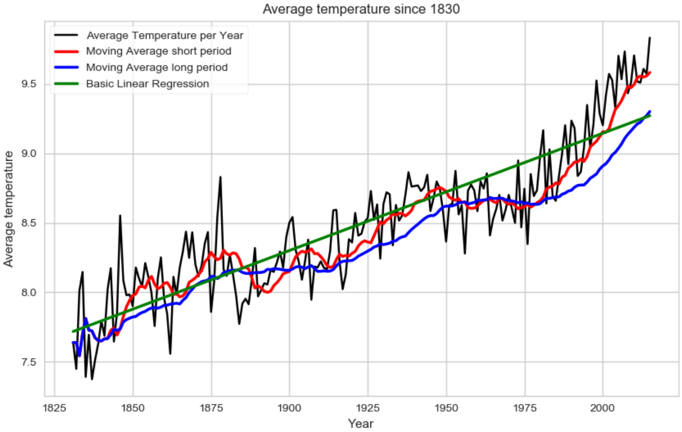

Figure 1: Yearly average temperature plotted together with moving averages and a basic linear regression line.

Models for analysis (I) - Linear Regression

- General setting:
  - $x = \text{Years} - \text{StartYear}$
  - $y = \text{Average temperature per year}$

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- General setting:
  - $x = \text{Years} - \text{StartYear}$
  - $y = \text{Average temperature per year}$
- Very basic Model:

$$y = b + ax + \epsilon, \quad (a, b) \sim \mathcal{N}(\mu, \Sigma) \quad \epsilon \sim \mathcal{N}(0, \sigma)$$

- Leads to a multivariate-Gamma distribution as a prior.
- Conjugate Model: can be solved analytically.

Models for analysis (II) - Polynomial Regression of degree 3

- Mathematical Model:

$$y = (a_0, a_1, a_2, a_3)\left(1, x, x^2, x^3\right)^T + \sigma S_d$$
$$(a_0, a_1, a_2, a_3) \sim \mathcal{N}(\mu, \Sigma), \quad \sigma \sim \mathrm{Exp}(\lambda), \quad d \sim \Gamma(k)$$

- Student Noise to account for outliers.
- More complex model but MLE solution known.
- σ and d need to be positive.

Models for analysis (III) - Linear Regression with Breaks

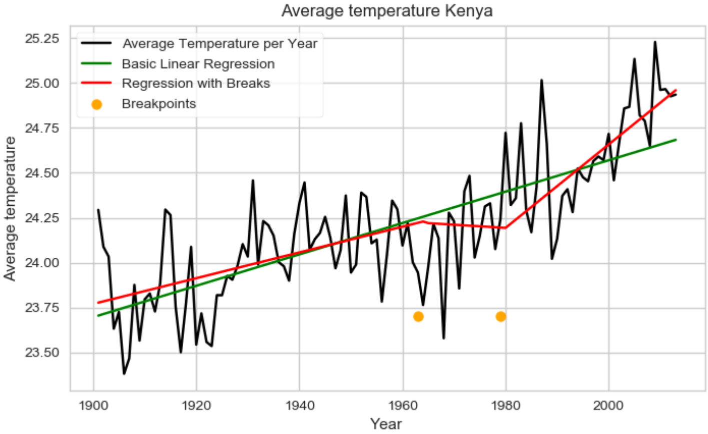

Figure 2: Example of Model III for the yearly average temperature in Kenya since 1900.

Models for analysis (III) - Linear Regression with Breaks

- Mathematical Model:

$$y = a_0 + \begin{cases} a_1 x & \text{if} \quad x < B_1 \\ a_1 B_1 + a_2 x & \text{if} \quad B_1 < x < B_2 \\ a_1 B_1 + a_2 B_2 + a_3 x & \text{if} \quad B_2 < x \end{cases} \quad + \sigma S_d$$

$$(a_0, a_1) \sim \mathcal{N}(\mu, \Sigma), \quad a_2, a_3 \sim \mathcal{N}(\mu, \tau) \quad \sigma \sim \mathrm{Exp}(\lambda), \quad d \sim \Gamma(k)$$

$$B_1 \sim \mathrm{Uniform}(0, \mathrm{Endpoint}), B_2 \sim \mathrm{Uniform}(B_1, \mathrm{Endpoint})$$

- Student Noise to account for outliers.
- σ and d need to be positive.
- B_1 and B_2 need to satisfy $0 \leq B_1 \leq B_2 \leq \mathrm{Endpoint}$.

Methods for approximation (I) - Laplace Approximation

- General Setting:
  - Unnormalised posterior $\tilde{f}(\theta|d)$
- Idea: $\tilde{f}(\theta|d) \sim C \exp\left(-\frac{1}{2}(\theta-\mu)^T \Sigma^{-1}(\theta-\mu)\right)$

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- Approximate mean as mode of distribution.

$$\mu = \bar{\theta} = \mathrm{argmax}_\theta \left(\tilde{f}\left(\bar{\theta}|d\right)\right)$$

- Approximate Covariance as log curvature at the mode.

$$\Sigma = \left(-H_\theta\left(\log \tilde{f}\left(\bar{\theta}|d\right)\right)\right)^{-1}$$

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- Approximate Covariance as log curvature at the mode.

$$\Sigma = \left(-H_\theta\left(\log\ \tilde{f}\left(\bar{\theta}|d\right)\right)\right)^{-1}$$

  - How to find the Hessian?
- It follow $C = \tilde{f}\left(\bar{\theta}|d\right)$.

Methods for approximation (II) - Gaussian Variational Approximation

- Idea: $f(\theta|d) \sim \mathcal{N}(\mu, \Sigma)$
  - Minimize $\mathrm{KL}(g(\theta), f(\theta|d))$
  - Maximise

$$ELBO(\mu, \Sigma) = -\mathbb{E}\left(\phi(\mu + \exp(L)\tau)\right) + \frac{d}{2}\log(2\pi e) + Tr(L),$$

$$L = \frac{\log(\Sigma)}{2}, \quad \theta = \mu + S_\Sigma \tau, \quad S_\Sigma S_{Sigma}^T = \Sigma$$

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- This requires $\nabla_\theta ELBO$ and $\nabla_L ELBO$.
- Use stochastic gradient descent to find optimal μ, L.
- Two possible methods have been implemented.

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  2. Generate proposal $P_{prop} = P_n + \lambda\eta_n$, where η_n need to be symmetric.
     1. Accept or reject proposal if probability rises.
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  3. Other proposals possible.
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- Stepsize is critical!

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- Not very good for forecasting because it is sluggish.

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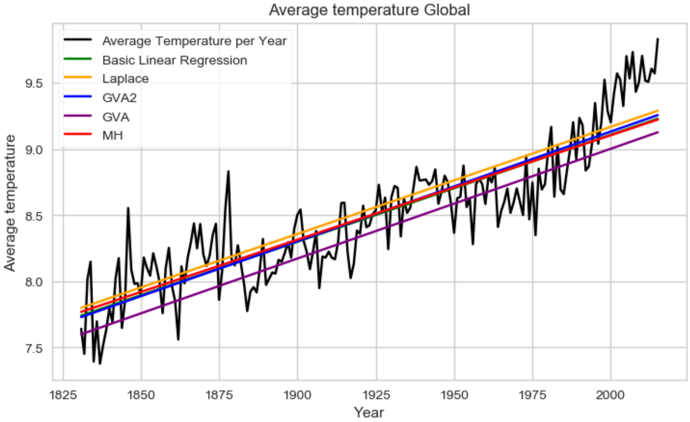
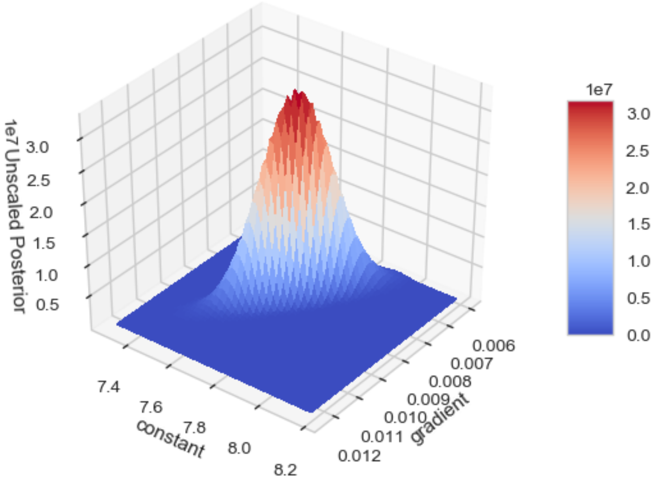


Figure 3: Basic Bayesian linear regression with different approximation methods.

Comparison of Models - Basic Linear Regression



Comparison of Models - Polynomial Regression

- MH yield very good results in a decent amount of time.
- Laplace and GVA take long to arrive at acceptable results.
- Variance for coefficients is very small.
  - Problems
    1. σ and d might not be normally distributed.
    2. Calamity of multimodality.
    3. Posterior is steep.
    4. Optimisation issues.

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- Not very good for forecasting because of overfitting.

Comparison of Models - Polynomial Regression

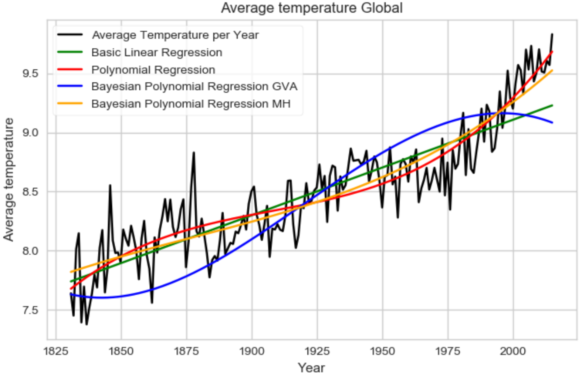

Figure 5: Result of polynomial regression for different approximation methods.

Comparison of Models - Polynomial Regression

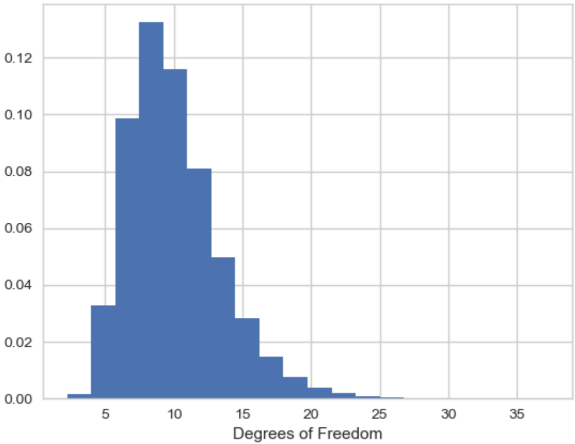

Figure 6: Histogram for parameter d obtained using the MH-algorithm.

Comparison of Models - Polynomial Regression

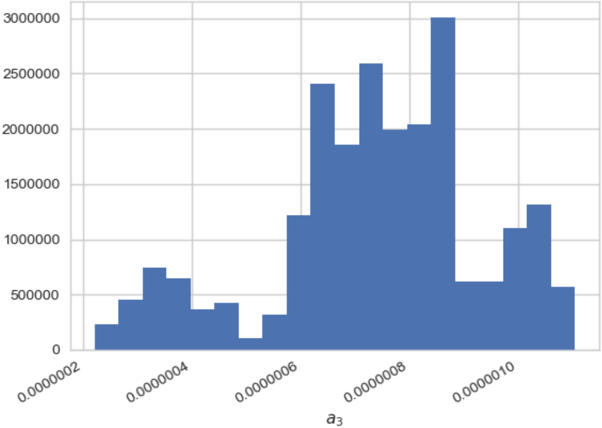

Figure 7: Histogram for coefficient of cubic term obatained using MH-algorithm with 1,000,000 samples.

Comparison of Models - Linear Regression with Breaks

- All methods yield acceptable results. MH has best performance.
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- Variance for coefficients is very small.
- Similar problems as for polynomial Regression but not as severe.
- Sum of Squares if less than for polynomial regression.
- Model is easier to compute than polynomial regression.
- Best model for forecasting.

Comparison of Models - Forecasting

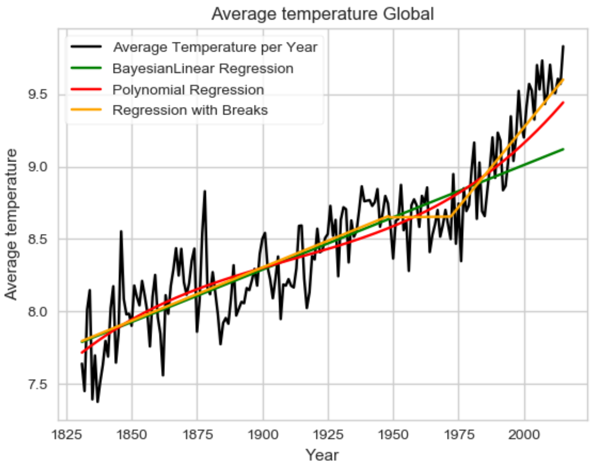

Figure 8: Different Models plotted when climate data until 1990 is considered.

Conclusions

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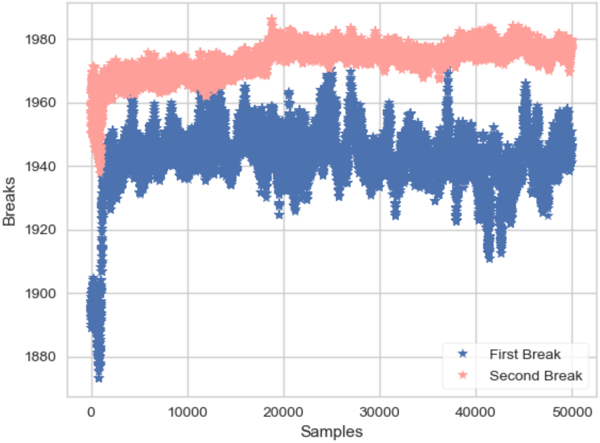

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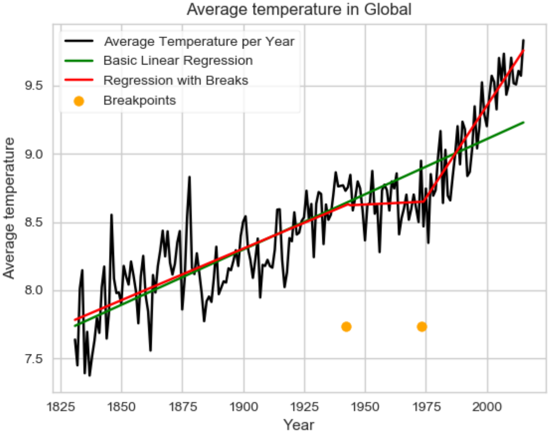
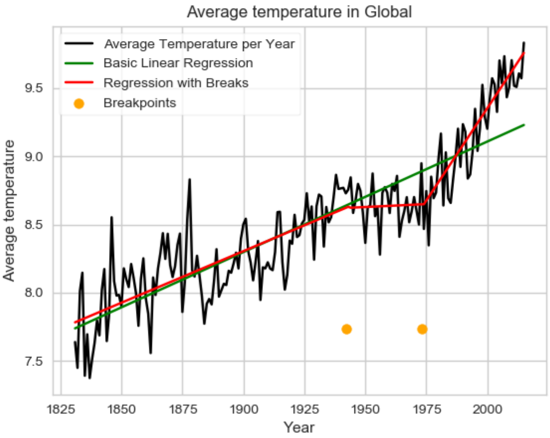
3. Model III - Linear Regression with breaks works best for the given data.

Conclusion



Average temperature in Global
Average Temperature per Year
Basic Linear Regression
Regression with Breaks
Breakpoints
Average temperature
9.5
9.0
8.5
8.0
7.5
1825
1850
1875
1900
1925
1950
1975
2000
Year

Conclusion



It allows to see the man-made climate change.