# ASTR 231: Problem Set 5

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**Problem 1.** Low mass white dwarfs have an equation of state that is independent of temperature, since they are entirely supported by electron degeneracy pressure. The fact that they are low mass means that relativistic effects do not need to be taken into account because the momenta of the supporting particles (electrons) are small enough. In the case of non-relativistic (complete) degeneracy, the equation of state can be written as:

$$P = K\rho^{\frac{5}{3}}$$

The value of K in this case is given in the Lecture Notes for Lecture 14.

a) Integrate the equations of stellar structure (hydrostatic equilibrium and mass conservation) to determine the mass and radius of a white dwarf that has a central density of $10^5$ gm $\mathrm{cm}^{-3}$.

b) Find the mean density of this star.

c) Give its mass in terms of solar masses and its radius in terms of both solar radii and Earth radii.

[Note: For the numerical integration it is fine to use a simple "Newton's method". You can vary the integration step size to see how it affects results. If any of you wish to do a more sophisticated integration, say using the Runga-Kutta or other method, you are most welcome to do so, but it is not necessary.]

**Answer 1.** a) In order to solve for the mass and radius of a given non-relativistic white dwarf, we must integrate the mass conservation and hydrostatic equilibrium equations, and the equation of state shown above. I use the following equations:

$$\frac{dM}{dR} = 4\pi R^2 \rho \tag{1}$$

and

$$\frac{dP}{dR} = -\frac{GM\rho}{R^2} \tag{2}$$

I also use the equation of state provided, where

$$K = \left(\frac{h^2}{5m_e}\right)\left(\frac{3}{8\pi}\right)^{\frac{2}{3}}\left(\frac{1}{2m_h}\right)^{\frac{5}{3}}$$

I integrate each of these simultaneously, giving each value a realistic starting value as if it were the very center of the star, and moving outwards in radius by steps of dr (Newton's method). My intial central density is the one given, my initial pressure is calculated using the equation of state above, and my initial mass and radius are 0. I write code where each of the equations described above are explicitly defined. Equation (1) above is called SphericalMass, it's a function of r and $\rho$, and it returns a value dM. Equation (2) above is called Hydrostatic, it's a function of M, r, and $\rho$, and it returns a value dP. I give each equation the most recent values of M, r, and $\rho$ as arguments to get the change in mass and pressure (dM and dP respectively), and solve for a new mass and pressure using my value dr as follows:

$$M_{new} = M_{last} + dr(dM)$$

$$P_{new} = P_{last} + dr(dP)$$

I use the new value for pressure to calculate my new central density, as follows:

$$\rho_{new} = \left(\frac{P_{new}}{K}\right)^{\frac{3}{5}}$$

I then iterate again, stepping by my value dr. For this problem, I found that a value dr = 9000 gave me enough points to make a meaningful mass estimate, without over-exerting my computer or causing long run times.

When I complete this integration, I find that the mass and radius of the white dwarf are:

$$M = 3.082 \text{ x } 10^{29} \text{ kg}$$

$$R = 1.58 \text{ x } 10^{4} \text{ km}$$

b) I find the mean density of this star by doing a simple mass over volume calculation, using the mass and radius I calculate in part a.

$$\rho_{mean} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\rho_{mean} = 1.8 \text{ x } 10^{7} \text{ kg m}^{-3}$$

$$\rho_{mean} = 1.8 \text{ x } 10^{4} \text{ g cm}^{-3}$$

c) Here, I simply convert my mass and radius to solar mass and radius, as well as Earth radius, using accepted values for the mass and radius of the sun, and the radius of the Earth. I get:

$$M = 0.15 \text{ M}_{\odot}$$

$$R = 0.02 \text{ R}_{\odot}$$

$$R = 2.48 \text{ R}_{Earth}$$

**Problem 2.** Now consider the case of a very high mass white dwarf, in which relativistic effects result in a somewhat different equation of state, namely the one applicable to complete ultra-relativistic degeneracy. From the lecture notes, we have in this case:

$$P = K\rho^{\frac{4}{3}}$$

Again, see the lecture notes for Lecture 14 to get the value of K, which is different from the value for non-relativistic degeneracy.

a) As before, integrate the equations of hydrostatic equilibrium and the mass equation to determine how the mass and radius of a star depend on central density over the range of $10^9$ gm cm$^{-3}$ to $10^{15}$ gm cm$^{-3}$. Note that those higher densities are about equal to the density of a neutron!

b) What is the maximum possible mass of a white dwarf star, according to your calculations? Note that you have calculated the Chandrasekhar mass limit for a white dwarf.

**Answer 2.** a) I do the integration for this problem just as I did for problem 1a, setting the initial central density differently each time so as to get a good range. The only change between the two is the value K, which changes to:

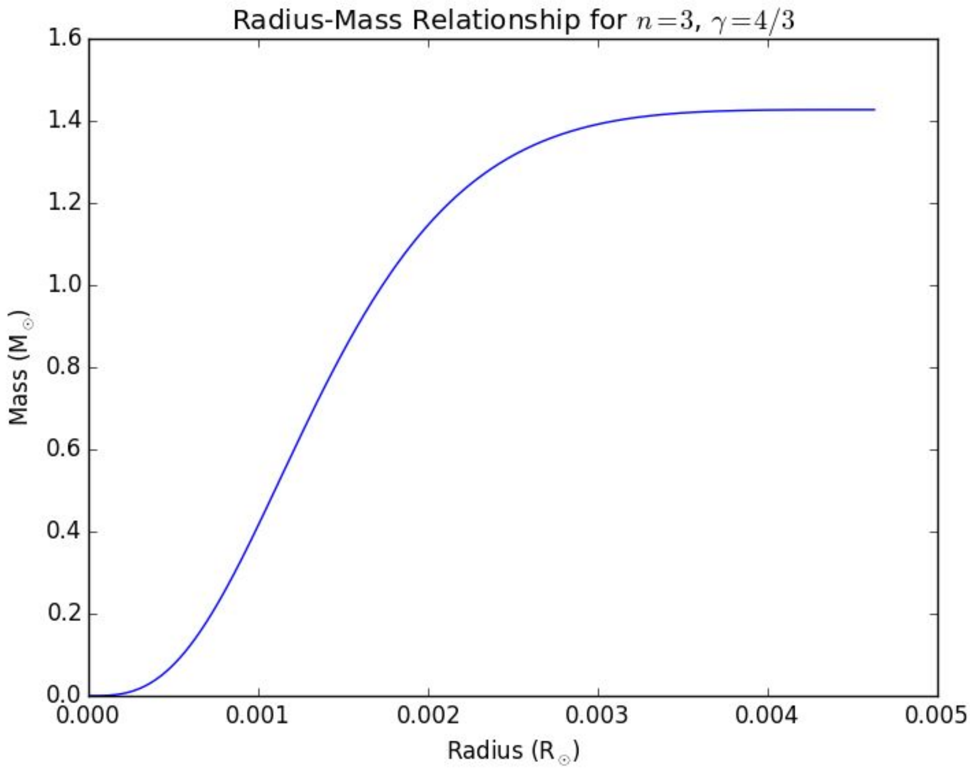
$$K = \left(\frac{h^2}{5m_e}\right)\left(\frac{3}{8\pi}\right)^{\frac{1}{3}}\left(\frac{1}{2m_h}\right)^{\frac{4}{3}}$$

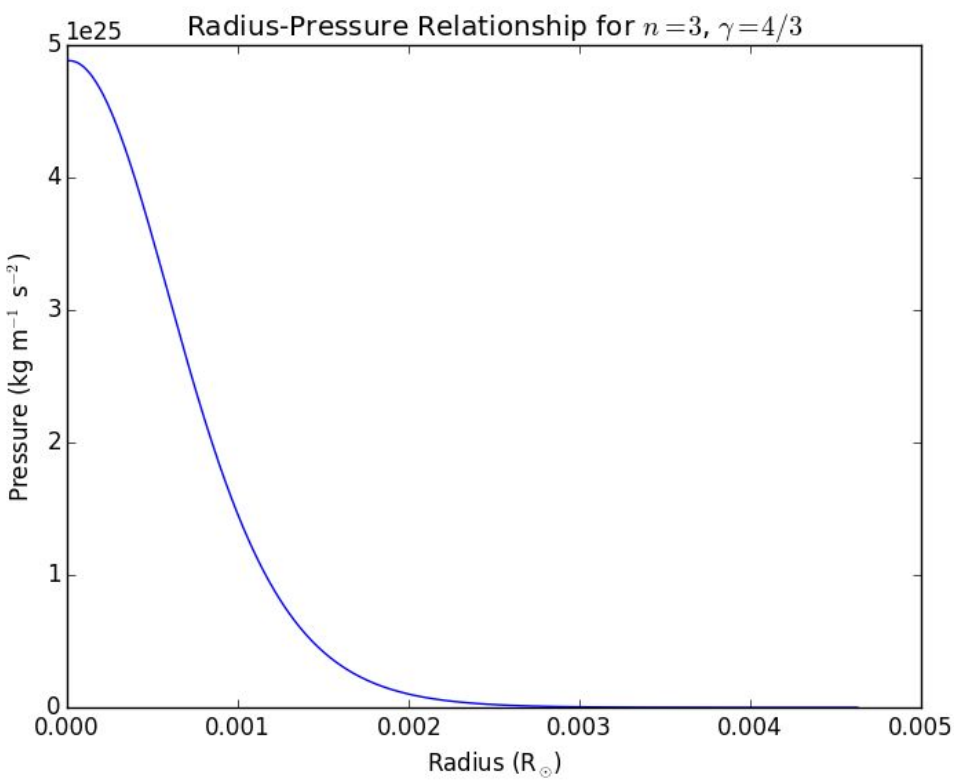
I do the calculation for $\rho = 10^9$, $10^{10}$, $10^{11}$, $10^{12}$, $10^{13}$, $10^{14}$, and $10^{15}$ g cm$^{-3}$. For the higher densities, I reduce my step size to allow for more points. The values I calculate for each can be found in the table below:

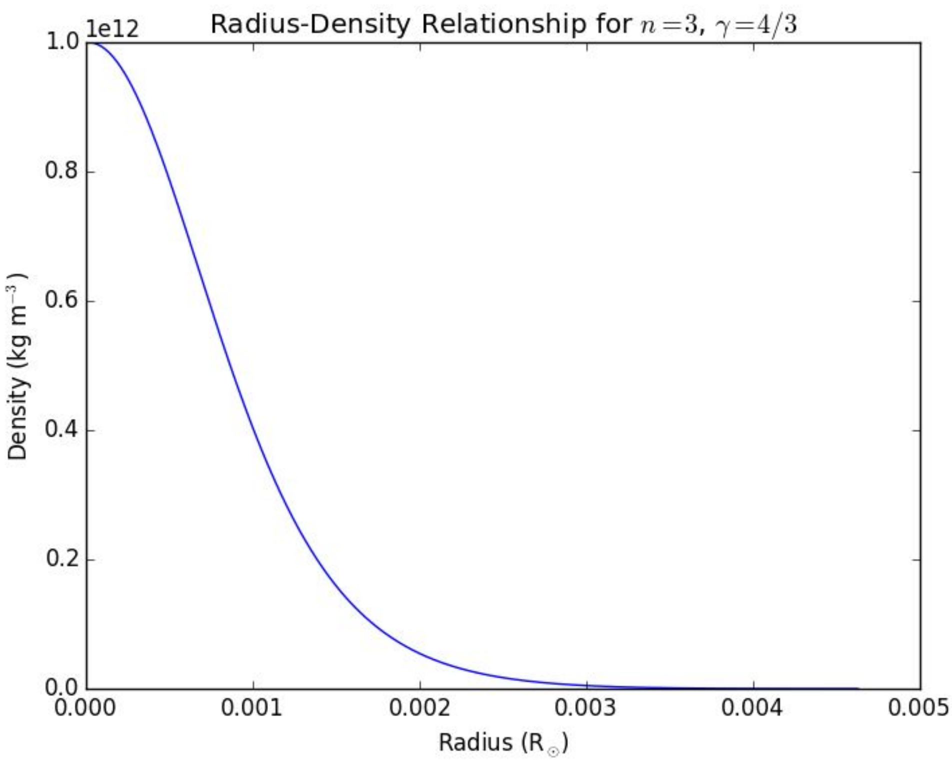
| $\rho$ (g cm$^{-3}$) | Mass ($M_\odot$) | Radius ($R_\odot$) |
|---|---|---|
| $10^9$ | 1.4 | 0.004 |
| $10^{10}$ | 1.4 | 0.002 |
| $10^{11}$ | 1.4 | 0.0008 |
| $10^{12}$ | 1.4 | 0.0004 |
| $10^{13}$ | 1.4 | 0.0002 |
| $10^{14}$ | 1.4 | 8.9 x $10^{-5}$ |
| $10^{15}$ | 1.4 | 3.5 x $10^{-5}$ |

b) We can see from these calculations that there is a maximum mass that a white dwarf can have, namely 1.4 $M_\odot$. This value that I calculate agrees with the accepted value of the Chandrasekhar mass limit for a white dwarf.

**Note:** For each of these, I also chose to plot my mass, radius, and density calculations at each step as a function of radius, just to ensure that the shape of the graph looked accurate. I used this as a final check for my integration (on top of making sure that the Chandrasekhar limit I calculated agreed with the accepted value). An example of those graphs (for $\rho = 10^9$ g cm$^{-3}$) are shown below. My code is also attached.


Radius-Mass Relationship for $n=3$, $\gamma=4/3$
Mass ($M_\odot$)
Radius ($R_\odot$)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
0.000
0.001
0.002
0.003
0.004
0.005


Radius-Pressure Relationship for $n=3$, $\gamma=4/3$
1e25
Pressure (kg $m^{-1}$ $s^{-2}$)
Radius ($R_\odot$)
0
1
2
3
4
5
0.000
0.001
0.002
0.003
0.004
0.005



Radius-Density Relationship for $n = 3$, $\gamma = 4/3$
1e12
1.0
0.8
0.6
0.4
0.2
0.0
Density (kg m$^{-3}$)
0.000
0.001
0.002
0.003
0.004
0.005
Radius ($R_\odot$)

```
# Homework 5
# Hannah Fritze

import numpy as np
import math
from scipy.integrate import quad
from matplotlib import pyplot as plt
import pandas as pd


######## Problem 1 ##########


def SphericalMass(r,rho):
    dm = 4.*np.pi*(r**2.)*rho
    return dm

def Hydrostatic(m,rho,r):
    dP = -G*m*rho/(r**2.)
    return dP

Msolar = 1.988 * 10.**30. #mass of the sun in kg
Rsolar = 6.957*10.**8. #radius of the sun in m
Rearth = 6.371008 * 10.**6. #radius of the Earth in meters
G = 6.67*10**(-11) #m^(3) kg^(-1) s^(-2)

h = 6.62607004*10.**(-34.) # m^(2) kg s^(-1)
me = 9.109383*10.**(-31) #mass of an electron in kg
mh = 1.674 * 10.**(-27.) #mass of hydrogen in kg
K = (h**2./(5.*me)) * ((3./(8.*np.pi))**(2./3.))*((1./(2.*mh))**(5./3.))
print K
i = 0
m = np.zeros(1000)
P = np.zeros(1000)
rho = np.zeros(1000)
r = np.zeros(1000)
r[0] = 0.00001
dr = 90000.
n = 3./2.
m[0] = 0.
rho[0] = 1.*10.**8. #kg m^(-3)
P[0] = K*rho[0]**(5./3.)
while P[i] > 0:
    m[i+1] = m[i] + dr*(SphericalMass(r[i],rho[i]))
    P[i+1] = P[i] + dr*Hydrostatic(m[i],rho[i],r[i])
    r[i+1] = r[i] + dr
    rho[i+1] = (P[i+1]/K)**(n/(n+1))
    i = i+1
m = m[:177]
r = r[:177]
P = P[:177]
rho = rho[:177]
m = m/Msolar
r = r/Rsolar
plt.figure(1)
plt.plot(r,rho)
```

```
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Density (kg m$^{-3}$)')
plt.title('Radius-Density Relationship for $n = 3/2$, $\gamma = 5/3$')
plt.show()

plt.figure(2)
plt.plot(r,P)
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Pressure (kg m$^{-1}$ s$^{-2}$)')
plt.title('Radius-Pressure Relationship for $n = 3/2$, $\gamma = 5/3$')
plt.show()

plt.figure(3)
plt.plot(r,m)
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Mass (M$_{\odot}$)')
plt.title('Radius-Mass Relationship for $n = 3/2$, $\gamma = 5/3$')
plt.show()

print "cutoff is: ", m.argmax()
print "the mass of this white in solar masses is dwarf is: ", max(m)
m = m*Msolar
print "the mass of this white dwarf in kg is: ", max(m)
print "the radius of this white dwarf in Solar radii is: ", max(r)
r = r*Rsolar
print "the radius of this white dwarf in meters is: ", max(r)
meandesnsity = max(m)/(4./3. * np.pi * (max(r)**3.))
print "mean desnity in kg per cubic meter is: ", meandesnsity
print "mean desnity in g per cubic centimeter is: ", meandesnsity/1000.
r = r/Rearth
print "the radius of this white dwarf in Earth radii is: ", max(r)

###### Problem 2 ########
i = 0.
m = np.zeros(5000)
P = np.zeros(5000)
rho = np.zeros(5000)
r = np.zeros(5000)
c = 2.998*10.**8. #speed of light in m s^(-1)
K =(h*c/4.) * (3./(8.*np.pi))**(1./3.)*(1./(2.*mh))**(4./3.)
r[0] = 0.000001
dr = 5000.
n = 3.
m[0] = 0.
rho[0] = 1.*10.**12. #kg m^(-3)
P[0] = K*rho[0]**((n+1.)/n)
while P[i] > 0:
    m[i+1] = m[i] + dr*(SphericalMass(r[i],rho[i]))
    P[i+1] = P[i] + dr*Hydrostatic(m[i],rho[i],r[i])
    r[i+1] = r[i] + dr
    rho[i+1] = (P[i+1]/K)**(n/(n+1))
    i = i+1

mplot = m/Msolar
rplot = r/Rsolar
print rplot
```

```
print "radius of the white dwarf for n=3 is :", max(rplot)
print "mass of the white dwarf for n= 3 is: ", max(mplot)
rhoplot = rho[:645]
Pplot = P[:645]
mplot = mplot[:645]
rplot = rplot[:645]
plt.figure(4)
plt.plot(rplot,rhoplot)
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Density (kg m$^{-3}$)')
plt.title('Radius-Density Relationship for $n = 3$, $\gamma = 4/3$')
plt.show()

plt.figure(5)
plt.plot(rplot,Pplot)
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Pressure (kg m$^{-1}$ s$^{-2}$)')
plt.title('Radius-Pressure Relationship for $n = 3$, $\gamma = 4/3$')
plt.show()

plt.figure(6)
plt.plot(rplot,mplot)
plt.xlabel('Radius (R$_{\odot}$)')
plt.ylabel('Mass (M$_{\odot}$)')
plt.title('Radius-Mass Relationship for $n = 3$, $\gamma = 4/3$')
plt.show()
```