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Theory of stellar convection: removing the mixing-length parameter

S. Pasetto,[1★] C. Chiosi,[2] M. Cropper[1] and E. K. Grebel[3]

[1] *Department of Space & Climate Physics, University College London, Mullard Space Science Laboratory, Holmbury St. Mary, Dorking Surrey, UK*
[2] *Department of Physics & Astronomy, 'Galileo Galilei', University of Padua, I-35122 Padova, Italy*
[3] *Astronomisches Rechen-Institut, Zentrum für Astronomie der Universität Heidelberg, Mönchhofstr. 12-14, D-69120 Heidelberg, Germany*

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ABSTRACT

Stellar convection is customarily described by Mixing-Length Theory, which makes use of the mixing length-scale to express the convective flux, velocity, and temperature gradients of the convective elements and stellar medium. The mixing length-scale is taken to be proportional to the local pressure scaleheight, and the proportionality factor (the mixing-length parameter) must be determined by comparing the stellar models to some calibrator, usually the Sun. No strong arguments exist to suggest that the mixing-length parameter is the same in all stars and at all evolutionary phases. The aim of this study is to present a new theory of stellar convection that does not require the mixing-length parameter. We present a self-consistent analytical formulation of stellar convection that determines the properties of stellar convection as a function of the physical behaviour of the convective elements themselves and of the surrounding medium. This new theory is formulated starting from a conventional solution of the Navier–Stokes/Euler equations, i.e. the Bernoulli equation for a perfect fluid, but expressed in a non-inertial reference frame comoving with the convective elements. In our formalism, the motion of stellar convective cells inside convectively unstable layers is fully determined by a new system of equations for convection in a non-local and time-dependent formalism. We obtain an analytical, non-local, time-dependent subsonic solution for the convective energy transport that does not depend on any free parameter. The theory is suitable for the outer convective zones of solar type stars and stars of all mass on the main-sequence band. The predictions of the new theory are compared with those from the standard mixing-length paradigm for the most accurate calibrator, the Sun, with very satisfactory results.

Key words: Sun: fundamental parameters – Sun: interior – stars: evolution – stars: fundamental parameters.

1 INTRODUCTION

In stellar interiors convection plays an important role: together with radiation and conduction, it transports energy throughout a star, and it chemically homogenizes the regions affected by convective instability. Therefore, convection significantly affects the structures and evolutionary histories of stars. For example, the centre of main-sequence stars slightly more massive than the Sun and above is dominated by convective transport of energy. In stars less massive than about 0.3 $M_\odot$ the whole structure becomes fully convective. The outer layers of stars of any mass are convective towards the surface. Very extended convective envelopes exist in red-giant-branch and asymptotic-giant-branch stars. Pre-main-sequence stars are fully convective along the Hayashi line. Finally, convection is present in the pre-supernova stages of Type I and II supernovae, and even during the collapse phase of Type II supernovae (e.g. Meakin & Arnett 2007; Arnett & Meakin 2011; Arnett, Meakin & Viallet 2014; Smith & Arnett 2014). In most cases, convection in the cores and inner shells does not pose serious difficulties to our understanding of the structure of the stars because the large thermal capacity of convective elements results in the degree of 'superadiabaticity' being so small that for any practical purpose the temperature gradient of the medium in the presence of convection can be set equal to the adiabatic value, unless evaluations of the velocities and distances travelled by convective elements are required, e.g. in presence of convective overshooting (see for instance the early studies by Maeder 1975a,b; Bressan, Chiosi & Bertelli 1981). Describing convection in the outer layers of a star is by far more difficult and uncertain. Convective elements in this region have low thermal capacity, so that the superadiabatic approximation can no longer be applied, and the temperature gradient of the elements and surrounding medium must be determined separately to exactly know the amount of energy carried by convection and radiation (e.g. Weiss et al. 2004; Kippenhahn, Weigert & Weiss 2013).

A suitable description of convection is therefore essential to determine stellar structure. The universally adopted solution is the

★ E-mail: s.pasetto@ucl.ac.uk

Mixing-Length Theory (MLT) of convection, a simplified analytical formulation of the problem. The MLT stands on the works of Biermann (1951) and Böhm-Vitense (1958) which are based on earlier works on the concept of convective motion by Prandtl (1925). In this standard approach, the motion of convective elements is related to the mean free path l_m that a generic element is *supposed* to travel at any given depth inside the convectively unstable regions of a star (e.g. Kippenhahn & Weigert 1994, chapter 7). The mean free path l_m is assumed to be proportional to the natural distance scale h_P given by the pressure stratification of the star. The proportionality factor is however poorly known and constrained. The mixing-length (ML) parameter Λ_m, defined by $l_m \equiv \Lambda_m h_P$, must be empirically determined. Nevertheless, the knowledge of this parameter is of paramount importance in correctly determining the convective energy transport, and hence the radius and effective temperature of a star. This critical situation explains the many versions of convection theory that can be found when investigated in different regions and evolutionary phases of a star such as the overshooting from core or envelopes zones (e.g. Bressan et al. 1981; Claret 2007; Deng & Xiong 2008), the helium semi-convection in low- and intermediate-mass stars $m < 5\,\mathrm{M}_\odot$ (e.g. Castellani et al. 1985; Bressan et al. 1993), the time-dependent convection in the carbon deflagration process in Type I supernovae (e.g. Nomoto, Sugimoto & Neo 1976), the studies on the efficiency of convective overshooting (e.g. Bressan et al. 2013), and the effects of rotation (e.g. Maeder, Georgy & Meynet 2008) to mention just a few.

Examining the classical formulation of the MLT presented in any textbook, see for instance Hofmeister, Kippenhahn & Weigert (1964), Cox & Giuli (1968) and their modern versions (Kippenhahn et al. 2013; Weiss et al. 2004, respectively), we note that the MLT reduces to the energy conservation principle supplemented by an estimate of the mean velocity of convective elements. In a convective region the total energy flux (φ) is the sum of the convective flux (φ_{cnv}) and the radiative flux (φ_{rad}); the total flux is set proportional to a fictitious radiative gradient ∇_{rad} [1] (which is always known once the total flux coming from inside is assigned, typically case in stellar interiors); the true radiative flux φ_{rad} is proportional to the real gradient of the medium ∇; and the convective flux φ_{cnv} is proportional to the difference between the gradient of the convective elements and the gradient of the medium ($\nabla_e - \nabla$). By construction, the convective flux is also proportional to the mass of an ideal convective element, i.e. the amount of matter crossing the unit area per unit time with the mean velocity of convective elements. These elements may have any shape, mass, velocity and lifetime, and may travel different distances before dissolving into the surrounding medium, releasing their energy excess and inducing mixing in the fluid. However, all this ample variety of possibilities is simplified to an ideal element of averaged dimensions, lifetime, mean velocity and distance travelled before dissolving: the so-called ML l_m (and associated ML parameter Λ_m). As far as the velocity is concerned, this is estimated from the work done by the buoyancy force over the distance l_m, a fraction of which is supposed to go into kinetic energy of the convective elements. Since in this problem the number of unknowns exceeds the number of equations (flux conservation and velocity), two more suitable relations are usually added. These are firstly the ratio between the excess of energy in the bubble just before dissolving, to the energy radiated away (lost) during the lifetime, and secondly the excess rate of energy generation minus the excess rate of energy loss by radiation in the element relative to the surroundings. These are all functions of ∇, ∇_e and ∇_{ad}, see e.g. Cox & Giuli (1968). Now the number of unknowns, i.e. φ_{rad}, φ_{cnv}, ∇, ∇_e, is equal to the number of equations and the problem can be solved once l_m or Λ_m are assigned. In this way the complex fluid-dynamic situation is reduced to an estimate of the mean element velocity simply derived from the sole buoyancy force, neglecting other fluid-dynamic forces that can shape the motion of convective elements as function of time and surrounding medium.

We present here a new description of stellar convection that provides a simple and yet dynamically complete fully analytical integration of the hydrodynamic equations, matching the existing literature results based on the classical MLT, but without making use of any ML parameter Λ_m.

The plan of the paper is as follows. In Section 2, we formulate the problem within the mathematical framework we intend to adopt. In Section 3, we define the concept of a scalar field of the velocity potential for expanding/contracting convective elements. In Sections 4 and 5 we formulate the equation governing the two degrees of freedom of our dynamical system: Section 4 formulates the equation of motion (EoM) for a convective element as seen by a non-inertial frame of reference comoving with it, and presents two lemmas that are functional to our aim; in Section 5 we solve the EoM of a convective element expressed in the comoving frame of reference. In Section 6, we present the predictions of our theory. First, we formulate the basic equations of stellar convection showing that the ML parameter is no longer required. Then we apply the new formalism to the case of the Sun. Finally, in Section 7, we present some concluding remarks highlighting the novelty and the power of the new theory.

2 FORMULATION OF THE PROBLEM

Inside a convective unstable layer, upward (downward) displacements of convective cells continually occur. The upwardly displaced elements are hotter and lighter than their surroundings, at the same pressure, so that heat exchange results in energy release to the surrounding interstellar medium. For downward displacements the result is the opposite: convective cells sink when they have lower temperatures than their surroundings, and are heated on some length-scale. A mathematical formalism for this process is presented in Section 6.1. Here we focus on the motion of a single convective element.

Our starting point is represented by the classical solution of the Navier–Stokes equations for an incompressible perfect fluid where no electromagnetic forces are taken into account (e.g. Chandrasekhar 1961). We approach the mechanics of the convection by approximating the stellar fluid as a perfect fluid, i.e. a fluid of density ρ in which a suitable equation of state (EoS) links density $\rho = \rho(P, T, \mu)$ with pressure $P = P(\boldsymbol{x}; t)$, temperature $T = T(\boldsymbol{x}; t)$ and molecular weight $\mu = \mu(\boldsymbol{x}; t)$ at a given instant t and position $\boldsymbol{x}$ inside a star (see also footnote 3). Perfect fluids are intrinsically unstable and turbulent, therefore the higher the Reynolds numbers characterizing the fluid the better the above approximation holds. It is well known (e.g. Chandrasekhar 1961) that in stellar interiors where turbulence prevails over viscosity in the Navier–Stokes equations, the inertial term $\langle \rho \boldsymbol{v}_0, \nabla_{\boldsymbol{x}} \boldsymbol{v}_0 \rangle$ dominates over the

[1] Throughout the paper, we will introduce several logarithmic temperature gradients with respect to pressure $\frac{\mathrm{d}\log T}{\mathrm{d}\log P}$, shortly indicated as ∇. Each of these gradients is also identified by a subscript such as ∇_e, ∇_ξ, ∇_{ad}, ∇_{rad} depending on the circumstances. Finally, the symbol ∇ with no subscript is reserved for the ambient temperature gradient with respect to pressure across a star.

viscous one, $-\eta\underline{\Delta}_x\boldsymbol{v}_0$. Here $\boldsymbol{v}_0$ is the stellar fluid velocity, η the viscosity coefficient, $\langle *, * \rangle$ the standard innerproduct between two generic vectors, and ∇_x and $\underline{\Delta}_x$ are the gradient and Laplacian operators, respectively, for an inertial reference system of coordinate $S_0(O, \boldsymbol{x})$ centred in O at the centre of the star with direction vector $\boldsymbol{x}$. Then, if in the EoM) for the stellar plasma we neglect the contribution of the magnetic field $\boldsymbol{B}$, i.e. the term $\frac{\boldsymbol{j}}{c} \times \boldsymbol{B}$ where $\boldsymbol{j} = \rho\boldsymbol{v}_0$ is the charge-current-density (and $* \times *$ is the cross-product between two generic vectors) the corresponding Euler's equation, $\frac{\partial \rho \boldsymbol{v}_0}{\partial t} + \langle \nabla, \boldsymbol{P} + \rho\boldsymbol{v}_0\boldsymbol{v}_0 \rangle - \rho \sum_i n_i \boldsymbol{F}_i = 0$, together with the continuity equation, $\frac{\partial \rho}{\partial t} + \langle \nabla, \rho\boldsymbol{v}_0 \rangle = 0$ and accounting for the relation $\langle \nabla, \rho\boldsymbol{v}_0\boldsymbol{v}_0 \rangle = \boldsymbol{v}_0\rho\nabla \cdot \boldsymbol{v}_0 + \rho\boldsymbol{v}_0 \cdot \nabla\boldsymbol{v}_0$, reads

$$\rho\frac{\partial \boldsymbol{v}_0}{\partial t} + \langle \nabla_x, \boldsymbol{P} \rangle + \langle \rho\boldsymbol{v}_0, \nabla_x\boldsymbol{v}_0 \rangle - \sum_i n_i \boldsymbol{F}_i = 0. \tag{1}$$

In equation (1), $\boldsymbol{P}$ is the pressure tensor, $\boldsymbol{F}$ the force acting on every particle of the fluid, n_i the number density of every type of fluid particle (with the above assumption that no electric field $\boldsymbol{E}$ enters the plasma EoM). This is a partial differential equation (PDE) where the quantities involved, say Q, are functions of time t and position $\boldsymbol{x}$, $Q = Q(\boldsymbol{x}; t)$ in the given inertial reference frame $S_0(O, \boldsymbol{x})$. Hereafter, we omit writing this dependence explicitly to simplify the notation (unless specified otherwise for the sake of better understanding). Stellar interiors on macroscopic scale are well represented by a perfect fluid in local thermodynamical equilibrium, i.e. each elemental component, n_i of the fluid is isotropic, homogeneous, in mechanical equilibrium and obeying the conditions of detailed balance with any other component n_j. Therefore, we can then simplify the pressure tensor to a scalar $\langle \nabla_x, \boldsymbol{P} \rangle = \nabla_x P$, and because the force acting on the fluid particle is non-diffusive, i.e. in our case the gravity $\boldsymbol{F}_i = m_i\boldsymbol{g}$ on the particles of the ith species, we assume that $\sum_i n_i \boldsymbol{F}_i = \sum_i m_i n_i \boldsymbol{g} = (\sum_i m_i n_i)\boldsymbol{g} = \rho\boldsymbol{g}$. All this further simplifies equation (1) to

$$\rho\frac{\partial \boldsymbol{v}_0}{\partial t} + \nabla_x P + \langle \rho\boldsymbol{v}_0, \nabla_x\boldsymbol{v}_0 \rangle - \rho\boldsymbol{g} = 0. \tag{2}$$

We proceed further with an additional simplification by assuming that the stellar fluid is incompressible and irrotational on large distance scales. The concept of a large distance scale for incompressibility and irrotationality is defined here from a heuristic point of view: This length should be large enough to contain a significant number of convective elements so that a statistical formulation is possible when describing the mean convective flux of energy (see below), but small enough so that the distance travelled by the convective element is short compared to the typical distance over which significant gradients in temperature, density, pressure, etc. can develop (i.e. those gradients are locally small). These assumptions stand at the basis of every stellar model integration in the literature, and are fully compatible with making use of the simple concept of a potential flow (e.g. Landau & Lifshitz 1959, chapter 1): $\nabla_x \times \boldsymbol{v}_0 = 0 \Leftrightarrow \exists\Phi_{v_0} \mid \boldsymbol{v}_0 = \nabla_x\Phi_{v_0}$ with Φ_{v_0} the velocity potential. In particular, with the help of the vector relation $\langle \boldsymbol{v}_0, \nabla_x\boldsymbol{v}_0 \rangle = \frac{1}{2}\nabla_x\langle \boldsymbol{v}_0, \boldsymbol{v}_0 \rangle - \boldsymbol{v}_0 \times (\nabla_x \times \boldsymbol{v}_0)$ and remembering that the curl of a gradient is null, $\nabla_x \times \boldsymbol{v}_0 = \nabla_x \times \nabla_x\Phi_{v_0} = 0$, we are able to write equation (2) as $\nabla_x(\frac{\partial\Phi_{v_0}}{\partial t} + \frac{P}{\rho} + \frac{\|\boldsymbol{v}_0\|^2}{2} + \Phi_g) = 0$ where the relation between gravitational force and gravitational potential $\boldsymbol{g} = -\nabla_x\Phi_g$ has been adopted. The symbol $\|*\|$ indicates the standard Euclidian norm of a generic vector. Finally, the integration of the previous equation leads to the Bernoulli's equation in an inertial reference system S_0 centred at the centre of the star. With the formalism developed here

$$\frac{\partial\Phi_{v_0}}{\partial t} + \frac{P}{\rho} + \frac{\|\boldsymbol{v}_0\|^2}{2} + \Phi_g = f(t). \tag{3}$$

This is one of the basic equations describing the stellar plasma in which convection is at work. A more complete treatment would include diffusion and turbulence. However, as the main goal here is to derive the mechanics of convection from simple principles, the present approach is adequate for our aims. Diffusion and turbulence can eventually be included using the same formalism in a future study. In the context of thermal convection, it is worth recalling that the Boussinesq (Spiegel & Veronis 1960) and anelastic (Gough 1969) approximations would be valuable alternatives worth being investigated. Nevertheless, for the aims of this study the potential flow approximation turns out to be fully satisfactory at an extremely high degree of precision as our numerical investigation in Section 6.3 will confirm.

After these preliminary remarks, we are now in the position to state the queries that we intend to address as follows: the main target of stellar convection is to find a solution of equation (3) linking the physical quantities characterizing the stellar interiors such as pressure, density, temperature, velocities, etc. and the mechanics governing the motion of the convective elements as functions of the fundamental temperature gradients with respect to pressure, i.e. the radiative gradient $\nabla_{\rm rad}$, the adiabatic gradient $\nabla_{\rm ad}$, the local gradient of the star $\nabla \equiv |\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}|$, the convective element gradient ∇_e and the molecular weight gradient $\nabla_\mu \equiv |\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}|$.

The task is difficult because of the large number of variables involved to describe the physics of the convective element and of the stellar interiors, both of which poorly known. Mathematically, the problem translates into a system of algebraic differential equations (ADEs). In the MLT, the solution of this ADE is simplified to an algebraic system of equations by introducing a statistical description of the motion, size, lifetime, etc. of the convective elements. In this way, the complicated pattern of possible convective elements is reduced to a mean element whose dimensions and path are simply supposed to be $l_m = \Lambda_m h_P$, where Λ_m is a parameter to be fixed by comparing real stars (the Sun) to stellar models. Once Λ_m is calibrated is this way, it is assumed to be the same for all stars of any mass, chemical composition, and evolutionary stage. This is indeed a strong assumption.

In what follows we propose and formulate an alternative approach to this theory and apply it to recover well-established results of the theory of stellar structure and observational properties of our best calibrator, the Sun, but without making use of any adjustable parameter.

The approach is based on the addition of an equation for the motion of the convective elements to the classical system of algebraic equations for the convective energy transport. The whole system of algebraic/differential equations (DEs) is solved by considering together the evolution of the generalized coordinates (i.e. independents or Lagrangian coordinates associated with the degree of freedoms) of the radius and position of a convective element. This result is achieved by means of a series of theorems, corollaries and lemmas that analyse the different physical and mathematical aspects of the problem.

Before starting our analysis, in order to avoid a possible misunderstanding of the real meaning of some of our analytical results, it might be wise to call attention to a formal aspect of the mathematical notation we have adopted. For some quantities Q function of time or space or both, $Q(\boldsymbol{x};t)$, we look at their asymptotic behaviour by formally taking the limits

$$Q^{\infty} \equiv \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^{\infty}\\ t\to\infty}} Q\left(\boldsymbol{x};t\right) = Q\left(\boldsymbol{x}^{\infty};\infty\right). \tag{4}$$

This does not mean that we are taking temporal intervals infinitely long, rather that we are considering time long enough so that the asymptotic trend of the quantity Q is reached but still short enough so that the physical properties of the whole system have not changed significantly, such as that the star still exists. In analogy, with the notation $\boldsymbol{x}^{\infty}$ we refer to a location far away from the system considered (e.g. the convective element in consideration) but at a given location $\boldsymbol{x} = \boldsymbol{x}^{\infty}$ that is still *inside* the star, i.e. where $\rho = \rho(\boldsymbol{x}^{\infty}) \neq 0$, $T = T(\boldsymbol{x}^{\infty}) \neq 0$, etc.

3 VELOCITY-POTENTIAL SCALAR FIELD FOR EXPANDING/CONTRACTING CONVECTIVE ELEMENTS

The formalism we will develop refers to stars in hydrostatic equilibrium under the effect of their own gravity[2] which applies to the vast majority. We limit ourselves to consider the onset of convection either in central cores, intermediate shells or external envelopes in the same conditions usually described by the classical MLT.

In addition to the natural inertial reference frame S_0 whose origin is fixed at the centre of the star (at rest by definition), we now introduce a non-inertial reference frame comoving with a generic convective element. The new system is named S_1: $(O', \boldsymbol{\xi})$, origin and position vector respectively, to distinguish it from the inertial reference frame S_0. Even though at first glance, this approach may look awkward, because the most intuitive way of thinking about the motion of a body (in our case a convective element) is the translational motion with respect to the static observer, we show that this way of thinking yields the desired mathematical expression for the EoM of the convective element and eventually allows us to eliminate the ML parameter.

Assuming spherical symmetry, in S_0 we define the equation of a generic convective element of radius r_e by means of the time-dependent relation $\|\boldsymbol{r}\| - r_e(t) = 0$ because the element is expected to expand/contract and rise/sink during its lifetime evolution, and where we indicated with $\|\boldsymbol{r}\| = \|\boldsymbol{x} - \boldsymbol{x}_{O'}\|$ the radius vector of the element centred on $\boldsymbol{x}_{O'}$ at the instant $t = \hat{t}$. In S_1, we identify a point in the surface of the convective element by $\boldsymbol{\xi}_e$, then the radius of the convective element is $\xi_e \equiv \|\boldsymbol{\xi}_e\|$. Note that $\xi_e = \xi_e(t)$. This is shown in Fig. 1, where the relation between the position vector in S_0 and S_1 is shown.

We define two velocity potentials, as introduced in Section 1, denoted by $\Phi^{I}_{\boldsymbol{v}_0}$ and $\Phi^{II}_{\boldsymbol{v}_0}$. $\Phi^{I}_{\boldsymbol{v}_0}$ is defined to satisfy the Laplace equation $\Delta_x \Phi^{I}_{\boldsymbol{v}_0} = 0$ in S_0 in the case when the element is at rest, and this has particular solution $\Phi^{I}_{\boldsymbol{v}_0} = -\frac{\dot{r}_e r_e^2}{\|\boldsymbol{r}\|}$ (e.g. Landau & Lifshitz 1966). The potential velocity $\boldsymbol{v}^{I}_0 = v_r \hat{\boldsymbol{e}}_r = \frac{\partial \Phi^{I}_{\boldsymbol{v}_0}}{\partial r}\hat{\boldsymbol{e}}_r = \frac{\partial}{\partial r}(-\frac{\dot{r}_e r_e^2}{\|\boldsymbol{r}\|})$ $\hat{\boldsymbol{e}}_r = \frac{\dot{r}_e r_e^2}{\|\boldsymbol{r}\|^2}\hat{\boldsymbol{e}}_r$. Here the dot indicates the time derivative and $\hat{\boldsymbol{e}}_r$ is the

[2]It has long been known that the very external regions of some types of stars, e.g. pulsating stars, may deviate from rigorous hydrostatic equilibrium. Furthermore, some stars may experience evolutionary phases far from hydrostatic equilibrium, e.g. the collapsing core of massive stars or the accretion phase of protostars. In both cases, convection may set in. Although these situations would represent an interesting field of investigation, these objects are not considered in the present study.

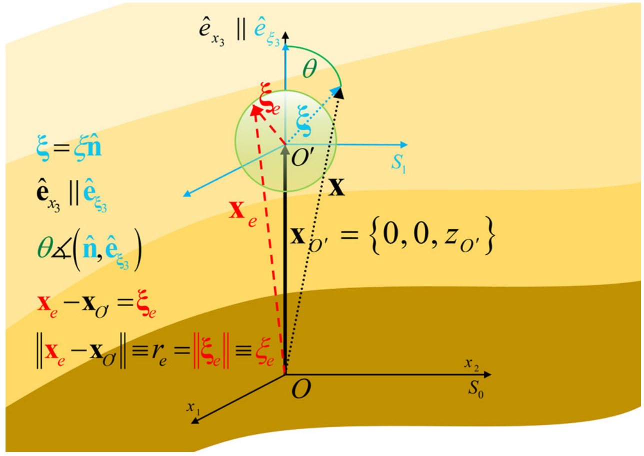

Figure 1. Schematic representation of the quantities involved in the two reference systems introduced in the text. The star is supposed to be stratified in hydrostatic equilibrium layers (each of the monochromatic background colours) and the convective elements are living inside these layers. In green is an example of an expanding convective element. The reference frame S_0 (in black) is fixed at the centre of the star (supposed to be at rest or in translational motion), and the frame S_1 (in blue) is the non-inertial system of reference and it is located with the convective element along the z-axis of the S_0 system of reference, $\boldsymbol{x}_{O'} = \{0, 0, z_{O'}\}$. The geometry employed is completely general and derived from a simplified version of fig. 1 of Pasetto & Chiosi (2009). The position vectors of the two system of reference are in black for the system S_0 and blue for S_1, respectively; they take red colour only when they refer to surface of the sphere instantaneously matching the location and size of the convective element considered. In green it is the angle between the direction of motion and the S_1 position vector.

normal radial vector of a polar coordinate system centred on $\boldsymbol{x}_O$. The boundary conditions for Dirichlet's problem are (i) far away from the convective element $\lim_{\|\boldsymbol{r}\|\to\infty} \boldsymbol{v}^{I}_0 = 0$, and (ii) the kinematic boundary conditions at the surface of the sphere the surrounding plasma cannot flow through it, i.e. the velocity has to be purely tangential so that velocity component locally perpendicular to the surface must be zero $\langle \boldsymbol{v}^{I}_0, \hat{\boldsymbol{e}}_r\rangle_{r_e} = 0$.

When the convective element is moving with velocity $\boldsymbol{v}$ in S_0, the kinematic boundary condition is replaced by the relative velocity between the fluid and the element in motion throughout the stellar plasma $\langle \boldsymbol{v}_0 - \boldsymbol{v}, \hat{\boldsymbol{e}}_r\rangle = 0$ at $\|\boldsymbol{r}\| = r_e$ to get the more general result for the potential flow $\Phi^{II}_{\boldsymbol{v}_0}$, where $\Phi^{II}_{\boldsymbol{v}_0} = -\frac{1}{2}\frac{r_e^3}{\|\boldsymbol{r}\|^2}\langle \boldsymbol{v}, \hat{\boldsymbol{e}}_r\rangle$ (to distinguish it from $\Phi^{I}_{\boldsymbol{v}_0}$) with velocity given by $\boldsymbol{v}^{II}_0 = \nabla_x \Phi^{II}_{\boldsymbol{v}_0} = \frac{r_e^3}{2\|\boldsymbol{r}\|^3}(3\langle \boldsymbol{v}, \hat{\boldsymbol{e}}_r\rangle\hat{\boldsymbol{e}}_r - \boldsymbol{v})$ (e.g. Landau & Lifshitz 1966).

We move now our point of view to the non-inertial reference system S_1 comoving with the convective element and with its axes always aligned with S_0: i.e. $\boldsymbol{x} = \boldsymbol{x}_{O'} + \boldsymbol{\xi}$ to get $\Phi'^{II}_{\boldsymbol{v}_0} = -\langle \boldsymbol{v}, \boldsymbol{\xi}\rangle(1 + \frac{1}{2}\frac{\xi_e^3}{\|\boldsymbol{\xi}\|^3})$ for the potential flow past the convective element. This is obtained by simply superimposing a translational potential vector, say $\Phi^{t}_{\boldsymbol{v}_0} = -\langle \boldsymbol{v}, \boldsymbol{\xi}\rangle$ to the classical potential vector of the flow past a sphere in S_0. Φ' indicates the potential flow passing the sphere at rest in S_1 [centred in O' located at $\boldsymbol{x}_{O'} = \boldsymbol{x}_{O'}(t)$ in S_0] and with a radius vector $\boldsymbol{\xi}(t) = \boldsymbol{x} - \boldsymbol{x}_{O'}(t)$. Now the fluid potential velocity is $\boldsymbol{v}'^{II}_0 = \nabla_x \Phi'^{II}_{\boldsymbol{v}_0} = \frac{\xi_e^3}{2\|\boldsymbol{\xi}\|^3}(3\langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle\hat{\boldsymbol{n}} - \boldsymbol{v}) - \boldsymbol{v}$ with the desired boundary condition $\lim_{\|\boldsymbol{\xi}\|\to\infty} \boldsymbol{v}'^{II}_0 = \boldsymbol{v}^{\infty} = -\boldsymbol{v}$ and $\hat{\boldsymbol{n}}$, the direction of the vector $\boldsymbol{\xi}$, given by $\hat{\boldsymbol{n}} = \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}$. Moreover, thanks to the linear character of the Laplace equation $\underline{\Delta}_x \Phi^{\rm tot} = \underline{\Delta}_x \sum_i \Phi^i = \sum_i \underline{\Delta}_x \Phi^i = 0$ for any assigned scalar potential Φ, we lump together the potential flows $\Phi'^{\rm tot}_{v_0} \equiv \Phi'^{I}_{v_0} + \Phi'^{II}_{v_0}$ for moving and contracting/expanding elements in order to obtain in S_1 the total potential flow outside the element surface as

$$\Phi' = -\langle \boldsymbol{v}, \boldsymbol{\xi} \rangle \left(1 + \frac{1}{2}\frac{\xi_e^3}{\|\boldsymbol{\xi}\|^3}\right) - \frac{\dot{\xi}_e \xi_e^2}{\|\boldsymbol{\xi}\|}, \quad (5)$$

where $\Phi' \equiv \Phi'^{\rm tot}_{v_0}$ for the sake of simplicity. The corresponding velocity in S_1 is obtained again as before by computing the gradient $\boldsymbol{v'}_0 \equiv \boldsymbol{v'}^{\rm tot}_0 = \nabla_{\xi} \Phi'$:

$$\begin{aligned}\boldsymbol{v'}_0 &= \left.\frac{\xi_e^3}{2\|\boldsymbol{\xi}\|^3}\left(3\langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle \hat{\boldsymbol{n}} - \boldsymbol{v}\right) - \boldsymbol{v} + \frac{\dot{\xi}_e \xi_e^2}{\|\boldsymbol{\xi}\|^2}\hat{\boldsymbol{n}}\right|_{\|\boldsymbol{\xi}\|=\xi_e} \\ &= \left.\frac{3}{2}\left(\langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle \hat{\boldsymbol{n}} - \boldsymbol{v}\right) + \dot{\xi}_e \hat{\boldsymbol{n}}\right|_{\|\boldsymbol{\xi}\|=\xi_e}, \end{aligned} \quad (6)$$

where in order to get the final expression we have evaluated the equation at the surface of the convective element $\|\boldsymbol{\xi}\| = \xi_e$ and simplified it. It is simple to check that this equation yields correct results at the element surface, $\|\boldsymbol{\xi}\| = \xi_e$ once written in spherical coordinates (with θ the angle between $\hat{\boldsymbol{e}}_z$ and $\hat{\boldsymbol{\xi}}$). For the radial component centred on O' we obtain $v'_{0,r} = -v(1 - \frac{\xi_e^3}{\|\boldsymbol{\xi}\|^3})\cos\theta + \frac{\dot{\xi}_e \xi_e^2}{\|\boldsymbol{\xi}\|^2}$ by computing the Laplacian. It follows from this that $v'_{0,r} = \dot{\xi}_e$ at $\|\boldsymbol{\xi}\| = \xi_e$ because in S_1 the convective element does not move with respect to the fluid and the only residual velocity in the fluid is the result of the expansion/contraction. In contrast, for the component $v'_{0,\theta} = v(1 + \frac{\xi_e^3}{2\|\boldsymbol{\xi}\|^3})\sin\theta$, at $\|\boldsymbol{\xi}\| = \xi_e$ we get $v'_{0,\theta} = \frac{3}{2} v \sin\theta$ which is maximum at $\theta = \{\frac{\pi}{2}, \frac{3\pi}{2}\}$. Moreover, for $\|\boldsymbol{\xi}\| \to \infty$ we get $v'_{0,r} = -v\cos\theta$ and $v'_{0,\theta} = v\sin\theta$ so that at the stagnation point, $\theta = 0$, we obtain $v'_{0,r} = -v$ and $v'_{0,\theta} = 0$ as required by construction of the boundary conditions in the integration of equation (5) (in S_1 the convective cell is at rest and the fluid flows along a direction opposite to the actual motion of the element in S_0). Finally, we compute the time derivative of the potential of equation (5) for use in the sections below and we recall that the surface of the convective element $\|\boldsymbol{\xi}\| = \xi_e$. The derivative is tedious but straightforward, and after a little algebra, we get

$$\left.\frac{\partial \Phi'}{\partial t}\right|_{\|\boldsymbol{\xi}\|=\xi_e} = -\frac{3}{2}\xi_e \langle \boldsymbol{A}, \hat{\boldsymbol{n}}\rangle - \frac{3}{2}\dot{\xi}_e \langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle - \ddot{\xi}_e \xi_e - 2\dot{\xi}_e^2, \quad (7)$$

where the relative acceleration of the two reference frames is indicated with $\boldsymbol{A}$. This quantity will be examined in detail in Section 5. The above analysis has given us two basic relationships, i.e. equations (6) and (7) that determine the EoMs of the convective elements to be presented and discussed below.

4 EQUATION GOVERNING THE EXPANSION/CONTRACTION OF A CONVECTIVE ELEMENT IN S_1

The goal of this section is to prove a relation connecting the evolution of the expansion rate of the convective element to the upward/downward motion inside the star. This important relation is obtained as a corollary of a more general theorem once two independent lemmas are considered.

4.1 Pressure–radius relation for expanding–contracting spheres in a non-inertial reference frame

We want to prove the existence of a relation connecting the two Lagrangian coordinates that describe our system: $\|\boldsymbol{x}_{O'}\|$, the location of the convective element and $\|\boldsymbol{\xi}_e\|$, the size of the convective element. Being these independent variables, the relation for which we are searching has to involve the physical quantities describing the environment in a given reference frame, which we choose to be S_1.

Instead of the classical approach reviewed in Section 1 proceeding from the Euler equation to Bernoulli's equation given by equation (3), we start from deriving the pressure acting on a convective element in the non-inertial reference system S_1 (defined in Section 3). This problem has been recently discussed by Pasetto et al. (2012) based on a previous approach presented by Pasetto & Chiosi (2009) (see their section 3.1) extended to include a Navier–Stokes fluid-dynamics equation treatment. However, the (simpler) version of Pasetto et al. (2012) was developed for plasmas of much higher temperatures than the typical ones in the stellar interiors, i.e. for the hot coronal plasma of the Milky Way, and did not consider the temporal evolution of the inner border of the fluid[3]. The formalism developed there can however be adapted to the case of convective elements, with the new velocity potential Φ' and associated velocity already given in equations (5) and (6) respectively, as follows.

Theorem: pressure–radius relation for an expanding/contracting sphere in an external environment. We prove that the pressure and radius temporal evolution of an expanding/contracting convective element retaining its spherical shape is related by the following equation

$$\begin{aligned}&\frac{v^2}{2}\left(\frac{9}{4}\sin^2\theta - 1\right) - v\dot{\xi}_e \frac{3}{2}\cos\theta + \left(\frac{P}{\rho} + \Phi_g\right) = \\ &+ A\xi_e\left(\frac{3}{2}\cos\theta - \cos\phi\right) + \ddot{\xi}_e \xi_e + \frac{3}{2}\dot{\xi}_e^2, \end{aligned} \quad (8)$$

holding in S_1, where $A = \|\boldsymbol{A}\|$ is the norm of the acceleration, ϕ the angle between the direction of motion of the fluid as seen from S_1 and the acceleration direction, and θ has been already introduced before as $\theta \angle (\boldsymbol{v}, \boldsymbol{\xi})$.

Proof: We start using equation 7 of Pasetto et al. (2012) written with the notation here set out. This is based on the same hypotheses (without the Young–Laplace treatment of the surface tension) and is

$$\frac{\partial \Phi'}{\partial t} + \frac{P}{\rho} + \frac{\left\|\boldsymbol{v'}_0\right\|^2}{2} = f(t) - \Phi_g - \langle \boldsymbol{A}, \boldsymbol{\xi}\rangle. \quad (9)$$

With the boundary condition for the hydrostatic equilibrium $\frac{P}{\rho} = -\Phi_g$ of the star, in the limit of $\|\boldsymbol{\xi}\| \to \infty$, it is easy to prove that we can fix the arbitrary function to be $f(t) = \lim_{\|\boldsymbol{\xi}\|\to\infty} \frac{\|\boldsymbol{v'}_0\|^2}{2}$. Using now equation (6) we obtain $\lim_{\|\boldsymbol{\xi}\|\to\infty} \boldsymbol{v'}_0 = \lim_{\|\boldsymbol{\xi}\|\to\infty} \frac{\xi_e^3}{2\|\boldsymbol{\xi}\|^3}(3\langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle \hat{\boldsymbol{n}} - \boldsymbol{v}) - \boldsymbol{v} = \boldsymbol{v}^{\infty} = -\boldsymbol{v}$ which means that

$$f(t) = \frac{\|\boldsymbol{v}\|^2}{2}. \quad (10)$$

[3] In S_1 a convective element can be identified either by an 'external' surface delimiting its volume or the inner border of the external fluid containing the convective element itself. In this case, we can speak also of an external border for the fluid, typically at $+\infty$ or far away from the convective element (e.g. Batchelor 2000).

Inserting equations (6) and (7) in equation (9), after some algebraic manipulations, we obtain the following equation:

$$\frac{1}{2}\left\|\frac{3}{2}\left(\langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle \hat{\boldsymbol{n}} - \boldsymbol{v}\right) + \dot{\xi}_e \hat{\boldsymbol{n}}\right\|^2 - \frac{\|\boldsymbol{v}\|^2}{2} - \frac{3}{2}\dot{\xi}_e \langle \boldsymbol{v}, \hat{\boldsymbol{n}}\rangle + \left(\frac{P}{\rho} + \Phi_g\right) - \frac{1}{2}\langle \boldsymbol{A}, \hat{\boldsymbol{n}}\rangle - \ddot{\xi}_e \xi_e - \frac{3}{2}\dot{\xi}_e^2 = 0. \tag{11}$$

To simplify further this equation we exploit spherical coordinates, motivated by the assumption of spherical symmetry made for the convective elements and the whole star (which retains its spherical shape during its existence). With the aid of this, at the surface of a convective element we write

$$\frac{\partial \Phi'}{\partial t} = -\frac{3}{2}A\xi_e \cos\phi - \frac{3}{2}\dot{\xi}_e v \cos\theta - \ddot{\xi}_e \xi_e - 2\dot{\xi}_e^2,$$

$$\frac{\|\boldsymbol{v}'_0\|^2}{2} = \frac{1}{2}\left(\dot{\xi}_e^2 + \frac{9}{4}v^2 \sin^2\theta\right).$$

Finally, by including these equations in equation (11) we complete the proof of equation (8) (Q.E.D.).

This is a rather complex PDE that links the fundamental quantities to which the generic convective element is subjected within the plasma inside the star. Nevertheless, despite its correctness, this equation is practically useless in this form because of its complexity. It is numerically solvable, but the lack of initial conditions to constrain the motion of the element does not allow us a complete coverage of the parameter space for the left-hand side (LHS) of equation (8). Nevertheless, it is the cornerstone of the new theory we are proposing. In order to achieve a deeper insight into its physical meaning we need to proceed with a further assumption.

4.2 The velocity-space expansion factor

As a convective element expands during the upward motion, its surface acts as a piston compressing the surrounding medium and the perturbation rapidly reaches the sound speed, v_{s} (e.g. Landau & Lifshitz 1959). Under the approximations made for equations (1) and (3), i.e. excluding attenuation by shear, bulk or relaxation viscosity, neglecting for the moment the heat conductivity, and limiting ourselves to the case of convective elements moving with velocities smaller than the sound speed, we obtain the following condition:

$$\varepsilon \equiv \frac{v}{\dot{\xi}_e} \ll 1 \forall t > \hat{t}, \tag{12}$$

i.e. the relative velocity between the convective element and the intrastellar medium $v = \|\boldsymbol{v}\|$ is much smaller than its expansion velocity $\dot{\xi}_e = \left\|\dot{\boldsymbol{\xi}}_e\right\|$. This is a reasonable assumption for the stars and phases that we want to consider. A simple, largely intuitive justification of equation (12) is provided by the following arguments: an ascending bubble must first counteract the gravity and push the surrounding medium, this second effect occurring at nearly constant gravity; therefore $v \ll \dot{\xi}_e$. In contrast, a descending bubble is accelerated by the gravity while being squeezed by the surrounding medium at nearly constant gravity and therefore its radius shrinks faster than the descending motion, also in this case $v \ll \dot{\xi}_e$. Trans/supersonic motions of the convective cells (e.g. expected in red supergiants), $v \sim v_{\mathrm{s}}$, require a fully compressive model that is beyond the aims of this paper. We will show in Section 6 that the theory developed under the approximation of equation (12) leads to correct predictions for the properties of the Sun.

Lemma 1: Pressure–radius relation for rapidly expanding/contracting sphere in an external environment. We prove that in the case that a sphere is expanding/contracting more rapidly than its translational motion, then the following approximate relation holds:

$$\frac{P}{\rho} + \Phi_g = A\xi_e\left(\frac{3}{2}\cos\theta - \cos\phi\right) + \ddot{\xi}_e \xi_e + \frac{3}{2}\dot{\xi}_e^2, \tag{13}$$

where pressure, density and potential are evaluated at the convective element surface.

Proof: We start by considering the result of the previous Theorem in the form expressed by equation (11). We are assuming that condition expressed by equation (12) holds in the same environment where the Theorem is considered. Dividing both sides of equation (11) by $\dot{\xi}_e^2$ when $\dot{\xi}_e \neq 0$, i.e. formally when $\hat{t} \neq 0$, we can find a time $\hat{t}$ so that for $t > \hat{t} \neq 0$ we have

$$\left(\frac{v}{\dot{\xi}_e}\right)^2 \frac{1}{2}\left(\frac{9}{4}\sin^2\theta - 1\right) \ll A\frac{\xi_e}{\dot{\xi}_e^2}\left(\frac{3}{2}\cos\theta - \cos\phi\right) + \frac{\ddot{\xi}_e \xi_e}{\dot{\xi}_e^2},$$

and

$$\left(\frac{v\dot{\xi}_e}{\dot{\xi}_e^2}\right)^2 \frac{5}{2}\cos\theta \ll A\frac{\xi_e}{\dot{\xi}_e^2}\left(\frac{3}{2}\cos\theta - \cos\phi\right) + \frac{\ddot{\xi}_e \xi_e}{\dot{\xi}_e^2},$$

thus yielding

$$\frac{P}{\rho} + \Phi_g = \frac{A}{2}\|\boldsymbol{\xi}\| \frac{\xi_e^3}{\|\boldsymbol{\xi}\|^3}\cos\theta + \frac{\xi_e}{\|\boldsymbol{\xi}\|}\left(\ddot{\xi}_e \xi_e + 2\dot{\xi}_e^2\right) - \frac{1}{2}\dot{\xi}_e^2 \left(\frac{\xi_e^2}{\|\boldsymbol{\xi}\|^2}\right)^2 \Bigg|_{\|\boldsymbol{\xi}\| = \xi_e}, \tag{14}$$

where pressure, density and potential are evaluated at the convective element surface, and with $\hat{t}$ we do not refer to any 'initial time' for the existence of a generic convective cell when $\dot{\xi}_e \sim v$, but rather to any time at which equation (12) is fully satisfied. Simplifying and exploiting spherical coordinates we obtain equation (13). This proves the Lemma 1 (Q.E.D.).

Initially, the expansion rate is not necessarily faster than the bubble speed and equation (12) is satisfied only asymptotically for t larger than a given $\hat{t}$ that can depend on the stellar properties. The acceleration term A has to be retained because the condition equation (12) can relate our two Lagrangian variables only by integration/derivation, but we have not yet obtained this relation as a function of the Lagrangian variables. This prevents us from performing an integration or derivation of equation (12) being not yet explicit the relation between acceleration and velocity: we will see only in Section 5 that indeed it is $A = A(\boldsymbol{x}, \boldsymbol{v}, \dot{\xi}_e)$, and the correct relation between acceleration, position, motion and expansion will be worked out only in equation (60) in relation with the radiative and adiabatic gradients.

We move now to a realistic situation. We consider the case in which a convective element moves radially upward throughout the external zones of a star and the acceleration and velocity are colinear, and finally we exclude the possibility of convective overshooting that will be considered in a forthcoming paper (Pasetto et al. 2014). Since the mathematical simplification of equation (11) brought by equation (12) is of paramount importance, we must fully understand its physical implication and meaning of it. This theory of convection is based on the assumption of non-local equilibrium, i.e. we assume that the interstellar plasma on the surface of the expanding/contracting convective element while moving outward/inward slightly deviates from strict hydrostatic equilibrium. The condition of rigorous hydrostatic equilibrium is met by the star only at larger distances from the surface of a convective element, as already done for equation (10). In this way, the property of hydrostatic equilibrium to which we refer by pushing to infinity the limit $\boldsymbol{x} \to \infty$ in equation (9) in order to fix equation (10), has the physical meaning of 'far away' from the convective element surface, but still 'close enough' to retain the density ρ as constant on global stellar scale. We refer to these mathematically asymptotic but local values for the pressure as P^{∞}, with ρ^{∞} and Φ_g^{∞} as already mentioned in equation (4). The corresponding equation of hydrostatic equilibrium reads

$$\left.\frac{\nabla_x P(\boldsymbol{\xi})}{\rho(\boldsymbol{\xi})}\right|_{\|\boldsymbol{\xi}\|\to\infty} = \left.\boldsymbol{g}(\boldsymbol{\xi})\right|_{\|\boldsymbol{\xi}\|\to\infty} \left.\frac{\nabla_x P(\boldsymbol{\xi})}{\rho(\boldsymbol{\xi})}\right|_{\|\boldsymbol{\xi}\|\to\infty}$$
$$= -\nabla\Phi_g(\boldsymbol{\xi})\big|_{\|\boldsymbol{\xi}\|\to\infty}\frac{P^{\infty}}{\rho^{\infty}} + \Phi_g^{\infty} = 0, \tag{15}$$

where the third equation holds by integration of the second at equilibrium (i.e. $\frac{\partial}{\partial t} = 0$) (see for instance Weiss et al. 2004; Kippenhahn et al. 2013)

We consider now the unlikely situation in which the convective element moves outward travelling through the entire star preserving its identity until it reaches the outer layers of the star. In the comoving reference S_1 the element surface expands until it reaches the equilibrium with the surrounding medium (note that this situation is also in strong contradiction with the standard formulation of the MLT). Thus, the element reaches the kinetic limit $v \gg \dot{\xi}_e$ opposite to that considered in equation (13), i.e. the element surface no longer expands and in S_1 is in static equilibrium (or in S_0 the element rises with constant ξ_e). In this case, the element is able to travel long distances keeping its size unchanged (apart from an initial phase of oscillations at the surface not to be mistaken with the Brunt–Vaisala oscillations of the element position in the layers of a star stable against convection)[4]. This situation does not apply here because it is ruled out by the conditions of equation (12).

We now call $A^{\infty} = A(\boldsymbol{x}^{\infty}; t)$ the direction-dependent relative acceleration between S_1 and S_0 to the same limit where equation (12) holds and we omit now the explicit dependence on the velocity space. The behaviour of this term is complicated and requires a careful treatment for which we reserve all of Section 5. We *assume* here that this term is approximately constant for the physical system under consideration and we will provide a rigorous proof of this assumption in Section 5. Under this hypothesis we prove the following.

Corollary 1: The asymptotic expansion equation for the convective element. In a stellar layer where $A^{\infty}\cong$const., the expansion of the convective element is governed asymptotically in the time evolution by the following equation:

$$\ddot{\xi}_e \xi_e + \frac{3}{2}\dot{\xi}_e^2 + \frac{A^{\infty}\xi_e}{2} = 0. \tag{16}$$

Proof: When considering equation (15) it is simple to prove that the LHS of equation (13) cancels: $\frac{\rho}{P^{\infty}}\left(\frac{P}{\rho} + \Phi_g\right) = \frac{\rho}{P^{\infty}}\left(\frac{P+\rho\Phi_g}{\rho}\right) = \frac{\rho}{P^{\infty}}\left(\frac{P-P^{\infty}}{\rho}\right) = \frac{P}{P^{\infty}} - 1$, which goes to zero as $t > \hat{t}$ and $\|\boldsymbol{\xi}\| \to \infty$ because $P \to P^{\infty}$ independently from any angular dependence. Hence Theorem equation (8) with Lemma 1 and this consideration results in the corollary equation (16) and we conclude (Q.E.D.).

The equation of this corollary governs the temporal asymptotic behaviour of the convective element. Its solution is a difficult task achieved in the next section.

[4]Note that in such a case it might be necessary to include the surface tension by means of the Young–Laplace equation that must be included in the EoM. This is not the limit of interest for us.

4.3 Solution of the equation for a convective element in S_1

As equation (16) governs the asymptotic evolution of any convective element, it is important to cast it in a dimensionless form and derive its most general solution. Even though equation (16) looks relatively simple, actually it is not, because of its high non-linearity. Indeed it contains two non-linear terms for the dependent variable, $\ddot{\xi}_e\xi_e$ and $\frac{3}{2}\dot{\xi}_e^2$, and must be coupled with another DE for the acceleration $\boldsymbol{A}^{\infty}$ to form a system of two coupled PDEs. To cope with this difficulty, we start recasting equation (16) by means of dimensionless variables

$$\chi \equiv \frac{\xi_e}{\xi_0} \quad \text{and} \quad \tau \equiv \frac{\mathrm{t}}{\mathrm{t}_0}, \tag{17}$$

so that for any given initial size ξ_0 of a convective element at the initial time $t = 0$ in units of t_0 we have

$$\chi(0) = 1 \quad \text{and} \quad \frac{\mathrm{d}\chi(0)}{\mathrm{d}\tau} = 0, \tag{18}$$

according to which we have assumed that a generic convective element of any arbitrary size starts expanding with zero expansion velocity. We remark that this choice for the initial conditions is arbitrary. As we are interested in the asymptotic behaviour of the solution (for $\tau \gg \hat{\tau}$ with $\hat{\tau} \equiv \frac{\hat{t}}{t_0}$), any other initial conditions, such as $\chi(\hat{\tau}) = \chi_0$ and $\frac{\mathrm{d}\chi(\hat{\tau})}{\mathrm{d}\tau} > 0$, would yield the same results. Therefore, $\hat{\tau}$ can be chosen arbitrarily close to '0' and considered as a dimensionless parameter. With these assumptions we rewrite equation (16) as

$$\chi\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} + \frac{3}{2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\right)^2 + \frac{A^{\infty}}{2}\frac{t_0^2}{\xi_0}\chi = 0. \tag{19}$$

In this equation, the normalized acceleration is a function of the time and position, the dependences of which will be investigated in detail in Section 5. In the previous section, we have seen the relation between the condition equation (12) and the reduced spatial motion travelled by a convective element. Here we assumed that

$$A^{\infty}(\chi;\tau) \cong \text{const.} \tag{20}$$

to solve equation (19), deferring a rigorous proof of this assumption to a devoted corollary in the next section. At this point, one could try to find a numerical solution of the equations as functions of time and space provided the temporal and spatial evolution of the acceleration is known. However, this way of proceeding would not improve significantly the theory of convection. This goal can be achieved by pushing the analytical analysis of the problem further. We continue to equation (19) in fully non-dimensional form by assuming

$$\frac{A^{\infty}}{2}\frac{t_0^2}{\xi_0} \equiv \frac{1}{2}\frac{A^{\infty}}{A_0^{\infty}} \equiv -2. \tag{21}$$

The reason for the last equality to -2 will become clear later on: it simply allows us to account for the fact that in S_1 the acceleration of the convective element is due only to the surface expansion and to the opposite motion of the intrastellar fluid on the surface of the convective element itself. The factor of 2 is introduced for mathematical convenience. Now

$$\chi\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} + \frac{3}{2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\right)^2 - 2\chi = 0, \tag{22}$$

whose solution is obtained in Appendix A (being simply a mathematical problem). The χ's asymptotic solution of interest for

$t \to \infty$ (the interested reader can look at Appendix A) is

$$\chi(\tau) = \frac{1}{8^2}\frac{(\sqrt{\pi}\Gamma(-1/8) - \Gamma(3/8)\tau)^2}{\Gamma(3/8)^2} + \mathrm{O}(\chi)^3$$

$$\simeq \frac{1}{8^2}\tau^2 - \frac{\sqrt{\pi}\Gamma(-1/8)}{32\Gamma(3/8)}\tau + \frac{\pi\Gamma(-1/8)^2}{64\Gamma(3/8)^2}. \qquad (23)$$

The χ asymptotic dependence is $\sim\tau^2$ plus lower order correction terms, i.e. quadratic in τ for $t \to \infty$ (see our cautionary remark made at the beginning of the analysis at the bottom of Section 2). As a consequence of this, also the time averaged value $\bar{\chi}(\tau) = \frac{1}{\tau}\int_0^\tau \chi(\tau')\mathrm{d}\tau'$ will grow with the same temporal dependence. This is an extremely important result that will play a key role in the proof of the independence of the stellar convection from any free parameter (see Section 6.2 below). These results fully determine the relationship between the motion and the expansion (contraction) of a convective element.

Finally, it is easy to prove that once we are interested in the integration of a star in a phase of non-hydrostatic equilibrium, i.e. where $\frac{P^\infty}{\rho^\infty} + \Phi_{\boldsymbol{g}}^\infty \neq 0$, equation equation (13) is again integrated numerically with $\frac{P(x)}{\rho} + \Phi_{\boldsymbol{g}}(\boldsymbol{x})$ being a known term valid everywhere in the stellar model. Thus, the solution to equation (13) can equally be recovered by simple translation of the solution for χ presented above, e.g. with a renormalization to $\frac{P^\infty}{\rho^\infty} - \Phi_g^\infty / A^\infty \equiv -2$ being A^∞ bounded in spherical coordinates $(\frac{3}{2}\cos\theta - \cos\phi) \in [-\frac{5}{2}, \frac{5}{2}]$.

With this approximation, we are simplifying the system of two coupled PDEs, one for the expansion radius ξ_e and one for the acceleration $\boldsymbol{A}^\infty = \ddot{\boldsymbol{x}}_{O'}$ of a convective element, to a new system of two coupled DEs; these being equation (22) and an equation for the acceleration to be developed now. The next issue to address is therefore now to prove a corollary on the acceleration in equation (20). This is the subject of the Section 5.

5 THE ACCELERATION OF A CONVECTIVE ELEMENT

So far we have dealt with the first degree of freedom, the convective element radius ξ_e. In this section, we examine the forces acting on the convective element and ruling its motion, i.e. we deal with the second degree of freedom of our system, the motion of the convective element initially located at a generic position inside the star, $\boldsymbol{x}_{O'}$. The PDE equation (8) or its approximated Ordinary Differential Equation (ODE) equation (16) has to be supplied with a second ODE describing the motion of a convective element throughout the stellar medium. This yields a system of two equations with two unknowns, i.e. $\xi_e = \xi_e(t)$ and $\boldsymbol{x}_{O'} = \boldsymbol{x}_{O'}(t)$. We present the detailed derivation of these equations from basic principles in order to explain the precise founding hypotheses of these equations. The arguments that lead to this result hold only in the context of the Theorem in Section 4. In the same hypothesis of this theorem the following corollary holds.

Corollary 2: Acceleration of the convective element. In a stellar layer under the assumption of Theorem equation (8) the acceleration of a convective element is given by

$$\boldsymbol{A} = -\boldsymbol{g}\frac{m_e - M}{m_e + \frac{M}{2}} - \frac{10}{3}\pi\xi_e^2\rho\boldsymbol{v}\dot{\xi}_e. \qquad (24)$$

Proof: The total force acting on a convective element in S_1 is determined by the total pressure acting on its surface $\boldsymbol{F}_P = -\int P\hat{\boldsymbol{n}}\,\mathrm{d}\Omega$ and the weight $m_e\boldsymbol{g}$ of the element. The corresponding EoM derived from the Newtonian law $\boldsymbol{F} = m\boldsymbol{A} = m\dot{\boldsymbol{v}} = m\ddot{\boldsymbol{x}}_{O'}$ is the equation to be integrated together with equation (8). In the reference frame S_1 we can use this force balance to express the pressure force $\boldsymbol{F}_P$ and in turn relate it to the acceleration. We may write

$$\begin{aligned} -\int P\hat{\boldsymbol{n}}d\Omega &= -\rho\int\left(\frac{v^2}{2}\left(1 - \frac{9}{4}\sin^2\theta\right) + \frac{5}{2}v\dot{\xi}_e\cos\theta\right. \\ &\quad \left. + \frac{A\xi_e}{2}\cos\theta + \ddot{\xi}_e\xi_e + \frac{3}{2}\dot{\xi}_e^2 - \Phi_g\right)\hat{\boldsymbol{n}}\,\mathrm{d}\Omega \\ &= -\rho\int\left(\frac{v^2}{2}\left(1 - \frac{9}{4}\sin^2\theta\right) + \frac{5}{2}v\dot{\xi}_e\cos\theta\right. \\ &\quad \left. + \frac{A\xi_e}{2}\cos\theta + \ddot{\xi}_e\xi_e + \frac{3}{2}\dot{\xi}_e^2 - \Phi_g\right)\mathcal{J}\hat{\boldsymbol{n}}\,\mathrm{d}\Omega \\ &\quad +\rho\int\nabla\Phi_{\boldsymbol{g}}\mathrm{d}^3\xi_e \\ &= -\frac{2}{3}\pi\xi_e^3\rho\boldsymbol{A} - \frac{10}{3}\pi\xi_e^2\rho\boldsymbol{v}\dot{\xi}_e + \frac{4}{3}\pi\xi_e^3\rho\boldsymbol{g}, \end{aligned} \qquad (25)$$

where the integral is carried out over the sphere representing the convective element at each instant on the differential form (the solid angle) $\mathrm{d}\Omega$. θ is the angle already defined after equation (11), $\mathcal{J}$ is the Jacobian of the transformation from Cartesian to spherical coordinates, and $\hat{\boldsymbol{n}}$ is the unit vector of the position vector. Finally, on the right-hand side (RHS) of the equation, third line, simple trigonometric integrals have been computed and the Gauss theorem has been used to work out explicitly the result. This equation accounts for the buoyancy of the convective element $\frac{4}{3}\pi\xi_e^3\rho\boldsymbol{g}$, the inertial term of the fluid displaced by the movement of the convective cell, i.e. the reaction mass $\frac{1}{2}\frac{4}{3}\pi\xi_e^3\rho \equiv \frac{M}{2}$, and a new extra term $-\frac{10}{3}\pi\xi_e^2\rho\boldsymbol{v}\dot{\xi}_e$ arising from the changing size of the convective element: the larger the convective element, the stronger the buoyancy effect and the larger is the velocity acquired by the convective element. These terms have to be included in the Newtonian EoM. Adding now the effect of the gravity on the convective element of mass m_e we get the general expression for the acceleration $\boldsymbol{A}$ in S_1 as in equation (24) (Q.E.D.).

It is worth calling attention to the correction to the force balance that is required in order to properly include the inertia that a convective element experiences during its motion across the stellar fluid. More precisely, a convective element in motion experiences a drag force produced by the different density of the fluid it is moving through. In this way, we naturally reconcile the correct physics with the D'Alambert paradox intrinsic to the velocity-potential theory approximation. If in equation (24) we change sign to recast it in S_0 instead of S_1 we recover standard results from fluid dynamics for the force balance, e.g. equation 6.8.20 of Batchelor (2000). Examining equation (24) we note that for $m_e = M$, we have $\boldsymbol{A} = 0$ as indeed expected if there is no overdensity (no convection). For $m_e \ll M$, $\boldsymbol{A} \simeq -2\boldsymbol{g}$, which means that the convective element is reaching a limiting acceleration, i.e. the case excluded in the convection regime. For $m_e \gg M$, the fluid scarcely affects the initial acceleration of a convective element. This apparently means that the approximation in equation (20) does not hold, because the expansion rate of the bubble $\dot{\xi}_e \neq 0$ as clearly assumed in the previous section and so apparently $A \neq 0$. To show that this is not the case is simple by taking into account again equation (12) and we examine this as a case of interest.

The case of interest: rapidly expanding convective element. As simple application of this corollary we look at the case of a convective element rising along the vertical direction in the reference frame S_0 with $\boldsymbol{g} = \{0, 0, -g\}$ and $g > 0$, and considering the same approximation used for equation (12) and notation used for equations (15) and (16) we get

$$m_e A_z^\infty = -\frac{2}{3}\pi\xi_e^3\rho A_z^\infty - \frac{10}{3}\pi\xi_e^2\rho v\dot{\xi}_e - \frac{4}{3}\pi\xi_e^3\rho g + m_e g \Leftrightarrow$$

$$A_z^\infty = g\frac{m_e - M}{m_e + \frac{M}{2}} - \frac{10}{3}\pi\xi_e^2\rho v\dot{\xi}_e$$

$$\cong g\frac{m_e - M}{m_e + \frac{M}{2}}, \tag{26}$$

where for $\dot{\xi}_e \neq 0$ we divide and multiply by $\dot{\xi}_e^2$ as already done for equation (13) to eliminate the term $\frac{v\dot{\xi}_e}{\dot{\xi}_e^2}$ and to formulate equation (26) as an asymptotic expansion of order $O(\frac{A_z^\infty}{g})$. This simple exercise proves that at the same degree of approximation under which equation (19) holds also equation (20) holds – as was left to prove. We see also that the convective element will rise when $m_e < M \Rightarrow m_e - M < 0$, i.e. when A in S_1 is negative so that the sign adopted in equation (21) remains fully justified as originally adopted.

The last step required to integrate the EoM for a convective element within a convective layer and it deals with the instability conditions. The following auxiliary lemma proves a self-standing result that once included in the previous corollary will allow us to mathematically close the set of equations and to conclude the theory.

Lemma 2: Linear response of the convective element to the stellar pressure gradients. We prove that the response, i.e. the motion, of the convective element to the forces applied on it, i.e. equation (26), is given in linear regime by

$$A_z^\infty \simeq g\frac{\nabla_e - \nabla + \frac{\varphi}{\delta}\nabla_\mu}{\frac{3h_P}{2\delta\Delta z} + \left(\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right)} \tag{27}$$

to the leading order on $O(\frac{\Delta x}{h_P})$.

Proof: We need to express the masses term in equation (26) as a function of the fundamental logarithmic gradients introduced in Section 1. This step is indeed necessary owing to the presence of the new term $m_e + \frac{M}{2}$ in the denominator of equation (26). For a small displacement of the convective element, say $\Delta\boldsymbol{x}_{O'}$ the density can be expanded as

$$\rho = \rho|_{\boldsymbol{x}_{O'}} + \nabla_{\boldsymbol{x}}\rho|_{\boldsymbol{x}_{O'}}\Delta\boldsymbol{x}_{O'} + \ldots$$

$$\rho_e = \rho_e|_{\boldsymbol{x}_{O'}} + \nabla_{\boldsymbol{x}}\rho_e|_{\boldsymbol{x}_{O'}}\Delta\boldsymbol{x}_{O'} + \ldots, \tag{28}$$

in which all terms of quadratic order in $\Delta\boldsymbol{x}_{O'}$ and higher orders are neglected. The subscript 0 refers to the equilibrium position of the convective cell. Because the volume occupied by the convective element is the same as displaced in the fluid, the relation equation (26) can easily be translated to an equivalent one in the density. In its numerator, assuming that $[\rho_e - \rho]_{\boldsymbol{x}_{O'}} = 0$, we get the standard approximation (e.g. Kippenhahn & Weigert 1994) to the first order $\rho_e - \rho \simeq [\nabla_{\boldsymbol{x}}\rho_e - \nabla_{\boldsymbol{x}}\rho_e]_{\boldsymbol{x}_{O'}}\Delta\boldsymbol{x}_{O'}$ for the displacement of a convective element. However, because of the terms at the denominator we get the more complicated expression

$$\rho_e + \frac{\rho}{2} \simeq \frac{3}{2}\rho|_{\boldsymbol{x}_{O'}} + \left[\nabla_{\boldsymbol{x}}\rho_e + \frac{\nabla_{\boldsymbol{x}}\rho}{2}\right]_{\boldsymbol{x}_{O'}}\Delta\boldsymbol{x}_{O'}. \tag{29}$$

This equation requires the density gradients which do not appear naturally in the equations of stellar structure. Therefore, we express them as a function of temperature T with the help of the EoS. Now, we must recast the correspondent instability criteria starting from the EoS for a perfect fluid $\rho = \rho(P, T, \mu)$ (see Section 1) that in its differential form reads $\frac{\mathrm{d}\rho}{\rho} = \alpha\frac{\mathrm{d}P}{P} + \delta\frac{\mathrm{d}T}{T} + \varphi\frac{\mathrm{d}\mu}{\mu}$. Here $\{\alpha, \delta, \varphi\} \equiv \{\frac{\partial\ln\rho}{\partial\ln P}, -\frac{\partial\ln\rho}{\partial\ln T}, \frac{\partial\ln\rho}{\partial\ln\mu}\}$, where the standard notation has been used. Thus, we get

$$\nabla_x\rho_e + \frac{\nabla_x\rho}{2} = \left[\frac{\rho\alpha}{P}\nabla_x P - \frac{\rho\delta}{T}\nabla_x T + \frac{\rho\varphi}{\mu}\nabla_x\mu\right]_e$$

$$+\frac{1}{2}\left(\frac{\rho\alpha}{P}\nabla_x P - \frac{\rho\delta}{T}\nabla_x T + \frac{\rho\varphi}{\mu}\nabla_x\mu\right). \tag{30}$$

After some manipulation, and assuming that in a small displacement the change of molecular weight μ, $\mathrm{d}\mu = 0$ for the moving element, we are able to simplify the RHS as

$$\mathrm{RHS} = \rho_e\left[\frac{\alpha}{P}\nabla_x P\right]_e - \rho_e\left[\frac{\delta}{T}\nabla_x T\right]_e$$

$$+\frac{\rho}{2}\frac{\alpha}{P}\nabla_x P - \frac{\rho}{2}\frac{\delta}{T}\nabla_x T + \frac{\rho}{2}\frac{\varphi}{\mu}\nabla_x\mu. \tag{31}$$

We consider now the first and the third element in the previous equation (31). While in the case of the derivation of the Schwarzschild criteria the first two terms in the previous equation simplify because $P_e - P = 0$ in any adiabatic expansion, this is no longer the case here. We have

$$\rho_e\left[\frac{\alpha}{P}\nabla_x P\right]_e + \frac{\rho}{2}\frac{\alpha}{P}\nabla_x P \cong \rho\alpha\left(\frac{1}{P_e}\nabla_x P_e + \frac{1}{2}\frac{1}{P}\nabla_x P\right)$$

$$= \frac{\rho\alpha}{P}\left(1 + \frac{1}{2}\right)\nabla_x P$$

$$= \frac{3}{2}\frac{\rho\alpha}{P}\nabla_x P, \tag{32}$$

which is used to simplify equation (31) as

$$\mathrm{RHS} = \frac{3}{2}\frac{\rho\alpha}{P}\nabla_x P - \rho\left[\frac{\delta}{T}\nabla_x T\right]_e - \frac{\rho}{2}\frac{\delta}{T}\nabla_x T + \frac{\rho}{2}\frac{\varphi}{\mu}\nabla_x\mu. \tag{33}$$

Now, by introducing the pressure scalelength, with $\boldsymbol{x} = \{0, 0, z\}$, we can further expand the previous equation (33) as

$$\mathrm{RHS} = \frac{\rho}{h_P}\frac{3\alpha}{2P}\left(-P\frac{\mathrm{d}z}{\mathrm{d}P}\right)\frac{\mathrm{d}P}{\mathrm{d}z} - \frac{\rho}{h_P}\left[\left(-P\frac{\mathrm{d}z}{\mathrm{d}P}\right)\frac{\delta}{T}\frac{\mathrm{d}T}{\mathrm{d}z}\right]_e$$

$$[5pt] \quad -\frac{\rho}{h_P}\left(-P\frac{\mathrm{d}z}{\mathrm{d}P}\right)\frac{1}{2}\frac{\delta}{T}\frac{\mathrm{d}T}{\mathrm{d}z} + \frac{\rho}{h_P}\left(-P\frac{\mathrm{d}z}{\mathrm{d}P}\right)\frac{1}{2}\frac{\varphi}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}z}. \tag{34}$$

If we introduce the logarithmic derivative to write this equation as a function of the standard temperature gradients:

$$\mathrm{RHS} = -\frac{\rho}{h_P}\frac{3\alpha}{2} + \left[\delta\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right]_e + \frac{1}{2}\delta\frac{\rho}{h_P}\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} - \frac{\rho}{h_P}\frac{\varphi}{2}\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}$$

$$= \frac{\rho\delta}{h_P}\left(-\frac{3\alpha}{2\delta} + \nabla_e + \frac{1}{2}\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right), \tag{35}$$

and recalling that $\frac{\alpha}{\delta} = \frac{\partial\ln\rho/\partial\ln P}{-(\partial\ln\rho/\partial\ln T)} = -\frac{\partial\ln T}{\partial\ln P} = -\nabla$ we can write equation (30) after simple algebraic manipulation as

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}z}\right|_e + \frac{1}{2}\frac{\mathrm{d}\rho}{\mathrm{d}z} = \frac{\rho\delta}{h_P}\left(\nabla_b + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right).$$

It is straightforward to now compute the acceleration as a function of the fundamental logarithmic gradients. By considering the motion of the convective elements along the vertical direction z, once we introduce this term in equation (29) and we consider equation (26) after algebraic manipulation we conclude with equation (27) (Q.E.D.).

It is interesting now to call attention to some aspects of the acceleration equation (27) with respect to the classical formulation in the literature.

(i) In regions of homogeneous chemical composition the acceleration reduces to

$$A_z^\infty \simeq g\,\frac{\nabla_e - \nabla}{\frac{3h_P}{2\delta\Delta z} + (\nabla_e + 2\nabla)}. \tag{36}$$

It is then immediately evident how this new instability criterion induces exactly the Schwarzschild instability zones ($\nabla_e - \nabla < 0$) as the denominator of equation (36) is always positive by definition! This is a very important result because it allows us to extend the Schwarzschild and/or Ledoux criteria for instability: even with the new criterion, the convective zones occur exactly in the same regions predicted by the Schwarzschild criterion.

(ii) The zones of chemical inhomogeneity should be treated according to the instability criterion given by equation (27). In any case, we point out that relation (27) is a second order Taylor expansion on the small parameter, $\varepsilon = \frac{2\delta\Delta z}{3h_P}$, which to the first order yields the classical results for the acceleration of a convective element $A_z^\infty = g\frac{2}{3}\frac{\delta}{h_P}(\nabla_e - \nabla + \frac{\varphi}{\delta}\nabla_\mu)\Delta z + \mathrm{O}(\frac{2\delta\Delta z}{3h_P})^2$ corrected by the factor 2/3 for the presence of the inertial mass with respect to classical results. Furthermore, it incorporates the Ledoux condition. The onset and effects of convection in the presence of a gradient in molecular weight are highly debated subjects that are not addressed here (e.g. Kippenhahn & Weigert 1994; Maeder 2009).

In Section 6.2, we will show that retaining the second order terms is the key action in order to eliminate of the ML thanks to equation (8).

By proving corollary 2 and lemma 2, we have concluded the new theory. We obtained indeed the relevant equations to govern the evolution of a convective element in an unstable convective layer inside a star. In a non-inertial reference frame S_1, we developed a framework where the two equations (16) and (27) describe the evolution of the two degrees of freedom used to describe our physical system. We apply now our theory of convection in stellar interiors to a few key tests to show its potential capability.

6 RESULTS OF THE THEORY

In this section, we present some results and predictions of our theory. We compute a few selected physical quantities of interest and then we consider their temporal evolution in the integration of a stellar model (in our case, the Sun).

Before proceeding further, in the spirit of the cautionary remark made at the end of Section 2, we comment on the time limits that we are going to consider. As already pointed out in Section 4.3, once the condition of instability to convection is matched at a certain location $\boldsymbol{x} + \mathrm{d}\boldsymbol{x}$ inside the star, convective elements are born and initially their radius ξ_e and surface in turn do not necessarily expand faster than their vertical motion v, i.e. initially it is $\frac{v}{\dot{\xi}_e} = O(1)$ as $t \to 0$. Therefore, as the condition of equation (12) cannot be satisfied, we must start our time integration from $t > t_{\min}$ (this is shown in detail in Fig. A1 in Appendix A). Different arguments apply to the upper temporal limit. Integrating the last row of equation (26) we see that $v \propto t$ as $t \to \infty$ (this is indeed also consistent with the spatial series expansion of equation (28) retained to the first order), but at the time $\chi \propto t^2$ and hence $\xi_e \propto t^2$ as $t \to \infty$ and finally $\dot{\xi}_e \propto t$ as $t \to \infty$. Therefore, the condition $\frac{v}{\dot{\xi}_e} = O(1)$ as $t \to \infty$ cannot be satisfied as required by equation (12). To cope with this and maintain the standard notation $\lim_{t\to\infty} Q(t) = Q^\infty$ at the same time, we take a suitable time interval $t \in \,]t_{\min}, t_{\max}[$, where at $t_{\max}$ the convective elements still live in an ambient medium of constant intrastellar density (see equation 28) and acceleration (see equations 20 and 26). This approach has some similarity with the Boussinesq and anelastic approximations commonly used in other branches of research such as planetary and atmospheric sciences, oceanography, and geodynamics (e.g. Glatzmaier 2013). The novelty here is that for the first time all this has been formulated in the comoving frame of reference S_1.

6.1 The convective flux: from the single element to the collective description

In order to apply the theory in practice, we need to consider a collective description of the convective cells. Many options (both numerical and analytical) are nowadays available to stochastically describe a phenomena within the framework of a theory, nevertheless we will limit to the simple method of the moments (MoM). The reasons are twofold: first, we will see that by limiting our analysis to the mean stream of the convective cells (i.e. the first of a full hierarchy of moments) will leave us with extremely satisfactory results, and secondly, the MoM permits a more natural comparison with the MLT where only the mean velocity of convective elements is considered.

We consider an arbitrary but fixed surface S inside the star with infinitesimal element $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\mathrm{d}S$, where $\hat{\boldsymbol{n}}$ is the outward normal to the surface under consideration (i.e. any ideal surface through which the convective elements are free to flow). We assume that the number of convective elements passing through $\mathrm{d}\boldsymbol{S}$ at a given time is $n = f\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{p}$, where $f = f(\boldsymbol{x}, \boldsymbol{p}; t)$ is the unknown distribution function (DF) of the convective elements inside the star. Then for every scalar quantity of interest, say Q, the out/inward flux φ of Q is $\varphi \equiv \langle Qf\boldsymbol{V}, \mathrm{d}\boldsymbol{S}\mathrm{d}t\mathrm{d}^3\boldsymbol{p}\rangle$ with $\boldsymbol{V} \equiv \frac{\boldsymbol{p}}{m_e}$. This represents the amount of Q transported through $\mathrm{d}\boldsymbol{S}$ with a given momentum $\boldsymbol{p} \in [\boldsymbol{p}, \boldsymbol{p} + \mathrm{d}^3\boldsymbol{p}]$ during the time interval $\mathrm{d}t$. Therefore, $\langle\overline{Q\boldsymbol{V}}, \mathrm{d}\boldsymbol{S}\rangle\mathrm{d}t = \frac{1}{n}\int_{\boldsymbol{p}^3}\langle Q\boldsymbol{V}, \mathrm{d}\boldsymbol{S}\rangle f\mathrm{d}t\mathrm{d}^3\boldsymbol{p}$, where the overbar indicates the average of the quantity, and the amount of Q transported by any convective element with any $\boldsymbol{p}$ through $\mathrm{d}\boldsymbol{S}$ in $\mathrm{d}t$ is $n\langle\overline{Q\boldsymbol{V}}, \mathrm{d}\boldsymbol{S}\rangle\mathrm{d}t = \int_{\boldsymbol{p}^3}\langle Q\boldsymbol{V}, f\mathrm{d}\boldsymbol{S}\rangle\mathrm{d}t\mathrm{d}^3\boldsymbol{p}$. Hence the flux of Q is the amount of Q per unit area and unit time[5]

$$\boldsymbol{\varphi} = n\overline{Q\boldsymbol{V}}. \tag{37}$$

In order to calculate the *convective flux* we need to compute the amount of internal energy per unit area per unit time carried by the convective elements. In our previous computation, we assumed that in asymptotic regime the convective elements move adiabatically. Recalling that by definition the specific heat at constant pressure c_P is the amount of heat required to increase the temperature of a convective element of unit mass by 1°, we set $c_P \equiv \frac{1}{m_e}\frac{\mathrm{d}Q}{\mathrm{d}T}$ where the pressure $P = \mathrm{const.}$ and $m_e = 1$ and we have replaced the

[5] A preliminary investigation suggests that the present formalism (see the relationship between χ and ξ of Fig. A1, the connection between the size scale ξ_0 and the acceleration A, and finally the relationship between a star's gravitational stratification and ξ_0) could suggest the shape of the DF, f. However, we remind that despite only the infinite series of moments in a MoM method is mathematically equivalent to the underlying DF, for the sake of simplicity we prefer to adopt here the classical definition of convective flux based on the average (first order moment) velocity of convective elements. We defer a Monte Carlo approach to future studies of the DF.

'Q' symbol of 'quantity' defined above before with its meaning of 'heat'. Therefore,

$$\Delta Q \equiv m_e c_P \Delta T \tag{38}$$

is the heat excess of a convective element of mass m_e over the surroundings. We must use c_P, rather than c_V, here, in accordance with our assumption of pressure equilibrium, the heat exchange with the surrounding medium occurs at constant pressure at each level. Once the element has moved from its initial position $\boldsymbol{x}_1$ with temperature $T_{e,1} = T_e(\boldsymbol{x}_1)$ and ambient temperature $T_1 = T(\boldsymbol{x}_1)$ to a new position at distance $\boldsymbol{x}_2$ with temperature $T_{e,2} = T_e(\boldsymbol{x}_2)$ and ambient temperature $T_2 = T(\boldsymbol{x}_2)$, the heat stored in the element flows from this to the surrounding medium (or vice versa depending on the ratio between T_{e2} and T_2). The amount of heat flowing through d$\boldsymbol{S}$ in dt is written as $Q = c_P \Delta T \rho \langle \bar{\boldsymbol{V}}, \mathrm{d}\boldsymbol{S}\mathrm{d}t\rangle$, where $\rho \bar{\boldsymbol{V}}$ is the mass flux. Consequently, the convective flux (i.e. heat passing through the surface d$\boldsymbol{S}$ in dt) is

$$\boldsymbol{\varphi}_{\mathrm{cnv}} \equiv c_P \Delta T \rho \bar{\boldsymbol{V}}, \tag{39}$$

with $\bar{\boldsymbol{V}}$ being the average velocity of the convective elements during the time interval dt as seen in S_0. Here, we see that even if there is no net mass flux, the heat is anyway transported, because this flux in S_0 can be written as the sum

$$\begin{aligned}\boldsymbol{\varphi}_{\mathrm{cnv}} &= n\overline{m_e c_P \Delta T \boldsymbol{V}}\\ &= \rho c_P \overline{\Delta T \boldsymbol{v}} + \rho c_P \overline{\Delta T \boldsymbol{v}_0},\end{aligned} \tag{40}$$

where $\boldsymbol{v} = \boldsymbol{V} - \boldsymbol{v}_0$ is the peculiar velocity of a convective element in equation (5). If the mean velocity is zero, i.e. if there is no outflow/inflow mass flux carrying the heat, the energy transport owing to the heat carried by the convective elements $\boldsymbol{\varphi}_{\mathrm{cnv}} = \rho c_P \overline{\Delta T \boldsymbol{v}}$.

We can calculate a self-consistent expression for the velocity in S_1 from equation (11). If $\boldsymbol{v} = 0$ the heat flux carried by each convective element is null (the convective element is supposed to be spherical) because the same stagnation point will exist in the diametrically opposite side of the convective element. Therefore, what contributes to the convective flux is not the velocity of the stagnation points, but the velocity of the whole convective element, i.e. the velocity of the barycentre. Starting from the general expression for the velocity of a fluid element impacting the surface of a convective element given by equation (11), to get the sole motion of the centre of the convective element it is sufficient to set $\dot{\xi}_e = \ddot{\xi}_e = 0$. This means that neglecting the radial expansion/contraction, the convective element moves as a rigid body, therefore any points of its surface comove with the stagnation points, i.e. $\theta = 0$, $\phi = 0$. Finally, the condition of quasi hydrostatic equilibrium applies, i.e. $(\frac{P}{\rho} + \Phi_g) \cong 0$. Eventually, the square of the velocity is (remember the definition of equation 21 as acceleration of the fluid as seen from S_1 on to the convective element surface)

$$\begin{aligned}v^2 &= -A\xi_e\\ &= \frac{\nabla - \nabla_e + \frac{\varphi}{\delta}\nabla_\mu}{\frac{3h_P}{2\delta\Delta z} + \left(\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right)}\xi_e g,\end{aligned} \tag{41}$$

where we have used equation (27). From the previous equation we obtain

$$\frac{3h_P}{2\delta v \Delta t}\frac{v^2}{\xi_e} + \left(\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu\right)\frac{v^2}{\xi_e} = \left(\nabla - \nabla_e - \frac{\varphi}{\delta}\nabla_\mu\right) g, \tag{42}$$

which, once the limit $\epsilon = v/\dot{\xi}_e \longrightarrow 0$, see equation (12) and corollary 2, i.e. at $t \to \infty$ behaves as

$$\frac{v^2}{\xi_e} = \frac{\nabla - \nabla_e - \frac{\varphi}{\delta}\nabla_\mu}{\nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu} g. \tag{43}$$

This is a remarkable result that at a first sight may look surprising: the dominant term of the acceleration (in the denominator) $\frac{3h_P}{2\delta v \Delta t} > \nabla_e + 2\nabla - \frac{\varphi}{2\delta}\nabla_\mu$ does not affect the velocity at the regime in which we are going to integrate our solution of the Navier–Stokes equation, see equation (8).

This counter-intuitive result is mathematically consistent only asymptotically in time and valid only within the approximation adopted in equation (12). It will be numerically checked in the next section (see Fig. 4) where we will see that the ratio of the two terms of the LHS of equation (42) shows a maximum not yet investigated in the literature. It means that the mechanical evolution of a convective element (for instance its expansion) is dominated more by the local gradients of temperature over the pressure and less by its location inside the star $\boldsymbol{x_e}$. What we may learn from the above analysis is that the transfer of energy by convection is governed more by the expansion of the convective cells than by their upward/downward motions. It follows from this, that the properties of convection are mainly driven by local rather than large-scale physics. This lends support to the hypotheses already implicit in the MLT. However, we remark that the above locality does not contradict the spatial changing of temperature gradients (∇, ∇_e, etc.) and gravitational force across the star.

Finally, recalling that $\Delta T = \frac{T}{h_P}(\nabla - \nabla_e) v \Delta t$ (e.g. equation 6.19 in Kippenhahn et al. 2013), we can define the convective flux along the radial direction as [6]

$$\varphi_{\mathrm{cnv}} = \frac{1}{2} c_P \rho T \left(\nabla - \nabla_e\right) \frac{\bar{v}^2 \Delta t}{h_P}, \tag{44}$$

where the factor 1/2 comes from the fact that at each level approximately one-half of the matter is rising and one-half is descending (Weiss et al. 2004) to secure mass conservation locally. Finally, recalling that the fluid acceleration in S_0 is seen as a negative quantity, we can also write the flux as $\varphi_{\mathrm{cnv}} \propto c_P \rho T (\nabla_e - \nabla)\sqrt{A\xi_e}\frac{\Delta x_{O'}}{2h_P}$, and we see that this equation is equivalent to the standard formalism, e.g. equation 7.7 of Kippenhahn & Weigert (1994), if we use the velocity derived from their heuristic considerations because the term $(A\xi_e)^{1/2}$ takes the dimension of a distance $[D]$ over a velocity $\frac{[D]}{[T]}$; indeed $[(A\xi_e)^{1/2}] = (\frac{[D]}{[T]^2}[D])^{1/2} = \frac{[D]}{[T]}$ as required by the dimensional analysis.

6.2 The basic equations of stellar convection without the mixing-length parameter

The ultimate result we are seeking is a self-consistent solution of the equations for the convective transport of energy inside a convective layer, without making use of adjustable parameters such as the ML Λ_m. For simplicity we present here the case of a chemically homogeneous medium $\nabla_\mu = 0$, since in any case the major role

[6] This is derived from equation (41) using this formalism and recalling the fundamental theorem of calculus for a Lipschitzian function, according to which $\Delta \boldsymbol{x}_{O'}$ differs from $\boldsymbol{v}\Delta t$ by the quadratic terms $\mathrm{O}(\Delta t)^2$, i.e. beyond the approximation made in the Schwarzschild–Ledoux criterion or equation (36).

of the convection is indeed to homogenize gradients of chemical composition. The general case $\nabla_\mu(\boldsymbol{x}) \neq 0$ comes then trivially with a suitable change of variables as will be more evident a posteriori. The key result, and ultimately one of the goals of this paper is to prove that if we call the body of variables (temperature, pressure, density, etc.) defining the physical state of stellar interiors at a given position $\boldsymbol{x}$ the 'stellar system', the following holds:

Theorem of the uniqueness of the stellar convection. *The radiative $\nabla_{\rm rad}$, the adiabatic $\nabla_{\rm ad}$, the local gradient of the star ∇, and the convective element gradient ∇_e are in a one-to-one correspondence (a bijection) with the stellar system in which they are embedded.*

Proof: To prove the assertion of this theorem we need to solve the equation of stellar convection without any free parameter (e.g. the ML Λ_m) thus unequivocally assigning to each location inside a star its own characteristic convection. In other words, we are going to describe the stellar convection not as a one-parameter family of solutions (i.e. the ML parameter Λ_m to be fixed by external constraints) but with a unique solution of the system of equations governing stellar convection. We start by extending the present formalism to include a few fundamental theoretical tools. A convective cell of mass m_e, volume υ_e and radius ξ_e, once it has acquired a positive excess of temperature $|\Delta T| = T(\nabla - \nabla_e)\frac{\Delta x_{O'}}{h_P}$ with respect to its surroundings, radiates energy into the stellar medium with a flux $\varphi_{\rm rad} = \frac{4acT^3}{3\kappa\rho}|\nabla_{\hat{\boldsymbol{n}}}T|$, where $a = 7.5657 \times 10^{-16}\,{\rm Jm^{-3}K^{-4}}$ is the radiation-density constant, κ the mean absorption coefficient, or opacity, and c the speed of light. The radiative loss per unit of time $\frac{{\rm d}Q_{\rm loss}}{{\rm d}t}$ from the convective element from this radiative flux and its adiabatic expansion causes a temperature decrease, simply because from equation (38) ${\rm d}Q_{\rm loss} = -m_e c_P {\rm d}T \Rightarrow \dot{Q}_{\rm loss} = \rho_e \upsilon_e c_P \langle \nabla_{\boldsymbol{x}} T, \dot{\boldsymbol{x}}\rangle$. We then relate the radiative loss $\frac{{\rm d}Q_{\rm loss}}{{\rm d}t} = \frac{8acT^3}{3\kappa\rho}T(\nabla - \nabla_e)\frac{|\Delta \boldsymbol{x}|}{h_P}\frac{S}{2\xi_e}$ to the temperature gradient $||\nabla_{\boldsymbol{x}} T|| = -\frac{1}{\rho V c_P \upsilon}\frac{{\rm d}Q_{\rm loss}}{{\rm d}t}$ using the formalism of Section 4 by recalling that $\frac{\Sigma}{2V\xi_e} = \frac{3}{2\xi_e^2}$ (with V and Σ volume and surface of the convective element) to obtain the relation

$$\frac{\nabla_e - \nabla_{\rm ad}}{\nabla - \nabla_e} = \frac{4acT^3}{\kappa\rho^2 c_P}\frac{\Delta t}{\xi_e^2}, \tag{45}$$

which represents another equation to solve together with those we have developed.

Furthermore, in addition to the convective flux one should consider the flux carried by radiation and conduction. The radiative flux is ubiquitous and no other comments are necessary. Suffice to recall here that it depends on the temperature gradient existing in the stellar medium and the so-called Rosseland mean opacity. Conduction has an important role in the degenerate cores of red giants and advanced stages of intermediate-mass and massive stars, and dominates in the isothermal cores of white dwarfs and neutron stars. The conductive flux can be expressed by the same relation for the radiative flux provided the opacity is suitably redefined. In the following, we will limit ourselves to the case of normal (main-sequence) stars and therefore leave conduction aside. However, to consider the possibility of including the conductive flux, we indicate with $\varphi_{\rm rad|cnd}$ either the radiative flux alone or the radiative and conductive fluxes lumped together with the mean opacity κ suitably redefined (see Kippenhahn et al. 2013, for all details). Therefore, in our simplified situation, the total flux is the sum of the radiative and convective terms $\varphi_{\rm rad} + \varphi_{\rm cnv}$.

We define now the gradient $\nabla_{\rm rad}$ that would be necessary to transport the *total* flux by radiation alone as

$$\begin{aligned}\varphi_{\rm rad} + \varphi_{\rm cnv} &= \frac{4acG}{3}\frac{T^4 m}{\kappa p \|\boldsymbol{x}_{O'}\|^2}\nabla_{\rm rad}\\ &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla_{\rm rad}.\end{aligned} \tag{46}$$

Denoting with ∇ the ambient gradient in presence of radiation and convection the amount of energy carried by the sole radiation (or radiation + conduction) is

$$\begin{aligned}\varphi_{\rm rad|cnd} &= \frac{4acG}{3}\frac{T^4 m}{\kappa P \|\boldsymbol{x}_{O'}\|^2}\nabla\\ &= \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla.\end{aligned} \tag{47}$$

We recollect here the system of six equations equations (47), (46), (45), (44), (43), and (21) (considered with the non-dimensional numerical solution of χ, equation 20, and with its dimensional form equation 17) of the six unknowns $\{\varphi_{\rm rad|cnd}, \varphi_{\rm cnv}, \upsilon, \nabla_e, \nabla, \xi_e\}$ that we solve as a function of position inside the star $\boldsymbol{x}$ and for $t \to \infty$, once the quantities $\{P, T, \rho, l, m, c_P, \nabla_{\rm ad}, \nabla_{\rm rad}, g\}$ or $\{P, T, \rho, l, m, c_P, \nabla_{\rm ad}, \nabla_{\rm rad}, \nabla_\mu, g\}$ are locally known as function of $\boldsymbol{x}$. In the case of a chemically homogeneous layer unstable to convection, the system of equations for $t \to \infty$ is[7]

$$\begin{cases}\varphi_{\rm rad|cnd} = \frac{4acG}{3}\frac{T^4 m}{\kappa P \|\boldsymbol{x}_{O'}\|^2}\nabla\\ \varphi_{\rm rad|cnd} + \varphi_{\rm cnv} = \frac{4acG}{3}\frac{T^4 m}{\kappa P \|\boldsymbol{x}_{O'}\|^2}\nabla_{\rm rad}\\ \frac{\bar{\upsilon}^2}{\bar{\xi}_e} = \frac{(\nabla - \nabla_e)}{(\nabla_e + 2\nabla)} g\\ \varphi_{\rm cnv} = \frac{1}{2}\rho c_p T\,(\nabla - \nabla_e)\,\frac{\bar{\upsilon}^2 t_0 \tau}{h_p}\\ \frac{\nabla_b e - \nabla_{\rm ad}}{\nabla - \nabla_e} = \frac{4acT^3}{\kappa\rho^2 c_p}\frac{t_0\tau}{\bar{\xi}_e^2}\\ \bar{\xi}_e = \left(\frac{t_0}{2}\right)^2 \frac{\nabla - \nabla_b}{\nabla_b + 2\nabla} g\,\bar{\chi}(\tau),\end{cases} \tag{48}$$

where the last equation is the convective element equation studied in Section 4.3, time averaged (see equations 21 and 17 and Section 6.1). To prove the theorem we need to show that the asymptotic behaviour of this system of equations is time independent, i.e. we do not need to introduce any temporal time-scale (or any arbitrary spatial scale l_m as required by the MLT), i.e. the solution of the system is a unique manifold. This will induce an asymptotic behaviour in the numerical solution of the system, which will be presented in Section 6.3.

The solution of this set of algebraic equations leads to a manifold that determines the gradients for which we are seeking, i.e. ∇_e and ∇. We proceed to this solution in the next section, here we prove that the relation between these two gradient is unique as follows. We substitute the first equation into the second one to reduce the number of equations from six to five. We then substitute its result

[7]Since we are interested only in the asymptotic behaviour of the system we can insert equation (36) already accounting for its asymptotic behaviour, see the remarks on equations (36) and (43) for chemically homogeneous layers. However, when performing the numerical integration presented in Section 6.3, all terms will be included.

into the third equation thus obtaining a set of four equations

$$\begin{cases} \varphi_{\mathrm{cnv}} = \frac{4acT^4}{3\kappa h_P \rho}\left(\nabla_{\mathrm{rad}} - \nabla\right) \\ \varphi_{\mathrm{cnv}}\left(\nabla_e + 2\nabla\right) = \frac{\rho c_P T \tau g}{h_P}\bar{\xi}_e(\nabla - \nabla_e)^2 \\ \frac{\nabla_e - \nabla_{\mathrm{ad}}}{\nabla - \nabla_e} = \frac{4acT^3}{\kappa\rho^2 c_P}\frac{\tau}{\bar{\xi}_e^2} \\ \bar{\xi}_e = \frac{g}{4}\frac{\nabla - \nabla_e}{\nabla_e + 2\nabla}\bar{\chi}. \end{cases} \tag{49}$$

Inserting now the first equation into the second one and defining two auxiliary quantities depending only on the local properties of the star

$$k \equiv \frac{acT^3}{\kappa\rho^2 c_P} \quad \text{and} \quad g_4 \equiv \frac{g}{4}, \tag{50}$$

we get

$$\begin{cases} \frac{(\nabla - \nabla_e)^2}{(\nabla_e + 2\nabla)(\nabla_{\mathrm{rad}} - \nabla)}\bar{\xi}_e = \frac{k}{3\tau g_4} \\ \frac{\nabla_e - \nabla_{\mathrm{ad}}}{\nabla - \nabla_e}\bar{\xi}_e^2 = \tau k \\ \frac{\nabla_e + 2\nabla}{\nabla - \nabla_e}\bar{\xi}_e = g_4\bar{\chi}. \end{cases} \tag{51}$$

Furthermore, taking the ratio of the first to the third equation and the ratio of second equation to the square of the third one, after some algebraic manipulations we get

$$\begin{cases} \frac{(\nabla - \nabla_e)^3}{(\nabla_e + 2\nabla)^2(\nabla_{\mathrm{rad}} - \nabla)} = \frac{1}{3\tau}\frac{k}{g_4^2\chi} \\ \frac{(\nabla_e - \nabla_{\mathrm{ad}})(\nabla - \nabla_e)}{(\nabla_e + 2\nabla)^2} = \frac{\tau}{\chi}\frac{k}{g_4^2\bar{\chi}}. \end{cases} \tag{52}$$

For each layer inside the convectively unstable region, we define a few auxiliary variables

$$W \equiv \nabla_{\mathrm{rad}} - \nabla_{\mathrm{ad}} > 0, \tag{53}$$

and

$$\eta \equiv \nabla - \nabla_{\mathrm{ad}}, \tag{54}$$

$$Y \equiv \nabla - \nabla_e. \tag{55}$$

Using these expressions we can write

$$\begin{aligned} \nabla_{\mathrm{rad}} - \nabla &= W - \eta \\ \nabla_e - \nabla_{\mathrm{ad}} &= \eta - Y \\ \nabla_e + 2\nabla &= -Y + 3\left(\eta + \nabla_{\mathrm{ad}}\right). \end{aligned} \tag{56}$$

Finally, equation (52) yields the most important relation and result of our study. Merging the two equations we get

$$\frac{Y^2}{(W - \eta)(\eta - Y)} = \frac{1}{3}\frac{\bar{\chi}}{\tau^2}. \tag{57}$$

which we need to solve for $\tau \to \infty$. But recalling equation (23), the asymptotic temporal dependence of this relation (RHS $\to$ const. for $\tau \to \infty$) establishes that convection inside stars *does not depend on time evolution and/or any spatial scale parameter, to first order* (i.e. it is independent from any the ML/mixing time):

$$\frac{Y^2}{(W - \eta)(\eta - Y)} = \mathrm{const.} \tag{58}$$

This equation in the space of W, η and Y describes a surface containing the manifold of all possible solutions. This manifold is graphically represented in Fig. 2, where W, η and Y are replaced by their definitions: equations (53), (54) and (55) and the RHS constant have been evaluated at some arbitrary layer inside the convective region. It is worth recalling here that of the four temperature gradients

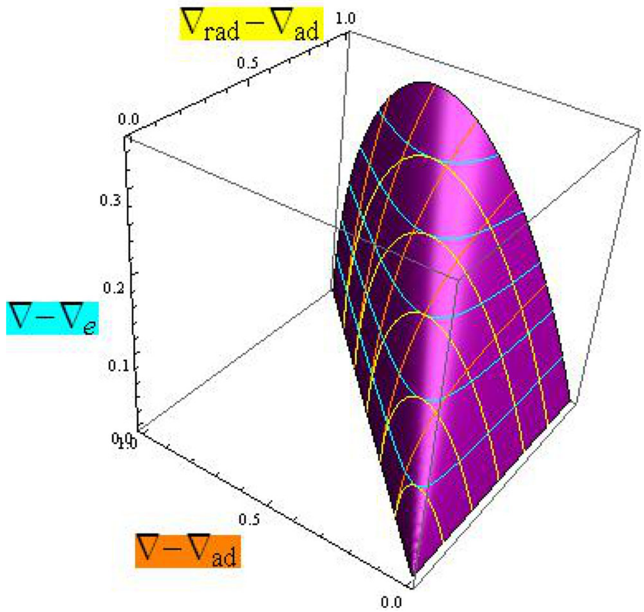

Figure 2. The manifold represent the surface of all the possible solution for the convection equation. Surfaces of constant W, η and Y are plotted but labelled with their corresponding definitions: W in yellow, η in orange and Y in blue. The purple manifold solution is computed for all the numerical values of interest for stars on the LHS of the Hayashi line in the Herstsprung Russel (HR) diagram.

that are involved, i.e. ∇_{rad}, ∇_{ad}, ∇_e, ∇, the adiabatic gradient ∇_{ad} is always known given the thermodynamical state of the medium, and the radiative gradient ∇_{rad} is known once the total flux is specified (this is the typical case of convection in the outer layers, where the MLT and/or the present theory are best suited). We are left with the unknown gradients ∇_e and ∇ asymptotically related by a unique relation, i.e. all the unknowns of the system are in a one-to-one correspondence without any free parameter (Q.E.D.).

It goes without saying that a different free-parameter manifold can be worked out to prove the theorem, and the constant at RHS of equation (58), ∇_{rad} and ∇_{ad} depend on the position inside the star, so that each layer has its own values for ∇_e and ∇. The study of the solution for all the unknowns of our original system is presented in the next section.

Finally, in the case of a chemically non-homogeneous medium, $\nabla_\mu \neq 0$, after changing the definition of Y in equation (55) to

$$Y \equiv \nabla - \nabla_e + \frac{\varphi}{\delta}\nabla_\mu, \tag{59}$$

we obtain a solution manifold in the form of equation (58) (but with a different constant). Thus, an analogous theorem holds in the case of a chemically non-homogeneous convectively unstable layer. This represents the mathematical proof of the recent finding in numerical investigations by Tanner, Basu & Demarque (2013).

6.3 Numerical solution: comparing the new theory with the classical MLT

The previous theorem immediately suggests a time-independent behaviour for the functions that are solutions of our system of equations for stellar convection. Thus, we expect a numerical integration of the system of equations

$$\begin{cases} \varphi_{\mathrm{rad|cnd}} = \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla \\ \varphi_{\mathrm{rad|cnd}} + \varphi_{\mathrm{cnv}} = \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla_{\mathrm{rad}} \\ \frac{\bar{v}^2}{4\xi_e} = \frac{\nabla-\nabla_e}{\frac{3h_P}{2\delta\bar{v}\tau}+\nabla_e+2\nabla} g_4 \\ \varphi_{\mathrm{cnv}} = \frac{1}{2}\rho c_P T\left(\nabla-\nabla_e\right)\frac{\bar{v}^2\tau}{h_P} \\ \frac{\nabla_e-\nabla_{\mathrm{ad}}}{\nabla-\nabla_e} = \frac{4acT^3}{\kappa\rho^2 c_P}\frac{\tau}{\bar{\xi}_e^2} \\ \bar{\xi}_e = \frac{\nabla-\nabla_e}{\frac{3h_P}{2\delta\bar{v}\tau}+\nabla_e+2\nabla} g_4 \bar{\chi}(\tau) \end{cases} \tag{60}$$

to present an asymptotic behaviour in time for at least some of the function solutions. In equation (60) the last equation is obtained from equation (43) with $\nabla_\mu = 0$ and the help of equation (17). Note that in the third and sixth equation of the system, the term $(3h_p/2\delta\bar{v}\tau)$ is retained. Although an algebraic solution of this system (containing six equations and six unknowns as function of the time) can be worked out with an algebraic manipulator, it is exceptionally long and does not add a better comprehension of the physics we are investigating. For this reason, we present here a numerical investigation that enlightens the features of the system we presented and deferring to a future study the investigation of the complete ADE system (Pasetto et al. 2014).

At the same time, this will prove that we obtain correct numerical values for the gradients studied and that we can directly compare our results with the standard ones of the literature based on complete stellar models calculated with any of the sophisticated codes of stellar structure in the literature, e.g. the classical Göttingen code developed by Hofmeister et al. (1964), the many versions of this developed over the years by the Padova group, e.g. Chiosi & Summa (1970) with semiconvection, Bressan et al. (1981) with ballistic convective overshoot from the core, Alongi et al. (1991) with envelope overshoot, Deng, Bressan & Chiosi (1996a,b) and Salasnich, Bressan & Chiosi (1999) with turbulent diffusion, finally the many revision and improvements described in Bertelli et al. (1994, 1995, 2008); Bertelli & Nasi (2001); Bertelli et al. (2003, 2009), and the Garching version developed by Weiss & Schlattl (2008, GARSTEC).

Using the library of complete stellar models of Bertelli et al. (2009) for different values of the stellar mass and chemical composition, calculated with the standard MLT ($\Lambda_m = 1.64$), we select a typical model for the Sun on which we can test the new theory of convection in a very simple way while we leave an extended numerical investigation of a different stellar models to a future study (Pasetto et al. 2014). Indeed the Sun is the best place to test the new theory of convection because for it we have the most complete information (see e.g. Bonaca et al. 2012, and references therein). The solar model provides the mass, luminosity, pressure, density, temperature, opacity, chemical composition and many other physical quantities throughout the Sun and we have precise data on the total luminosity, effective temperature and radius in addition to surface abundances and a rather precise estimate of the age.

In particular the Sun's model provides us with ∇_{rad} and ∇_{ad} the two gradients that are needed to start the analysis and that do not depend on the convection theory in use. It is worth recalling here that ∇_{ad} in presence of ionization, as it occurs in the external layers of a star, is a complicated function of EoS, temperature, density, degree of ionization, etc. that cannot be approximated by simple analytical expressions. It becomes a simple function of the EoS only when ionization is complete (Weiss et al. 2004). The model of the Sun we are using includes ionization and takes it into account when calculating ∇_{ad}.

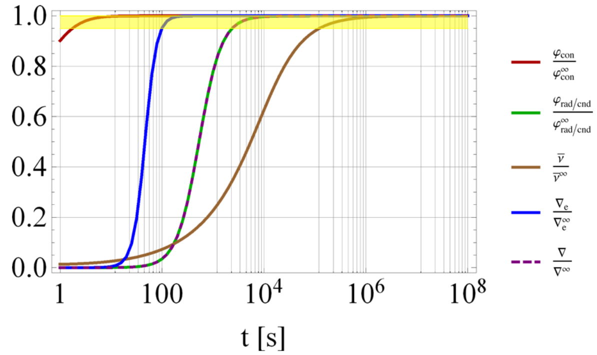

Figure 3. Temporal dependence of the mean velocity of a convective element in the Sun at the layer $z = \frac{98}{100}\mathrm{R}_\odot$. Remarkably, the velocity reaches the asymptotic value within a rather short time-scale (of the order of a month or so). The yellow bar shows the 5 per cent region of the asymptotic value of the various quantities on display.

Using this model, we calculate ∇ and ∇_e, velocities, etc. both according to the new formalism for convection and also to the standard MLT in the Bertelli et al. (2009) model adopting for MLT the current estimate for Λ_m in the Sun, i.e. $\Lambda_m = 1.64$.

Given the gradients ∇_{ad} and ∇_{rad}, together with the function $\bar{\chi}(\tau)$, in each layer in the external convective regions of the Sun, we solve the system (60) as a function of τ (i.e. time) and position to derive the gradients ∇ and ∇_e, the mean velocity $\bar{v}$, and the convective flux φ_{cnv}. Since the term $(3h_p/2\delta\bar{v}\tau)$ is retained, the system will relax to that of equations (48) only after a certain time interval has elapsed. The best way of evaluating how long the time interval necessary to reach the asymptotic behaviour is to plot the time dependence of the velocity, the temperature gradients ∇ and ∇_e, and the convective fluxes φ_{cnv}, $\varphi_{\mathrm{rad/cnd}}$ (Fig. 3). In particular, in Fig. 3 the yellow bar indicates when the asymptotic values are reached within 5 per cent of their limit value. These curves plotted refer to the layer $z = \frac{98}{100}\mathrm{R}_\odot$, i.e. a shell representative of the external convective region of the Sun. All the quantities of interest reach the asymptotic value on a time-scale of one month or even shorter.

Note that with our theory the path travelled by the bubble is of course not defined, nor has it any physical meaning, being a theory developed in a system of reference S_1 where the bubble is at rest by definition.

Finally, it might be of interest to estimate the typical lifetime t_* of a bubble by taking the ratio of the natural scalelength h_p to typical expansion velocity of a convective cell $t_* \equiv \frac{h_P}{\dot{\xi}_e}$ that at $z = \frac{98}{100}\mathrm{R}_\odot$ is $t_* \sim 3.2$ h.

Now we verify the assumption leading to our equation (42) with a direct numerical integration of this equation as a function of time, using the input physics of Sun model. If we define the ratio between the first and second term in the LHS of equation (42) computed with the average velocity of the convective elements:

$$\bar{\mathrm{Q}}_\odot \equiv \frac{\frac{3h_P}{\bar{v}\Delta t 2\delta}}{\nabla_e + 2\nabla}, \tag{61}$$

we have seen that the asymptotic behaviour expected as $t \to \infty$ requires this ratio to converge to zero. The same trend is expected for the average behaviour of the convective elements. What is not expected a priori from a simple asymptotic expansion is the maximum that we see in Fig. 4. This maximum is the result of the opposite time dependence of the numerator and denominator: while the denominator progressively increases towards its asymptotic value

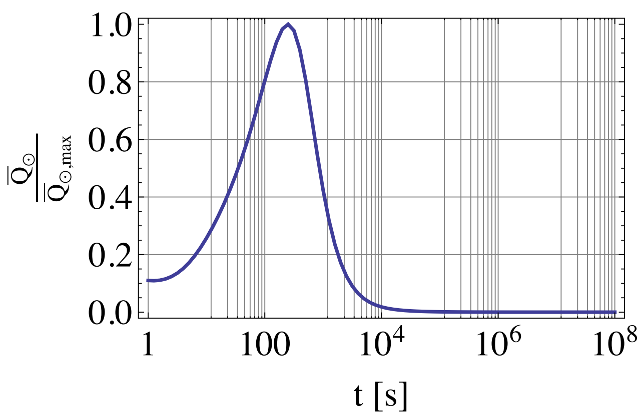

Figure 4. Q parameter for the model of the Sun in an arbitrary but fixed point in the Sun at the layer $z = \frac{98}{100}\mathrm{R}_\odot$. The plot was normalized to the value of the maximum.

Table 1. Numerical results at the layer $\boldsymbol{x}_{O'} = \frac{98}{100}\mathrm{R}_\odot$ inside the Sun's model. The differences between our theory and MLT at any other arbitrary but fixed unstable convective layer are of the same order.

	Units	Equation (60)	Equations (62) with $\Lambda_m = 1.64$
$\varphi_{\mathrm{rad\vert cnd}}$	erg s^{-1}	61.6629	61.7020
φ_{cnv}	erg s^{-1} × 10^7	6.428 55	6.428 54
h_P	m × 10^6	2.138 52	2.138 52
∇_{rad}		295 187.0	295 187.0
∇_e		0.283 10	0.283 10
∇		0.283 15	0.283 32
$\bar{v}$	m s^{-1}	184.042	208.913

as shown in Fig. 3, the numerator monotonically decreases with time.

In column (3) Table 1 we list the results we have obtained from the system of equation (60) limited to the layer we have selected ($z = \frac{98}{100}\mathrm{R}_\odot$). Any other convectively unstable layer would have shown similar results. A numerical investigation of the consequences of the present theory and a complete upgrade of the stellar models in Bertelli et al. (2008, 2009) is deferred to the forthcoming paper Pasetto et al. (2014).

Now we compare our results with those obtained from the standard MLT of stellar convection represented by the system of equations

$$\begin{cases} \varphi_{\mathrm{rad|cnd}} = \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla \\ \varphi_{\mathrm{rad|cnd}} + \varphi_{\mathrm{cnv}} = \frac{4ac}{3}\frac{T^4}{\kappa h_P \rho}\nabla_{\mathrm{rad}} \\ \bar{v}^2 = g\delta\left(\nabla - \nabla_e\right)\frac{l_m^2}{8h_P} \\ \varphi_{\mathrm{cnv}} = \rho c_P T\sqrt{g\delta}\frac{l_m^2}{4\sqrt{2}}h_P^{-3/2}(\nabla-\nabla_e)^{3/2} \\ \frac{\nabla_e-\nabla_{\mathrm{ad}}}{\nabla-\nabla_e} = \frac{6acT^3}{\kappa\rho^2 c_P l_m \bar{v}}, \end{cases} \tag{62}$$

in which l_m contains the ML parameter Λ_m. The derivation and solution of this system of equations can be found in any classical textbook of stellar structure (e.g. Kippenhahn et al. 2013; Weiss et al. 2004). We limit ourselves to note that in this classical system we have five equations instead of six, see equation (48). If we adopt the same model of the Sun we have used before to solve the system equation (48) with the extra value of Λ_m tuned on the Sun, we obtain the results presented in column (4) of Table 1. The results are practically coincident with those from this new theory.

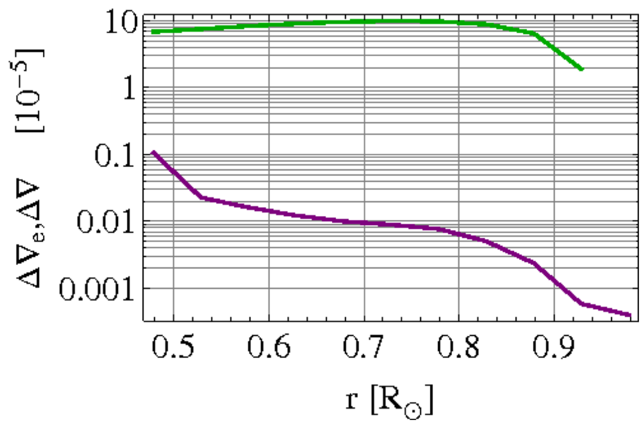

Figure 5. Normalized difference function evaluated for ∇_{e} ($\Delta\nabla_{\mathrm{e}}$, purple line) and ∇ ($\Delta\nabla$, green line). MLT values were computed assuming $\Lambda_m = 1.64$.

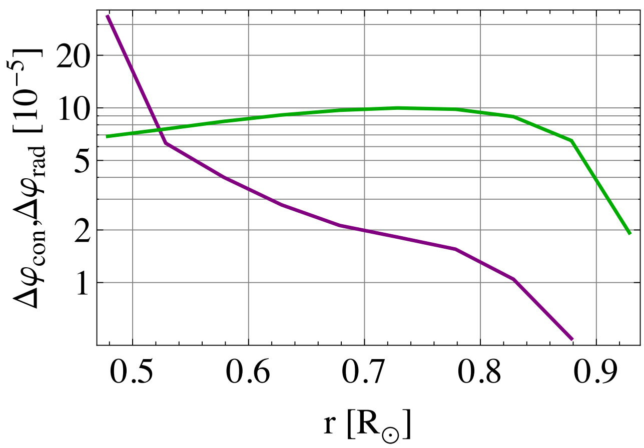

Figure 6. Normalized difference function evaluated for convective ($\Delta\varphi_{\mathrm{cnv}}$, purple line) and radiative flux ($\Delta\varphi_{\mathrm{rad}}$, green line). Same Λ_m has been adopted as in Fig. 5.

The comparison between our theory and the standard MLT predictions can then be extended over the entire convective region inside the Sun. We define a normalized difference function as $\Delta\Xi(\boldsymbol{x}) \equiv \left|\frac{\Xi_{\mathrm{MLT}}(\boldsymbol{x})-\Xi_{\mathrm{new}}(\boldsymbol{x})}{\Xi_{\mathrm{new}}(\boldsymbol{x})}\right|$, where $\Xi = \nabla$, ∇_e, etc., i.e. for every function of interest we compute the difference of its values obtained with the standard MLT, Ξ_{MST}, and our new approach Ξ_{new}. The results are plotted in Fig. 5 for $\Xi = \nabla$ and $\Xi = \nabla_{\mathrm{e}}$ and in Fig. 6 for $\Xi = \varphi_{\mathrm{cnv}}$ and $\Xi = \varphi_{\mathrm{rad}}$.

As it is evident from these figures, the normalized differences between the two theoretical predictions are of the order of $O(10^{-5})$ over all the stellar radii of interest. This result holds independently from the stellar model adopted.

Furthermore, we show in Fig. 7 that the conditions that form the foundation of our theory (equation 12) are fully satisfied. The sound speed profile of our model, which is part of our equation (12), is shown in Fig. 7. The result is closely consistent with the models available from the literature (e.g. Weiss et al. 2004) or computed from Bertelli et al. (2008).

The different velocities of the convective elements predicted by the two theories may have some implications on the extension and efficiency of convective overshooting. The subject

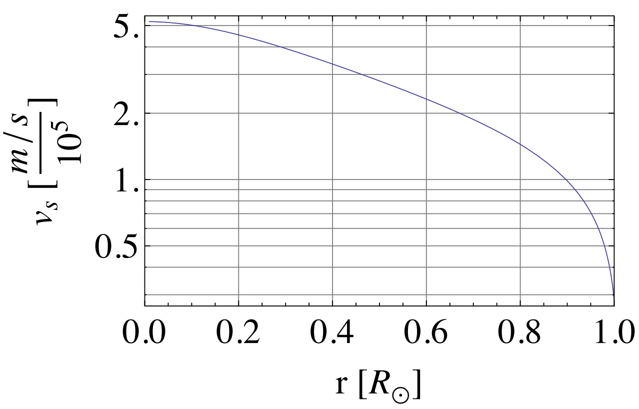

Figure 7. Profile of the sound velocity in the adopted model of the Sun and limited to the outer layers.

will be investigated in a forthcoming paper (Pasetto et al. 2014). Moreover, the meaning of the time-scales also differs. As far as the lifetime of the typical convective element is concerned at a chosen stellar radius, we evaluate this for the MLT as customary with $t_* = \frac{\Lambda_m h_P}{v} \sim 2.3$ d.

Finally, it is worth remarking that from the mathematical point of view, the results do not depend on the particular stellar model we have examined. Indeed, it would suffice to assign reasonable values to the gradients $\nabla_{\rm rad}$ and $\nabla_{\rm ad}$ and all other quantities to verify that *both systems of equations, i.e. equation (62) or (48), would lead to the same solutions even though the equations are different in form and physical meaning*. This is confirmed by the entries of Table 1 and *by the difference functions* shown in Figs 5 and 6.

The results we have obtained come out in a simple and straightforward way. The physical foundations of the theory are simple and free of ad hoc assumptions. Equally, this is the case for the mathematical developments that are carried out to reach the final result.

7 CONCLUSION AND CRITICAL COMMENTS

It has taken almost one century to develop a theory for stellar convection and energy transport without the ML parameter. In this first study, we have presented a new simple theory of stellar convection that does not contain adjustable parameters such as the ML. The whole solution (temperature gradients of the medium and convective elements, the distances travelled by typical elements, their velocities and lifetimes, the convective flux, etc.) are all determined by the physical conditions inside the stars. We consider this to be a significant advance.

We have formulated the equations of fluid dynamics in the potential-flow approximation. A posteriori it is evident that this is advantageous, simply because for a body rapidly expanding from rest it is a good approximation every time the inertia forces are larger than the viscous ones (at least on a time-scale of the order of $\tau \sim O(\frac{\Delta z}{\xi})$). This justifies this approximation and the description of the mechanics of the convective elements that follows.

We summarize here the major features and the major achievements of our theory. The approach is based on the addition of an equation for the motion of the convective elements to the classical system of algebraic equations for the convective energy transport. The motion of a convective element is described by the vertical displacement of its barycentre and relative expansion (contraction) of its radius, and the inertia of the fluid mass displaced by the convective element is accounted for. Consequently, the acceleration imparted to the convective elements in addition to the buoyancy force takes into account effects that in the standard MLT are neglected, i.e. the inertial term of the fluid displaced by the movement and expansion (contraction) of the convective cell, and an extra term arising from the changing size of the convective element (the larger the convective element the stronger is the buoyancy effect and the larger the acquired velocity and vice versa). This results in a new and more complicated term of the acceleration $\propto \frac{\nabla-\nabla_e}{\nabla+2\nabla} \propto \nabla - \nabla_e$, agreeing with the Schwarzschild criterion.

It is found that the best reference frame to describe the system is the one comoving with the element. Our treatment of the fluid dynamics governing the motion of the convective elements allows us to remove any preliminary assumptions about the size and path of the convective elements and these now arise as natural outputs of our theory.

No external calibration of parameters is required: the solution of the equations governing stellar convection is unique, in the sense that it is fully determined by physical properties of the medium. This is best shown in our Fig. 3 which represents the numerical and graphical visualization of the Theorem of Uniqueness. The solution of the system we build up behaves asymptotically, so no ML/time is required. It is required only to wait that amount of time for which, within a given layer, the solution becomes stable to the required precision (in our case the yellow strip of Fig. 3).

The whole system of ADEs is further simplified to an algebraic system by decoupling the evolution of the generalized coordinates of the radius and position of a convective element. This result is achieved by means of a series of theorems, corollaries and lemmas that permit the analysis of the different mathematical and physical aspects of the problem, always retaining the necessary rigour to trace progress to the final result. The new theory applied to the external convection in the Sun has been proven to yield results (convective fluxes, temperature gradients ∇ and ∇_e, velocity and size of the convective elements) as good as those that are currently obtained with the standard MLT upon having calibrated the ML parameter. The size and path of a convective element will change with the position inside the convective region, the evolution of the star, i.e. the particular phase under consideration, and finally the stellar mass itself.

We have two final comments. First, we comment briefly on the reasons why it was necessary to develop the theory in the non-inertial reference frame S_1 comoving with the convective element instead of the more natural frame S_0 comoving with the star. The flow past a sphere is indeed a well-studied topic of fluid dynamics (too large to be reviewed here!) and recently the Lagrange formalism has become particularly suitable to address this kind of problem: see e.g. Tuteja et al. (2010, and references therein) for a recent review and discussion. Unfortunately, this approach does not yield equations (8) and (16), in the Theorem and companion Corollary discussed in Section 4 which are required to derive the acceleration term in which the properties of the convection element are related to the depth inside the star. To compute the kinetic term of the energy we would require to evaluate the integral $T = \frac{1}{2}\rho \int \|\boldsymbol{v}_0\| \mathrm{d}^3\boldsymbol{x}$ which for the potential flow of equation (5) turns out to diverge. This would force us to work at the limit condition $\lim_{\xi\to\infty} \boldsymbol{v}_0 = 0$ for equation (6) and with a suitable potential energy $E_P = E_P(z, R)$ in the two generalized coordinates z and R as defined above in S_0. The resulting Lagrange equations under the approximation of equation (12) reduce to a system of decoupled equations instead of equation (16) which in contrast retains the desired coupling between the generalized coordinates. At this point, the only viable solution is instead to write a Lagrangian for the non-inertial system, and this is indeed what was derived in Pasetto & Chiosi (2009) (their section 3.1), which represents our starting point.

Secondly, we compare the new theory with recent statistical analyses of turbulent convection in stars by Mocák et al. (2014, and references therein). In brief, adopting the so-called Reynolds-Averaged Navier Stokes framework in spherical geometry, developed by the authors over the years (e.g. Mocák et al. 2014) they present results for convection occurring in the stellar interiors and evolutionary phases of typical stars. These works represent an ideal tool to set up *numerical* experiments of stellar convection (from 1D to 3D models). However, an *analytical* approach provides an understanding of the process in a way that a numerical one does not. In a forthcoming study, we will present an extended survey of the impact on stellar models and a direct integration of the ADE system composed by equation (48) with equation (16) to extend the present formalism and theory to the case of overshooting, where the path travelled by the convective element has specifically to be computed (Pasetto et al. 2014).

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APPENDIX A: MATHEMATICAL SOLUTION OF THE NON-DIMENSIONAL EXPANSION RATE EQUATION

Despite its elegance, equation (22) that we report here,

$$\chi \frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} + \frac{3}{2}\left(\frac{\mathrm{d}\chi}{\mathrm{d}\tau}\right)^2 - 2\chi = 0, \tag{A1}$$

is a *non-linear* DE, and so there are no general techniques available in the literature to solve it. Nevertheless, as the ODE equation (22) does not contain explicitly the independent variable, τ, a convenient change of variables is performed by introducing $\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \eta$. In our case, we have $\frac{\mathrm{d}^2\chi}{\mathrm{d}\tau^2} = \frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \frac{\mathrm{d}\eta}{\mathrm{d}\chi}\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \frac{\mathrm{d}\eta}{\mathrm{d}\chi}\eta$. Therefore, equation (22) becomes

$$\chi\eta\frac{\mathrm{d}\eta}{\mathrm{d}\chi} + \frac{3}{2}\eta^2 - 2\chi = 0, \tag{A2}$$

which is a lower order DE, the solution of which is simply

$$\eta(\chi) = \pm\frac{\sqrt{c_1 + \chi^4}}{\chi^{3/2}}. \tag{A3}$$

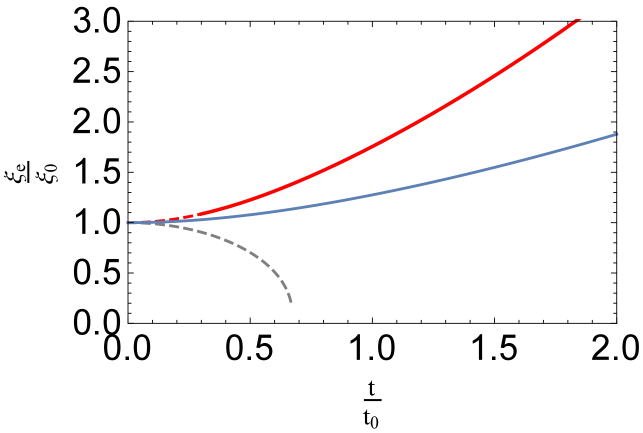

Figure A1. Integration of equation (A5), for convection-unstable (χ, thick red), time averaged ($\bar{\chi}(\tau)$, thick blue) and stable layers (dashed grey). The red line is partially dashed to remind the reader that equation (A5) holds only on its asymptotic expansion, say for $t > \hat{t}$ for an arbitrary chosen $\hat{t} = 0.3t/t_0$ in the figure (see Section 4.3 for detailed discussion) with monotonic character evident from equation (23). The black semicircle over $\{\chi, \tau\} = \{1.0, 0.0\}$ is excluded.

Using the original variable $\frac{d\chi}{d\tau} = \eta$ and equation (18) we obtain

$$\frac{d\chi}{d\tau} = \pm\frac{\sqrt{\chi^4 - 1}}{\chi^{3/2}}, \tag{A4}$$

where the positive sign is for the expanding convective elements and the negative sign for the contracting ones. The solution of this equation exists for $\chi > 1$ strictly. In the case of an expanding convective element, a solution is always possible. By separating the variables, we get

$$\int_1^X \frac{\chi^{3/2}}{\sqrt{\chi^4 - 1}} d\chi = \int_0^T d\tau, \tag{A5}$$

with $X = \frac{\xi_e(T)}{\xi_0}$, whose solution for the LHS is obtained by means of two changes of variable, first $y = \chi^4$ and then $z = \frac{y-1}{X^4-1}$. We must exclude the case $X = 1$ which means that the initial and final sizes of the convective elements are equal. Therefore, in all other cases $X \neq 1$, the solution is

$$\int_1^X \frac{\chi^{3/2}}{\sqrt{\chi^4 - 1}} d\chi = 2(X^4 - 1)^{1/2} {}_2F_1\left(\frac{3}{8}, \frac{1}{2}; \frac{3}{2}; 1 - X^4\right), \tag{A6}$$

where we made use of the standard definition of the Hypergeometrical Function ${}_2F_1(a, b; c, z)$ (e.g. Lebedev 1972)

$${}_2F_1(a, b; c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}. \tag{A7}$$

The $\Gamma(c)$, $\Gamma(b)$ and $\Gamma(c - b)$ are the Euler Gamma functions whose values of interest for equation (A6) are $\Gamma(1) = 1$, $\Gamma(3/2) = \sqrt{\pi}/2$, and $\Gamma(1/2) = \sqrt{\pi}$. This pretty and elegant solution is particularly suitable for numerical implementations thanks to the large body of literature on the ${}_2F_1(a, b; c; x)$ functions. Most importantly, the intersection with the solution of the RHS of equation (A5) can be proven to be *bijective* thus representing a unique solution of the equation for a convecting element in S_1. The solution obtained from equation (A6) and equation (A5) for unstable and stable convective regions is plotted in Fig. A1. *We remind the reader that the solution is no longer valid for small or null values of* τ*, but only in the limit of very large* τ *and hence* t*, mathematically* $t \to \infty$, i.e. only when the condition for equation (12) is satisfied. Of course this does not represent a difficulty owing to the large possible freedom that we have on the time-scale t_0 (say, seconds, days, to years over the time-scale of the stellar existence, from Myr to Gyr).

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