THE ASTROPHYSICAL JOURNAL, Vol. 160, April 1970


# BINARY FORMATION BY THREE-BODY COLLISIONS*

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*Received 1969 May 12; revised 1969 August 25*

## ABSTRACT

Keck's "variational method" is used to obtain an expression for the rate of formation of binaries due to stellar three-body collisions, under quite general conditions. In the solar neighborhood this rate is too small to account for any binaries. In globular clusters it could account for a few. The mass dependence and separation dependence of the three-body rate is also not in agreement with observations in the solar neighborhood, disproving the possibility that binaries were formed by this process but under different galactic circumstances. The general rate expression is also applicable to other problems, such as comet capture by a star or asteroid capture by a planet. An estimate of the latter, using available data, indicates that the rate is somewhat too low to account for Jupiter's outer moons if the third body is another asteroid. The third body might be a more massive satellite, however; this possibility is promising.

## I. INTRODUCTION

In this paper we apply the "variational method" of Keck (1960, 1962, 1967; Keck and Carrier 1965; Makin and Keck 1963; Mansbach and Keck 1969) to the calculation of the rate of binary-star formation through a process of three-body collisions.

This problem has been considered before. In an early work, Jeans (1922) mentioned two-body collisions as a possible source of binaries, but offered no mechanism by which the two stars could lose enough energy to remain bound. Chandrasekhar (1944) considered statistical fluctuations of the gravitational field and showed that binaries, once formed, will not be rapidly dissociated through the action of other stars; however, he did not discuss formation of binaries through such action. Lynden-Bell (1968) has remarked that the rate of formation of tight ($\sim$1 a.u.) binaries through three-body collisions is minute. Huang (1967) also stated that the rate of three-body collisions is too small to account for many binary stars.

In this paper we obtain, by a conceptually simple method, a formula expressing this rate of binary-star formation under quite general conditions. The validity of this method has been demonstrated for the analogous plasma problem by comparison with experiment (Mansbach and Keck 1969). We avoid the divergences which appear in the binary equilibrium distribution function at small and large binding energies, such as were encountered, for example, by Lynden-Bell (1968); and we have some justification from the plasma problem for our treatment of the divergence at large distances of the third star, which arises because of the long-range nature of the gravitational interaction.

Unfortunately, the algebra involved in the variational method becomes quite unmanageable for problems involving more than three stars. Therefore, it cannot be extended to provide an analytic treatment of binary formation from the disintegration of small clusters, such as has been investigated by van Albada (1968), who used numerical techniques.

## II. DERIVATION OF THE GENERAL RATE

The collision process we consider consists of an encounter of three unbound stars in which stars 1 and 2 give up some of their relative energy to star 3, and are left in a gravitationally bound state. We assume that we can neglect the rest of the Galaxy during

* This research was supported in part by U.S.A.F. contracts OSR-368-69 and AF 19628-69-C-0074.

the course of such an encounter. We view these encounters in an eighteen-dimensional phase space in which each three-body system is represented by a point, which moves along some trajectory as the system evolves in time. Every selection (including relabeling) of three stars in the Galaxy is represented by such a moving point. Moreover, we identify the regions in this phase space which correspond to stars 1 and 2 being gravitationally bound or free. The rate of binary formation, then, is simply the inward flux of points across a surface which separates these regions:

$$\Re_S = -\frac{1}{V}\int_S \rho \boldsymbol{v}\cdot\boldsymbol{n}dS\,. \tag{1}$$

Here $\rho$ is the density of points in phase space, $\boldsymbol{v}$ is the (eighteen-component) phase-space velocity, $\boldsymbol{n}$ is the outward unit normal to the surface $S$, and $V$ is the spatial volume.

For $\rho$ we use a Maxwell-Boltzmann equilibrium distribution function $\rho_e$, and we integrate over only that part of $S$ for which $\boldsymbol{v}\cdot\boldsymbol{n} < 0$. This is equivalent to using $\rho = \rho_e$ for unbound systems and $\rho = 0$ for bound systems in that no trajectories are counted which cross from the bound to the free region. The resulting one-way equilibrium rate may be used as an estimate of the true rate, since (1) $\rho = \rho_e$ is a good representation of

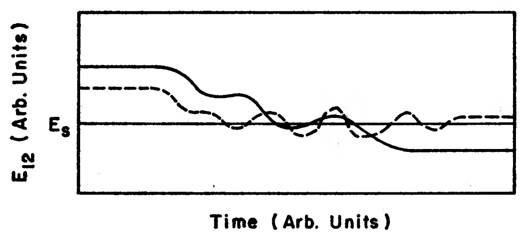


FIG. 1.—Qualitative plot of $E_{12}$ versus time during a three-body collision. Note that $E_{12}$ becomes constant in time when the third star gets very far away. *Solid line*, a typical successful recombination; *dashed line*, an unsuccessful collision.

the motion of free stars (Chandrasekhar 1942) and (2) the cascading process of successive three-body collisions (Keck and Carrier 1965; Mansbach and Keck 1969) is a posteriori negligible. For stars of masses $m_1$, $m_2$, $m_3$, we have

$$\rho = \rho_e = \prod_{i=1}^{3}\left[\frac{n(m_i)}{(2\pi m_i kT)^{3/2}}\right]e^{-H/kT}\,, \tag{2}$$

where $n(m_i)$ is the number density (per unit mass) of free stars of mass $m_i$, $H$ is the total energy of the three-body system, and $kT$ is the average "thermal energy" of the stellar distribution, i.e., the average kinetic energy due to the peculiar velocities of the stars.

We define

$$E_{12} = \tfrac{1}{2}\mu_{12}v_{12}^2 - \frac{Gm_1m_2}{r_{12}}\,, \tag{3}$$

the relative energy of the 1–2 pair. Here $\mu_{12}$ is the reduced mass of the pair, $r_{12}$ their separation, and $v_{12}$ their relative velocity. In order for a pair of stars to combine into a bound system, $E_{12}$ must start out positive and end up sufficiently negative for the pair to be considered bound, i.e., the orbit size should be significantly less than the typical interstellar distance. Thus every phase-space trajectory that represents a successful three-body recombination collision must cross the surface $S$ defined by $E_{12} = E_S$, where $E_S$ is some suitably chosen negative energy. Note, however, that such a trajectory may cross more than once, and that some trajectories cross into the bound region but then recross and so do not end up bound (see Fig. 1). Note that by integrating only over

$E_{12} = E_S$ we avoid the divergences in the bound-pair equilibrium distribution function which occur, at $r_{12} = 0$ because the binding energy blows up when we treat the stars as point masses, and at $r_{12} \to \infty$ because there are so many bound states available to the system.

As normally happens in Coulomb problems, however, the integral in equation (1) still diverges logarithmically in the distance of star 3, and some cutoff procedure must be invoked. Makin and Keck (1963) introduced a cutoff on the dimensionless parameter $\omega\tau$ at $\omega\tau = 1$. Here

$$\omega = \frac{(-2E_S)^{3/2}}{G m_1 m_2 \mu_{12}{}^{1/2}} \tag{4}$$

is the orbital frequency of the 1–2 pair and

$$\tau = \frac{r_{13} + r_{23}}{2v_3} \sin \gamma \approx \frac{b}{v_3} \tag{5}$$

is a measure of the collision time. (Here $b$ is the impact parameter of star 3 relative to the center of mass of the 1–2 system: $b = r_3 \sin \gamma \approx \frac{1}{2}(r_{13} + r_{23}) \sin \gamma$ in the region where the cutoff is invoked. The quantity $r_{13}$ is the separation between stars 1 and 3,

TABLE 1

VALUE OF $\alpha$ TO GIVE AGREEMENT WITH RIGOROUS PROCEDURE

| $\epsilon$ | $\alpha$ |
|---|---|
| −5 | 0.47 |
| −3 | 0.80 |
| −1 | 1.15 |
| 0 (extrapolated) | 1.3 |

$r_3$ is the distance of star 3 from the center of mass of the 1–2 pair, and $v_3$ is the corresponding velocity; $\cos \gamma = v_3 \cdot r_3 / v_3 r_3$.) The quantity $2\pi\omega\tau$ is the number of orbits traversed by the pair during the passage of star 3. The reasoning is that for $\omega\tau \gg 1$ the collisions become adiabatic, each star of the bound pair alternately gaining and losing energy to the free star as the bound stars orbit about their center of mass. Thus distant collisions tend to cross the surface a great number of times, a single trajectory contributes to the integral each time it crosses, and very often it ends above the surface, anyway. We expect that the true rate, which takes multiple crossings into account, converges without a cutoff. Mansbach and Keck (1969) have indeed verified this in the plasma case by determining the entire trajectories of a statistically large number of points crossing the surface. This method, while more rigorous, requires a considerable amount of computer time. What is done here instead is first to repeat the plasma calculation of Makin and Keck, but by using a cutoff at $\omega\tau = \alpha$ instead of $\omega\tau = 1$. The value of $\alpha$ is then adjusted to give agreement with the more rigorous results of Mansbach and Keck (1969). In effect we omit as many successful trajectories with $\omega\tau > \alpha$ as there are unsuccessful ones with $\omega\tau < \alpha$. This value of $\alpha$ we then apply to the astrophysical problem. In fact, $\alpha$ varies somewhat with $\epsilon = E_S/kT$ (see Table 1), and we extrapolate graphically to $\epsilon \approx 0$ which is relevant in our case. We thus obtain $\alpha = 1.3$. The exact value is not critical, as there is only a mild logarithmic dependence on $\alpha$.

We perform the integrations in equation (1), using the coordinates defined by Keck (1960) and the step-by-step procedure described by Mansbach and Keck (1969) to get the rate per (unit mass range)$^3$:

$$\Re_S = \frac{n(m_1)n(m_2)n(m_3)4\pi 2^{1/2} G^{1/2} m_3 (G m_1 m_2)^{9/2}}{(kT)^{3/2}(-E_S)^3(m_1 + m_2)^{1/2}} e^{-\epsilon}\phi_S(\epsilon, \alpha; m_i) , \tag{6}$$

where $\epsilon = E_S/kT$. This derivation is outlined in more detail in the Appendix. An exact expression for $\phi_S$ appears there, also.

The final integrations for the function $\phi_S$ have been done numerically on the computer. The analytic form given below is obtained from approximate integration techniques, with the coefficient and power of the third term and the value of the constant term fitted to the computed numbers:

$$\begin{aligned}\phi_S \approx\ & 0.28 - 0.24 \log_{10} |\epsilon| + 1.25 \left[ |\epsilon| \frac{m_3(m_1 + m_2)}{2m_1m_2} \right]^{0.8} \\ & + 0.47 \log_{10}\left(\frac{a}{1.3}\right) + 0.24 \log_{10}\left[\tfrac{4}{3} \frac{m_1m_2}{m_3} \frac{(m_1 + m_2 + m_3)}{(m_1 + m_2)^2}\right] .\end{aligned} \tag{7}$$

The analytic representation is excellent for $|\epsilon| m_3(m_1 + m_2)/(2m_1m_2) \leq 0.1$, which is the range of interest.

The choice of $\epsilon$ proceeds somewhat differently here than in the plasma case. There we were interested in the net rate from free ions and electrons to ground-state atoms through a sequence of three-body collisions ("cascade process"). The one-way rate $\mathfrak{R}_S(\epsilon)$ is an overestimate to this rate for *any* $\epsilon$, since it does not subtract the effect of reexcitations during subsequent collisions. (This is in addition to the recrossings during a single collision, which we have taken care of by our cutoff procedure.) Thus we could choose $\epsilon$ to be the point at which $\mathfrak{R}_S(\epsilon)$ has its minimum (Makin and Keck 1963). This point has been described as a rate-limiting "bottleneck" in the cascade recombination process. Above this bottleneck, electrons can be excited and de-excited many times before finally crossing the bottleneck, and below it also; but fewer electrons will be reexcited back up across the bottleneck. A closer analysis (Keck and Carrier 1965; Mansbach and Keck 1969) supports this argument, but shows that this minimum one-way rate is still 3 times too large, because of reexcitations across the bottleneck.

In the astrophysical case, however, the bottleneck energy corresponds to binaries with a separation of the order of 1 a.u., far less than most visual binaries. Nevertheless, the collision rate is a posteriori so small that we can neglect reexcitations, so that for *any* $\epsilon < 0$, $\mathfrak{R}_S(\epsilon)$ gives a reasonable value for the rate of formation of all binaries with $E_{12} < \epsilon kT$, provided our distribution function $\rho$ is accurate. Moreover, if $|\epsilon| \ll 1$, most of the contribution to the rate comes from initial configurations with $E_{12} > 0$, and our use of an equilibrium distribution is justified.

We define $E_S = -Gm_1m_2/2r_s$, where $r_s$ is the average separation of the binary, since this is the parameter that would appear in statistical surveys. In integrating over the mass distribution we wish to fix $r_s$ rather than $E_S$, thereby calculating the rate of formation of binaries with separation less than $r_s$. The quantity $r_s$ is to be the maximum separation for which binaries are readily identifiable. However, it must be small enough so that the average cutoff distance $r_c$, at which $\omega_T$ is typically around $a$, is less than the average interstellar distance, in order that the neglect of surrounding stars can be justified. Setting $r_c = \frac{1}{2}(r_{13} + r_{23})$, $v_3 = (2kT/\langle m\rangle)^{1/2}$, $m_1 = m_2 = m_3 = \langle m\rangle$, and $\sin\gamma = \frac{1}{2}$ in equations (4) and (5) gives

$$r_c \sim 2ar_s^{3/2}(kT/G\langle m\rangle^2)^{1/2} . \tag{8}$$

## III. RESULTS

We first integrate equation (6) over the mass distributions, keeping $r_s$ constant, to get

$$\mathfrak{R} = \tfrac{1}{2}\int \mathfrak{R}_S dm_1dm_2dm_3 = 16\pi n^3 r_s^3 G^2 (kT)^{-3/2} \left\langle \frac{m_3(m_1m_2)^{3/2}}{(m_1 + m_2)^{1/2}} e^{-\epsilon} \phi_S(\epsilon, a; m_i) \right\rangle , \tag{9}$$

or approximately

$$\mathfrak{R} \approx 16\pi n^3 r_s^3 G^2 \langle m\rangle^{7/2} (kT)^{-3/2} e^{-\langle\epsilon\rangle} \langle\phi_S\rangle . \tag{10}$$

Here

$$\langle\epsilon\rangle = -G\langle m\rangle^2/2r_s kT\,, \tag{11}$$

$$\langle\phi_S\rangle \approx 0.28 - 0.24\log_{10}|\langle\epsilon\rangle| + 1.25|\langle\epsilon\rangle|^{0.8} + 0.47\log_{10}\left(\frac{a}{1.3}\right), \tag{12}$$

$$n = \int n(m)dm \tag{13}$$

is the number density of stars, and

$$\langle m\rangle = \int n(m)m\,dm/n \tag{14}$$

is the average stellar mass. $\mathfrak{R}$ includes a factor of $\frac{1}{2}$ which arises because each trajectory $(m_1, m_2)$ has been counted also as $(m_2, m_1)$.

To get a rate valid in the solar neighborhood, we use $n = 0.080/\mathrm{pc}^3$ (Chiu 1968), $\langle m\rangle = 0.5\,M_\odot$ (Chiu 1968; Chandrasekhar 1942) (both of these figures apply to stars of absolute magnitude brighter than 14.5). The separation $r_s$ is chosen to be 0.01 pc ($\sim$2000 a.u.) which is quite reasonable observationally, and close to the maximum allowable under the condition that $r_c$ shall be less than interstellar distances. The quantity $kT$ is estimated to be $3.5 \times 10^{45}$ ergs, by applying the rms peculiar velocity by spectral type (Norton and Inglis 1966) to data on mass distribution (Chiu 1968). This value of $kT$ corresponds to an average peculiar velocity $(2kT/\langle m\rangle)^{1/2} = 26$ km sec$^{-1}$, and gives a value for $\langle\epsilon\rangle = -0.00031$. These values give

$$\mathfrak{R} = 2.1 \times 10^{-86}\ (\mathrm{cm}^3\ \mathrm{sec})^{-1}\,, \tag{15}$$

or less than one binary so formed in the Galaxy during its lifetime.

The rate in the core of a globular cluster is somewhat larger. Using a core of uniform density with a population of $5 \times 10^4$ and a radius of 20 pc (half the cluster radius), we estimate $kT = 1.7 \times 10^{43}$ ergs. If we also include more loosely bound binaries, up to $r_s = 0.04$ pc (8000 a.u.), we obtain

$$\mathfrak{R} = 1.7 \times 10^{-77}\ (\mathrm{cm}^3\ \mathrm{sec})^{-1}\,, \tag{16}$$

a value corresponding to an average of six binaries per cluster.

In answer to the suggestion that binaries may have been formed by this three-body process but during earlier stages of galactic evolution, when densities and kinetic energies may have been very different, we can compute the mass distribution that would obtain in cases where the three-body process is dominant. The functional dependence of such mass distributions on the masses is essentially independent of the conditions under which binary formation took place, since the three-body rate per unit mass $\mathfrak{R}_S$ (see eq. [6]) factors into a function of the mass distribution only, times a function of the other conditions which is approximately independent of mass:

$$\mathfrak{R}_S \approx n(m_1)n(m_2)n(m_3)\frac{m_3(m_1m_2)^{3/2}}{(m_1+m_2)^{1/2}} \times 16\pi r_s^3 G^2(kT)^{-3/2}e^{-\langle\epsilon\rangle}\langle\phi_S\rangle\,. \tag{17}$$

Thus we can calculate the dependence on mass of the various distributions which appear in statistical surveys, as suitable integrals of $\mathfrak{R}_S$ over the remaining masses.

For example, the distribution of secondary mass $m_2$ for a given primary mass $m_1$ $(m_1 > m_2)$ would be

$$\psi(m_2) = \int \mathfrak{R}_S dm_3 \propto \frac{m_2^{3/2}}{(m_1+m_2)^{1/2}}n(m_2) \propto \frac{\mu^{3/2}}{(1+\mu)^{1/2}}n(m_1\mu)\,, \tag{18}$$

where $\mu = m_2/m_1$. This is plotted in Figure 2 for several mass distributions. It is clearly quite sensitive to the mass distribution function $n(m)$. For comparison with the data we use the following approximate distribution function, which we choose so as to agree with the numbers in Chiu (1968, p. 63) (all masses are in solar units):

$$n(m) \propto \begin{cases} m^{-3} & (0.6 < m < 4)\,, \\ 2.8m^{-1} & (0.2 < m < 0.6)\,. \end{cases} \tag{19}$$

We also express the result as a function $\Phi(\Delta m)$ of the difference $\Delta m$ in visual magnitude between the primary and secondary, to agree in form with the observational statistics. For this we use the same mass-luminosity relation as Kuiper (1935*a*):

$$\log m = 0.405 - 0.086\, M_v \tag{20}$$

so that $\log \mu = -0.086\, \Delta m$. This gives

$$\Phi(\Delta m) \propto \begin{cases} 10^{+0.13\,\Delta m}/(1 + 10^{-0.086\,\Delta m})^{1/2}\,, & \text{for} \quad \Delta m < 11.6 \log (m_1/0.6)\,; \\ 2.8m_1^2 10^{-0.043\,\Delta m}/(1 + 10^{-0.086\,\Delta m})^{1/2}\,, & \text{for} \quad \Delta m > 11.6 \log (m_1/0.6)\,. \end{cases} \tag{21}$$

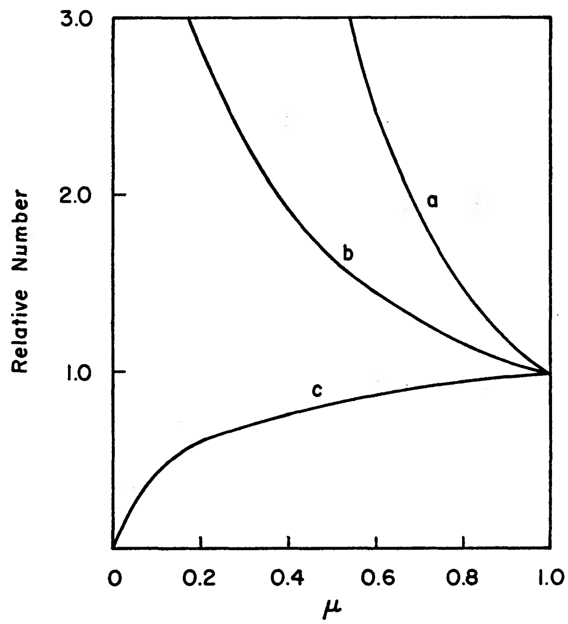


FIG. 2.—Relative number of binaries with secondaries of mass $m_2$ for a given primary mass $m_1$, as a function of $\mu = m_2/m_1$. (*a*) $n(m) \propto m^{-3}$; (*b*) $n(m) \propto n^{-2}$; (*c*) $n(m) \propto m^{-1}$. Arbitrary normalization.

This is plotted for several values of $m_1$ in Figure 3, along with the observational data on visual binaries given by Kuiper (1935*a*). We see that the agreement is not particularly good. However, Kuiper's data are admittedly not very accurate; they are based on a small sample (25 massive primaries, 225 intermediate main sequence, etc.), and his data have been adjusted in an attempt to correct for systematic observational errors. Furthermore, there is considerable uncertainty both in the mass distribution function and in the mass-luminosity relationship.

(It should be mentioned that we do not include data on spectroscopic binaries, since the three-body rate is yet smaller for such close pairs.)

We can similarly obtain the mass dependence of the fraction of all stars of a given mass which occur as primaries of binary systems (whose secondaries have masses greater than 0.2). As before, the overall magnitude of this fraction is left arbitrary; it depends on the history of galactic conditions. But the mass dependence does not depend on these conditions. Thus we get

$$f(m_1) = \frac{1}{n(m_1)} \int_{0.2}^{m_1} dm_2 \int dm_3 \Re_S \propto \int_{0.2}^{m_1} \frac{m_1^{3/2} m_2^{3/2}}{(m_1 + m_2)^{1/2}}\, n(m_2) dm_2\,. \tag{22}$$

The integral can be done analytically by using the mass distribution given above (eq. [19]). The result is plotted in Figure 4. Quantitative observational data were not available for comparison.

The distribution of binaries as a function of their separation can be similarly obtained. The number of binaries of separation $r_s$ is roughly proportional to $r_s^2$, as can be seen from equation (10). This definitely contradicts the data in the solar neighborhood (Kuiper 1935*b*).

## IV. OTHER APPLICATIONS

The general expression for the three-body rate (eq. [6]) is also applicable to other problems, such as comet capture by a star or capture of planetoids by a massive planet.

An order-of-magnitude estimate of the three-body rate for the capture of asteriods by Jupiter can be made, although the data on asteroids in Jupiter's vicinity are very incomplete. Jupiter's outer moonlets are at a distance slightly less than $2.4 \times 10^{12}$ cm from the planet, and we take $r_s$ to be this number. They are roughly eighteenth magnitude at opposition, a value which indicates a diameter of around 10 km and a mass of about

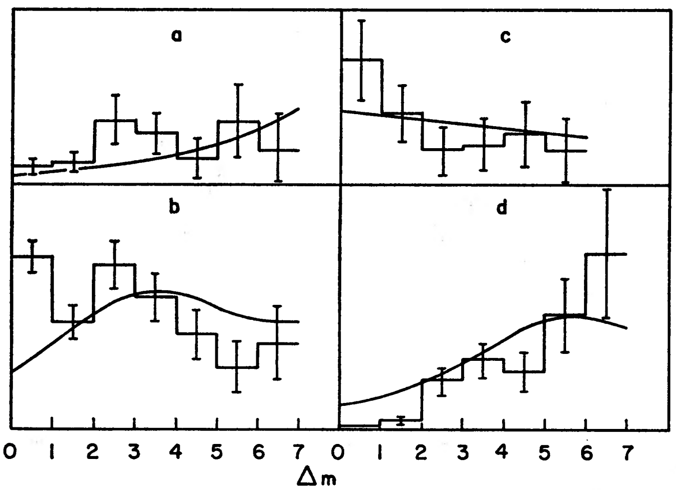


FIG. 3.—Comparison of our prediction of the three-body rate with observational data (Kuiper 1935*a*). Number of visual binaries is plotted against magnitude difference $\Delta m$ between primary and secondary, for several classes of primary star. Histogram plots are Kuiper's data; error bars are given by him also. (*a*) Spectral type O–B3; (*b*) type B5–dK0; (*c*) type dK; (*d*) type gG8–gM. *Smooth lines*, three-body predictions (see text). (*a*) $m_1 = 16$; (*b*) $m_1 = 1.0$; (*c*) $m_1 = 0.6$; (*d*) $m_1 = 1.6$. They have been normalized to give a least-squares fit to the data.

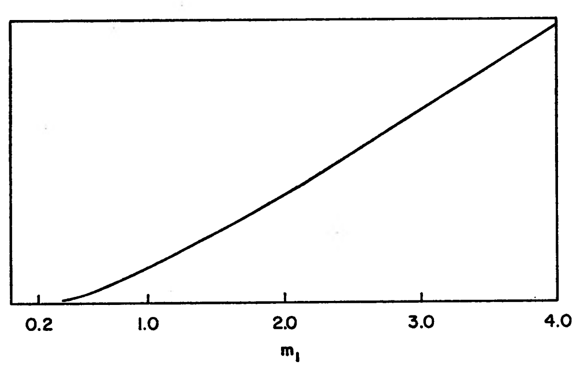


FIG. 4.—Ratio of the number of binaries with primary mass $m_1$ to total number of stars of mass $m_1$, as a function of $m_1$. Arbitrary normalization.

$10^{18}$ g. For the purpose of this approximation we take $n(m_i) = n_i\delta(m - m_i)$. The number of asteroids in the vicinity of Jupiter's orbit (the Trojan asteroids) of this size or larger has been estimated to be around 100 (Kuiper *et al.* 1958). If these are all in a torus of radius $r_s$ about Jupiter's orbit, we obtain an estimate of the asteroid density $n_2$ (also $n_3$) of about $10^{-38}$ cm$^{-3}$. This is an order of magnitude smaller than the average density of all asteroids of this size over the whole asteroid belt. For $kT$ we should probably use not the orbital energy but rather the relative energy of asteroids all moving in similar orbits; this could be perhaps 5 percent of the orbital energy (based on a dispersion of semimajor axes of the known Trojan orbits of only 0.2 a.u.), or about $5 \times 10^{28}$ ergs. Then

$$\Re_p = \int \frac{\Re_S dm_2 dm_3}{n(m_1)} \sim 2 \times 10^{-19} \text{ sec}^{-1}, \tag{23}$$

about 2 orders of magnitude too low to account for Jupiter's outer satellites.

Asteroid capture can also occur, however, if the third body is a true satellite, such as Ganymede. It absorbs energy from the incident asteroid by changing its orbit; this change is slight, since the mass of the true satellite is much greater than that of the asteroid. This process, with the third body bound, should be separately investigated by using the "variational method."

An estimate using equation (6) of this paper may be made, however, as follows. The mass $m_3$ is the mass of Ganymede, $1.5 \times 10^{26}$ g; $v_3$ is its orbital velocity, $1 \times 10^6$ cm sec$^{-1}$. We choose the density $n_3$ to be such that the flux $n_3 v_3$ across a semicircle of radius $r_s$ intersecting all Jovian satellite orbits should equal 1 per Ganymede orbital period. This gives $n_3 = 1.6 \times 10^{-37}$ cm$^{-3}$. The $kT$ which appears in equation (6) refers to particle 2, the asteroid, as can be ascertained from the derivation of the rate expression. We use the above value of $5 \times 10^{28}$ ergs. The function $\phi_S$ is estimated from the exact form in the Appendix to have the order of magnitude of $(-\epsilon_m)^{1/2}$ for large negative $\epsilon_m$, as is now the case ($\epsilon_m = \epsilon m_3[m_1 + m_2]/2m_1m_2$). Then

$$\Re_p \sim 10^{-5} \text{ sec}^{-1}, \tag{24}$$

many orders of magnitude too large. No doubt the use of the "Ganymede flux" in place of $n_3$ is at fault; a separate rate integral, for the third body already bound, would remedy this. Also, the values of the asteroid density and relative velocity are inaccurate. There might also be a large rate of dissociation. However, we interpret our result as an indication that it is this mechanism, with a true satellite as the third body, that is the effective one. Clearly, a more exact treatment is desirable.

## APPENDIX

### INTEGRATION OF THE RATE INTEGRAL

Detailed explanations appear in Mansbach and Keck (1969). The following is an abbreviated outline of the method. The rate integral is defined in equation (1), and $\rho$ is given above in equation (2). We have

$$v_i = \frac{dx_i}{dt} = \pm \frac{\partial H}{\partial \tilde{x}_i} \tag{A1}$$

(Hamilton's equations), where $\tilde{x}_i$ is canonically conjugate to $x_i$ and where

$$H = \Sigma \frac{p_i^2}{2m_i} - \Sigma \frac{Gm_im_j}{r_{ij}}. \tag{A2}$$

Also,

$$n_i = \frac{\partial E_{12}}{\partial x_i} \Big/ \left|\frac{\partial E_{12}}{\partial x_i}\right| = \frac{\partial E_{12}}{\partial x_i}\frac{\Delta h}{\Delta E_{12}}, \tag{A3}$$

where $\Delta h$ is the distance between the surfaces at $E_{12}$ and $E_{12} + \Delta E_{12}$. Then

$$dhdS = \prod_{i=1}^{18} dx_i \tag{A4}$$

provided the $x_i$ form a canonical set of coordinates. In particular, we use Delaunay coordinates (see Keck 1960 for their use in this problem), and take $p_{12}$ to be the dependent variable in the equation for the surface $S$. We now have

$$\begin{aligned}\Re_S = -\frac{1}{V}\Sigma_{\mathrm{sign}(p_{12})}\int_S \rho\frac{1}{\mu_{12}}\left\{-p_{12}\frac{\partial V_3}{\partial r_{12}} - \frac{l_{12}}{r_{12}^2}\frac{\partial V_3}{\partial \omega_{12}}\right\}\left|\frac{\partial p_{12}}{\partial E_{12}}\right| \\ \times\, dr_{12}dl_{12}dm_{12}d\omega_{12}d\phi_{12}dr_3dp_3dl_3dm_3d\omega_3d\phi_3d^3X_{\mathrm{CM}}d^3P_{\mathrm{CM}}\,,\end{aligned} \tag{A5}$$

where

$$p_{12} = \mathrm{sign}\,(p_{12})\left(2\mu_{12}E_S + 2\mu_{12}\frac{Gm_1m_2}{r_{12}} - \frac{l_{12}^2}{r_{12}^2}\right)^{1/2}, \tag{A6}$$

$$V_3 = -\frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}, \tag{A7}$$

and the surface $S$ is further restricted by the conditions

$$\{\text{quantity in braces}\} < 0 \tag{A8}$$

(downward crossings only),

$$\tfrac{1}{2}\mu_3v_3^2 + V_3 > 0 \tag{A9}$$

so that the third star was free before the encounter, and

$$\omega\tau = \frac{(-2E_S)^{3/2}(r_{13}+r_{23})\sin\gamma}{2Gm_1m_2\mu_{12}^{1/2}v_3} < 1\,, \tag{A10}$$

as discussed in the text.

We can immediately integrate over the center-of-mass coordinates, and also $\phi_3$, $\omega_3$. We transform $(\omega_{12}, m_{12}, \phi_{12})$ to $(\alpha_1, \beta_{12}, \beta_3)$ to enable us to evaluate the derivatives inside the braces. $(\beta_3, -\alpha_1, -\beta_{12})$ are the Euler angles of $(r_{12}, l_{12})$ with respect to $(r_3, l_3)$. We need

$$\frac{\partial\alpha_1}{\partial\omega_{12}} = -\cos\beta_{12} \tag{A11}$$

and the Jacobian

$$\frac{\partial(\omega_{12}, m_{12}, \phi_{12})}{\partial(\alpha_1, \beta_{12}, \beta_3)} = l_{12}\sin\alpha_1\,. \tag{A12}$$

The braces are evaluated in terms of $\alpha_1$, but we write them here in terms of $\theta_1$ and $\theta_2$, the exterior angles at stars 1 and 2, in the triangle defined by the three stars:

$$\begin{aligned}\{\} = \Big\{-p_{12}\left(-\frac{Gm_1m_3}{2r_{13}^2}\cos\theta_1 - \frac{Gm_2m_3}{2r_{23}^2}\cos\theta_2\right) \\ + \frac{l_{12}}{r_{12}}\cos\beta_{12}\left(\frac{Gm_1m_3}{2r_{13}^2}\sin\theta_1 - \frac{Gm_2m_3}{2r_{23}^2}\sin\theta_2\right)\Big\}\,.\end{aligned} \tag{A13}$$

We further transform $(p_3, l_3)$ into $(v_3, \gamma)$, which are defined by

$$p_3 = \mu_3 v_3 \cos\gamma\,, \qquad l_3 = \mu_3 v_3 r_3 \sin\gamma\,; \tag{A14}$$

and $(r_3, \alpha_1)$ into $(r_{13}, r_{23})$. This last introduces the constraint (triangle inequality)

$$|r_{13} - r_{23}| \leq r_{12} \leq r_{13} + r_{23}\,. \tag{A15}$$

We now perform the integrations over $m_3$, $\beta_3$, $\gamma$, $\beta_{12}$, and $l_{12}$, the sum over sign $(p_{12})$, and the integral over $r_{12}$, in that order, subject to the constraints in equations (A8), (A9), (A10), and (A15). We express the result in terms of the dimensionless quantities

$$\begin{aligned} \epsilon &= E_S/kT\,, \qquad \rho_+ = \frac{-E_S}{Gm_1m_2}(r_{13} + r_{23})\,, \\ \rho_- &= \frac{-E_S}{Gm_1m_2}(r_{13} - r_{23})\,, \qquad \text{and} \qquad x = \tfrac{1}{2}\mu_3 v_3^2/kT\,. \end{aligned} \tag{A16}$$

Then

$$\Re_S = \frac{n(m_1)n(m_2)n(m_3)4\pi 2^{1/2}Gm_3(Gm_1m_2)^4\mu_{12}^{1/2}}{(kT)^{3/2}(-E_S)^3}\,e^{-\epsilon}\phi_S(\epsilon, a;\, m_i)\,, \tag{A17}$$

where

$$\phi_S = \tfrac{1}{2}\int_0^\infty d\rho_+ \int_{-\rho_m}^{\rho_m} d\rho_- e^{-u}\Gamma^*(u, x_0)\, f(\rho_+, \rho_-)[\rho_+^2 - \rho_-^2]^{-1/2}\,. \tag{A18}$$

Here

$$\rho_m = \min\{\rho_+, 1\}\,, \tag{A19}$$

$$u = \frac{4\epsilon_m}{\rho_+^2 - \rho_-^2}\left[\rho_+ + \frac{m_1 - m_2}{m_1 + m_2}\rho_-\right] \qquad (u < 0)\,, \tag{A20}$$

$$\epsilon_m = \frac{m_3(m_1 + m_2)}{2m_1m_2}\,\epsilon \qquad (\epsilon_m < 0)\,, \tag{A21}$$

and

$$a_m = a\left[\frac{2(m_1 + m_2 + m_3)}{3(m_1 + m_2)}\right]^{1/2}\,. \tag{A22}$$

(Note that for $m_1 = m_2 = m_3$ we have $a_m = a$, $\epsilon_m = \epsilon$, the second term drops out of $u$, and we have symmetry in $\rho_- \leftrightarrow -\rho_-$.) Also,

$$x_0 = -\tfrac{4}{3}\epsilon_m\rho_+^2/a_m^2 \qquad (x_0 > 0)\,, \tag{A23}$$

$$\Gamma^*(u, x_0) = \int_{-u}^\infty e^{-x}x^{1/2}\psi(x, x_0)dx\,, \tag{A24}$$

$$\psi(x, x_0) = \begin{cases} 1 & (x > x_0)\,, \\ 1 - \left(1 - \dfrac{x}{x_0}\right)^{1/2} & (x < x_0)\,, \end{cases} \tag{A25}$$

$$a^2 = \frac{4\rho_-^4}{\rho_+^2 - \rho_-^2} + 3\rho_-^2\,, \tag{A26}$$

and

$$\begin{aligned} f(\rho_+, \rho_-) = {} & -\tfrac{2}{3}(\rho_m^2 + a^2)^{3/2} + \tfrac{2}{3}(\rho_-^2 + a^2)^{3/2} + \rho_m(\rho_m^2 + a^2)^{1/2} \\ & - |\rho_-|(\rho_-^2 + a^2)^{1/2} + a^2 \log_e\left[\frac{\rho_m + \sqrt{(\rho_m^2 + a^2)}}{|\rho_-| + \sqrt{(\rho_-^2 + a^2)}}\right]. \end{aligned} \tag{A27}$$

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