p-Adic Numbers and Krasner's Lemma

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Plan for Today

1. Construction of the p-adic numbers and their properties
2. Reminder of field theory and Krasner's lemma
3. Krasner's lemma in Lean
4. Lean implementation and main takeaways

Norms and Induced Metrics

Let K be a field. A function $\|\cdot\| : K \to \mathbb{R}_{\geq 0}$ is called a **norm** (or absolute value) on K if it satisfies, for all $x, y \in K$:

- **Non-degeneracy:** $\|x\| = 0 \iff x = 0$,
- **Multiplicativity:** $\|xy\| = \|x\| \cdot \|y\|$,
- **Triangle inequality:** $\|x + y\| \leq \|x\| + \|y\|$.

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A norm is called **non-Archimedean** if

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}.$$

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Let p be a fixed prime number. Define

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The p-**adic norm** on $\mathbb{Q}$ is then defined as:

$$|x|_p := \begin{cases} p^{-\operatorname{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

This is a non-Archimedean norm.

Examples

- $|27|_3 = 3^{-\operatorname{ord}_3(27)} = 3^{-3} = \frac{1}{27}$,
- $\left|\frac{81}{2}\right|_3 = 3^{-(\operatorname{ord}_3(81) - \operatorname{ord}_3(2)} = 3^{-4} = \frac{1}{81}$,
- $\left|\frac{1}{243}\right|_3 = 3^{-(\operatorname{ord}_3(1) - \operatorname{ord}_3(243))} = 3^5 = 243$.

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Observation: The p-adic concept of size is very different from our usual understanding.

Ostrowski's Theorem

Theorem: Every absolute value $\|\cdot\|$ on $\mathbb{Q}$ is equivalent to exactly one of the following:

- The trivial absolute value, given by $\|x\|_{\mathrm{triv}} = 1$ for $x \neq 0$.
- The usual absolute value $|\cdot|$.
- A p-adic norm $|\cdot|_p$ for some prime p.

Completeness

Recall that a metric space is called complete if every Cauchy sequence is convergent. Completeness is one of the most fundamental properties of metric spaces.

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The space $(\mathbb{Q}, |\cdot|_p)$ is not complete. Its **completion** is denoted $\mathbb{Q}_p$, the p-**adic numbers**.

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- $\iota(\mathbb{Q})$ is dense in $\mathbb{Q}_p$.
- ι is an isometry, that is $|\iota(x)|_p = |x|_p$ for all $x \in \mathbb{Q}$.

Construction of $\mathbb{Q}_p$

Step 1: Consider the set of all Cauchy sequences in $(\mathbb{Q}, |\cdot|_p)$:

$$\mathcal{C}_p := \{(a_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}} \mid (a_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence } \}.$$

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Defining the norm on $\mathbb{Q}_p$: For a class $[(a_n)_{n\in\mathbb{N}}] \in \mathbb{Q}_p$, define:

$$|[(a_n)_{n\in\mathbb{N}}]|_p := \lim_{n\to\infty} |a_n|_p$$

Construction of $\mathbb{Q}_p$

Operations on $\mathbb{Q}_p$:

$$[(a_n)_{n\in\mathbb{N}}] + [(b_n)_{n\in\mathbb{N}}] := [(a_n + b_n)_{n\in\mathbb{N}}],$$
$$[(a_n)_{n\in\mathbb{N}}][(b_n)_{n\in\mathbb{N}}] := [(a_n b_n)_{n\in\mathbb{N}}].$$

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All of these definitions are well-defined, which is checked by routine arguments, and our desired properties are fulfilled.

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- $\mathbb{Q}_p$ is locally compact.
- $\mathbb{Q}_p$ is **not** algebraically closed.

Algebraic Closure and Completeness

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Question: In the p-adic world, can we ever reach a field that is *both* complete and algebraically closed?

Answer: Yes, we can. Completing $(\overline{\mathbb{Q}_p}, |\cdot|_p)$ yields a non-Archimedean normed field $(\mathbb{C}_p, |\cdot|_p)$ which is complete and algebraically closed.

3-adic Visualization Animation

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- An algebraic element $x \in L$ is called **separable** if the minimal polynomial $P_x \in K[X]$ only has simple roots (in some field where it splits).
- We denote $K(x)$ the smallest subfield of L containing $K \cup \{x\}$.

Krasner's Lemma

Theorem: Let $a, b \in \overline{\mathbb{Q}_p}$. Suppose that for every conjugate $a_i \neq a$ of a in $\overline{\mathbb{Q}_p}$ (over $\mathbb{Q}_p$) it holds that

$$|b - a|_p < |a_i - a|_p.$$

Then $\mathbb{Q}_p(a) \subseteq \mathbb{Q}_p(b)$.

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Krasner's lemma can be used to prove that $\mathbb{C}_p$ is algebraically closed.

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$$\sigma\colon K(a) \to K(a_i)$$

such that $\sigma|_K = \mathrm{id}_K$ and $\sigma(a) = a_i$.

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such that $\sigma|_K = \mathrm{id}_K$ and $\sigma(a) = a_i$. We will see later that $|\cdot|_p$ is invariant under isomorphisms, i.e. $|\sigma(x)|_p = |x|_p$ for every $x \in K(a)$.

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This implies that

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We conclude that

$$\begin{aligned} |a_i - a|_p = |a_i - b + b - a|_p &\le \max\{|a_i - b|_p, |b-a|_p\} \\ &= |b-a|_p < |a_i - a|_p, \end{aligned}$$

which is a contradiction.

Krasner's Lemma in Lean

```
theorem lemma_krasner {p : ℕ} [Fact (Nat.Prime p)] (a b : AlgebraicClosure ℚ_p)
(h : ∀x ∈ AlgebraicClosure ℚ_p, x ≠ a ∧ IsConjRoot ℚ_p a x → PAdicNormExt(b − a) <
PAdicNormExt(x − a)) :
adjoin ℚ_p ({a} : Set (AlgebraicClosure ℚ_p)) ≤ adjoin ℚ_p ({b} : Set
(AlgebraicClosure ℚ_p)) :=
```

Lean	**Explanation**
`have ha : a adjoin Q_p ({b} : Set (AlgebraicClosure Q_p)) := lemma_main a b h`	*We prove that a belongs to $K = Qp(b)$ using the 'main_lemma'*
`adjoin_of_mem_adjoin a b ha`	*We explain why that is enough to deduce that $Qp(a)$ is contained in $Qp(b)$*

Main Lemma in Lean

lemma lemma_main $\{p : \mathbb{N}\}$ [Fact (Nat.Prime p)] (a b : AlgebraicClosure $\mathbb{Q}_p$)
(h : $\forall x \in$ AlgebraicClosure $\mathbb{Q}_p$, $a \neq x \wedge$ IsConjRoot $\mathbb{Q}_p$ a $x \to$ PAdicNormExt$(b - a) <$ PAdicNormExt$(x - a)$) :
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Lean	Explanation
`have h1 : (c : AlgebraicClosure Q_p), a ≠ c ∧ IsConjRoot K a c := conj_lemma K a h0`	*Get a Galois conj.*
`have h2 : ∃ (σ : AlgebraicClosure Q_p [K] AlgebraicClosure Q_p), σ a = c ∧ ∀ x ∈ K, σ x = x := sigma_isom K a c h_conj_in_K`	*Get an isom. from the conj.*

Norm Invariance

```
have h4 : PAdicNormExt (b - a) = PAdicNormExt (c - b) := calc
```

`PAdicNormExt (b - a) = PAdicNormExt (σ (b - a)) := h_norm_inv`	*Norm invariance*
`= PAdicNormExt (σ b - σ a) := Lin_of_sigma`	*Linearity*
`= PAdicNormExt (b - σ a) := by rw [sigma_b]`	*b is fixed*
`= PAdicNormExt (b - c) := by rw [h_sigma1]`	*a is sent to c*
`= PAdicNormExt (-(b - c)) := PAdicNormExt_mult_minus (b - c)`	*Norm inv -1*
`= PAdicNormExt (c - b) := neg_sub_norm`	*Norm sym.*

Contradiction Step

```
have h5 : PAdicNormExt (c - a) < PAdicNormExt (c - a) := calc
```

`PAdicNormExt (c - a) = PAdicNormExt ((c - b) + (b - a)) := by rw [sub_add_sub_cancel]`	*Add and subtract*
`_ ≤ max (PAdicNormExt (c - b)) (PAdicNormExt (b - a)) := PAdicNormExt_non_arch (c - b) (b - a)`	*Non-arch triangle ineq.*
`= PAdicNormExt (b - a) := max_is_b_sub_a`	*By h4*
`< PAdicNormExt (c - a) := h c a_c_IsConj_in_Q_p`	*Our assumption*

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- For $x \in K$, we have that $N_{L/K}(x) = x^{[L:K]}$.
- If M/L is another finite field extension, then $N_{M/K} = N_{L/K} \circ N_{M/L}$.

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Hence we can extend the norm on K to a norm on the algebraic closure $\bar{K}$.

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- If K is complete, then this extension is unique.
- If K is complete, non-Archimedian and locally compact, the extended norm is also non-Archimedian. Again, the non-Archimedian triangle inequality is a little tricky to check.

Implementation in Lean – Key Points and Takeaways

- Many parts of this were already implemented in Lean 3 (approx. 5000 lines of code) in a more general context, but it has not been migrated to Lean 4 as of yet.
- **Intermediate fields:** Sometimes it's best to work under a much bigger field than you "need" to avoid complications from type mismatches and coercions.