Lecture 5

Revenue-Maximizing Auctions

Lectures 2–4 focused on the design of mechanisms that maximize, exactly or approximately, the social welfare of the outcome. Revenue is generated in such mechanisms only as a side effect, a necessary evil to incentivize agents to report truthfully their private information. This lecture studies mechanisms that are designed to raise as much revenue as possible, and characterizes the expected revenue-maximizing mechanism with respect to a prior distribution over agents' valuations.

Section 5.1 explains why reasoning about revenue maximization is harder than welfare maximization, and introduces Bayesian environments. Section 5.2 is the heart of this lecture, and it characterizes expected revenue-maximizing mechanisms as "virtual welfare maximizers." Section 5.3 describes how this theory was used to boost sponsored search revenue at Yahoo. Section 5.4 proves a technical lemma needed for the characterization in Section 5.2.

5.1 The Challenge of Revenue Maximization

5.1.1 Spoiled by Social Welfare Maximization

There are several reasons to begin the study of mechanism design with the objective of maximizing social welfare. The first reason is that this objective is relevant to many real-world scenarios. For instance, in government auctions (e.g., to sell wireless spectrum; see Lecture 8), the primary objective is welfare maximization. Revenue is also a consideration in such auctions, but it is usually not the first-order objective. Also, in competitive markets, a rule of thumb is that a seller should focus on welfare maximization, since otherwise a competitor will (thereby stealing their customers).

The second reason to start with social welfare maximization is pedagogical: social welfare is special. In every single-parameter environment, there is a DSIC mechanism that, for every profile of private valuations, assuming truthful bids, computes the welfare-maximizing outcome (cf., Exercise 4.1).[1] Such a mechanism optimizes the social welfare as effectively as if all of the private information was known in advance—the DSIC constraint is satisfied for free. This amazingly strong performance guarantee, called an "ex post" guarantee, cannot generally be achieved for other objective functions.

5.1.2 One Bidder and One Item

The following trivial example is illuminating. Suppose there is one item and only one bidder, with a private valuation v. With only one bidder, the space of direct-revelation DSIC auctions is small: they are precisely the *posted prices*, meaning take-it-or-leave-it offers.[2] With a posted price of $r \geq 0$, the auction's revenue is either r (if $v \geq r$) or 0 (if $v < r$).

Maximizing social welfare in this setting is trivial: just set $r = 0$, so that the auction always awards the item to the bidder for free. This optimal posted price is *independent of* v.

Suppose we wanted to maximize revenue. How should we set r? If we telepathically knew v, we would set $r = v$. But with v private to the bidder, what should we do? It is not clear how to reason about this question.

The fundamental issue is that the revenue-maximizing auction varies with the private valuations. With a single item and bidder, a posted price of 20 will do very well on inputs where v is 20 or a little larger, and terribly when v is less than 20 (while smaller posted prices will do better). Welfare maximization, for which there is an input-independent optimal DSIC mechanism, is special indeed.

5.1.3 Bayesian Analysis

To compare the revenue of two different auctions, we require a model to compare trade-offs across different inputs. The classical approach is to use *average-case* or *Bayesian* analysis. We consider a model comprising the following ingredients:

[1] This holds even more generally; see Lecture 7.

[2] These are the deterministic DSIC auctions. An auction can also randomize over posted prices, but the point of this example remains the same.

- A single-parameter environment (Section 3.1). We assume that there is a constant M such that $x_i \leq M$ for every i and feasible solution $(x_1, \ldots, x_n) \in X$.

- Independent distributions $F_1, \ldots, F_n$ with positive and continuous density functions $f_1, \ldots, f_n$. We assume that the private valuation v_i of participant i is drawn from the distribution F_i.[3] We also assume that the support of every distribution F_i belongs to $[0, v_{\max}]$ for some $v_{\max} < \infty$.[4]

A key assumption is that the mechanism designer knows the distributions $F_1, \ldots, F_n$.[5] The realizations $v_1, \ldots, v_n$ of agents' valuations are private, as usual. Since we focus on DSIC auctions, where agents have dominant strategies, the agents do not need to know the distributions $F_1, \ldots, F_n$.[6]

In a Bayesian environment, it is clear how to define the "optimal" mechanism—it is the one that, among all DSIC mechanisms, has the highest expected revenue (assuming truthful bids). The expectation is with respect to the given distribution $F_1 \times F_2 \times \cdots \times F_n$ over valuation profiles.

5.1.4 One Bidder and One Item, Revisited

With our Bayesian model, single-bidder single-item auctions are easy to reason about. The expected revenue of a posted price r is simply

$$\underbrace{r}_{\text{revenue of a sale}} \cdot \underbrace{(1 - F(r))}_{\text{probability of a sale}} .$$

Given a distribution F, it is usually a simple matter to solve for the best posted price r. An optimal posted price is called a *monopoly*

[3] The distribution function $F_i(z)$ denotes the probability that a random variable with distribution F_i has value at most z.

[4] The results of this lecture hold more generally if every distribution F_i has finite expectation.

[5] In practice, these distributions are typically derived from data, such as bids in past auctions.

[6] In mechanisms without dominant strategies, such as first-price single-item auctions, the standard approach is to consider "Bayes-Nash equilibria"; see Problem 5.3 for details. Bayes-Nash equilibrium analysis assumes a "common prior," meaning that all of the agents know the distributions $F_1, \ldots, F_n$.

price of the distribution F. Since DSIC mechanisms are posted prices (and distributions thereof), posting a monopoly price is a revenue-maximizing auction. For instance, if F is the uniform distribution on $[0, 1]$, so that $F(x) = x$ on $[0, 1]$, then the monopoly price is $\frac{1}{2}$, achieving an expected revenue of $\frac{1}{4}$.

5.1.5 Multiple Bidders

The plot thickens already with two bidders, where the space of DSIC auctions is more complicated than the space of posted prices. For example, consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$. We could of course run a second-price auction (Section 2.4); its expected revenue is the expected value of the smaller bid, which is $\frac{1}{3}$ (Exercise 5.1(a)).

We can also supplement a second-price auction with a *reserve price*, analogous to the opening bid in an eBay auction. In a second-price auction with reserve r, the allocation rule awards the item to the highest bidder, unless all bids are less than r, in which case no one gets the item. The corresponding payment rule charges the winner (if any) the second-highest bid or r, whichever is larger. From a revenue standpoint, adding a reserve price r is both good and bad: you lose revenue when all bids are less than r, but when exactly one bid is above r the reserve price boosts the revenue. With two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$, adding a reserve price of $\frac{1}{2}$ raises the expected revenue of a second-price auction from $\frac{1}{3}$ to $\frac{5}{12}$ (Exercise 5.1(b)). Can we do better? Either by using a different reserve price, or with a entirely different auction format?

5.2 Characterization of Optimal DSIC Mechanisms

The primary goal of this lecture is to give an explicit description of an optimal (i.e., expected revenue-maximizing) DSIC mechanism for every single-parameter environment and distributions $F_1, \ldots, F_n$.

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5.2.1 Preliminaries

We can simplify the problem by applying the revelation principle from last lecture (Theorem 4.3). Since every DSIC mechanism is equivalent to—and hence has the same expected revenue as—a direct-revelation DSIC mechanism $(\mathbf{x}, \mathbf{p})$, we can restrict our attention to such mechanisms. We correspondingly assume truthful bids (i.e., $\mathbf{b} = \mathbf{v}$) for the rest of the lecture.

The expected revenue of a DSIC mechanism $(\mathbf{x}, \mathbf{p})$ is, by definition,

$$\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right], \tag{5.1}$$

where the expectation is with respect to the distribution $\mathbf{F} = F_1 \times \cdots \times F_n$ over agents' valuations. It is not clear how to directly maximize the expression (5.1) over the space of DSIC mechanisms. We next work toward a *second* formula for the expected revenue of a mechanism. This alternative formula only references the allocation rule of a mechanism, and not its payment rule, and for this reason is far easier to maximize.

5.2.2 Virtual Valuations

The second formula for expected revenue uses the important concept of virtual valuations. For an agent i with valuation distribution F_i and valuation v_i, her *virtual valuation* is defined as

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}. \tag{5.2}$$

The virtual valuation of an agent depends on her own valuation and distribution, and not on those of the other agents. For example, if F_i is the uniform distribution on $[0, 1]$, with $F_i(z) = z$ for $z \in [0, 1]$, then $f_i(z) = 1$, and $\varphi_i(z) = z - \frac{1-z}{1} = 2z - 1$ on $[0, 1]$. A virtual valuation is always at most the corresponding valuation, and it can be negative. See Exercise 5.2 for more examples.

Virtual valuations play a central role in the design of expected revenue-maximizing auctions. But what do they mean? One way to interpret the formula

$$\varphi_i(v_i) = \underbrace{v_i}_{\text{what you'd like to charge}} - \underbrace{\frac{1 - F_i(v_i)}{f_i(v_i)}}_{\text{information rent earned by agent}}$$

is to think of v_i as the maximum revenue obtainable from agent i, and the second term as the inevitable revenue loss caused by not knowing v_i in advance, known as the *information rent*. A second interpretation of $\varphi_i(v_i)$ is as the slope of a "revenue curve" at v_i, where the revenue curve plots the expected revenue obtained from an agent with valuation drawn from F_i as a function of the probability of a sale. Problem 5.1 elaborates on this interpretation.

5.2.3 Expected Revenue Equals Expected Virtual Welfare

The following lemma is the workhorse of our characterization of optimal auctions. We give the proof, which is really just some calculus, in Section 5.4.

Lemma 5.1 *For every single-parameter environment with valuation distributions* $F_1, \ldots, F_n$*, every DSIC mechanism* $(\mathbf{x}, \mathbf{p})$*, every agent* i*, and every value* $\mathbf{v}_{-i}$ *of the valuations of the other agents,*

$$\mathbf{E}_{v_i \sim F_i}[p_i(\mathbf{v})] = \mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\mathbf{v})]\,. \tag{5.3}$$

That is, the expected payment of an agent equals the expected virtual value earned by the agent. This identity holds only in expectation over v_i, and not pointwise.[7]

Taking Lemma 5.1 as given, we have the following important result.

Theorem 5.2 (Exp. Revenue Equals Exp. Virtual Welfare) *For every single-parameter environment with valuation distributions* $F_1, \ldots, F_n$ *and every DSIC mechanism* $(\mathbf{x}, \mathbf{p})$,

$$\underbrace{\mathbf{E}_{\mathbf{v} \sim \mathbf{F}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right]}_{\textit{expected revenue}} = \underbrace{\mathbf{E}_{\mathbf{v} \sim \mathbf{F}}\left[\sum_{i=1}^{n} \varphi_i(v_i) \cdot x_i(\mathbf{v})\right]}_{\textit{expected virtual welfare}}. \tag{5.4}$$

[7] For example, virtual valuations can be negative while payments are always nonnegative.

Proof: Taking the expectation, with respect to $\mathbf{v}_{-i} \sim \mathbf{F}_{-i}$, of both sides of (5.3) we obtain

$$\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}[p_i(\mathbf{v})] = \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}[\varphi_i(v_i) \cdot x_i(\mathbf{v})] .$$

Applying the linearity of expectation (twice) then gives

$$\begin{aligned}
\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^n p_i(\mathbf{v})\right] &= \sum_{i=1}^n \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}[p_i(\mathbf{v})] \\
&= \sum_{i=1}^n \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}[\varphi_i(v_i) \cdot x_i(\mathbf{v})] \\
&= \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(\mathbf{v})\right] ,
\end{aligned}$$

as desired. ■

The second term in (5.4) is our second formula for the expected revenue of a mechanism, and we should be pleased with its simplicity. If we replaced the $\varphi_i(v_i)$'s by v_i's, then we would be left with an old friend: the expected *welfare* of the mechanism. For this reason, we refer to $\sum_{i=1}^n \varphi_i(v_i) \cdot x_i(\mathbf{v})$ as the *virtual welfare* of a mechanism on the valuation profile $\mathbf{v}$. Theorem 5.2 implies that maximizing expected revenue over the space of DSIC mechanisms reduces to maximizing expected virtual welfare over the same space.

5.2.4 Maximizing Expected Virtual Welfare

It is shocking that a formula as simple as (5.4) holds. It says that even though we only care about payments, we can focus on an optimization problem that concerns only the mechanism's allocation rule. This second form is far more operational, and we proceed to determine the mechanisms that maximize it.

How should we choose the allocation rule $\mathbf{x}$ to maximize the expected virtual welfare

$$\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^n \varphi_i(v_i) x_i(\mathbf{v})\right] ? \tag{5.5}$$

We have the freedom of choosing $\mathbf{x}(\mathbf{v})$ for each valuation profile $\mathbf{v}$, and have no control over the input distribution $\mathbf{F}$ or the virtual values $\varphi_i(v_i)$. Thus, the obvious approach is to *maximize pointwise*: separately for each $\mathbf{v}$, we choose $\mathbf{x}(\mathbf{v})$ to maximize the virtual welfare $\sum_{i=1}^n \varphi_i(v_i)x_i(\mathbf{v})$ obtained on the input $\mathbf{v}$, subject to feasibility of the allocation. We call this the *virtual welfare-maximizing allocation rule*. This is the same as the welfare-maximizing allocation rule of (4.1) and (4.2), except with agents' valuations replaced by their virtual valuations (5.2).

For example, in a single-item auction, where the feasibility constraint is $\sum_{i=1}^n x_i(\mathbf{v}) \leq 1$ for every $\mathbf{v}$, the virtual welfare-maximizing rule just awards the item to the bidder with the highest virtual valuation. Well, not quite: recall that virtual valuations can be negative—for instance, $\varphi_i(v_i) = 2v_i - 1$ when v_i is uniformly distributed between 0 and 1—and if every bidder has a negative virtual valuation, then the virtual welfare is maximized by not awarding the item to anyone.[8]

The virtual welfare-maximizing allocation rule maximizes the expected virtual welfare (5.5) over *all* allocation rules, monotone or not. The key question is: *Is the virtual welfare-maximizing rule monotone?* If so, then by Myerson's lemma (Theorem 3.7) it can be extended to a DSIC mechanism, and by Theorem 5.2 this mechanism has the maximum possible expected revenue.

5.2.5 Regular Distributions

Monotonicity of the virtual welfare-maximizing allocation rule depends on the valuation distributions. The next definition identifies a sufficient condition for monotonicity.

Definition 5.3 (Regular Distribution) A distribution F is *regular* if the corresponding virtual valuation function $v - \frac{1-F(v)}{f(v)}$ is nondecreasing.

If every agent's valuation is drawn from a regular distribution, then with consistent tie-breaking, the virtual welfare-maximizing allocation rule is monotone (Exercise 5.5).

[8]Recall from the single-bidder example in Section 5.1 that maximizing expected revenue entails not always selling the item.

For example, the uniform distribution on $[0, 1]$ is regular because the corresponding virtual valuation function is $2v - 1$. Many other common distributions are also regular (Exercise 5.3). Irregular distributions include many multi-modal distributions and distributions with extremely heavy tails.

With regular valuation distributions, we can extend the (monotone) virtual welfare-maximizing allocation rule to a DSIC mechanism using Myerson's lemma (Theorem 3.7). This is an expected revenue-maximizing DSIC mechanism.[9]

Virtual Welfare Maximizer

Assumption: the valuation distribution F_i of every agent is regular (Definition 5.3).

1. Transform the (truthfully reported) valuation v_i of agent i into the corresponding virtual valuation $\varphi_i(v_i)$ according to (5.2).
2. Choose the feasible allocation $(x_1, \ldots, x_n)$ that maximizes the virtual welfare $\sum_{i=1}^{n} \varphi_i(v_i) x_i$.[10]
3. Charge payments according to Myerson's payment formula (see (3.5) and (3.6)).[11]

We call this mechanism the *virtual welfare maximizer* for the given single-parameter environment and valuation distributions.

Theorem 5.4 (Virtual Welfare Maximizers Are Optimal) *For every single-parameter environment and regular distributions $F_1, \ldots, F_n$, the corresponding virtual welfare maximizer is a DSIC mechanism with the maximum-possible expected revenue.*

A stunning implication of Theorem 5.4 is that revenue-maximizing mechanisms are almost the same as welfare-maximizing

[9] With additional work, the results of this lecture can be extended to irregular valuation distributions. See the Notes for details.

[10] The simplest way to break ties is lexicographically with respect to some fixed total ordering over the feasible outcomes.

[11] If every x_i can only be 0 or 1, then these payments are particularly simple: every winner pays the infimum of the bids at which she would continue to win, holding others' bids fixed.

mechanisms, and differ only in using virtual valuations in place of valuations. In this sense, revenue maximization reduces to welfare maximization.

Remark 5.5 (Bayesian Incentive Compatible Mechanisms) Generalizing the derivations in Section 3.4 and this section yields a substantially stronger version of Theorem 5.4: the mechanism identified in the theorem maximizes expected revenue not only over all DSIC mechanisms but more generally over all "Bayesian incentive compatible (BIC)" mechanisms. A BIC mechanism for valuation distributions $F_1, \ldots, F_n$ is one in which truthful revelation forms a Bayes-Nash equilibrium (see Problem 5.3 for a definition). Every DSIC mechanism is BIC with respect to every choice of $F_1, \ldots, F_n$. Since optimizing expected revenue over all BIC mechanisms yields a DSIC mechanism, the DSIC property comes for free. The revelation principle (Theorem 4.3) can be adapted to BIC mechanisms (Problem 5.4), implying that, under the assumptions of Theorem 5.4, no Bayes-Nash equilibrium of any mechanism (e.g., first-price auctions) results in expected revenue larger than that earned by the optimal DSIC mechanism.

5.2.6 Optimal Single-Item Auctions

Theorem 5.4 gives a satisfying solution to the problem of expected revenue-maximizing mechanism design, in the form of a relatively explicit and easy-to-implement optimal mechanism. However, these mechanisms are not easy to interpret. Do they ever simplify to familiar mechanisms?

Let's return to single-item auctions. Assume that bidders are *i.i.d.*, meaning that they have a common valuation distribution F and hence a common virtual valuation function φ. Assume also that F is *strictly regular*, meaning that φ is a strictly increasing function. The virtual-welfare-maximizing mechanism awards the item to the bidder with the highest nonnegative virtual valuation, if any. Since all bidders share the same increasing virtual valuation function, the bidder with the highest virtual valuation is also the bidder with the highest valuation. This allocation rule is the same as that of a second-price auction with a reserve price of $\varphi^{-1}(0)$. By Theorem 3.7(b), the payment rules also coincide. Thus, for any number of i.i.d. bidders and a strictly regular valuation distribution, eBay (with a suitable opening bid) is the optimal auction format! Returning to the setting described at the end of Section 5.1, with all valuations distributed uniformly on $[0, 1]$, the second-price auction with reserve $\frac{1}{2} = \varphi^{-1}(0)$ is optimal. Given the richness of the DSIC auction design space, it is astonishing that such a simple and practical auction pops out as the theoretically optimal one.

5.3 Case Study: Reserve Prices in Sponsored Search

So how does all this optimal mechanism design theory get used, anyway? This section discusses a 2008 field experiment that explored whether or not the lessons of optimal auction theory could be used to increase sponsored search revenue at Yahoo.

Recall from Section 2.6 our model of sponsored search auctions. Which such auction maximizes the expected revenue, at least in theory? If we assume that bidders' valuations-per-click are drawn i.i.d. from a regular distribution F with virtual valuation function φ, then the optimal auction considers only bidders who bid at least the reserve price $\varphi^{-1}(0)$, and ranks these bidders by bid (from the best slot to the worst). See Exercise 5.8.

What had Yahoo been doing, up to 2008? First, they were using relatively low reserve prices—initially \$.01, later \$.05, and then \$.10. Perhaps more naively, they were using the same reserve price of \$.10 across all keywords, even though some keywords surely warranted higher reserve prices than did others (e.g., "divorce lawyer" versus "pizza"). How would Yahoo's revenue change if reserve prices were updated, independently for each keyword, to the theoretically optimal ones?

In the first step of the field experiment, a lognormal valuation distribution was fitted to past bidding data for approximately 500,000 different keywords.[12] The qualitative conclusions of the experiment appear to be independent of the details of this step, such as the particular family of valuation distributions chosen.

[12]Since Yahoo, like other search engines, uses a non-DSIC auction based on the GSP auction (Problem 3.1), one cannot expect the bids to be truthful. In this field experiment, valuations were reversed engineered from bids under the assumption that bidders are playing the equilibrium that is outcome-equivalent to the dominant-strategy outcome of the revenue-maximizing DSIC sponsored search auction (Exercise 5.8).

In the second step, theoretically optimal reserve prices were computed for each keyword, assuming that valuations were distributed according to the fitted distributions. As expected, the optimal reserve price varied significantly across keywords. There were plenty of keywords with a theoretically optimal reserve price of \$.30 or \$.40. Yahoo's uniform reserve price was much too low, relative to the advice provided by optimal auction theory, on these keywords.

The obvious experiment is to try out the theoretically optimal (and generally higher) reserve prices to see how they do. Yahoo's top brass wanted to be a little more conservative, though, and set the new reserve prices to be the average of the old ones (\$.10) and the theoretically optimal ones.[13] And the change worked: auction revenues went up several percent (of a very large number). The new reserve prices were especially effective in markets that are valuable but "thin," meaning not very competitive (less than six bidders). Better reserve prices were credited by Yahoo's president as the biggest reason for higher search revenue in Yahoo's third-quarter report in 2008.

*5.4 Proof of Lemma 5.1

This section sketches a proof of Lemma 5.1, that the expected (over $v_i \sim F_i$) revenue obtained from an agent i equals the expected virtual value that she earns. As a starting point, recall Myerson's payment formula (3.6)

$$p_i(v_i, \mathbf{v}_{-i}) = \int_0^{v_i} z \cdot x_i'(z, \mathbf{v}_{-i}) dz$$

for the payment made by agent i in a DSIC mechanism with allocation rule $\mathbf{x}$ on the valuation profile $\mathbf{v}$. We derived this equation assuming that the allocation function $x_i(z, \mathbf{v}_{-i})$ is differentiable. By standard advanced calculus, the same formula holds more generally for an arbitrary monotone function $x_i(z, \mathbf{v}_{-i})$, including piecewise constant functions, for a suitable interpretation of the derivative $x_i'(z, \mathbf{v}_{-i})$ and the corresponding integral. Similarly, all of the following proof

[13]Both in theory and empirically, this more conservative change accounts for most of the revenue increase. There are usually diminishing returns to revenue as the reserve price approaches the theoretical optimum, providing flexibility near the optimal price. The intuition for this principle is that the derivative of the expected revenue with respect to the reserve price is 0 at the optimal point.

steps, which make use of calculus maneuvers like integration by parts, can be made fully rigorous for arbitrary bounded monotone functions without significant difficulty. We leave the details to the interested reader.[14]

Equation (3.6) states that payments are fully dictated by the allocation rule. Thus, at least in principle, we can express the expected revenue of an auction purely in terms of its allocation rule, with no explicit reference to its payment rule. Will the resulting revenue formula be easier to maximize than the original one? It's hard to know without actually doing it, so let's do it.

Step 1: Fix an agent i. By Myerson's payment formula, we can write the expected (over $v_i \sim F_i$) payment by i for a given value of $\mathbf{v}_{-i}$ as

$$\begin{aligned}\mathbf{E}_{v_i \sim F_i}[p_i(\mathbf{v})] &= \int_0^{v_{\max}} p_i(\mathbf{v}) f_i(v_i) dv_i \\ &= \int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i.\end{aligned}$$

The first equality exploits the independence of agents' valuations—the fixed value of $\mathbf{v}_{-i}$ has no bearing on the distribution F_i from which v_i is drawn.

This step is exactly what we knew was possible in principle—rewriting the expected payment in terms of the allocation rule. For this to be useful, we need some simplifications.

Step 2: Whenever you have a double integral (or double sum) that you don't know how to interpret, it's worth reversing the integration order. Reversing the order of integration in

$$\int_0^{v_{\max}} \left[\int_0^{v_i} z \cdot x_i'(z, \mathbf{v}_{-i}) dz \right] f_i(v_i) dv_i$$

yields

$$\int_0^{v_{\max}} \left[\int_z^{v_{\max}} f_i(v_i) dv_i \right] z \cdot x_i'(z, \mathbf{v}_{-i}) dz,$$

which simplifies to

$$\int_0^{v_{\max}} (1 - F_i(z)) \cdot z \cdot x_i'(z, \mathbf{v}_{-i}) dz,$$

[14] For example, every bounded monotone function is integrable, and is differentiable except at a countable set of points.

suggesting that we're on the right track.

Step 3: Integration by parts is also worth trying when massaging an integral into a more interpretable form, especially if there's an obvious derivative hiding in the integrand. We again get some encouraging simplifications:

$$
\begin{aligned}
&\int_0^{v_{\max}} \underbrace{(1 - F_i(z)) \cdot z}_{g(z)} \cdot \underbrace{x_i'(z, \mathbf{v}_{-i})}_{h'(z)}\, dz \\
&= \underbrace{(1 - F_i(z)) \cdot z \cdot x_i(z, \mathbf{v}_{-i})|_0^{v_{\max}}}_{=0-0} \\
&\qquad - \int_0^{v_{\max}} x_i(z, \mathbf{v}_{-i}) \cdot (1 - F_i(z) - z f_i(z)) dz \\
&= \int_0^{v_{\max}} \underbrace{\left(z - \frac{1 - F_i(z)}{f_i(z)} \right)}_{=\varphi_i(z)} x_i(z, \mathbf{v}_{-i}) f_i(z) dz. \qquad (5.6)
\end{aligned}
$$

Step 4: We can interpret (5.6) as an expected value, with z drawn from the distribution F_i. Recalling the definition (5.2) of virtual valuations, this expectation is $\mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\mathbf{v})]$. Summarizing, we have

$$
\mathbf{E}_{v_i \sim F_i}[p_i(\mathbf{v})] = \mathbf{E}_{v_i \sim F_i}[\varphi_i(v_i) \cdot x_i(\mathbf{v})],
$$

as desired.

The Upshot

☆ Unlike welfare-maximizing mechanisms, the revenue-maximizing mechanism for an environment varies with the (private) valuations.

☆ In the average-case or Bayesian approach to comparing different mechanisms, each agent's valuation is drawn independently from a distribution known to the mechanism designer. The optimal mechanism is the one with the highest expected revenue with respect to these distributions.

☆ The expected revenue of a DSIC mechanism can be expressed purely in terms of its allocation rule, using the important concept of virtual valuations (5.2).

☆ A distribution is regular if the corresponding virtual valuation function is nondecreasing. Many common distributions are regular.

☆ With regular valuation distributions, the optimal mechanism is a virtual welfare maximizer, which for each valuation profile chooses an outcome with maximum virtual welfare.

☆ In a single-item auction with bidders' valuations drawn i.i.d. from a regular distribution, the optimal auction is a second-price auction with a reserve price.

☆ The lessons learned from the theory of optimal mechanism design were used in 2008 to increase Yahoo's sponsored search revenue by several percent.

Notes

The model and main results of this lecture are due to Myerson (1981), as are the mentioned extensions to irregular valuation distributions and to Bayesian incentive compatible mechanisms (Remark 5.5). Myerson (1981) also notes the crucial importance of the independence assumption on agents' valuations, an observation that is developed further by Crémer and McLean (1985). With irregular distributions, the virtual welfare-maximizing allocation rule is not monotone, and it is necessary to solve for the monotone allocation rule with the maximum expected virtual welfare. This can be done by "ironing" virtual valuation functions to make them monotone, while at the same time preserving the virtual welfare of the mechanisms that matter. See Hartline (2016) for a textbook treatment of these extensions.

The field experiment with reserve prices in Yahoo sponsored search auctions (Section 5.3) is reported by Ostrovsky and Schwarz (2009). The revenue curve interpretation of virtual valuations in Problem 5.1 is due to Bulow and Roberts (1989). Problem 5.2 is from Azar et al. (2013). Problem 5.3 is closely related to the property of "revenue equivalence," identified already in Vickrey (1961); see Krishna (2010) for an excellent exposition.

Exercises

Exercise 5.1 Consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$.

(a) Prove that the expected revenue obtained by a second-price auction (with no reserve) is $\frac{1}{3}$.

(b) Prove that the expected revenue obtained by a second-price auction with reserve $\frac{1}{2}$ is $\frac{5}{12}$.

Exercise 5.2 Compute the virtual valuation function of the following distributions.

(a) The uniform distribution on $[0, a]$ with $a > 0$.

(b) The exponential distribution with rate $\lambda > 0$ (on $[0, \infty)$).

(c) The distribution given by $F(v) = 1 - \frac{1}{(v+1)^c}$ on $[0, \infty)$, where $c > 0$ is some constant.

Exercise 5.3 Which of the distributions in Exercise 5.2 are regular (Definition 5.3)?

Exercise 5.4 A valuation distribution meets the *monotone hazard rate (MHR)* condition if its *hazard rate* $\frac{f_i(v_i)}{1-F_i(v_i)}$ is nondecreasing in v_i.[15]

[15] For intuition behind the MHR condition, consider waiting for a light bulb to burn out. Given that the bulb hasn't burned out yet, the probability that it burns out right now is increasing in the amount of time that has elapsed.

(a) Prove that every distribution meeting the MHR condition is regular.

(b) Which of the distributions in Exercise 5.2 satisfy the MHR condition?

Exercise 5.5 Prove that for every single-parameter environment and regular valuation distributions $F_1, \dots, F_n$, the virtual-welfare-maximizing allocation rule is monotone in the sense of Definition 3.6. Assume that ties are broken lexicographically with respect to some fixed total ordering over the feasible outcomes.

Exercise 5.6 *(H)* For the valuation distribution in Exercise 5.2(c), with $c = 1$, argue that the expected revenue of an auction does *not* necessarily equal its expected virtual welfare. How do you reconcile this observation with Theorem 5.2?

Exercise 5.7 Consider a k-unit auction (Example 3.2) in which bidders' valuations are drawn i.i.d. from a regular distribution F. Describe an optimal auction. Which of the following does the reserve price depend on: k, n, or F?

Exercise 5.8 Repeat the previous exercise for sponsored search auctions (Example 3.3).

Exercise 5.9 Consider a single-parameter environment and regular valuation distributions $F_1, \dots, F_n$. For $\alpha \in [0, 1]$, call a DSIC mechanism an *α-approximate virtual welfare maximizer* if it always selects a feasible allocation with virtual welfare at least α times the maximum possible. Prove that the expected revenue of an α-approximate virtual welfare maximizer is at least α times that of an optimal mechanism.

Exercise 5.10 In the sponsored search auction case study in Section 5.3, raising reserve prices was particularly effective for valuable keywords (typical valuations-per-click well above the old reserve price of \$.10) that had few bidders (6 or less). Give at least two examples of keywords that you think might have these properties, and explain your reasoning.

Problems

Problem 5.1 This problem derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density function f on the interval $[0, v_{\max}]$, with $v_{\max} < +\infty$.

For a single bidder with valuation drawn from F, for $q \in [0,1]$, define $V(q) = F^{-1}(1-q)$ as the (unique) posted price that yields a probability q of a sale. Define $R(q) = q \cdot V(q)$ as the expected revenue obtained from a single bidder when the probability of a sale is q. The function $R(q)$, for $q \in [0,1]$, is the *revenue curve* of F. Note that $R(0) = R(1) = 0$.

(a) What is the revenue curve for the uniform distribution on $[0,1]$?

(b) Prove that the slope of the revenue curve at q (i.e., $R'(q)$) is precisely $\varphi(V(q))$, where φ is the virtual valuation function for F.

(c) Prove that a distribution is regular if and only if its revenue curve is concave.

Problem 5.2 *(H)* Consider a single bidder with valuation drawn from a regular distribution F that satisfies the assumptions of Problem 5.1. Let p be the *median* of F, meaning the value for which $F(p) = \frac{1}{2}$. Prove that the price p earns at least 50% of the expected revenue of the optimal posted price for F.

Problem 5.3 This problem introduces the Bayes-Nash equilibrium concept and compares the expected revenue earned by first- and second-price single-item auctions.

First-price auctions have no dominant strategies, and we require a new concept to reason about them. For this problem, assume that bidders' valuations are drawn i.i.d. from a commonly known distribution F. A *strategy* of a bidder i in a first-price auction is a predetermined plan for bidding—a function $b_i(\cdot)$ that maps her valuation v_i to a bid $b_i(v_i)$. The semantics are: "when my valuation is v_i, I will bid $b_i(v_i)$." We assume that bidders' strategies are common knowledge, with bidders' valuations (and hence induced bids) private as usual.

A strategy profile $b_1(\cdot), \cdots, b_n(\cdot)$ is a *Bayes-Nash equilibrium* if every bidder always bids optimally given her information—if for every bidder i and every valuation v_i, the bid $b_i(v_i)$ maximizes i's expected utility, where the expectation is with respect to the distribution over others bids induced by F and $\mathbf{b}_{-i}$.

(a) Suppose F is the uniform distribution on $[0, 1]$. Verify that setting $b_i(v_i) = v_i(n-1)/n$ for every i and v_i yields a Bayes-Nash equilibrium.

(b) Prove that the expected revenue of the seller (over $\mathbf{v}$) at this equilibrium of the first-price auction is exactly the expected revenue of the seller in the truthful outcome of a second-price auction.

(c) *(H)* Extend the conclusion in (b) to every continuous and strictly increasing distribution function F on $[0, 1]$.

Problem 5.4 This problem uses first-price auctions to illustrate the extension of the revelation principle (Theorem 4.3) to Bayesian incentive compatible mechanisms (Remark 5.5).

(a) Let $F_1, \ldots, F_n$ be valuation distributions and $b_1, \ldots, b_n$ a Bayes-Nash equilibrium of a first-price auction, as defined in Problem 5.3. Prove that there exists a single-item auction M' such that truthful bidding is a Bayes-Nash equilibrium and, for every valuation profile $\mathbf{v}$, the truthful outcome of M' is identical to the equilibrium outcome of the first-price auction.

(b) A first-price auction is "prior-independent" in that its description makes no reference to bidders' valuation distributions. (See also Section 6.4.) Is the auction M' in part (a) prior-independent in this sense?