Math-Comp 553: Algorithmic Game Theory. *Due: Wednesday 16th October 2019*

Assignment 2

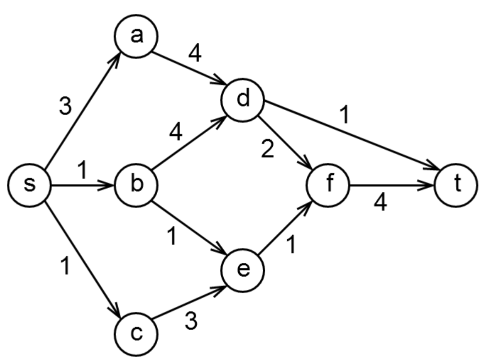
1. *Vickrey-Clarke-Groves Mechanisms.*

   (a) Consider a public project where the government offers three options:
   (i) no project, N, is undertaken (ii) a low quality project, L, is implemented at a cost c_L, or (iii) a high quality project, H, is implemented at a cost c_H. Assume that $c_H = 15$, $c_L = 3$ and that there are three players whose values for the three options are:

	N	L	H
I	0	0	10
II	4	6	8
III	-5	2	11

   With VCG, what is the outcome and what are the payments of the players?

   (b) Consider the following directed network. You wish to purchase a unit of bandwidth from s to t (i.e. you need to buy a directed path from s to t in the network). The cost of providing a unit of bandwidth along each arc is shown; for example, the cost of arc (s, a) is \$3. However, each arc in the network is owned by a different company; thus, these arc costs are private information and are not known to you. You decide to use a VCG mechanism (i.e. a VCG procurement auction) to purchase the bandwidth. What is the outcome, and what are the payments you make to each firm (arc)?



2. *Nash Equilibria.*
   In this question you fill in the missing details from the proof seen in class of Nash's existence theorem.

   (a) Course Textbook (Roughgarden), p296: Exercise 20.5.

   (b) Course Textbook (Roughgarden), p296: Exercise 20.6.

3. *Correlated Equilibria.*

   (a) For a 2-player game, give a linear program that will find the *fairest* correlated equilibrium, namely the one that maximises the minimum payoff of any player.

   (b) Draw the payoff-space produced by correlated equilibria in the following 2-player game, and find the fairest correlated equilibrium.

(P_1, P_2)	L	R
T	(4,4)	(1,6)
B	(6,1)	(-3,-3)

4. *Treasure Discovery Games.*
   Take a bipartite graph $G = (A \cup B, E)$. There is a set of k players, and each player can choose any node in A to locate at. At each node $b_j \in B$ there is a piece of treasure worth v_j dollars. Once the players have chosen their locations, the piece of treasure at b_j will be *discovered* if and only if there is at least one player located at a node in A adjacent to b_j. If there is exactly one player i located at a node adjacent to b_j then player i will *win* that piece of treasure. If more than one player are adjacent to b_j then that treasure is disputed and no player will win it – we may assume such treasure becomes public property. Thus, the social objective is to maximise the value of treasure that is found (i.e. treasure that is adjacent to at least one player). The private objective of player i is to maximise the value of the treasure it wins (i.e. treasure that it is adjacent to but that no other player is adjacent to).

   (a) Show by a potential function argument that this game has a Pure strategy Nash equilibrium.

   (b) Use the potential function to give a bound on the Price of Stability.

5. *Cost-Sharing Games.*
   Suppose we have k players and player i wants to open a connection (path) between vertex s_i and vertex t_i in a network $G = (V, E)$. It costs c_e to open an edge (link) $e \in E$, but this cost is shared equally amongst all the players that use it. (If no-one uses it then the edge remains closed and there is no cost). Thus, the strategy of player i is simply to select a path P_i in G between s_i and t_i. Given a set of strategies $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ the cost to player i is

$$\sum_{e \in P_i} \frac{c_e}{n_e(\mathcal{P})}$$

   where n_e is the total number of players using edge e under these strategies $\mathcal{P}$.

   (a) What is the social objective for this game?

   (b) Show by a potential function argument that this game has a Pure strategy Nash equilibrium.

   (c) Use the potential function to give a bound on the Price of Stability.

6. *Weighted Atomic Selfish Routing.*
   Course Textbook (Roughgarden), p184: Exercise 13.5.

7. **[Math 553 Bonus Question]** *Sperner's Lemma.*
   Suppose we have a rectangular cake. There are three people and we must cut the cake into 3 pieces, using two cuts parallel to the horizontal axis. Given such a cutting, each player will choose their favourite piece (this choice can depend upon factors other than size, e.g. what toppings fall on those pieces, etc). Use Sperner's Lemma[1] to show that there is some way to cut the cake so there each player prefers a different piece.

8. **[Comp 553 Bonus Question]** *Implementing a VCG Auction.*
   Consider the following auction. There are 10 bidders and 8 items. The bidders have a non-negative value for each item, and their value for any set of items is the *maximum* of the values for the items in the set. The value of the empty set for any bidder is 0. The following table gives the value of each item for each bidder. (This table is available in text form on the MyCourses webpage, where the i^{th} row contains only bidder i's values for each item.) Write a program to compute the VCG allocation and prices. Your answer must include your program along with the allocation and prices.

Bidder	1	2	3	4	5	6	7	8
1	26	8	59	5	42	17	6	34
2	11	18	53	9	40	22	17	35
3	19	25	50	24	49	23	21	31
4	2	3	52	3	45	14	21	38
5	1	23	54	28	47	17	14	33
6	22	27	57	27	43	19	23	36
7	21	19	55	28	46	16	5	32
8	20	12	56	18	41	16	10	39
9	2	4	58	28	48	26	15	30
10	20	9	51	10	44	20	6	37

[1] You may assume that preference sets are closed: any piece that is preferred for a convergent sequence of cut-sets is preferred at the limiting cut-set.