Lecture 3

Myerson's Lemma

Last lecture advocated a two-step approach to the design of auctions that are DSIC, welfare maximizing, and computationally efficient (Section 2.6.4). The first step assumes truthful bids and identifies how to allocate items to bidders to maximize the social welfare. For instance, in sponsored search auctions, this step is implemented by assigning the ith highest bidder to the ith best slot. The second step derives the appropriate selling prices, to render truthful bidding a dominant strategy. This lecture states and proves *Myerson's lemma*, a powerful and general tool for implementing this second step. This lemma applies to sponsored search auctions as a special case, and Lectures 4 and 5 provide further applications.

Section 3.1 introduces single-parameter environments, a convenient generalization of the mechanism design problems introduced in Lecture 2. Section 3.2 rephrases the three steps of sealed-bid auctions (Section 2.2) in terms of allocation and payment rules. Section 3.3 defines two properties of allocation rules, implementability and monotonicity, and states and interprets Myerson's lemma. Section 3.4 gives a proof sketch of Myerson's lemma; it can be skipped on a first reading. Myerson's lemma includes a formula for payments in DSIC mechanisms, and Section 3.5 applies this formula to sponsored search auctions.

3.1 Single-Parameter Environments

A good level of abstraction at which to state Myerson's lemma is *single-parameter environments*. Such an environment has some number n of agents. Each agent i has a private nonnegative valuation v_i, her value "per unit of stuff" that she acquires. Finally, there is a *feasible set* X. Each element of X is a nonnegative n-vector $(x_1, x_2, \ldots, x_n)$, where x_i denotes the "amount of stuff" given to agent i.

Example 3.1 (Single-Item Auction) In a single-item auction (Section 2.1), X is the set of 0-1 vectors that have at most one 1—that is, $\sum_{i=1}^{n} x_i \leq 1$.

Example 3.2 (k-Unit Auction) With k identical items and the constraint that each bidder gets at most one (Exercise 2.3), the feasible set is the set of 0-1 vectors that satisfy $\sum_{i=1}^{n} x_i \leq k$.

Example 3.3 (Sponsored Search Auction) In a sponsored search auction (Section 2.6), X is the set of n-vectors corresponding to assignments of bidders to slots, where each slot is assigned to at most one bidder and each bidder is assigned to at most one slot. If bidder i is assigned to slot j, then the component x_i equals the click-through rate α_j of her slot.

Example 3.4 (Public Project) Deciding whether or not to build a public project that can be used by all, such as a new bridge, can be modeled by the set $X = \{(0, 0, \ldots, 0), (1, 1, \ldots, 1)\}$.

Example 3.4 shows that single-parameter environments are general enough to capture applications different from auctions. At this level of generality, we refer to *agents* rather than bidders. We sometimes use the term *reports* instead of bids. A *mechanism* is a general procedure for making a decision when agents have private information (like valuations), whereas an *auction* is a mechanism specifically for the exchange of goods and money. See also Table 3.1.

auction	mechanism
bidder	agent
bid	report
valuation	valuation

Table 3.1: Correspondence of terms in auctions and mechanisms. An auction is the special case of a mechanism that is designed for the exchange of goods and money.

3.2 Allocation and Payment Rules

Recall that a sealed-bid auction has to make two choices: who wins and who pays what. These two decisions are formalized via an *allocation rule* and a *payment rule*, respectively. Here are the three steps:

1. Collect bids $\mathbf{b} = (b_1, \ldots, b_n)$ from all agents. The vector $\mathbf{b}$ is called the *bid vector* or *bid profile*.
2. **[allocation rule]** Choose a feasible allocation $\mathbf{x}(\mathbf{b}) \in X \subseteq \mathbb{R}^n$ as a function of the bids.
3. **[payment rule]** Choose payments $\mathbf{p}(\mathbf{b}) \in \mathbb{R}^n$ as a function of the bids.

Procedures of this type are called *direct-revelation mechanisms*, because in the first step agents are asked to reveal directly their private valuations. An example of an indirect mechanism is an iterative ascending auction (cf., Exercise 2.7).

With our quasilinear utility model, in a mechanism with allocation and payment rules $\mathbf{x}$ and $\mathbf{p}$, respectively, agent i receives utility

$$u_i(\mathbf{b}) = v_i \cdot x_i(\mathbf{b}) - p_i(\mathbf{b})$$

when the bid profile is $\mathbf{b}$.

We focus on payment rules that satisfy

$$p_i(\mathbf{b}) \in [0, b_i \cdot x_i(\mathbf{b})] \tag{3.1}$$

for every agent i and bid profile $\mathbf{b}$. The constraint that $p_i(\mathbf{b}) \geq 0$ is equivalent to prohibiting the seller from paying the agents. The constraint that $p_i(\mathbf{b}) \leq b_i \cdot x_i(\mathbf{b})$ ensures that a truthful agent receives nonnegative utility (do you see why?).[1]

3.3 Statement of Myerson's Lemma

Next are two important definitions. Both articulate a property of allocation rules.

[1] There are applications where it makes sense to relax one or both of these restrictions on payments, including those discussed in Exercise 2.5 and Problem 7.1.

Definition 3.5 (Implementable Allocation Rule) An allocation rule $\mathbf{x}$ for a single-parameter environment is *implementable* if there is a payment rule $\mathbf{p}$ such that the direct-revelation mechanism $(\mathbf{x}, \mathbf{p})$ is DSIC.

That is, the implementable allocation rules are those that extend to DSIC mechanisms. Equivalently, the projection of DSIC mechanisms onto their allocation rules is the set of implementable rules. If our aim is to design a DSIC mechanism, we must confine ourselves to implementable allocation rules—they form our "design space." In this terminology, we can rephrase the cliffhanger from the end of Lecture 2 as: is the welfare-maximizing allocation rule for sponsored search auctions, which assigns the ith highest bidder to the ith best slot, implementable?

For instance, consider a single-item auction (Example 3.1). Is the allocation rule that awards the item to the highest bidder implementable? Sure—we've already constructed a payment rule, the second-price rule, that renders it DSIC. What about the allocation rule that awards the item to the *second-highest* bidder? Here, the answer is not clear: we haven't seen a payment rule that extends it to a DSIC mechanism, but it also seems tricky to argue that no payment rule could conceivably work.

Definition 3.6 (Monotone Allocation Rule) An allocation rule $\mathbf{x}$ for a single-parameter environment is *monotone* if for every agent i and bids $\mathbf{b}_{-i}$ by the other agents, the allocation $x_i(z, \mathbf{b}_{-i})$ to i is nondecreasing in her bid z.

That is, in a monotone allocation rule, bidding higher can only get you more stuff.

For example, the single-item auction allocation rule that awards the item to the highest bidder is monotone: if you're the winner and you raise your bid (keeping other bids fixed), you continue to win. By contrast, awarding the item to the second-highest bidder is a non-monotone allocation rule: if you're the winner and you raise your bid high enough, you lose.

The welfare-maximizing allocation rule for sponsored search auctions (Example 3.3), with the ith highest bidder awarded the ith best slot, is monotone. When a bidder raises her bid, her position in the sorted order of bids can only increase, and this can only increase the click-through rate of her assigned slot.

We state Myerson's lemma in three parts; each is conceptually interesting and useful in later applications.

Theorem 3.7 (Myerson's Lemma) *Fix a single-parameter environment.*

(a) An allocation rule $\mathbf{x}$ *is implementable if and only if it is monotone.*

(b) If $\mathbf{x}$ *is monotone, then there is a unique payment rule for which the direct-revelation mechanism* $(\mathbf{x}, \mathbf{p})$ *is DSIC and* $p_i(\mathbf{b}) = 0$ *whenever* $b_i = 0$.

(c) The payment rule in (b) is given by an explicit formula.[2]

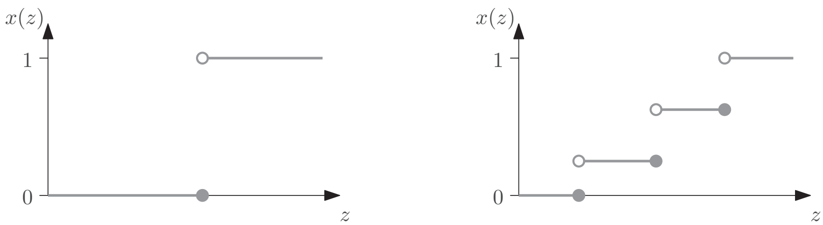
Myerson's lemma is the foundation on which we'll build most of our mechanism design theory. Part (a) states that Definitions 3.5 and 3.6 define exactly the same class of allocation rules. This equivalence is incredibly powerful: Definition 3.5 describes our design goal but is unwieldy to work with and verify, while Definition 3.6 is far more "operational." Usually, it's not difficult to check whether or not an allocation rule is monotone. Part (b) states that when an allocation rule is implementable, there is no ambiguity in how to assign payments to achieve the DSIC property—there is only one way to do it. Moreover, there is a relatively simple and explicit formula for this payment rule (part (c)), a property we apply to sponsored search auctions in Section 3.5 and to revenue-maximizing auction design in Lectures 5–6.

*3.4 Proof of Myerson's Lemma (Theorem 3.7)

Fix a single-parameter environment and consider an allocation rule $\mathbf{x}$, which may or may not be monotone. Suppose there is a payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is a DSIC mechanism—what could $\mathbf{p}$ look like? The plan of the proof is to use the stringent DSIC constraint to whittle the possibilities for $\mathbf{p}$ down to a single candidate. We establish all three parts of the theorem in one fell swoop.

[2]See formulas (3.5) and (3.6) for details and Section 3.5 for concrete examples.

Recall the DSIC condition: for every agent i, every possible private valuation v_i, every set of bids $\mathbf{b}_{-i}$ by the other agents, it must be that i's utility is maximized by bidding truthfully. For now, fix i and $\mathbf{b}_{-i}$ arbitrarily. As shorthand, write $x(z)$ and $p(z)$ for the allocation $x_i(z, \mathbf{b}_{-i})$ and payment $p_i(z, \mathbf{b}_{-i})$ of i when she bids z, respectively. Figure 3.1 gives two examples of a possible *allocation curve*, meaning the graph of such an x as a function of z.



(a) 0-1 monotone curve

(b) Piecewise constant monotone curve

Figure 3.1: Examples of allocation curves $x(\cdot)$.

We invoke the DSIC constraint via a simple but clever swapping trick. Suppose $(\mathbf{x}, \mathbf{p})$ is DSIC, and consider any $0 \le y < z$. Because agent i might well have private valuation z and is free to submit the false bid y, DSIC demands that

$$\underbrace{z \cdot x(z) - p(z)}_{\text{utility of bidding } z} \quad \ge \quad \underbrace{z \cdot x(y) - p(y).}_{\text{utility of bidding } y} \tag{3.2}$$

Similarly, since agent i might well have the private valuation y and could submit the false bid z, $(\mathbf{x}, \mathbf{p})$ must satisfy

$$\underbrace{y \cdot x(y) - p(y)}_{\text{utility of bidding } y} \quad \ge \quad \underbrace{y \cdot x(z) - p(z).}_{\text{utility of bidding } z} \tag{3.3}$$

Myerson's lemma is, in effect, trying to solve for the payment rule $\mathbf{p}$ given the allocation rule $\mathbf{x}$. Rearranging inequalities (3.2) and (3.3) yields the following "payment difference sandwich," bounding $p(y) - p(z)$ from below and above:

$$z \cdot [x(y) - x(z)] \le p(y) - p(z) \le y \cdot [x(y) - x(z)]. \tag{3.4}$$

The payment difference sandwich already implies that every implementable allocation rule is monotone (Exercise 3.1). Thus, we can assume for the rest of the proof that $\mathbf{x}$ is monotone.

Next, consider the case where x is a piecewise constant function, as in Figure 3.1. The graph of x is flat except for a finite number of "jumps." In (3.4), fix z and let y tend to z from above. Taking the limit $y \downarrow z$ from above in (3.4), the left- and right-hand sides become 0 if there is no jump in x at z. If there is a jump of magnitude h at z, then the left- and right-hand sides both tend to $z \cdot h$. This implies the following constraint on p, for every z:

$$\text{jump in } p \text{ at } z = z \cdot [\text{jump in } x \text{ at } z].$$

Combining this with the initial condition $p(0) = 0$, we've derived the following *payment formula*, for every agent i, bids $\mathbf{b}_{-i}$ by other agents, and bid b_i by i:

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j], \tag{3.5}$$

where $z_1, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(\cdot, \mathbf{b}_{-i})$ in the range $[0, b_i]$.

A similar argument applies when x is a monotone function that is not piecewise constant. For instance, suppose that x is differentiable.[3] Dividing the payment difference sandwich (3.4) by $y - z$ and taking the limit as $y \downarrow z$ yields the constraint

$$p'(z) = z \cdot x'(z).$$

Combining this with the condition $p(0) = 0$ yields the payment formula

$$p_i(b_i, \mathbf{b}_{-i}) = \int_0^{b_i} z \cdot \tfrac{d}{dz} x_i(z, \mathbf{b}_{-i}) dz \tag{3.6}$$

for every agent i, bid b_i, and bids $\mathbf{b}_{-i}$ by the other agents.

We reiterate that these payment formulas give the *only possible* payment rule that has a chance of extending the given allocation rule $\mathbf{x}$ into a DSIC mechanism. Thus, for every allocation rule $\mathbf{x}$,

[3] With some additional facts from calculus, the proof extends to general monotone functions. We omit the details.

there is at most one payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is DSIC (cf., part (b) of Theorem 3.7). But the proof is not complete: we still have to check that this payment rule works provided $\mathbf{x}$ is monotone! Indeed, we already know that even this payment rule cannot extend a non-monotone allocation rule to a DSIC mechanism.

We give a proof by picture that, if $\mathbf{x}$ is a monotone and piecewise constant allocation rule and $\mathbf{p}$ is defined by (3.5), then $(\mathbf{x}, \mathbf{p})$ is a DSIC mechanism. The same argument works more generally for monotone allocation rules that are not piecewise constant, with payments defined as in (3.6). This will complete the proof of all three parts of Myerson's lemma.

Figures 3.2(a)–(i) depict the utility of an agent when she bids truthfully, overbids, and underbids, respectively. The allocation curve $x(z)$ and the private valuation v of the agent is the same in all three cases. Recall that the agent's utility when she bids b is $v \cdot x(b) - p(b)$. We depict the first term $v \cdot x(b)$ as a shaded rectangle of width v and height $x(b)$ (Figures 3.2(a)–(c)). Using the formula (3.5), we see that the payment $p(b)$ can be represented as the shaded area to the left of the allocation curve in the range $[0, b]$ (Figures 3.2(d)–(f)). The agent's utility is the difference between these two terms (Figures 3.2(g)–(i)). When the agent bids truthfully, her utility is precisely the area under the allocation curve in the range $[0, v]$ (Figure 3.2(g)). When the agent overbids, her utility is this same area, minus the area above the allocation curve in the range $[v, b]$ (Figure 3.2(h)). When the agent underbids, her utility is a subset of the area under the allocation curve in the range $[0, v]$ (Figure 3.2(i)). Since the agent's utility is the largest in the first case, the proof is complete.

3.5 Applying the Payment Formula

The explicit payment rule given by Myerson's lemma (Theorem 3.7(c)) is easy to understand and apply in many applications. For starters, consider a single-item auction (Example 3.1) with the allocation rule that allocates the item to the highest bidder. Fixing a bidder i and bids $\mathbf{b}_{-i}$ by the others, the function $x_i(z, \mathbf{b}_{-i})$ is 0 up to $B = \max_{j \neq i} b_j$ and 1 thereafter. Myerson's payment formula for such piecewise constant functions, derived in Section 3.4 as equation (3.5),

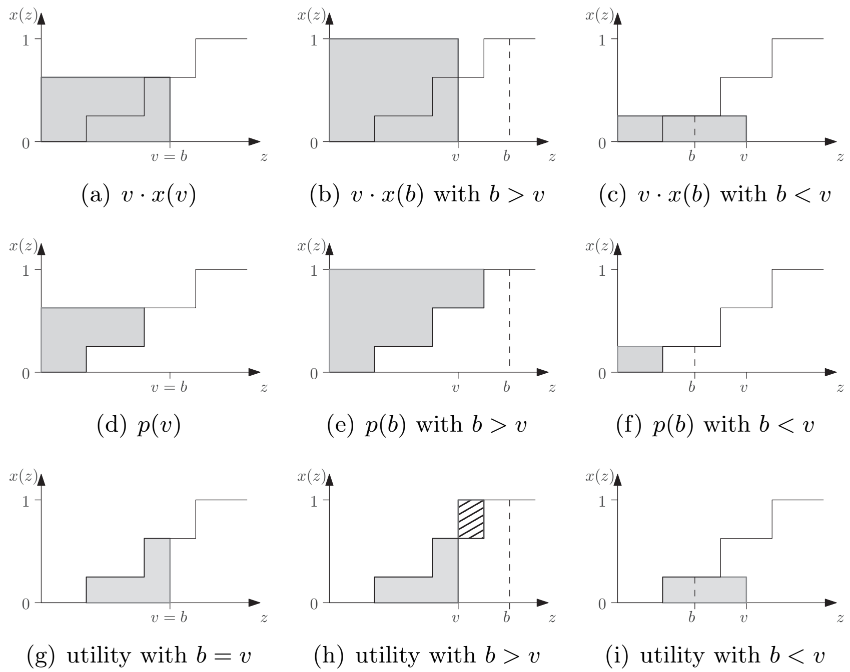


Figure 3.2: Proof by picture that the payment rule in (3.5), coupled with the given monotone and piecewise constant allocation rule, yields a DSIC mechanism. The three columns consider the cases of truthful bidding, overbidding, and underbidding, respectively. The three rows show the welfare $v \cdot x(b)$, the payment $p(b)$, and the utility $v \cdot x(b) - p(b)$, respectively. In (h), the solid region represents positive utility and the lined region represents negative utility.

is

$$p_i(b_i, \mathbf{b}_{-i}) = \sum_{j=1}^{\ell} z_j \cdot [\text{jump in } x_i(\cdot, \mathbf{b}_{-i}) \text{ at } z_j],$$

where $z_1, \ldots, z_\ell$ are the breakpoints of the allocation function $x_i(\cdot, \mathbf{b}_{-i})$ in the range $[0, b_i]$. For the highest-bidder single-item allocation rule, this is either 0 (if $b_i < B$) or, if $b_i > B$, there is a single breakpoint (a jump of 1 at B) and the payment is $p_i(b_i, \mathbf{b}_{-i}) = B$. Thus, Myerson's lemma regenerates the second-price payment rule as a special case.

Next, consider a sponsored search auction (Example 3.3), with

k slots with click-through rates $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Let $\mathbf{x}(\mathbf{b})$ be the allocation rule that assigns the ith highest bidder to the ith best slot, for $i = 1, 2, \ldots, k$. This rule is monotone and, assuming truthful bids, welfare maximizing (Exercise 2.8). Myerson's payment formula then gives the unique payment rule $\mathbf{p}$ such that $(\mathbf{x}, \mathbf{p})$ is DSIC. To describe it, consider a bid profile $\mathbf{b}$, and re-index the bidders so that $b_1 \geq b_2 \geq \cdots \geq b_n$. First focus on the highest bidder and imagine increasing her bid from 0 to b_1, holding the other bids fixed. The allocation $x_1(z, \mathbf{b}_{-1})$ increases from 0 to α_1 as z increases from 0 to b_1, with a jump of $\alpha_j - \alpha_{j+1}$ at the point where z becomes the jth highest bid in the profile $(z, \mathbf{b}_{-1})$, namely b_{j+1}. In general, Myerson's payment formula gives the payment

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1}(\alpha_j - \alpha_{j+1}) \tag{3.7}$$

for the ith highest bidder (where $\alpha_{k+1} = 0$).

Our sponsored search model assumes that bidders don't care about impressions (i.e., having their link shown), except inasmuch as it leads to a click. This motivates charging bidders per click rather than per impression. The per-click payment for the bidder i in slot i is simply that in (3.7), scaled by $\frac{1}{\alpha_i}$:

$$p_i(\mathbf{b}) = \sum_{j=i}^{k} b_{j+1} \frac{\alpha_j - \alpha_{j+1}}{\alpha_i}. \tag{3.8}$$

Thus, when a bidder's link is clicked, she pays a suitable convex combination of the lower bids.

By historical accident, the sponsored search auctions used by search engines are based on the "generalized second-price (GSP)" auction, which is a simpler version of the DSIC auction above. The GSP allocation rule also assigns the ith highest bidder to the ith best slot, but its payment rule charges this bidder a per-click payment equal to the $(i+1)$th highest bid. For a fixed set of bids, these per-click payments are generally higher than those in (3.8). Myerson's lemma asserts that the payment rule in (3.8) is the unique one that, when coupled with the GSP allocation rule, yields a DSIC mechanism. We can immediately conclude that the GSP auction is *not* DSIC. It still has a number of nice properties, however, and is equivalent to the

DSIC auction in certain senses. Problem 3.1 explores this point in detail.

The Upshot

☆ In a single-parameter environment, every agent has a private valuation per "unit of stuff," and a feasible set defines how much "stuff" can be jointly allocated to the agents. Examples include single-item auctions, sponsored search auctions, and public projects.

☆ The allocation rule of a direct-revelation mechanism specifies, as a function of agents' bids, who gets what. The payment rule of such a mechanism specifies, as a function of agents' bids, who pays what.

☆ An allocation rule is implementable if there exists a payment rule that extends it to a DSIC mechanism.

☆ An allocation rule is monotone if bidding higher can only increase the amount of stuff allocated to an agent, holding other agents' bids fixed.

☆ Myerson's lemma states that an allocation rule is implementable if and only if it is monotone. In this case, the corresponding payment rule is unique (assuming that bidding 0 results in paying 0).

☆ There is an explicit formula, given in (3.5) and (3.6), for the payment rule that extends a monotone allocation rule to a DSIC mechanism.

☆ Myerson's payment formula yields an elegant payment rule (3.8) for the payments-per-click in an ideal sponsored search auction.

Notes

Myerson's lemma is from Myerson (1981). The sponsored search payment formula (3.8) is noted in Aggarwal et al. (2006). Problem 3.1 is due independently to Edelman et al. (2007) and Varian (2007). Problem 3.2 is related to the "profit extractors" introduced by Goldberg et al. (2006) and the cost-sharing mechanisms studied by Moulin and Shenker (2001). Problem 3.3 is a special case of the theory developed by Moulin and Shenker (2001).

Exercises

Exercise 3.1 Use the "payment difference sandwich" in (3.4) to prove that if an allocation rule is not monotone, then it is not implementable.

Exercise 3.2 The proof of Myerson's lemma (Section 3.4) concludes with a "proof by picture" that coupling a monotone and piecewise constant allocation rule $\mathbf{x}$ with the payment rule defined by (3.5) yields a DSIC mechanism. Where does the proof-by-picture break down if the piecewise constant allocation rule $\mathbf{x}$ is not monotone?

Exercise 3.3 Give an algebraic proof that coupling a monotone and piecewise constant allocation rule $\mathbf{x}$ with the payment rule defined by (3.5) yields a DSIC mechanism.

Exercise 3.4 Consider the following extension of the sponsored search setting described in Section 2.6. Each bidder i now has a publicly known *quality* β_i, in addition to a private valuation v_i per click. As usual, each slot j has a CTR α_j, and $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_k$. We assume that if bidder i is placed in slot j, then the probability of a click is $\beta_i\alpha_j$. Thus bidder i derives value $v_i\beta_i\alpha_j$ from the jth slot.

Describe the welfare-maximizing allocation rule in this generalized sponsored search setting. Prove that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

Problems

Problem 3.1 Recall our model of sponsored search auctions (Section 2.6): there are k slots, the jth slot has a click-through rate

(CTR) of α_j (nonincreasing in j), and the utility of bidder i in slot j is $\alpha_j(v_i - p_j)$, where v_i is the (private) value-per-click of the bidder and p_j is the price charged per-click in slot j. The *Generalized Second-Price (GSP)* auction is defined as follows:

The Generalized Second Price (GSP) Auction

1. Rank advertisers from highest to lowest bid; assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_n$.
2. For $i = 1, 2, \ldots, k$, assign the ith bidder to the i slot.
3. For $i = 1, 2, \ldots, k$, charge the ith bidder a price of b_{i+1} per click.

The following subproblems show that the GSP auction always has a canonical equilibrium that is equivalent to the truthful outcome in the corresponding DSIC sponsored search auction.

(a) Prove that for every $k \geq 2$ and sequence $\alpha_1 \geq \cdots \geq \alpha_k > 0$ of CTRs, the GSP auction is not DSIC.

(b) *(H)* Fix CTRs for slots and values-per-click for bidders. We can assume that $k = n$ by adding dummy slots with zero CTR (if $k < n$) or dummy bidders with zero value-per-click (if $k > n$). A bid profile $\mathbf{b}$ is an *equilibrium* of GSP if no bidder can increase her utility by unilaterally changing her bid. Verify that this condition translates to the following inequalities, under our standing assumption that $b_1 \geq b_2 \geq \cdots \geq b_n$: for every i and higher slot $j < i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j);$$

and for every lower slot $j > i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}).$$

(c) A bid profile $\mathbf{b}$ with $b_1 \geq \cdots \geq b_n$ is *envy-free* if for every bidder i and slot $j \neq i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}). \qquad (3.9)$$

Verify that every envy-free bid profile is an equilibrium.[4]

(d) *(H)* A bid profile is *locally envy-free* if the inequality (3.9) holds for every pair of adjacent slots—for every i and $j \in \{i-1, i+1\}$. By definition, an envy-free bid profile is also locally envy-free. Prove that, for every set of strictly decreasing CTRs, every locally envy-free bid profile is also envy-free.

(e) *(H)* Prove that, for every set of values-per-click and strictly decreasing CTRs, there is an equilibrium of the GSP auction in which the assignments of bidders to slots and all payments-per-click equal those in the truthful outcome of the corresponding DSIC sponsored search auction.

(f) Prove that the equilibrium in (e) is the lowest-revenue envy-free bid profile.

Problem 3.2 This problem considers a k-unit auction (Example 3.2) in which the seller has a specific revenue target R. Consider the following algorithm that, given bids $\mathbf{b}$ as input, determines the winning bidders and their payments.

Revenue Target Auction

initialize a set S to the top k bidders
while there is a bidder $i \in S$ with $b_i < R/|S|$ **do**
  remove an arbitrary such bidder from S
allocate an item to each bidder of S (if any) at a price of $R/|S|$

(a) Give an explicit description of the allocation rule of the Revenue Target Auction, and prove that it is monotone.

(b) *(H)* Conclude from Myerson's lemma that the Revenue Target Auction is a DSIC mechanism.

[4] Why "envy-free?" Because if we write $p_j = b_{j+1}$ for the current price-per-click of slot j, then these inequalities translate to: "every bidder i is as happy getting her current slot at her current price as she would be getting any other slot and that slot's current price."

(c) Prove that whenever the DSIC and welfare-maximizing k-unit auction (Exercise 2.3) obtains revenue at least R, the Revenue Target Auction obtains revenue R.

(d) Prove that there exists a valuation profile for which the Revenue Target Auction obtains revenue R but the DSIC and welfare-maximizing auction earns revenue less than R.

Problem 3.3 This problem revisits the issue of collusion in auctions; see also Problem 2.2.

(a) Prove that the Revenue Target Auction in Problem 3.2 is *group-strategyproof*, meaning that no coordinated false bids by a subset of bidders can ever strictly increase the utility of one of its members without strictly decreasing the utility of some other member.

(b) Is the DSIC and welfare-maximizing k-unit auction group-strategyproof?