Math 236 Algebra 2 Assignment 4

Jonathan Pearce, 260672004

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Problem 1a.

$$x_3 + x_4 = 0 \implies x_3 = -x_4$$

$$x_1 + 3x_2 - x_4 = 0 \Leftrightarrow x_1 + 3x_2 + x_3 = 0$$

From these conditions, it can be seen that any element of this vector space is of the form $(a, b, -(a+3b), (a+3b))$ where $a, b \in \mathbb{F}$. Consider the set of vectors,

$$\{(0,1,-3,3), (-3,1,0,0)\}$$

Let $a, b \in \mathbb{F}$:

$$a(0,1,-3,3) + b(-3,1,0,0) = (0, a, -3a, 3a) + (-3b, b, 0, 0)$$

$$= (-3b, a+b, -(3a), 3a) = (-3b, a+b, -(-3b+3a+3b), (-3b+3a+3b))$$

$$= (-3b, a+b, -[(-3b)+3(a+b)], [(-3b)+3(a+b)])$$

Therefore $\{(0,1,-3,3), (-3,1,0,0)\}$ spans the vector space.
Now solve $a(0,1,-3,3) + b(-3,1,0,0) = 0$:

$$a*0 + b*(-3) = -3b = 0 \implies b = 0$$

$$a*1 + b*1 = a + b = 0 \implies a = -b = 0$$

Therefore $\{(0,1,-3,3), (-3,1,0,0)\}$ are linearly independent. This means that $\{(0,1,-3,3), (-3,1,0,0)\}$ is a basis for the vector space. The dimension of the vector space is 2.

Problem 1b.

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{pmatrix} = 0$$

$$\implies a = -2c, b = -2d$$

From these conditions, it can be seen that any element of this vector space is of the form $\begin{pmatrix} a & b \\ -\frac{1}{2}a & -\frac{1}{2}b \end{pmatrix}$ where $a, b \in \mathbb{F}$. Consider the set of vectors,

$$\left\{ \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} \right\}$$

Let $a, b \in \mathbb{F}$:

$$a\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} a & 0 \\ -\frac{1}{2}a & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & -\frac{1}{2}b \end{pmatrix} = \begin{pmatrix} a & b \\ -\frac{1}{2}a & -\frac{1}{2}b \end{pmatrix}$$

Therefore $\left\{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}\right\}$ spans the vector space.

Now solve $a\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} = 0$

$$a\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} a & b \\ -\frac{1}{2}a & -\frac{1}{2}b \end{pmatrix} = 0$$

$$\implies a = 0, b = 0$$

Therefore $\left\{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}\right\}$ are linearly independent. Hence, $\left\{\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}\right\}$ is a basis for the vector space. The dimension of the vector space is 2.

Problem 3. Prove Linear Independence:
Let $a_1, ..., a_m, b_1, ..., b_n \in \mathbb{F}$ Suppose,

$$a_1u_1 + ... + a_mu_m + b_1w_1 + ... + b_nw_n = 0$$

$$\implies a_1u_1 + ... + a_mu_m = -(b_1w_1 + ... + b_nw_n)$$

$$\implies a_1u_1 + ... + a_mu_m \in U \cap W, -(b_1w_1 + ... + b_nw_n) \in U \cap W$$

Since $U \oplus W$ is a direct sum, then $U \cap W = \{0\}$. Therefore,

$$a_1u_1 + ... + a_mu_m = 0, -(b_1w_1 + ... + b_nw_n) = 0$$

Since $u_1, ..., u_m$ is a basis for U and $w_1, ..., w_n$ is a basis for W, this means they are linearly independent and therefore $a_1 = ... = a_m = b_1 = ... = b_n = 0$. Therefore $u_1, ..., u_m, w_1, ..., w_n$ are linearly independent. Prove that $a_1, ..., a_m, b_1, ..., b_n$ spans V:
Since $V = U \oplus W$, then for any $v \in V$ there exists $u \in U$ and $w \in W$ such that $v = u + w$. Because $u_1, ...u_m$ is a basis for U and $w_1, ...w_n$ is a basis for W we can express $u = c_1u_1 + ... + c_mu_m$ and $v = d_1w_1 + ... + d_nw_n$ such that $c_1, ...c_m, d_1, ..., d_n \in \mathbb{F}$. Therefore,

$$v = u + w = c_1u_1 + ... + c_mu_m + d_1w_1 + ... + d_nw_n$$

Therefore $u_1, ..., u_m, w_1, ..., w_n$ span V and therefore $u_1, ..., u_m, w_1, ..., w_n$ is a basis for V.

Problem 6.

$$dim(U + W) = dimU + dimW - dim(U \cap W) = 10 - dim(U \cap W)$$

Since U and W are both subspaces of $\mathbb{R}^9$ then $U + W$ is a subspace of $\mathbb{R}^9$. Therefore,

$$dim(U + W) \leq dim\,\mathbb{R}^9 = 9$$

$$\implies 10 - dim(U \cap W) \leq 9$$

$$\implies dim(U \cap W) \geq 1$$

If $U \cap W = \{0\}$ then $dim(U \cap W) = 0$ since the subspace spanned by the zero vector has dimension zero. Therefore $U \cap W \neq \{0\}$