Assignment 3: Householder, Least Squares, SVD
Math 327/397 Winter 2019

Due: Friday Mar 22, 2019

Instructions

Submit a complete paper copy of your solutions in class or to the math department by 4:30pm on the due date. Questions 1 – 5 are for everyone. Question 6 is for Math 397 only.

Question 1: Householder QR Decomposition

(a) Let $u \in \mathbb{R}^m$ be a non-zero vector. Verify that for $v = u - \|u\|e_1$, where $e_1 \in \mathbb{R}^m$ is the first standard basis vector, the Householder matrix $H_v = I - 2\frac{vv^T}{v^Tv}$ maps u to $\|u\|e_1$. Is v unique? Justify. What v would allow H_v to map u to $-\|u\|e_1$?

(b) Show that if $H_v \in \mathbb{R}^{m\times m}$ is a Householder matrix for some $v \in \mathbb{R}^m$, then

$$H = \begin{bmatrix} I_{k\times k} & 0_{k\times m} \\ 0_{m\times k} & H_v \end{bmatrix}$$

is also a Householder matrix by finding $\bar{v}$ such that $H = H_{\bar{v}}$.

(c) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

Compute by hand the reduced QR decomposition of A using the Householder method. Ensure that R has non-negative diagonal elements. Keep at least four significant digits in your intermediate calculations.

(d) Assuming the input is full rank, write the pseudo-code for computing the Householder QR decomposition.

Question 2: Householder Efficiency

(a) Implement the Householder QR into a Matlab m-file in two different ways, one where you construct the matrix H_v explicitly using outer products, and another where H_v is implicit and you only use its action in intermediate calculations for Q and R.

(b) Run both implementations on A from Question 1 and paste the output.

(c) Write a Matlab function *compareHouseholderV1V2.m* which takes an input n, constructs a sequence of matrices $A = \text{randn}\ (100 * k, 50 * k)$ for k=1,2...,n, applies each version of Householder QR on these matrices, and, without displaying the output, calculates the time it took for each computation. Print the and also plot on the same graph the vector of times it took to compute your two versions of Householder using $n = 10$. (Hint: use Matlab's *tic* and *toc* commands.)

Question 3: Least Squares

Using the context of Question 2, suppose that running the Householder method using your efficient version gave you the following sequence of running times in seconds:

$$(0.0131, 0.0542, 0.1652, 0.5614, 1.6412, 2.9546, 4.9099, 8.3626, 10.9059, 15.4361)$$

(a) You believe that the running index k is related to these running times using a cubic polynomial. Formulate a linear least squares problem for this situation, clearly defining your variables.

(b) Using the normal equations, solve this least squares problem, showing all logical steps. You may use Matlab for any routine calculations. Plot the points in the table together with the model approximation.

(c) Repeat (b) using the QR decomposition method. You may use Matlab's *qr* function.

(d) According to your model, how much time do you expect your Householder algorithm to run for a 750×375 sized matrix? What about a 10000×5000 sized matrix?

Question 4: SVD

Let $A \in \mathbb{R}^{5\times 4}$ have full SVD

$$A = U\Sigma V^T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{2}{\sqrt{6}} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) What are $\|A\|_2$, $\|A\|_F$, $\|U\|_2$ and $\text{rank}(A)$?

(b) State the dimension and a basis (if nontrivial) for each of $\text{range}(A)$ and $\text{null}(A)$.

(c) State the reduced SVD of A.

(d) State A^TA as a product of three 4×4 matrices. What are the eigenvalues and singular values of A^TA?

(e) Let A_2 be the best rank-2 approximation to A in the 2-norm. What is $\|A - A_2\|_2$? State A_2. You do not need to compute the matrix, either state it as a sum of appropriative terms that you define or as a product of matrices.

Question 5: SVD and Least Squares

(a) If $A \in \mathbb{R}^{m\times n}$ is full column rank and $y \in \mathbb{R}^m$, write down the solution to the least squares problem $\min_{x\in\mathbb{R}^n} \|y - Ax\|_2$ using the SVD of A?

(b) What is the solution set in (a) if A is non-full column rank? Justify.

Question 6 (Math 397): SVD of Householder

Let $v \in \mathbb{R}^m$ be nonzero. What is the singular value decompositions of the Householder matrix $H_v = I - 2\frac{vv^T}{v^Tv}$? Justify.