Math 236 Algebra 2 Assignment 2

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Problem 1.

a.
- $0 = (0_1, 0_2)$ and $0_1 * 0_2 = 0$, Therefore, $0 \in U$
- $(x_1, y_1), (x_2, y_2) \in U$. Therefore $x_1y_1 = 0, x_2y_2 = 0$
  $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
  Consider $(x_1 + x_2)(y_1 + y_2) = x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2 = x_1y_2 + x_2y_1 \neq 0$
  Therefore $(x_1 + x_2, y_1 + y_2) \notin U$

Therefore U is not a subspace of V

b.
- $0 = (0_1, 0_2, 0_3, 0_4)$ and $0_1 \geq 0_2$, Therefore $0 \in U$
- $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in U$. Therefore $x_1 \geq x_2$ and $y_1 \geq y_2$
  $(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$
  Consider $x_1 \geq x_2$ and $y_1 \geq y_2 \implies x_1 + y_1 \geq x_2 + y_2$
  Therefore $(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \in U$
- $a \in \mathbb{F}$ and $(x_1, x_2, x_3, x_4) \in U$. Therefore $x_1 \geq x_2$
  $a(x_1, x_2, x_3, x_4) = (ax_1, ax_2, ax_3, ax_4)$
  Consider $x_1 \geq x_2 \implies ax_1 \geq ax_2$
  Therefore $(ax_1, ax_2, ax_3, ax_4) \in U$

Therefore U is a subspace of V

c.
- $0 = (0, cos(\frac{\pi}{2}) = (0, 0)$ and $0 \in \mathbb{R}, \frac{\pi}{2} \in \mathbb{R}$, Therefore, $0 \in U$
- $(x_1, cos(y_1)), (x_2, cos(y_2)) \in U$. Therefore $x_1, y_1, x_2, y_2 \in \mathbb{R}$
  $(x_1, cos(y_1)) + (x_2, cos(y_2)) = (x_1 + x_2, cos(y_1) + cos(y_2))$
  There is no formula such that $cos(\alpha) = cos(y_1) + cos(y_2)$ for $\alpha \in \mathbb{R}$
  Therefore $(x_1 + x_2, cos(y_1) + cos(y_2)) \notin U$

Therefore U is not a subspace of V

Problem 2. $V = \mathbb{R}^2$, suppose $U = \{(x, y) \in \mathbb{R}^2 \mid x, y \in \mathbb{Z}\}$
Let $(x_1, y_1), (x_2, y_2) \in U$. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in U$.
Let $(x, y) \in U \implies x, y \in \mathbb{Z} \implies -x, -y \in \mathbb{Z} \implies (-x, -y) \in U$ and $(x, y) + (-x, -y) = 0$
Therefore U is closed under addition and under taking additive inverses
Consider for example, $1.4 \in \mathbb{R}$ and $(0, 1) \in U$
$1.4(0, 1) = (0, 1.4) \notin U$ since $1.4 \notin \mathbb{Z}$. Therefore U is not a subspace of $\mathbb{R}^2$

Problem 3. Suppose U_1 and U_2 are subspaces of V
WLOG suppose $U_1 \subseteq U_2$, show $U_1 \cup U_2$ is a subspace of V

- $0 \in U_1 \implies 0 \in U_1 \cup U_2$
- $u_1, u_2 \in U_1 \cup U_2 \implies u_1, u_2 \in U_2 \implies u_1 + u_2 \in U_2 \implies u_1 + u_2 \in U_1 \cup U_2$
- $a \in \mathbb{F},\ u \in U_1 \cup U_2 \implies u \in U_2 \implies au \in U_2 \implies au \in U_1 \cup U_2$

Therefore $U_1 \cup U_2$ is a subspace of V.

If $\exists a \in U_1$ such that $a \notin U_2$ and $\exists b \in U_2$ such that $b \notin U_1$ i.e. U_1 not a subset of U_2 and vice versa. Show $U_1 \cup U_2$ is not a subspace of V by contradiction
Assume $U_1 \cup U_2$ is a subspace of V.

$$a \in U_1 \implies -a \in U_1 \ \textit{and} \ b \in U_2 \implies -b \in U_2$$

$$a, b \in U_1 \cup U_2 \implies a + b \in U_1 \cup U_2 \implies a + b \in U_1 \ \textit{or} \ a + b \in U_2$$

If $a + b \in U_1$ then $(a + b) + (-a) \in U_1 \implies b \in U_1$※ (contradiction)
If $a + b \in U_2$ then $(a + b) + (-b) \in U_2 \implies a \in U_2$※ (contradiction)
Therefore $U_1 \cup U_2$ is a subspace of V if and only if one subspace contains the other

Problem 4. Claim: The additive identity for subspaces is the trivial subspace $= \{0\}$
Proof: Let U be a subspace of V. By definition $0 \in U$. Therefore

$$U + \{0\} = U$$

Therefore $\{0\}$ is an additive identity of subspaces
Let W, U be a subspaces of V
The sum $U + W$ contains both U and W. The only way $U + W = \{0\}$ is if both U and W are $\{0\}$. Since if U or W contain an element that is not 0 then the sum $U + W$ will contain this element and therefore the sum will not equal $\{0\}$
Therefore the only subspace with an additive inverse is $\{0\}$

Problem 5. Suppose $W = \{(0, 0, z, a, b) \in \mathbb{F}^5 \mid z, a, b \in \mathbb{F}\}$
Suppose $\alpha \in U \cap W \implies \alpha \in W \implies \alpha = (0, 0, \beta, \delta, \epsilon)$
Since $\alpha \in U$ as well $\implies \alpha = (0, 0, 0 + 0, 0 - 0, 2 * 0) = (0, 0, 0, 0, 0)$
Therefore $U \cap W = \{(0, 0, 0, 0, 0)\}$ and $U + W$ is a direct sum. Now consider

$$(x, y, z, a, b) \in \mathbb{F}^5.$$

$$(x, y, z, a, b) = (x, y, x + y, x - y, 2x) + (0, 0, z - x - y, a - x + y, b - 2x) \in U + W$$

$\implies \mathbb{F}^5 \subseteq U + W$. Now consider

$$(x, y, x + y, x - y, 2x) \in U \ \textit{and} \ (0, 0, z, a, b) \in V$$

$(x, y, x + y, x - y, 2x) + (0, 0, z, a, b) = (x, y, x + y + z, x - y + a, 2x + b) \in \mathbb{F}^5$
$\implies U + W \subseteq \mathbb{F}^5$
$\implies U + W = \mathbb{F}^5$ and since $U + W$ is a direct sum, $U \oplus W = \mathbb{F}^5$

Problem 6. Statement is false. Counterexample:
Let $W := \{(x, y, 0) \in \mathbb{F}^3 | \, x, y \in \mathbb{F}\}$, $U_1 := \{(0, 0, z) \in \mathbb{F}^3 | \, z \in \mathbb{F}\}$ and $U_2 := \{(0, z, z) \in \mathbb{F}^3 | \, z \in \mathbb{F}\}$.

Let $A = (a, b, 0) \in W$ and $B = (0, 0, c) \in U_1$. If $A = B$, and therefore $A, B \in W \cap U_1$ then $a = 0, b = 0, c = 0$ Therefore $W \cap U_1 = \{0\}$ and $U_1 + W$ is a direct sum.

$$(x, y, z) \in \mathbb{F}^3. \, (x, y, z) = (0, 0, z) + (x, y, 0) = U_1 + W \implies \mathbb{F}^3 \subseteq U_1 + W$$
$$(0, 0, z) \in U_1, (x, y, 0) \in W. \, (0, 0, z) + (x, y, 0) = (x, y, z) \in \mathbb{F}^3 \implies U_1 + W \subseteq \mathbb{F}^3$$

$\implies U_1 + W = \mathbb{F}^3$ and since $U_1 + W$ is a direct sum, $U_1 \oplus W = \mathbb{F}^3$

$C = (0, d, d) \in U_2$. If $A = C$, and therefore $A, C \in W \cap U_2$ then $a = 0, b = d, d = 0 \implies b = 0$
Therefore $W \cap U_2 = \{0\}$ and $U_2 + W$ is a direct sum.

$$(x, y, z) \in \mathbb{F}^3. \, (x, y, z) = (0, z, z) + (x, y - z, 0) = U_2 + W \implies \mathbb{F}^3 \subseteq U_2 + W$$
$$(0, z, z) \in U_2, (x, y, 0) \in W. \, (0, z, z) + (x, y, 0) = (x, y + z, z) \in \mathbb{F}^3 \implies U_2 + W \subseteq \mathbb{F}^3$$

$\implies U_2 + W = \mathbb{F}^3$ and since $U_2 + W$ is a direct sum, $U_2 \oplus W = \mathbb{F}^3$
$\implies \mathbb{F}^3 = U_1 \oplus W = U_2 \oplus W$. However, $U_1 \neq U_2$. Therefore the problem statement is false.

Problem 7a. Consider $A \in U$. Since $A = A^t$ It follows,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\implies b = c$ and therefore

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

- $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, consider a, b and $d = 0$. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in U$ Therefore $0 \in U$
- $\begin{pmatrix} a & b \\ b & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix} \in U$. $\begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ b + b' & d + d' \end{pmatrix} \in U$
- $\lambda \in \mathbb{R}, \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in U$. $\lambda \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda b & \lambda d \end{pmatrix} \in U$
  Therefore U is a subspace of $M_2(\mathbb{R})$

Consider $A \in V$. Since $A = -A^t$ It follows,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix}$$

$\implies a = -a, \, d = -d \implies a = 0, d = 0$ and $c = -b$ therefore

$$A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

- $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, consider $b = 0$. $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in V$ Therefore $0 \in V$
- $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \begin{pmatrix} 0 & b' \\ -b' & 0 \end{pmatrix} \in V$. $\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} + \begin{pmatrix} 0 & b' \\ -b' & 0 \end{pmatrix} = \begin{pmatrix} 0 & b+b' \\ -(b+b') & 0 \end{pmatrix} \in V$
- $\lambda \in \mathbb{R}, \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in V$. $\lambda \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda b \\ -\lambda b & 0 \end{pmatrix} \in V$
  Therefore V is a subspace of $M_2(\mathbb{R})$

Problem 7b. let $A = \begin{pmatrix} a & c \\ c & d \end{pmatrix} \in U$. $B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in V$
If $A \in U \cap V$ and $B \in U \cap V$ Then $A = B$, and

$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \implies \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

$$\implies \begin{cases} c = b \\ c = -b \end{cases} \implies b = 0, c = 0$$

Therefore $U \cap V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \{0\}$ and $U + V$ is a direct sum. Now consider,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{b+c}{2} + \frac{b-c}{2} \\ \frac{b+c}{2} - \frac{b-c}{2} & d \end{pmatrix} = \begin{pmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{pmatrix} + \begin{pmatrix} 0 & \frac{b-c}{2} \\ -\frac{b-c}{2} & 0 \end{pmatrix} \in U + V$$

$\implies M_2(\mathbb{R}) \subseteq U + V$
Now consider,

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in U, \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} \in V$$

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} = \begin{pmatrix} a & b+c \\ b-c & d \end{pmatrix} \in M_2(\mathbb{R})$$

$\implies U + V \subseteq M_2(\mathbb{R})$
$\implies U + V = M_2(\mathbb{R})$ and since $U + V$ is a direct sum, $U \oplus V = M_2(\mathbb{R})$