Math 447 Assignment 1

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1.28

Let A be a random variable that represents the number of accidents in a day. Using the law of iterated expectation we have,

$$E(A) = E[E[A \mid \Lambda]]$$

$$E(A) = E[\Lambda]$$

$$E(A) = 1.5$$

Now using the law of iterated variance we have,

$$Var(A) = Var(E[A \mid \Lambda]) + E(Var[A \mid \Lambda])$$

$$Var(A) = Var(\Lambda) + E(\Lambda)$$

$$Var(A) = \frac{1}{12}(3-0)^2 + 1.5$$

$$Var(A) = 2.25$$

1.30

a)

$$E[Y \mid X] = E[g(X) \mid X] = g(X)$$

b)

$$Var(Y \mid X) = Var(g(X) \mid X) = E[(g(X) - E[g(X) \mid X])^2 \mid X]$$

$$= E[(g(X) - g(X))^2 \mid X] = E[0^2 \mid X] = 0$$

$$\implies Var(Y \mid X) = 0$$

1.36

Increasing the number of people initially infected with the disease reduces the duration of the disease. With the original values where 400 people were susceptible and 3 had the disease intially, by time step 20 there were roughly 1 or 2 people left with the disease. In these same experiments the number of people infected peaked around time step 10 and roughly 30 people were infected at that point. When I doubled the number of infected people to begin with the duration of the disease was reduced to about time step 15, and the peak came at time step 6 where about 40 people were infected. As you increase the number of intially infected people the duration of the disease reduces and the peak number of people infected becomes larger and occurs earlier in time.

1.37

```
#modified code from in class example

simulate_one_day = function(){
  lambda = runif(1,0,3)
  accidents = rpois(1,lambda)
  return(data_frame(total=sum(accidents)))
}

iter = 10000
carCrash_df = data_frame(Iter=1:iter) %>% group_by(Iter) %>%
  do(simulate_one_day())

carCrash_df %>% ungroup() %>% summarise(mean=mean(total),var=var(total),sd=sd(total), q2
5 = quantile(total,0.25), q75 = quantile(total,0.75)) %>% gather(value = value,key=stat)
```

```
## Warning: attributes are not identical across measure variables;
## they will be dropped
```

```
## # A tibble: 5 x 2
##   stat  value
##   <chr> <dbl>
## 1 mean   1.53
## 2 var    2.31
## 3 sd     1.52
## 4 q25    0
## 5 q75    2.00
```

2.1

a)

$$P(X_7 = 3 \mid X_6 = 2) = 0.6$$

b)

$$P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3)$$

$$= P(X_9 = 2 \mid X_7 = 3)$$

$$= (P^2)_{32}$$

$$P^2 = P * P$$

$$= \begin{pmatrix} 0.19 & 0.27 & 0.54 \\ 0.18 & 0.28 & 0.54 \\ 0.18 & 0.27 & 0.55 \end{pmatrix}$$

$$\implies (P^2)_{32} = 0.27$$

$$\implies P(X_9 = 2 \mid X_1 = 2, X_5 = 1, X_7 = 3) = 0.27$$

c)

$$P(X_0 = 3 \mid X_1 = 1) = P(X_1 = 1 \mid X_0 = 3) = 0.3$$

d)

From before we have P^2,

$$(\alpha P^2) = (0.2, 0.3, 0.5) \begin{pmatrix} 0.19 & 0.27 & 0.54 \\ 0.18 & 0.28 & 0.54 \\ 0.18 & 0.27 & 0.55 \end{pmatrix}$$

$$(\alpha P^2) = (0.182, 0.273, 0.545)$$

$$E(X_2) = \sum_{j=1}^{3} j * P(X_2 = j)$$

$$P(X_2 = j) = (\alpha P^2)_j$$

Therefore,

$$E(X_2) = \sum_{j=1}^{3} j * (\alpha P^2)_j$$

$$= 1 * 0.182 + 2 * 0.273 + 3 * 0.545$$

$$E(X_2) = 2.363$$

2.5

a)

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b)

$$P(X_7 = 1 \mid X_0 = 3, X_2 = 2, X_4 = 2)$$

$$= P(X_7 = 1 \mid X_4 = 2)$$

$$= (P^3)_{21}$$

$$P^3 = P * P * P$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{7}{16} & 0 & \frac{9}{16} \\ \frac{7}{64} & 0 & \frac{57}{64} & 0 \\ 0 & \frac{19}{64} & 0 & \frac{45}{64} \\ \frac{1}{16} & 0 & \frac{15}{16} & 0 \end{pmatrix}$$

$$\implies (P^3)_{21} = \frac{19}{64}$$

$$\implies P(X_7 = 1 \mid X_0 = 3, X_2 = 2, X_4 = 2) = \frac{19}{64}$$

c)

$$P(X_3 = 1, X_5 = 3)$$

$$= P(X_4 = 2 \mid X_3 = 1) P(X_5 = 3 \mid X_4 = 2)$$

$$= (\frac{3}{4})(\frac{3}{4})$$

$$= \frac{9}{16}$$

2.8

$$\begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{2} & 0 & \frac{1}{3} \\ \frac{1}{10} & \frac{1}{5} & \frac{1}{5} & \frac{1}{10} & \frac{2}{5} \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{pmatrix}$$

2.12

Suppose at time t there are a blue balls in the left urn and b red balls in the left urn, thus $a + b = k$. This process is a markov chain because the transistion probabilities at each discrete time step only depend on the distribution of balls at the previous time step, moments further in the past do not effect the probability of each transisition. Thus the Bernoulli–Laplace model of diffusion demonstrates the markov property.

There are 3 scenarios with regards to the number of blue balls in the left urn.

$$P(\text{a goes up 1}) = \frac{b}{k} * \frac{k-a}{k} = \frac{k-a}{k} * \frac{k-a}{k} = \left(\frac{k-a}{k}\right)^2$$

$$P(\text{a stay the same}) = \frac{b}{k} * \frac{k-b}{k} + \frac{a}{k} * \frac{k-a}{k} = \frac{k-a}{k} * \frac{a}{k} + \frac{a}{k} * \frac{k-a}{k} = 2\frac{a(k-a)}{k^2}$$

$$P(\text{a goes down 1}) = \frac{a}{k} * \frac{k-b}{k} = \frac{a}{k} * \frac{a}{k} = \left(\frac{a}{k}\right)^2$$

Therefore the transistion probability matrix is as follows,

$$P = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
\frac{1}{k^2} & \frac{2(k-1)}{k^2} & \frac{(k-1)^2}{k^2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & \frac{4}{k^2} & \frac{4(k-2)}{k^2} & \frac{(k-2)^2}{k^2} & 0 & \dots & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{9}{k^2} & \frac{6(k-2)}{k^2} & \frac{(k-3)^2}{k^2} & \dots & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{(k-2)^2}{k^2} & \frac{4(k-1)}{k^2} & \frac{4}{k^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(k-1)^2}{k^2} & \frac{2(k-1)}{k^2} & \frac{1}{k^2} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0
\end{pmatrix}$$