McGill University
Department of Mathematics and Statistics
MATH 243 Analysis 2, Winter 2017
Assignment 6

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1 and 2.**

This assignment is due **Tuesday, February 21, at 2:30pm** in class. **Late assignments will not be accepted!**

1. Let $F : [-1, 1] \to \mathbb{R}$,

$$F(x) := \begin{cases} x^2 \cos(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that F is differentiable but that F' is not Riemann integrable on $[-1, 1]$.

2. Let $f : [0, 1] \to \mathbb{R}$ be continuous and let $\displaystyle\int_0^x f = \int_x^1 f$ for all $x \in [0, 1]$. Prove that f is constantly equal to 0 on $[0, 1]$.

3. Let f be Riemann integrable on $[a, b]$. Prove that $\displaystyle\left|\int_a^b f\right| \leq \int_a^b |f|$.

4. Let f and g be Riemann integrable on $[a, b]$. Prove that

$$\left|\int_a^b fg\right| \leq \int_a^b |fg| \leq \sqrt{\int_a^b f^2 \cdot \int_a^b g^2}$$

This inequality is called the *Cauchy-Schwarz inequality* for Riemann integrals.

Hint: Follow the outline given below.

(a) Let t be a positive constant. Deduce from $\displaystyle\int_a^b (tf \pm g)^2 \geq 0$ that $\displaystyle 2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \frac{1}{t}\int_a^b g^2$.

(b) Deduce from (a) that if $\displaystyle\int_a^b f^2 = 0$ then $\displaystyle\int_a^b fg = 0$.

(c) Deduce from (a) that $\displaystyle\left|\int_a^b fg\right| \leq \sqrt{\int_a^b f^2 \cdot \int_a^b g^2}$ by setting $\displaystyle t := \sqrt{\left(\int_a^b g^2\right) / \left(\int_a^b f^2\right)}$ if $\displaystyle\int_a^b f^2 \neq 0$.

(d) Combine part (c) with problem 3 to prove the Cauchy-Schwarz inequality.

—Please turn over!—

5. Let $f, g, F, G : [0,1] \to \mathbb{R}$, where

$$f(x) := \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases},$$

$g(x) := xf(x)$, $F(x) := \int_0^x f$ and $G(x) := \int_0^x g$. (Note that f is Riemann integrable on $[0,1]$ by assignment 4; g is Riemann integrable on $[0,1]$ as a product of two Riemann integrable functions.)

(a) Prove that G is differentiable on $[0,1]$ but that $G'(c) \neq g(c)$ whenever g is discontinuous at c.

(b) Prove that F is differentiable on $[0,1]$. Prove furthermore that there exist $c, d \in [0,1]$ such that f is discontinuous at c and d and $F'(c) = f(c)$ but $F'(d) \neq f(d)$.

Remark: This shows that if the indefinite integral F of a Riemann integrable function f is differentiable at a point c where f is discontinuous, then $F'(c)$ may or may not equal $f(c)$.