McGill University
Department of Mathematics and Statistics

MATH 243 Analysis 2, Winter 2017
Assignment 1

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1 and 4.**

This assignment is due **Tuesday, January 17, at 2:30pm** in class. **Late assignments will not be accepted!**

1. Let I be an interval, let $c \in I$ and let $f, g : I \to \mathbb{R}$ be differentiable at c. By Carathéodory's theorem there exist functions $\varphi, \psi : I \to \mathbb{R}$ which are continuous at c and with $\varphi(c) = f'(c)$ and $\psi(c) = g'(c)$ such that for all $x \in I$ we have

$$f(x) = f(c) + \varphi(x)(x - c)$$
$$g(x) = g(c) + \psi(x)(x - c)$$

Use these representations to prove the product rule i.e. prove that $f \cdot g$ is differentiable at c and that $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$. No other method will be accepted!

2. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^{1/3}$. Prove that f is *not* differentiable at 0.

Hint: Consider $g \circ f$ where $g : \mathbb{R} \to \mathbb{R}$, $g(x) := x^3$.

3. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(a) Show that f is differentiable at 0 and find $f'(0)$.

(b) Show that f is not differentiable at any $c \neq 0$.

Remark: You may use, without proof, that both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Prove that f is differentiable on $\mathbb{R}$.

(b) Prove that f' is unbounded on $[-1, 1]$.

(c) Conclude from (b) or prove otherwise that f' is discontinuous at 0.

5. (Long) Consider the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^n & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where $n \in \mathbb{N}$. Prove that $f \in C^{n-1}(\mathbb{R}) \setminus C^n(\mathbb{R})$.