447 Assignment 3

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4.6

a.

$$G_X(s) = \sum_{k=0}^{\infty} s^k a_k$$
$$= (1-p) + sp$$

It follows,

$$G_Z(s) = E[s^Z]$$
$$= E[s^{X_1+X_2+...+X_N}]$$
$$= E[\prod_{i=1}^{N} s^{X_i}]$$
$$= \prod_{i=1}^{\lambda} E[s^{X_i}]$$
$$= G_{X_1}(s) * ... * G_{X_\lambda}(s)$$
$$= [G_X(s)]^\lambda$$
$$G_Z(s) = [1-p+sp]^\lambda$$

b.

We have, $G_Z(s) = [1-p+sp]^\lambda$

$$P(X=0) = G(0) = (1-p)^\lambda$$

Consider the jth derivative of $G(s)$

$$G^{(j)}(s) = (\lambda)(\lambda-1)...(\lambda-j+1)p^j(1-p+sp)^{\lambda-j}$$

Therefore,

$$P(X=j) = \frac{G^{(j)}(0)}{j!} = \frac{(\lambda)(\lambda-1)...(\lambda-j+1)p^j(1-p)^{\lambda-j}}{j!}$$
$$= \binom{\lambda}{j} p^j (1-p)^{\lambda-j}$$

Therefore Z has binomial probability distribution with parameters λ and p.

4.12

a.

$$\mu = \sum_{k=0}^{\infty} k a_k$$
$$= 0 * \frac{1}{4} + 1 * \frac{1}{4} + 2 * \frac{1}{2}$$
$$= 1.25$$

b.

$$G(s) = \sum_{k=0}^{\infty} s^k a_k$$
$$= \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}$$

c.

$$s = \frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}$$
$$0 = (2s-1)(s-1)$$
$$s = 1, s = \frac{1}{2}$$
$$\Rightarrow e = \frac{1}{2}$$

d.

$$G_2(s) = G(G_1(s)) = G(G(s))$$
$$= G(\frac{1}{4} + \frac{s}{4} + \frac{s^2}{2})$$
$$= \frac{s^4}{8} + \frac{s^3}{8} + \frac{9s^2}{32} + \frac{s}{8} + \frac{11}{32}$$

e.

$$P(Z_2 = 0) = G_2(0)$$
$$= \frac{11}{32}$$

4.14

$$\mu = \sum_{k=0}^{\infty} k a_k$$
$$= 0 * p + 1 * (1 - p - q) + 2 * q$$
$$= 1 - p + q$$

In the supercritical case $\mu > 1$

$$1 - p + q > 1$$
$$-p + q > 0$$
$$q > p$$

Therefore this process is supercritical when $q > p$

$$G(s) = \sum_{k=0}^{\infty} s^k a_k$$
$$= p + s(1 - p - q) + s^2(q)$$

$$s = G(s)$$
$$s = p + s(1 - p - q) + s^2(q)$$
$$0 = p + s(-p - q) + s^2(q)$$
$$\Rightarrow s = 1, s = \frac{p}{q}$$

Since in the supercritical case $p < q$ then $\frac{p}{q} < 1$. Therefore $e = \frac{p}{q}$

4.28

a.

We have, $Z_n = \left(\sum_{i=1}^{Z_{n-1}} X_i\right) + W_n$. Therefore,

$$G_n(s) = E(s^{Z_n})$$
$$= E(s^{\left(\sum_{i=1}^{Z_{n-1}} X_i\right) + W_n})$$
$$= E(s^{\sum_{i=1}^{Z_{n-1}} X_i} * s^{W_n})$$

By the indepednce of X_i's and W_n,

$$= E(s^{\sum_{i=1}^{Z_{n-1}} X_i})E(s^{W_n})$$
$$= G_{n-1}(G(s))H_n(s)$$

b.

We have the following generating functions:

$$G(s) = q + ps$$

where $q = 1 - p$

$$H(s) = e^{-\lambda(1-s)}$$

Therefore,

$$\begin{aligned} G_1(s) &= G_0(G(s))H(s) \\ &= (q+ps)e^{-\lambda(1-s)} \end{aligned}$$

$$\begin{aligned} G_2(s) &= G_1(G(s))H(s) \\ &= G_1(q+ps)H(s) \\ &= \left(e^{-\lambda(1-(q+ps))}(q+p(q+ps))\right)e^{-\lambda(1-s)} \\ &= e^{-\lambda(1-q-ps+1-s)}(q+pq+p^2s) \\ &= e^{-\lambda(1-s)(1+p)}(1-p^2+p^2s) \end{aligned}$$

$$\begin{aligned} G_3(s) &= G_2(G(s))H(s) \\ &= G_2(q+ps)H(s) \\ &= \left(e^{-\lambda(1-(q+ps))(1+p)}((1-p^2+p^2(q+ps)))\right)e^{-\lambda(1-s)} \\ &= e^{-\lambda((1-q-ps)(1+p)+(1-s))}(1-p^2+p^2(1-p)p^3s) \\ &= e^{-\lambda(1-s)(1+p+p^2)}(1-p^3+p^3s) \end{aligned}$$

$$\begin{aligned} G_n(s) &= e^{-\lambda(1-s)(1+p+p^2+\ldots+p^{n-1})}(1-p^n+p^ns) \\ &= e^{-\lambda(1-s)\frac{1-p^n}{1-p}}(1-p^n+p^ns) \end{aligned}$$

Before taking the limit of the generating function, note that $0 < p < 1$, therefore

$$\lim_{n\to\infty} p^n = 0$$

Therefore it follows,

$$\begin{aligned} &\lim_{n\to\infty} G_n(s) \\ &= \lim_{n\to\infty} e^{-\lambda(1-s)\frac{1-p^n}{1-p}}(1-p^n+p^ns) \\ &= e^{-\lambda\frac{1-s}{1-p}} \end{aligned}$$

This is the generating function of a Poisson distribution with parameter $\frac{\lambda}{1-p}$.

4.29

a.

```
G <- function(s) {
  return (0.8 + (s ^ 4)*0.1 + (s ^ 9)*0.1)
}
e_0 = runif(1, 0, 1)

e_prev = e_0
for (i in 1:20){
  e_n = G(e_prev)
  e_prev = e_n
}
e_n
```

```
## [1] 0.9150164
```

Therefore $e = 0.9150164$

b.

```
G <- function(s) {
  return ((1/11)*(1+s+(s^2)+(s^3)+(s^4)+(s^5)+(s^6)+(s^7)+(s^8)+(s^9)+(s^10)))
}
e_0 = runif(1, 0, 1)

e_prev = e_0
for (i in 1:20){
  e_n = G(e_prev)
  e_prev = e_n
}
e_n
```

```
## [1] 0.101138
```

Therefore $e = 0.101138$

c.

```
G <- function(s) {
  return (0.6 + (s ^ 3)*0.2 + (s ^ 6)*0.1 + (s ^ 12)*0.1)
}
e_0 = runif(1, 0, 1)

e_prev = e_0
for (i in 1:20){
  e_n = G(e_prev)
  e_prev = e_n
}
e_n
```

[1] 0.6700263

Therefore $e = 0.6700263$

4.32

```
set.seed(8)
p = 3/4
lamda = 1.2
trials = 1000 #number of time we simulate the branching process
n = 100
extentions = 0
totalPop = 0

G <- function(p) {
  if (runif(1) < p){
    return (1)
  }
  else{
    return (0)
  }
}
H <- function(lambda){
  return (rpois(1,lambda))
}

for (k in 1:trials){
  pop = 1
  for (i in 1:n){
    newpop = 0
    for (j in 1:pop){
      newpop = newpop + G(p)
    }
    imm = H(lamda)
    newpop = newpop + imm
    pop = newpop
  }
  totalPop = totalPop + pop
}
avgPop = totalPop/trials #average population size after 100 generations
avgPop
```

```
## [1] 4.875
```

In 4.28 we found that the limiting generating function, was the generating function of a Poisson distribution with parameter $\frac{\lambda}{1-p}$. In this simulation with the given parameters this would be a Poisson distribution with parameter $\frac{1.2}{1-(3/4)} = 4.8$. Therefore we would expect the population after n steps (where n is large, 100 in this example) to be on average 4.8 in size.

Numerically we computed an average population of 4.875. Therefore we have confirmed our result from 4.28 numerically.

5.1

State space = (C, T)

$$P = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{4}{5} & \frac{1}{5} \end{pmatrix}$$

Solving for the stationary distribution we obtain,

$$\pi = (\frac{16}{21}, \frac{5}{21})$$

We conclude that that after 1000 vehicles the toll booth collects,

$$\begin{aligned} \text{toll } &= \frac{16}{21} * (1000) * (1.5) + \frac{5}{21} * (1000) * (5) \\ &= \frac{49000}{21} = 2333.33 \end{aligned}$$

We can support this answer with a numerical simualtion of the environment described in the question.

```
set.seed(4)
money = 0
state = 1 #1 = car, 2 = truck
cost = c(1.5,5)
totalcars = 0
count = 0

for (i in 1:1000){
  if (state == 1){
    totalcars = totalcars + 1
    if(runif(1) < 1/4){
      state = 2
    }
  }else{
    if(runif(1) < 4/5){
      state = 1
    }
  }
  money = money + cost[state]
}
money
```

```
## [1] 2340
```

5.2

State space = $(0, 1, 2, 3, ..., k)$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & ... & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & ... & 0 & 1 & 0 \end{pmatrix}$$

Solving for the stationary distribution we obtain,

$$\pi = (\frac{1}{2k}, \frac{1}{k}, \frac{1}{k}, \frac{1}{k}, ..., \frac{1}{k}, \frac{1}{2k})$$

We conclude that the cost of the random walk is,

$$cost = \frac{1}{2k}*(10000)(k)+\frac{1}{k}*(10000)(-1)+\frac{1}{k}*(10000)(-1)+\frac{1}{k}*(10000)(-1)+...+\frac{1}{k}*(10000)(-1)+\frac{1}{2k}*(10000)(k)$$

$$= \frac{10000}{2} + \frac{10000(k-1)(-1)}{k} + \frac{10000}{2}$$

$$= \frac{10000}{k}$$

We can support this answer with a numerical simualtion of the environment described in the question. In our simulation we let $k = 10$, and thus we expect to gain 1000 dollars.

```
set.seed(2)
k = 10
totalmoney = 0
n = 100 #number of trials

for (j in 1:n){
  money = 0
  pos = sample(1:(k+1), 1)
  pos
  for (i in 1:10000){
    #on edges
    if (pos == 1){
      money = money + k
      pos = pos + 1
    }else if(pos == k+1){
      money = money + k
      pos = pos - 1
    }else{
      money = money - 1
      if(runif(1) < 0.5){
        pos = pos - 1
      }else{
        pos = pos + 1
      }
    }
  }
  totalmoney = totalmoney + money
}
totalmoney/n
```

```
## [1] 987.35
```