Lecture 2

Mechanism Design Basics

With this lecture we begin our formal study of mechanism design, the science of rule-making. This lecture introduces an important and canonical example of a mechanism design problem, the design of single-item auctions, and develops some mechanism design basics in this relatively simple setting. Later lectures extend the lessons learned to more complex applications.

Section 2.1 defines a model of single-item auctions, including the quasilinear utility model for bidders. After quickly formalizing sealed-bid auctions in Section 2.2 and mentioning first-price auctions in Section 2.3, in Section 2.4 we introduce second-price (a.k.a. Vickrey) auctions and establish their basic properties. Section 2.5 formalizes what we want in an auction: strong incentive guarantees, strong performance guarantees, and computational efficiency. Section 2.6 presents a case study on sponsored search auctions for selling online advertising, a "killer application" of auction theory.

2.1 Single-Item Auctions

We start our discussion of mechanism design with *single-item auctions*. Recall our overarching goal in this part of the course.

Course Goal 1 Understand how to design systems with strategic participants that have good performance guarantees.

Consider a seller with a single item, such as a slightly antiquated smartphone. This is the setup in a typical eBay auction, for example. There is some number n of (strategic!) bidders who are potentially interested in buying the item.

We want to reason about bidder behavior in various auction formats. To do this, we need a model of what a bidder wants. The first key assumption is that each bidder i has a nonnegative *valuation* v_i—her maximum willingness-to-pay for the item being sold. Thus bidder i wants to acquire the item as cheaply as possible, provided the selling price is at most v_i. Another important assumption is that this valuation is *private*, meaning it is unknown to the seller and to the other bidders.

Our bidder utility model, called the *quasilinear utility model*, is the following. If a bidder i loses an auction, her utility is 0. If the bidder wins at a price p, her utility is $v_i - p$. This is arguably the simplest natural utility model, and it is the one we focus on in these lectures.

2.2 Sealed-Bid Auctions

For the most part, we focus on a simple class of auction formats: *sealed-bid auctions*. Here's what happens:

1. Each bidder i privately communicates a bid b_i to the seller—in a sealed envelope, if you like.
2. The seller decides who gets the item (if anyone).
3. The seller decides on a selling price.

There is an obvious way to implement the second step—give the item to the highest bidder. This is the only selection rule that we consider in this lecture.[1]

There are multiple reasonable ways to implement the third step, and the choice of implementation significantly affects bidder behavior. For example, suppose we try to be altruistic and charge the winning bidder nothing. This idea backfires badly, with the auction devolving into a game of "who can name the highest number?"

2.3 First-Price Auctions

In a *first-price auction*, the winning bidder pays her bid. Such auctions are common in practice.

[1]When we study revenue maximization in Lectures 5 and 6, we'll see why other winner selection rules are important.

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First-price auctions are hard to reason about. First, as a participant, it's hard to figure out how to bid. Second, as a seller or auction designer, it's hard to predict what will happen. To drive this point home, imagine participating in the following first-price auction. Your valuation (in dollars) for the item for sale is the number of your birth month plus the day of your birth. Thus, your valuation is somewhere between 2 (for January 1) and 43 (for December 31). Suppose there is exactly one other bidder (drawn at random from the world) whose valuation is determined in the same way. What bid would you submit to maximize your expected utility? Would it help to know your opponent's birthday? Would your answer change if you knew there were two other bidders in the auction rather than one?[2]

2.4 Second-Price Auctions and Dominant Strategies

We now focus on a different single-item auction, also common in practice, which is much easier to reason about. What happens when you win an eBay auction? If you bid \$100 and win, do you pay \$100? Not necessarily: eBay uses a "proxy bidder" that increases your bid on your behalf until your maximum bid is reached, or until you are the highest bidder, whichever comes first. For example, if the highest other bid is only \$90, then you only pay \$90 (plus a small increment) rather than your maximum bid of \$100. *If you win an eBay auction, the sale price is essentially the highest other bid—the second highest overall.*

A *second-price* or *Vickrey* auction is a sealed-bid auction in which the highest bidder wins and pays a price equal to the second-highest bid. To state the most important property of second-price auctions, we define a *dominant strategy* as a strategy (i.e., a bid) that is guaranteed to maximize a bidder's utility, no matter what the other bidders do.

Proposition 2.1 (Incentives in Second-Price Auctions) *In a second-price auction, every bidder i has a dominant strategy: set the bid b_i equal to her private valuation v_i.*

Proposition 2.1 implies that second-price auctions are particularly easy to participate in. When selecting a bid, a bidder doesn't need

[2]For more on the theory of first-price auctions, see Problem 5.3.

to reason about the other bidders in any way—how many there are, what their valuations are, whether or not they bid truthfully, etc. This is completely different from a first-price auction, where it never makes sense to bid one's valuation—this guarantees zero utility—and the optimal amount to underbid depends on the bids of the other bidders.

Proof of Proposition 2.1: Fix an arbitrary bidder i, valuation v_i, and the bids $\mathbf{b}_{-i}$ of the other bidders. Here $\mathbf{b}_{-i}$ means the vector $\mathbf{b}$ of all bids, but with the ith component removed.[3] We need to show that bidder i's utility is maximized by setting $b_i = v_i$.

Let $B = \max_{j \neq i} b_j$ denote the highest bid by some other bidder. What's special about a second-price auction is that, even though there are an infinite number of bids that i could make, only two distinct outcomes can result. If $b_i < B$, then i loses and receives utility 0. If $b_i \geq B$, then i wins at price B and receives utility $v_i - B$.[4]

We conclude by considering two cases. First, if $v_i < B$, the maximum utility that bidder i can obtain is $\max\{0, v_i - B\} = 0$, and it achieves this by bidding truthfully (and losing). Second, if $v_i \geq B$, the maximum utility that bidder i can obtain is $\max\{0, v_i - B\} = v_i - B$, and it achieves this by bidding truthfully (and winning). ■

Another important property is that a truthful bidder—meaning one that bids her true valuation—never regrets participating in a second-price auction.

Proposition 2.2 (Nonnegative Utility) *In a second-price auction, every truthful bidder is guaranteed nonnegative utility.*

Proof: Losers receive utility 0. If a bidder i is the winner, then her utility is $v_i - p$, where p is the second-highest bid. Since i is the winner (and hence the highest bidder) and bid her true valuation, $p \leq v_i$ and hence $v_i - p \geq 0$. ■

Exercises 2.1–2.5 ask you to explore further properties of and variations on second-price auctions. For example, truthful bidding is the *unique* dominant strategy for a bidder in a second-price auction.

[3] This may be wonky notation, but it's good to get used to it.

[4] We're assuming here that ties are broken in favor of bidder i. You should check that Proposition 2.1 holds no matter how ties are broken.

2.5 Ideal Auctions

Second-price single-item auctions are "ideal" in that they enjoy three quite different and desirable properties. We formalize the first of these in the following definition.

Definition 2.3 (Dominant-Strategy Incentive Compatible) An auction is *dominant-strategy incentive compatible (DSIC)* if truthful bidding is always a dominant strategy for every bidder and if truthful bidders always obtain nonnegative utility.[5]

Define the *social welfare* of an outcome of a single-item auction by

$$\sum_{i=1}^{n} v_i x_i,$$

where x_i is 1 if i wins and 0 if i loses. Because there is only one item, we have the feasibility constraint that $\sum_{i=1}^{n} x_i \leq 1$. Thus, the social welfare is just the valuation of the winner, or 0 if there is no winner.[6] An auction is *welfare maximizing* if, when bids are truthful, the auction outcome has the maximum possible social welfare. The next theorem follows from Proposition 2.1, Proposition 2.2, and the definition of second-price auctions.

Theorem 2.4 (Second-Price Auctions Are Ideal) *A second-price single-item auction satisfies the following:*

(1) ***[strong incentive guarantees]*** *It is a DSIC auction.*

(2) ***[strong performance guarantees]*** *It is welfare maximizing.*

(3) ***[computational efficiency]*** *It can implemented in time polynomial (indeed, linear) in the size of the input, meaning the number of bits necessary to represent the numbers $v_1, \ldots, v_n$.*

[5]The condition that truthful bidders obtain nonnegative utility is traditionally considered a separate requirement, called *individual rationality* or *voluntary participation*. To minimize terminology in these lectures, we fold this constraint into the DSIC condition, unless otherwise noted.

[6]The sale price does not appear in the definition of the social welfare of an outcome. We think of the seller as an agent whose utility is the revenue she earns; her utility then cancels out the utility lost by the auction winner from paying for the item.

All three properties are important. From a bidder's perspective, the DSIC property makes it particularly easy to choose a bid, and levels the playing field between sophisticated and unsophisticated bidders. From the perspective of the seller or auction designer, the DSIC property makes it much easier to reason about the auction's outcome. Note that *any* prediction of an auction's outcome has to be predicated on assumptions about how bidders behave. In a DSIC auction, the only assumption is that a bidder with an obvious dominant strategy will play it. Behavioral assumptions don't get much weaker than that.[7]

The DSIC property is great when you can get it, but we also want more. For example, an auction that gives the item away for free to a random bidder is DSIC, but it makes no effort to identify which bidders actually want the item. The welfare maximization property states something rather amazing: even though the bidder valuations are a priori unknown to the seller, the auction nevertheless identifies the bidder with the highest valuation! (Provided bids are truthful, a reasonable assumption in light of the DSIC property.) That is, a second-price auction solves the social welfare maximization problem as well as if all of the bidders' valuations were known in advance.

Computational efficiency is important because, to have potential practical utility, an auction should run in a reasonable amount of time. For example, auctions for online advertising, like those in Section 2.6, generally need to run in real time.

Section 2.6 and Lectures 3–4 strive for ideal auctions, in the sense of Theorem 2.4, for applications more complex than single-item auctions.

2.6 Case Study: Sponsored Search Auctions

2.6.1 Background

A Web search results page comprises a list of organic search results—deemed relevant to your query by an algorithm like PageRank—and a list of sponsored links, which have been paid for by advertisers. (Go do a Web search now to remind yourself, preferably on a valuable keyword like "mortgage" or "attorney.") Every time you type a

[7]Non-DSIC auctions are also important; see Section 4.3 for a detailed discussion.

search query into a search engine, an auction is run in real time to decide which advertisers' links are shown, how these links are arranged visually, and what the advertisers are charged. It is impossible to overstate how important such *sponsored search auctions* have been to the Internet economy. Here's one jaw-dropping statistic: around 2006, sponsored search auctions generated roughly 98% of Google's revenue. While online advertising is now sold in many different ways, sponsored search auctions continue to generate tens of billions of dollars of revenue every year.

2.6.2 The Basic Model of Sponsored Search Auctions

We discuss next a simplistic but useful and influential model of sponsored search auctions. The items for sale are k "slots" for sponsored links on a search results page. The bidders are the advertisers who have a standing bid on the keyword that was searched on. For example, Volvo and Subaru might be bidders on the keyword "station wagon," while Nikon and Canon might be bidders on the keyword "camera." Such auctions are more complex than single-item auctions in two ways. First, there are generally multiple items for sale (i.e., $k > 1$). Second, these items are not identical. For example, if ads are displayed as an ordered list, then higher slots in the list are more valuable than lower ones, since people generally scan the list from top to bottom.

We quantify the difference between different slots using *click-through rates (CTRs)*. The CTR α_j of a slot j represents the probability that the end user clicks on this slot. Ordering the slots from top to bottom, we make the reasonable assumption that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. For simplicity, we also make the unreasonable assumption that the CTR of a slot is independent of its occupant. Everything we'll say about sponsored search auctions extends to the more general and realistic model in which each advertiser i has a "quality score" β_i (the higher the better) and the CTR of advertiser i in slot j is the product $\beta_i \alpha_j$ (e.g., Exercise 3.4).

We assume that an advertiser is not interested in an impression (i.e., being displayed on a page) per se, but rather has a private valuation v_i for each *click* on her link. Hence, the expected value derived by advertiser i from slot j is $v_i \alpha_j$.

2.6.3 What We Want

Is there an ideal sponsored search auction? Our desiderata are:

(1) DSIC. That is, truthful bidding should be a dominant strategy, and never leads to negative utility.

(2) Social welfare maximization. That is, the assignment of bidders to slots should maximize $\sum_{i=1}^{n} v_i x_i$, where x_i now denotes the CTR of the slot to which i is assigned (or 0 if i is not assigned to a slot). Each slot can only be assigned to one bidder, and each bidder gets only one slot.

(3) Computational efficiency. The running time should be polynomial (or even near-linear) in the size of the input $v_1, \ldots, v_n$. Remember that zillions of these auctions need to be run every day!

2.6.4 Our Design Approach

What's hard about auction design problems is that we have to design jointly two things: the choice of who wins what, and the choice of who pays what. Even in single-item auctions, it is not enough to make the "correct" choice to the first design decision (e.g., giving the item to the highest bidder)—if the payments are not just right, then strategic participants will game the system.

Happily, in many applications including sponsored search auctions, we can tackle this two-prong design problem one step at a time.

Step 1: Assume, without justification, that bidders bid truthfully. Then, how should we assign bidders to slots so that the above properties (2) and (3) hold?

Step 2: Given our answer to Step 1, how should we set selling prices so that the above property (1) holds?

If we efficiently solve both of these problems, then we have constructed an ideal auction. Step 2 ensures the DSIC property, which means that bidders will bid truthfully (provided each bidder with an obvious dominant strategy plays it). The hypothesis in Step 1 is then satisfied, so the outcome of the auction is indeed welfare-maximizing (and computable in polynomial time).

We conclude this lecture by executing Step 1 for sponsored search auctions. Given truthful bids, how should we assign bidders to slots to maximize the social welfare? Exercise 2.8 asks you to prove that the natural greedy algorithm is optimal (and computationally efficient): for $i = 1, 2, \ldots, k$, assign the ith highest bidder to the ith best slot.

Can we implement Step 2? Is there an analog of the second-price rule—sale prices that render truthful bidding a dominant strategy for every bidder? The next lecture gives an affirmative answer via Myerson's lemma, a powerful tool in mechanism design.

The Upshot

☆ In a single-item auction there is one seller with one item and multiple bidders with private valuations. Single-item auction design is a simple but canonical example of mechanism design.

☆ An auction is DSIC if truthful bidding is a dominant strategy and if truthful bidders always obtain nonnegative utility.

☆ An auction is welfare maximizing if, assuming truthful bids, the auction outcome always has the maximum possible social welfare.

☆ Second-price auctions are "ideal" in that they are DSIC, welfare maximizing, and can be implemented in polynomial time.

☆ Sponsored search auctions are a huge component of the Internet economy. Such auctions are more complex than single-item auctions because there are multiple slots for sale, and these slots vary in quality.

☆ A general two-step approach to designing ideal auctions is to first assume truthful bids and understand how to allocate items to maximize the social welfare, and second to design selling prices that turn truthful bidding into a dominant strategy.

Notes

The concept of dominant-strategy incentive-compatibility is articulated in Hurwicz (1972). Theorem 2.4 is from Vickrey (1961), the paper that effectively founded the field of auction theory. The model of sponsored search presented in Section 2.6 is due independently to Edelman et al. (2007) and Varian (2007). The former paper contains the mentioned jaw-dropping statistic. Problem 2.1 is closely related to the secretary problem of Dynkin (1963); see also Hajiaghayi et al. (2004).

The 2007 Nobel Prize citation (Nobel Prize Committee, 2007) presents a historical overview of the development of mechanism design theory in the 1970s and 1980s. Modern introductions to the field include Börgers (2015), Diamantaras et al. (2009), and chapter 23 of Mas-Colell et al. (1995). Krishna (2010) is a good introduction to auction theory.

Exercises

Exercise 2.1 Consider a single-item auction with at least three bidders. Prove that awarding the item to the highest bidder, at a price equal to the third-highest bid, yields an auction that is *not* DSIC.

Exercise 2.2 Prove that for every false bid $b_i \neq v_i$ by a bidder in a second-price auction, there exist bids $\mathbf{b}_{-i}$ by the other bidders such that i's utility when bidding b_i is strictly less than when bidding v_i.

Exercise 2.3 Suppose there are k identical copies of an item and $n > k$ bidders. Suppose also that each bidder can receive at most one item. What is the analog of the second-price auction? Prove that your auction is DSIC.

Exercise 2.4 Consider a seller that incurs a cost of $c > 0$ for selling her item—either because she has a value of c for retaining the item or because she would need to produce the item at a cost of c. The social welfare is now defined as the valuation of the winning buyer (if any) minus the cost incurred by the seller (if any). How would you modify the second-price auction so that it remains DSIC and welfare maximizing? Argue that your auction is *budget-balanced*, meaning that whenever the seller sells the item, her revenue is at least her cost c.

Exercise 2.5 Suppose you want to hire a contractor to perform some task, like remodeling a house. Each contractor has a cost for performing the task, which a priori is known only to the contractor. Give an analog of a second-price auction in which contractors report their costs and the auction chooses a contractor and a payment. Truthful reporting should be a dominant strategy in your auction and, assuming truthful bids, your auction should select the contractor with the smallest cost. The payment to the winner should be at least her reported cost, and losers should be paid nothing.

[Auctions of this type are called *procurement* or *reverse* auctions.]

Exercise 2.6 Compare and contrast an eBay auction with a sealed-bid second-price auction. (Read up on eBay auctions if you don't already know how they work.) Should you bid differently in the two auctions? State explicitly your assumptions about how bidders behave.

Exercise 2.7 You've probably seen—in the movies, at least—the call-and-response format of *open ascending single-item auctions*, where an auctioneer asks for takers at successively higher prices. Such an auction ends when no one accepts the currently proposed price, the winner (if any) is the bidder who accepted the previously proposed price, and this previous price is the final sale price.

Compare and contrast open ascending auctions with sealed-bid second-price auctions. Do bidders have dominant strategies in open ascending auctions?

Exercise 2.8 Recall the sponsored search setting of Section 2.6, in which bidder i has a valuation v_i per click. There are k slots with click-through rates (CTRs) $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. The social welfare of an assignment of bidders to slots is $\sum_{i=1}^{n} v_i x_i$, where x_i equals the CTR of the slot to which i is assigned (or 0 if bidder i is not assigned to any slot).

Prove that the social welfare is maximized by assigning the bidder with the ith highest valuation to the ith best slot for $i = 1, 2, \ldots, k$.

Problems

Problem 2.1 This problem considers *online* single-item auctions, where bidders arrive one-by-one. Assume that the number n of bidders is known, and that bidder i has a private valuation v_i. We consider auctions of the following form.

> **Online Single-Item Auction**
>
> For each bidder arrival $i = 1, 2, \ldots, n$:
> - if the item has not been sold in a previous iteration, formulate a price p_i and then accept a bid b_i from bidder i
> - if $p_i \leq b_i$, then the item is sold to bidder i at the price p_i; otherwise, bidder i departs and the item remains unsold

(a) Prove that an auction of this form is DSIC.

(b) Assume that bidders bid truthfully. Prove that if the valuations of the bidders and the order in which they arrive are arbitrary, then for every constant $c > 0$ independent of n, there is no deterministic online auction that always achieves social welfare at least c times the highest valuation.

(c) *(H)* Assume that bidders bid truthfully. Prove that there is a constant $c > 0$, independent of n, and a deterministic online auction with the following guarantee: for every unordered set of n bidder valuations, if the bidders arrive in a uniformly random order, then the expected welfare of the auction's outcome is at least c times the highest valuation.

Problem 2.2 Suppose a subset S of the bidders in a second-price single-item auction decide to collude, meaning that they submit their bids in a coordinated way to maximize the sum of their utilities. Assume that bidders outside of S bid truthfully. Prove necessary and sufficient conditions on the set S such that the bidders of S can increase their combined utility via non-truthful bidding.

Problem 2.3 We proved that second-price auctions are DSIC under the assumption that every bidder's utility function is quasilinear, with the utility of a bidder with valuation v_i winning the item at price p given by $v_i - p$. Identify significantly weaker assumptions on bidders' utility functions under which truthful bidding remains a dominant strategy for every bidder.