Math 236 Winter 2017

Due Friday, February 10

1. Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ in the standard bases by

$$T(x, y, z) = (x + y + z, 3x - 2z).$$

   (a) Find $M(T)$ with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

   (b) Find $M(T)$ with respect to the standard basis for $\mathbb{R}^3$ and the basis $((1, 1), (1, 2))$ for $\mathbb{R}^2$.

2. Define $f : \mathbb{F}^4 \to \mathbb{F}^2$ by

$$f(x_1, x_2, x_3, x_4) = (x_1 - 2x_3 + x_4, 3x_3 - 2x_4).$$

   Find a basis for the kernel and prove that it is a basis. Then, use the fundamental theorem of linear maps to find a basis of the image of f.

3. Suppose that V is finite-dimensional and that the dimension of V is odd. Show that there does not exist a linear map $T : V \to V$ such that

$$\ker(T) = \mathrm{range}(T).$$

4. Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R})$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbb{R})$ and a basis of $\mathcal{P}_2(\mathbb{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

5. (Ax 3.D.7) Suppose V and W are finite-dimensional. Let $v \in V$. Let

$$E = \{T \in \mathcal{L}(V, W) \mid Tv = 0\}.$$

   (a) Show that E is a subspace of $\mathcal{L}(V, W)$.

   (b) Suppose $v \neq 0$. What is $\dim E$? (Extended hint: If $v \neq 0$, we can extend it to a basis for V, $v, v_1, \ldots, v_n$. We can also fix a basis for W: $w_1, \ldots, w_m$. It may help to think of a linear map in terms of its matrix with respect to the bases we just listed. What condition on the matrix forces v to go to 0? What is a basis for this set of matrices?)

The following exercises are not required to be handed in but might be helpful.

- (Ax 3.B.13) Suppose T is a linear map from $\mathbb{F}^4 \to \mathbb{F}^2$ such that

$$\ker(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

  Prove that T is surjective.

- Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on V is not a necessarily a subspace of $\mathcal{L}(V)$.