Math 243 Analysis 2 Assignment 6

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Problem 2a. Derivative at 0:

$$\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}=\lim_{x\to 0}\frac{x^2cos(\frac{1}{x^2})-0}{x}=\lim_{x\to 0}xcos(\frac{1}{x^2})$$

Since $\mid cos(\frac{1}{x^2})\mid\leq 1$. It follows,

$$\mid xcos(\frac{1}{x^2})\mid\ \leq\ \mid x\mid$$

hence,

$$\lim_{x\to 0}-x\leq\lim_{x\to 0}xcos(\frac{1}{x^2})\leq\lim_{x\to 0}x$$

By the Squeeze Theorem,

$$\lim_{x\to 0}-x=0=\lim_{x\to 0}x\implies\lim_{x\to 0}xcos(\frac{1}{x^2})=0$$

$$\implies F'(0)=0$$

Let $c\in\mathbb{R}\backslash\{0\}$. Derivative at c: Differentiate F at $\mathbb{R}\backslash\{0\}$. There exists $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ such that $(g\cdot h)(x)=F(x)$. Suppose,

$$g(x)=x^2\ and\ h(x)=\begin{cases}cos(\frac{1}{x^2}), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Therefore by the product rule, $F'(x)=(g\cdot h)'(x)=g(x)h'(x)+h(x)g'(x)$
Differentiate g at $c\in\mathbb{R}$:

$$\lim_{x\to c}\frac{x^2-c^2}{x-c}=\lim_{x\to c}\frac{(x-c)(x+c)}{x-c}=\lim_{x\to c}(x+c)=2c$$

Therefore $g'(x)=2x$, $\forall x\in\mathbb{R}$

h can be expressed as the composition of two functions $\varphi\circ\psi$ where $\varphi=cos(x)$ and $\psi=\frac{1}{x^2}$
Therefore $h(x)=(\varphi\circ\psi)(x)=\varphi(\psi(x))$. From the chain rule,

$$h'(x)=(\varphi\circ\psi)'(x)=\varphi'(\psi(x))\cdot\psi'(x).$$

By definition $\varphi'(x) = -sin(x)$.
Differentiate ψ at $c \in \mathbb{R}\backslash\{0\}$:

$$\lim_{x\to c} \frac{\frac{1}{x^2} - \frac{1}{c^2}}{x - c} = \lim_{x\to c} \frac{\frac{c^2-x^2}{x^2c^2}}{x-c} = \lim_{x\to c} \frac{\frac{(c-x)(c+x)}{x^2c^2}}{x-c} = \lim_{x\to c} \frac{-(c+x)}{x^2c^2} = \frac{-2c}{c^4} = \frac{-2}{c^3}$$

Therefore $\psi'(x) = -2x^{-3}$, $\forall x \in \mathbb{R}\backslash\{0\}$.
Therefore $h'(x) = (\varphi \circ \psi)'(x) = \varphi'(\psi(x)) \cdot \psi'(x) = -sin(\frac{1}{x^2})(-2x^{-3}) \forall x \in \mathbb{R}\backslash\{0\}$
From the Product Rule,

$$F'(x) = (g \cdot h)'(x) = g(x)h'(x) + h(x)g'(x) = 2xcos(\frac{1}{x^2}) + x^2(-sin(\frac{1}{x^2}))(-2x^{-3}), \forall x \in \mathbb{R}\backslash\{0\}$$

$$\implies F'(x) = \begin{cases} 2xcos(\frac{1}{x^2}) + \frac{2}{x}sin(\frac{1}{x^2}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Therefore F is differentiable.
Consider the sequence $x_n := \sqrt{\frac{1}{2\pi n + \frac{\pi}{2}}}$. We have,

$$lim(x_n) = 0$$

$$F'(x_n) = 2x_n cos(\frac{1}{x_n^2}) + \frac{2}{x_n} sin(\frac{1}{x_n^2}) = 2\sqrt{\frac{1}{2\pi n + \frac{\pi}{2}}} cos(2\pi n + \frac{\pi}{2}) + \frac{2}{\sqrt{\frac{1}{2\pi n + \frac{\pi}{2}}}} sin(2\pi n + \frac{\pi}{2})$$

$$= 2\sqrt{2\pi n + \frac{\pi}{2}}$$

$$lim(F'(x_n)) = \infty$$

Therefore $F'(x)$ is unbounded on $[-1, 1]$. It follows that $F'(x)$ is not Riemann Integrable on [-1,1].

Problem 2.

$$\int_0^x f = \int_x^1 f \implies \int_0^x f = \int_0^1 f - \int_0^x f$$

Consider $x = 0$,

$$\int_0^0 f = \int_0^1 f \implies \int_0^1 f = 0$$

It follows,

$$\int_0^x f = -\int_0^x f$$

Since f is continuous on [0,1] it is Riemann Integrable on [0,1]. Therefore define the function F as follows

$$F(x) := \int_0^x f$$

$$F(x) = \int_0^x f = -\int_0^x f = -F(x)$$

From the Fundamental Theorem (Second Form) we have,

$$F'(x) = f$$

$$-F'(x) = -f$$

Therefore,

$$f = -f$$

We conclude $f(x) = 0 \, \forall x \in [0, 1]$.