Math 243 Analysis 2 Assignment 2

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Problem 2a. Derivative at 0:

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^4 sin(\frac{1}{x})-0}{x}=\lim_{x\to 0}x^3 sin(\frac{1}{x})$$

Since $\mid sin(\frac{1}{x})\mid\leq 1$. It follows,

$$\mid x^3 sin(\frac{1}{x})\mid\ \leq\ \mid x^3\mid$$

hence,

$$\lim_{x\to 0}-x^3\leq\lim_{x\to 0}x^3 sin(\frac{1}{x})\leq\lim_{x\to 0}x^3$$

By the Squeeze Theorem,

$$\lim_{x\to 0}-x^3=0=\lim_{x\to 0}x^3\implies\lim_{x\to 0}x^3 sin(\frac{1}{x})=0$$

$$\implies f'(0)=0$$

Let $c\in\mathbb{R}\backslash\{0\}$. Derivative at c:

Differentiate f at $\mathbb{R}\backslash\{0\}$. There exists $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ such that $(g\cdot h)(x)=f(x)$. Suppose,

$$g(x)=x^4\ and\ h(x)=\begin{cases}sin(\frac{1}{x}), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Therefore by the product rule, $f'(x)=(g\cdot h)'(x)=g(x)h'(x)+h(x)g'(x)$
Differentiate g at $c\in\mathbb{R}$:

$$\lim_{x\to c}\frac{x^4-c^4}{x-c}=\lim_{x\to c}\frac{(x^2-c^2)(x^2+c^2)}{x-c}=\lim_{x\to c}\frac{(x-c)(x+c)(x^2+c^2)}{x-c}=\lim_{x\to c}(x+c)(x^2+c^2)=4c^3$$

Therefore $g'(x)=4x^3$, $\forall x\in\mathbb{R}$

h can be expressed as the composition of two functions $\varphi\circ\psi$ where $\varphi=sin(x)$ and $\psi=\frac{1}{x}$
Therefore $h(x)=(\varphi\circ\psi)(x)=\varphi(\psi(x))$. From the chain rule,

$$h'(x)=(\varphi\circ\psi)'(x)=\varphi'(\psi(x))\cdot\psi'(x).$$

By definition $\varphi'(x) = cos(x)$.
Differentiate ψ at $c \in \mathbb{R}\backslash\{0\}$:

$$\lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \lim_{x\to c} \frac{\frac{c-x}{xc}}{x - c} = \lim_{x\to c} \frac{-1}{xc} = \frac{-1}{c^2}$$

Therefore $\psi'(x) = -x^{-2}$, $\forall x \in \mathbb{R}\backslash\{0\}$.
Therefore $h'(x) = (\varphi \circ \psi)'(x) = \varphi'(\psi(x)) \cdot \psi'(x) = cos(\frac{1}{x})(-x^{-2})\forall x \in \mathbb{R}\backslash\{0\}$
From the Product Rule,

$$f'(x) = (g \cdot h)'(x) = g(x)h'(x) + h(x)g'(x) = sin(\frac{1}{x})4x^3 + x^4cos(\frac{1}{x})(-x^{-2}), \forall x \in \mathbb{R}\backslash\{0\}$$

$$\implies f'(x) = \begin{cases} 4x^3sin(\frac{1}{x}) - x^2cos(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Prove the derivative is continuous:
Show that the function $j(x) = x$ is continuous on $\mathbb{R}$. Let $\varepsilon > 0$ and define $\delta := \varepsilon > 0$. Let $x \in \mathbb{R}$ and $\mid x - c \mid< \delta$.

$$\mid j(x) - j(c) \mid=\mid x - c \mid< \delta = \varepsilon$$

Therefore $j(x) = x$ is continuous on $\mathbb{R}$. Therefore a function $k(x) = x * x = x^2$ is continuous on $\mathbb{R}$ since it is the product of two continuous functions. From this we can conclude a function $l(x) = x^2 * x = x^3$ is continuous on $\mathbb{R}$ since it is the product of two continuous functions. Finally a function $m(x) = 4 * x^3 = 4x^3$ is continuous on $\mathbb{R}$ since it is the product of a continuous function and the integer $4 \in \mathbb{R}$.
Both cosine and sine are continuous functions on $\mathbb{R}$. A function that is defined as $a(x) = 1$ is continuous on $\mathbb{R}$, therefore $n(x) = \frac{a(x)}{j(x)} = \frac{1}{x}$ is continuous on $\mathbb{R}\backslash\{0\}$ since it is the quotient of two continuous functions and $j(x) \neq 0 \,\forall\, x \in \mathbb{R}\backslash\{0\}$. Therefore the composition of $sin(x)$ and $\frac{1}{x} = sin(\frac{1}{x})$ is continuous on $\mathbb{R}\backslash\{0\}$ and the composition of $cos(x)$ and $\frac{1}{x} = cos(\frac{1}{x})$ is continuous on $\mathbb{R}\backslash\{0\}$.
f' can be expressed as the product, sum and difference of these functions when $x \neq 0$. Therefore f' is continuous for all $x \in \mathbb{R}\backslash\{0\}$.
Continuity of f' at 0.

$$\lim_{x\to 0} f' = \lim_{x\to 0} 4x^3sin(\frac{1}{x}) - x^2cos(\frac{1}{x}) = \lim_{x\to 0} 4x^3sin(\frac{1}{x}) - \lim_{x\to 0} x^2cos(\frac{1}{x})$$

Using the Squeeze Theorem for both limits in the same way as above. We get that both limits equal 0. Since $f(0) = 0$, f' is continuous at 0, which means that f' is continuous on $\mathbb{R}$.

Problem 2b.

$$g(x) = 2x^4 + f(x) = \begin{cases} 2x^4 + x^4sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 2x^4 + 0, & \text{if } x = 0 \end{cases} = \begin{cases} x^4(2 + sin(\frac{1}{x})), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Suppose $x \in \mathbb{R}\backslash\{0\}$. Then,

$$x^4(2 + sin(\frac{1}{x})) \geq x^4(2 + (-1)) = x^4 > 0$$

$$\implies g(0) \leq g(x) \ \forall x \in \mathbb{R}$$

Therefore g has an absolute minimum at 0.

There exists $n \in \mathbb{N}$ such that $n \geq 2$ and $-\delta \leq -\frac{1}{2\pi n} \leq 0 \Leftrightarrow -\frac{1}{2\pi n} \in (-\delta, 0)$

$$g'(x) = 8x^3 + 4x^3 sin(\frac{1}{x}) - x^2 cos(\frac{1}{x})$$

$$g'(-\frac{1}{2\pi n}) = 8(-\frac{1}{2\pi n})^3 + 4(-\frac{1}{2\pi n})^3 sin(\frac{1}{-\frac{1}{2\pi n}}) - (-\frac{1}{2\pi n})^2 cos(\frac{1}{-\frac{1}{2\pi n}})$$

$$= -8(\frac{1}{2\pi n})^3 - 4(\frac{1}{2\pi n})^3 sin(-2\pi n) - (\frac{1}{2\pi n})^2 cos(-2\pi n) = -8(\frac{1}{2\pi n})^3 - (\frac{1}{2\pi n})^2 < 0$$

Therefore g is decreasing at $-\frac{1}{2\pi n}$.

Now take $\frac{1}{2\pi n} \in (0, \delta)$:

$$g'(\frac{1}{2\pi n}) = 8(\frac{1}{2\pi n})^3 + 4(\frac{1}{2\pi n})^3 sin(\frac{1}{\frac{1}{2\pi n}}) - (\frac{1}{2\pi n})^2 cos(\frac{1}{\frac{1}{2\pi n}})$$

$$= 8(\frac{1}{2\pi n})^3 + 4(\frac{1}{2\pi n})^3 sin(2\pi n) - (\frac{1}{2\pi n})^2 cos(2\pi n) = 8(\frac{1}{2\pi n})^3 - (\frac{1}{2\pi n})^2 = (\frac{1}{2\pi n})^2(\frac{8}{2\pi n} - 1)$$

$$(\frac{1}{2\pi n})^2 > 0 \ \forall n$$

Therefore in order for $g' > 0$, $\frac{8}{2\pi n} - 1 > 0 \ \forall n$

$$\frac{8}{2\pi n} - 1 > 0 \Leftrightarrow \frac{8}{2\pi} > n \Leftrightarrow \frac{4}{\pi} > n$$

Since $2 > \frac{4}{\pi} > 1$. This is a contradiction to the condition that n must be greater than or equal to 2. Therefore g is not increasing on $(0, \delta)$. Therefore there does not exist any $\delta > 0$ such that g is decreasing on $(-\delta, 0)$ and increasing on $(0, \delta)$.

Problem 5. f is differentiable on $[0, 2]$ therefore f is continuous on $[0, 2]$.
Since $f(0) = 0$ and $f(1) = 2$ $\exists x \in (0, 1)$ such that $f(x) = 1$. Considering this point and since $f(2) = 1$ it follows from the Mean Value Theorem that there exists a point $c \in (x, 2)$ such that $f'(c) = 0$. Furthermore since $f(1) = 2 > 1$ then there exists a local maximum in $(x, 2)$. Assume that the local maximum is at point c.

Problem 5a. Considering the points $f(0) = 0$ and $f(1) = 2$ it follows from the Mean Value Theorem that there exists $d \in (0, 1)$ such that $f'(d) = 2$. Now by the Intermediate Value Theorem, considering $f'(c) = 0$ and $f'(d) = 2$ there exists $\alpha \in (0, 2)$ in between c and d (i.e $c < \alpha < d$ or $d < \alpha < c$) such that $f'(\alpha) = \frac{1}{2}$

Problem 5b. Considering the points $f(1) = 2$ and $f(2) = 1$ it follows from the Mean Value Theorem that there exists $e \in (1, 2)$ such that $f'(e) = -1$. Now by the Intermediate Value Theorem, considering $f'(c) = 0$ and $f'(e) = -1$ there exists $\beta \in (0, 2)$ in between c and e (i.e $c < \beta < e$ or $e < \beta < c$) such that $f'(\beta) = -\frac{1}{2}$