ANSWERS TO SELECTED ODD-NUMBERED EXERCISES

Solutions for Chapter 1

1.1 (b) X_t is the student's status at the end of year t.
State space (discrete): $\mathcal{S} = \{$Drop Out, Frosh, Sophomore, Junior, Senior, Graduate$\}$.
Index set (discrete): $I = \{0, 1, 2, \ldots\}$.
(e) X_t is the arrival time of student t.
State space (continuous): $[0, 60]$
Index set (discrete): $\{1, 2, \ldots, 30\}$.
(f) X_t is the order of the deck of cards after t shuffles.
State space (discrete): Set of all orderings of the deck (52! elements).
Index set (discrete): $\{0, 1, 2, \ldots\}$

1.3

$$\begin{aligned}
\sum_{i=1}^{k} P(A|B_i \cap C)P(B_i|C) &= \sum_{i=1}^{k} \left(\frac{P(A \cap B_i \cap C)}{P(B_i \cap C)} \right) \left(\frac{P(B_i \cap C)}{P(C)} \right) \\
&= \sum_{i=1}^{k} \frac{P(A \cap B_i \cap C)}{P(C)} = \frac{1}{P(C)} \sum_{i=1}^{k} P(A \cap B_i \cap C) \\
&= \frac{P(A \cap C)}{P(C)} = P(A|C).
\end{aligned}$$

Introduction to Stochastic Processes with R, First Edition. Robert P. Dobrow.

1.5 (a) Uniform on $\{1, 2, 3, 4, 5, 6\}$.
(b) Uniform on $\{2, 3, 4, 5, 6\}$.

1.7 Let X denote the time until the rat finds the cheese. Let 1, 2, and 3 denote each door, respectively. Then,

$$E(X) = E(X|1)P(1) + E(X|2)P(2) + E(X|3)P(3)$$

$$= (2 + E(X))\frac{1}{3} + (3 + E(X))\frac{1}{3} + (1)\frac{1}{3}$$

$$= 2 + E(X)\frac{2}{3}.$$

Thus, $E(X) = 6$ minutes.

1.11 Let x_k be the probability of reaching n when the gambler's fortune is k. As in Example 1.10.

$$x_k = x_{k+1}p + x_{k-1}q, \text{ for } 1 \le k \le n-1,$$

with $x_0 = 0$ and $x_n = 1$, which gives

$$x_{k+1} - x_k = (x_k - x_{k-1})\frac{p}{q}, \text{ for } 1 \le k \le n-1.$$

It follows that

$$x_k - x_{k-1} = \cdots = (x_1 - x_0)(p/q)^{k-1} = x_1(p/q)^{k-1}, \text{ for all } k.$$

This gives $x_k - x_1 = \sum_{i=2}^{k} x_1(p/q)^{k-1}$, or

$$x_k = \sum_{i=1}^{k} x_1(p/q)^{k-1} = x_1\frac{1-(p/q)^k}{1-p/q}.$$

For $k = n$, this gives

$$1 = x_n = x_1\frac{1-(p/q)^n}{1-p/q}.$$

Thus, $x_1 = (1 - p/q)/(1 - (p/q)^n)$, which gives

$$x_k = \frac{1-(p/q)^k}{1-(p/q)^n}, \text{ for } k = 0, \dots, n.$$

1.13 (a) $f_{Y|X}(y|x) = 2y/(1 - x^2)$, for $x < y < 1$.
(b) The conditional distribution is uniform on $(0, y)$.

1.15 The area of the circle is π. The equation of the circle is $x^2 + y^2 = 1$. The joint density is

$$f(x, y) = \frac{1}{\pi}, \quad \text{for} - 1 < x < 1, -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}$$

Integrating out the y term gives the marginal density

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2\sqrt{1 - x^2}}{\pi}, \text{ for } -1 < x < 1.$$

The conditional density is

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{1/\pi}{2\sqrt{1 - x^2}/\pi} = \frac{1}{2\sqrt{1 - x^2}},$$

for $-1\sqrt{1 - x^2} < y < \sqrt{1 - x^2}$. The conditional distribution of Y given $X = x$ is uniform on $\left(- \sqrt{1 - x^2}, \sqrt{1 - x^2}\right)$.

1.17 $E(X|X > 2) = 4.16525$.

1.21

$$\int_0^\infty P(T > t)\, dt = \int_0^\infty \int_t^\infty f(s)\, ds\, dt = \int_0^\infty \int_0^s f(s)\, dt\, ds$$
$$= \int_0^\infty sf(s)\, ds = E(T).$$

1.23 (b) For $m > n$,

$$\begin{aligned} E(S_m|S_n) &= E(S_n + X_{n+1} + \cdots + X_m|S_n) \\ &= E(S_n|S_n) + E(X_{n+1} + \cdots + X_m|S_n) \\ &= S_n + \sum_{i=n+1}^{m} E(X_i|S_n) = S_n + \sum_{i=n+1}^{m} E(X_i) \\ &= S_n + (m - n)\mu. \end{aligned}$$

1.27 Let T be the total amount spent at the restaurant. Then,

$$E(T) = 200(15) = \$3000,$$

and

$$Var(T) = 9(200) + 15^2(40^2) = 361800, \qquad SD(T) = \$601.50.$$

1.29 Yes.

Solutions for Chapter 2

2.1 (a) 0.6;
(b) $P^2_{32} = 0.27$;
(c) $P_{31}\alpha_3/(\alpha P)_1 = (0.3)(0.5)/(0.17) = 15/17 = 0.882$;
(d) $(0.182, 0.273, 0.545) \cdot (1, 2, 3) = 2.363$.

2.3 $P^3_{10} = 0.517$.

2.5 (a)

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

(b) 19/64.
(c) 0.103.

2.9

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 3/5 & 0 & 2/5 \\ 1/7 & 2/7 & 0 & 0 & 4/7 \\ 0 & 2/9 & 2/3 & 1/9 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 1/4 & 0 \end{pmatrix} \end{array}.$$

2.11 (b) $P^3_{0,5} = 0.01327$.

2.25

Socializing	Traveling	Milling	Feeding	Resting
0.148	0.415	0.096	0.216	0.125

Solutions for Chapter 3

3.1

$$\boldsymbol{\pi} = \left(\frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{15}\right).$$

3.3 $\boldsymbol{P}$ and $\boldsymbol{R}$ are regular. $\boldsymbol{Q}$ is regular for $0 < p < 1$.

3.5 (a) All non-negative vectors of the form (a, a, b, c, a), where $3a + b + c = 1$.

3.7 The transition matrix is doubly stochastic. The stationary distribution is uniform.

3.11 (a)

$$\pi_j = \begin{cases} 1/(2k), & \text{if } j = 0, k, \\ 1/k, & \text{if } j = 1, \ldots, k-1. \end{cases}$$

(b) 2,000 steps.

3.13 Communication classes are $\{4\}$ (recurrent, absorbing); $\{1, 5\}$ (recurrent); $\{2, 3\}$ (transient). All states have period 1.

$$\boldsymbol{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \\ 5 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} \quad 2 & \quad\; 3 & \quad\; 1 & \quad\; 5 & \quad\; 4 \end{array} \\ \begin{pmatrix} 1/2 & 1/6 & 1/3 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

3.17

$$\boldsymbol{P}^n = \begin{pmatrix} 1/2^n & 1 - 1/2^n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \lim_{n\to\infty} \boldsymbol{P}^n = \begin{pmatrix} 2 & +\infty \\ 0 & +\infty \end{pmatrix}.$$

3.19 Let p_x be the expected time to hit d for the walk started in x. By symmetry, $p_b = p_e$ and $p_c = p_e$. Solve

$$p_a = \frac{1}{2}(1 + p_b) + \frac{1}{2}(1 + p_c),$$

$$p_b = \frac{1}{2}(1 + p_a) + \frac{1}{2}(1 + p_c),$$

$$p_c = \frac{1}{4}(1 + p_b) + \frac{1}{4} + \frac{1}{4}(1 + p_a) + \frac{1}{4}(1 + p_c).$$

This gives $p_a = 10$.

3.23

$$\pi_i = 2(k + 1 - i)/(k(k + 1)), \text{ for } i = 1, \dots, k.$$

3.25 (a) For $k = 2$, $\boldsymbol{\pi} = (1/6, 2/3, 1/6)$. For $k = 3$, $\boldsymbol{\pi} = (1/20, 9/20, 9/20, 1/20)$.

3.29 Communication classes are: (i) $\{a\}$ transient; (ii) $\{e\}$ recurrent; (iii) $\{c, d\}$ transient; and (iv) $\{b, f, g\}$ recurrent. The latter class has period 2. All other states have period 1.

3.33 For all states i and j, and $m > 0$,

$$P^{N+m}_{ij} = \sum_k P^m_{ik} P^N_{kj}.$$

Since $P^N_{kj} > 0$ for all k, the only way the expression above could be zero is if $P^m_{ik} = 0$ for all k, which is not possible since $\boldsymbol{P}^m$ is a stochastic matrix whose rows sum to 1.

3.43 (a) The chain is ergodic for all $0 \le p, q \le 1$, except $p = q = 0$ and $p = q = 1$.
(b) The chain is reversible for all $p = q$, with $0 < p < 1$.

3.47

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/3 & 4/9 & 2/9 \\ 2 & 0 & 1/2 & 1/2 \\ 3 & 1/2 & 0 & 1/2 \end{array}$$

$$\boldsymbol{P} = \begin{pmatrix} 1/3 & 4/9 & 2/9 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}.$$

3.53 (a) The probability that A wins is $1/(2-p)$.
(c) $\alpha = 1270/6049 \approx 0.210$ and $\beta = 737/6049 \approx 0.122$. For the first method, A wins with probability 0.599. For the second method, A wins with probability 0.565.

3.61 Yes. T is a stopping time.

3.63

$$\boldsymbol{\pi} = (0.325, 0.207, 0.304, 0.132, 0.030, .003, .0003).$$

3.67

```
> allsixes <- function() {
+ i <- 0
+ ct <- 0
+ while (ct < 5)
+ { x <- sample(1:6,5-ct,replace=T)
+ sixes <- sum(x==6)
+ ct <- ct + sixes
+ i <- i+1 }
+ i
+ }
> sim <- replicate(10000, allsixes())
> mean(sim)
[1] 13.0873
```

Solutions for Chapter 4

4.1 $P_{0,j} = 1$, if $j = 0$, and 0, otherwise.

$$P_{1,j} = \begin{cases} a, & \text{if } j = 0, \\ b, & \text{if } j = 1, \\ c, & \text{if } j = 2. \end{cases}$$

For the second row of $\boldsymbol{P}$,

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ \hline 2 & a^2 & 2ab & 2ac+b^2 & 2bc & c^2 & 0 & \cdots \end{array}$$

4.3 From Exercise 4.2, $G_X(s) = e^{\lambda(s-1)}$ and $G_Y(s) = e^{\mu(s-1)}$. Then,

$$G_{X+Y}(s) = G_X(s)G_Y(s) = e^{\lambda(s-1)}e^{\mu(s-1)} = e^{(\lambda+\mu)s-1}.$$

Thus, $X + Y$ has a Poisson distribution with parameter $\lambda + \mu$.

4.7
$$\begin{aligned} G(s) &= (1-p) + ps^3. \\ \mu &= G'(1) = 3p. \\ \sigma^2 &= G''(1) + G'(1) - G'(1)^2 = 6p + 3p - (3p)^2 = 9p(1-p). \\ E\left(Z^4\right) &= \mu^4 = (3p)^4 = 81p^4. \\ Var\left(Z^4\right) &= 9p(1-p)(3p)^3(81p^4 - 1)/(3p-1). \end{aligned}$$

4.11
$$\begin{aligned} E(Z_n) &= G'1_n(1) = G'(G_{n-1}(1))G'_{n-1}(1) = G'(1)G'_{n-1}(1) \\ &= \mu G'_{n-1}(1) = \mu E(Z_{n-1}). \end{aligned}$$

The result follows by induction.

4.15 Solve $s = 1/4 + s/4 + s^2/4 + s^3/4$. $e = 0.414$.

4.17 (a)
$$e = \begin{cases} (1-p)/p, & \text{if } p > 1/2, \\ 1, & \text{if } p \le 1/2. \end{cases}$$

(b)
$$P_{2i,2j} = \binom{2i}{2j} p^{2j}(1-p)^{2i}.$$

4.21 (a) $\mu = c/(1-p)^2$
(c) 0.693, 0.803

4.23
$$G_Z(s) = \frac{1}{a_0}(G(s) - a_0).$$

4.29 (a)

```
> pgf <- function(s) {
+  0.8 + 0.1*s^4 + 0.1*s^9
+ }
> x <- 0.5  # initial value
> e <- pgf(x)
> for (i in 1:100) {
+  e <- pgf(e)
+ }
> e
[1] 0.9152025
```

(b) 0.101138

4.33

```
> branch <- function(n,lam) {  ## Poisson
+  z <- c(1,rep(0,n))
+  for (i in 2:(n+1)) {
+  z[i] <- sum(rpois(z[i-1],lam))
+  }
+  return(z)   }
   # Assume extinction occurs by 50th generation
> n <- 10000
> simlist <- replicate(n, sum(branch(50,0.60)))
> mean(simlist)
[1] 2.5308
> var(simlist)
[1] 9.594811
```

Solutions for Chapter 5

5.1 Let

$$\boldsymbol{P} = \begin{array}{c} \\ \text{Truck} \\ \text{Car} \end{array} \begin{array}{c} \begin{array}{cc} \text{Truck} & \text{Car} \end{array} \\ \begin{pmatrix} 1/5 & 4/5 \\ 1/4 & 3/4 \end{pmatrix} \end{array},$$

with stationary distribution $\boldsymbol{\pi} = (5/21, 16/21)$. By the strong law of large numbers the toll collected is about

$$1000\left(5\left(\frac{5}{21}\right) + 1.5\left(\frac{16}{21}\right)\right) = \$2333.33.$$

5.3 (a) Compute $(10000) \times \boldsymbol{\lambda}\boldsymbol{P}^2$, with $\boldsymbol{\lambda} = (0.6, 0.3, 0.1)$. This gives Car: 3,645, Bus: 4,165, Bike: 2,190.

For long-term totals, find the stationary distribution and compute $(10000) \times \boldsymbol{\pi}$ to get Car: 2083, Bus: 4583, Bike: 3333.

(b) Current: $271(0.6) + 101(0.3) + 21(0.1) = 195$ g. Long-term: $271(0.208) + 101(0.458) + 21(0.333) = 109.75$ g.

5.7 Assume that the chain is currently at state i. Let j be the proposal state, chosen uniformly on $\{0, 1, \ldots, n\}$. Let $U \sim$ Uniform(0, 1). Accept j as the next state of the chain if

$$U < \frac{\binom{n}{j} p^j (1-p)^{n-j}}{\binom{n}{i} p^i (1-p)^{n-i}} = \frac{i!(n-i)!}{j!(n-j)!}\left(\frac{p}{1-p}\right)^{j-i}.$$

Otherwise, stay at state i.

5.15

$$
\boldsymbol{P} = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\
\begin{pmatrix}
1/2 & 1/8 & 1/8 & 1/8 & 1/8 & 0 \\
1/8 & 1/2 & 1/8 & 0 & 1/8 & 1/8 \\
1/8 & 1/8 & 1/2 & 1/8 & 0 & 1/8 \\
1/8 & 0 & 1/8 & 1/2 & 1/8 & 1/8 \\
1/8 & 1/8 & 0 & 1/8 & 1/2 & 1/8 \\
0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/2
\end{pmatrix}
\end{array}.
$$

5.19

```
> trials <- 20000
> n <- 50
> p <- 1/4
> sim <- numeric(trials)
> for (k in 1:trials) {
+ state <- 0
+ # run chain for 60 steps to be near stationarity
+ for (i in 1:60) {
+ y <- sample(0:n,1)
+ acc <- factorial(state)*factorial(n-state)/
  (factorial(y)*factorial(n-y))
+   *(p/(1-p))^(y-state)
+ if (runif(1) < acc) state <- y
+ }
+ sim[k] <- if (state >= 10 & state <= 15) 1 else 0
+ }
> mean(sim)  # estimate of P(10 <= X <= 15)
[1] 0.6712
# exact probability
> pbinom(15,n,p)-pbinom(9,n,p)
[1] 0.6732328
```

Solutions for Chapter 6

6.1 (a) 0.048;
(b) 0.1898;
(c) 0.297.

6.3 (a) 0.082;
(b) 0.0257;
(c) 0.01299.

6.7 (a) 1/2;
(b) 1/4;
(c) 1/6.

6.9 Let X be geometrically distributed with parameter p. The cumulative distribution function for X is

$$P(X \leq x) = \sum_{k=1}^{x} P(X = k) = \sum_{k=1}^{x} (1-p)^{k-1} p = p \frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x.$$

This gives,

$$P(X > s+t | X > s) = \frac{P(X > s+t)}{P(X > s)} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t = P(X > t).$$

6.11 Let X be a memoryless, continuous random variable. Let $g(t) = P(X > t)$. By memorylessness,

$$P(X > t) = P(X > s+t | X > s) = \frac{P(X > s+t)}{P(X > s)}.$$

Thus, $g(s+t) = g(s)g(t)$. It follows that

$$g(t_1 + \cdots + t_n) = g(t_1) \cdots g(t_n).$$

Let $r = p/q = \sum_{i=1}^{p} (1/q)$. Then, $g(r) = (g(1/q))^p$. Also,

$$g(1) = g\left(\sum_{i=1}^{q} \frac{1}{q}\right) = g\left(\frac{1}{q}\right)^q,$$

or $g(1/q) = g(1)^{1/q}$. This gives

$$g(r) = g(1)^{p/q} = g(1)^r = e^{r \ln g(1)},$$

for all rational r. By continuity, for all $t > 0$, $g(t) = e^{-\lambda t}$, where

$$\lambda = -\ln\, g(1) = -\ln\, P(X > 1).$$

6.15 (a) 0.112.
(b) 0.472.
(c) 0.997.

6.17

$$E\left(\sum_{n=1}^{N_t} S_n^2\right) = \frac{\lambda t^3}{3}.$$

6.19 $E(T) = 88.74$.

6.25 The expected time of failure was 8:43 a.m. on the last day of the week.

6.27 (a) 0.747;
(b) 0.632;
(c) 0.20.

6.29 0.77.

6.35 $P(N_C = 0) = e^{(1-e^{-1})\pi} = 0.137.$

6.39 $P(N_1 = 1) = (e-2)/e = 0.264.$

6.41 The goal scoring Poisson process has parameter $\lambda = 2.68/90$. Consider two independent thinned processes, each with parameter $p\lambda$, where $p = 1/2$. By conditioning on the number of goals scored in a 90-minute match, the desired probability is

$$\sum_{k=0}^{\infty} \left(\frac{e^{-90\lambda/2}(90\lambda/2)^k}{k!} \right)^2 = \sum_{k=0}^{\infty} \left(\frac{e^{-1.34}1.34^k}{k!} \right)^2 = 0.259.$$

6.43 Mean and variance are 41.89.

Solutions for Chapter 7

7.3

$$\boldsymbol{Q} = \begin{pmatrix} -a & a/2 & a/2 \\ b/2 & -b & b/2 \\ c/2 & c/2 & -c \end{pmatrix}.$$

$$\boldsymbol{\pi} = \left(\frac{bc}{ac+bc+ab}, \frac{ac}{ac+bc+ab}, \frac{ab}{ac+bc+ab} \right).$$

7.7 (a)

$$\begin{aligned}
P'_{11}(t) &= -P_{11}(t) + 3P_{13}(t) \\
P'_{12}(t) &= -2P_{12}(t) + P_{11}(t) \\
P'_{13}(t) &= -3P_{13}(t) + 2P_{12}(t) \\
P'_{21}(t) &= -P_{21}(t) + 3P_{23}(t) \\
P'_{22}(t) &= -2P_{22}(t) + P_{21}(t) \\
P'_{23}(t) &= -3P_{23}(t) + 2P_{22}(t) \\
P'_{31}(t) &= -P_{31}(t) + 3P_{33}(t) \\
P'_{32}(t) &= -2P_{32}(t) + P_{31}(t) \\
P'_{33}(t) &= -3P_{33}(t) + 2P_{32}(t)
\end{aligned}$$

(b)

$$\boldsymbol{P}(t) = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 0 & 1/4 \\ -3/2 & 1 & 1/2 \\ 3/4 & 0 & 1/4 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 3 + e^{-4t} & 0 & 1 - e^{-4t} \\ 3 + 3e^{-4t} - 6e^{-2t} & 4e^{-2t} & 1 - 3e^{-4t} + 2e^{-2t} \\ 3 - 3e^{-4t} & 0 & 1 + 3e^{-4t}. \end{pmatrix}$$

7.11

$$\frac{d}{dt} e^{t\boldsymbol{A}} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n}{n!} \boldsymbol{A}^n = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{A}^n \frac{d}{dt} t^n = \sum_{n=1}^{\infty} \frac{1}{n!} \boldsymbol{A}^n n t^{n-1}$$

$$= \boldsymbol{A} \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)} \boldsymbol{A}^{n-1} = \boldsymbol{A} \sum_{n=0}^{\infty} \frac{t^n}{n!} \boldsymbol{A}^n = \boldsymbol{A} e^{t\boldsymbol{A}}.$$

The second equality is done similarly.

7.13 Taking limits on both sides of $\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{Q}$ gives that $\boldsymbol{0} = \boldsymbol{\pi}\boldsymbol{Q}$. This uses the fact that if a differentiable function $f(t)$ converges to a constant then the derivative $f'(t)$ converges to 0.

7.15 (a)

$$\boldsymbol{Q} = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \end{array}.$$

7.17 The population process is a Yule process. The distribution of X_8, the size of the population at $t = 8$, is negative binomial, with mean and variance

$$E(X_8) = 651{,}019 \quad \text{and } SD(X_8) = 325{,}509.$$

7.21 Make 4 an absorbing state. We have

$$(-\boldsymbol{V})^{-1} = \left(- \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 10 & 9 & 4 & 1 \\ 9 & 9 & 4 & 1 \\ 8 & 8 & 4 & 1 \\ 6 & 6 & 3 & 1 \end{pmatrix},$$

with row sums $(24, 23, 21, 16)$. The desired mean time is 23.

7.25 (a) $\boldsymbol{\psi} = (0.1, 0.3, 0.3, 0.3)$.
(b) We have that $(q_1, q_2, q_3, q_4) = (1, 1/2, 1/3, 1/4)$. The stationary distribution $\boldsymbol{\pi}$ is proportional to $(0.1, 0.3(2), 0.3(3), 0.3(4))$. This gives

$$\boldsymbol{\psi} = \frac{1}{2.8}(0.1, 0.6, 0.9, 1.2) = (0.036, 0.214, 0.321, 0.428).$$

7.27 (a) If the first dog has i fleas, then the number of fleas on the dog increases by one the first time that one of the $N - i$ fleas on the other dog jumps. The time of that jump is the minimum of $N - i$ independent exponential random variables with parameter λ. Similarly, the number of fleas on the first dog decreases by one when one of the i fleas on that dog first jumps.
(b) The local balance equations are $\pi_i(N - i)\lambda = \pi_{i+1}(i + 1)\lambda$. The equations are satisfied by the stationary distribution

$$\pi_k = \binom{N}{k}\left(\frac{1}{2}\right)^k, \text{ for } k = 0, 1, \ldots, N,$$

which is a binomial distribution with parameters N and $p = 1/2$.
(c) 0.45 minutes.

7.29 (b) The embedded chain transition matrix, in canonical form, is

$$\tilde{\boldsymbol{P}} = \begin{array}{c} \\ 1 \\ 5 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 5 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 3/5 & 0 & 2/5 & 0 \end{pmatrix} \end{array}.$$

By the discrete-time theory for absorbing Markov chains, write

$$\tilde{\boldsymbol{Q}} = \begin{pmatrix} 0 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 2/5 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{R}} = \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \\ 0 & 3/5 \end{pmatrix}.$$

The matrix of absorption probabilities is

$$(\boldsymbol{I} - \tilde{\boldsymbol{Q}})^{-1}\tilde{\boldsymbol{R}} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 5 \end{array} \\ \begin{pmatrix} 4/7 & 3/7 \\ 5/14 & 9/14 \\ 1/7 & 6/7 \end{pmatrix} \end{array}.$$

The desired probability is $3/7$.

7.31 The process is an M/M/2 queue with $\lambda = 2$, $\mu = 3$, and $c = 2$. The desired probability is

$$\pi_0 = \left(1 + \frac{2}{3} + \frac{1}{3}\right)^{-1} = \frac{1}{2}$$

7.33 (a) The long-term expected number of customers in the queue L is the mean of a geometric distribution on $0, 1, 2, \ldots$, with parameter $1 - \lambda/\mu$, which is $\lambda/(\mu - \lambda)$. If both λ and μ increase by a factor of k, this does not change the value of L.
(b) The expected waiting time is $W = L/\lambda$. The new waiting time is $L/(k\lambda) = W/k$.

7.37 (c) Choose N such that $P(Y > N) < 0.5 \times 10^{-3}$, where Y is a Poisson random variable with parameter $9 \times 0.8 = 7.2$. This gives $N = 17$.

Solutions for Chapter 8

8.3 (a) 0.013.
(b) $f_{X_2|X_1}(x|0) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for } -\infty < x < \infty.$
(c) 3.
(d) X_1.

8.5 (a) For the joint density of B_s and B_t, since

$$\{B_s = x, B_t = y\} = \{B_s = x, B_t - B_s = y - x\},$$

it follows that

$$f_{B_s,B_t}(x, y) = f_{B_s}(x) f_{B_{t-s}}(y - x) = \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} \frac{1}{\sqrt{2\pi(t - s)}} e^{-(y-x)^2/2(t-s)}.$$

(b) $E(B_s|B_t = y) = sy/t$ and $Var(B_s|B_t = y) = s(t - s)/t$.

8.7 One checks that the reflection is a Gaussian process with continuous paths. Furthermore, the mean function is $E(-B_t) = 0$ and the covariance function is

$$E((-B_s)(-B_t)) = E(B_s B_t) = \min\{s, t\}$$

8.11 0.11.

8.13 $E(X_s X_t) = st\mu^2 + \sigma^2 s.$

8.19 $E(M_t) = \sqrt{\frac{2t}{\pi}}$ and $Var(M_t) = (1 - 2/\pi)t.$

8.23 (a) $\arcsin(\sqrt{r/t}) / \arcsin(\sqrt{r/s}).$
(b) $\sqrt{s}/\sqrt{t}.$

8.25 0.1688.

8.27 Write $Z_{n+1} = \sum_{i=1}^{Z_n} X_i$. Then,

$$\begin{aligned} E\left(\frac{Z_{n+1}}{\mu^{n+1}} | Z_n, \dots, Z_0\right) &= \frac{1}{\mu^{n+1}} E\left(\sum_{i=1}^{Z_n} X_i | Z_n, \dots, Z_0\right) \\ &= \frac{1}{\mu^{n+1}} E\left(\sum_{i=1}^{Z_n} X_i | Z_n\right) \\ &= \frac{1}{\mu^{n+1}} Z^n \mu = \frac{Z_n}{\mu^n}. \end{aligned}$$

8.31 (a) $4/9$.
(b) 20.

8.35 $SD(T) = \sqrt{2/3}a^2$.

8.43 (a) Black–Scholes price is \$35.32.
(b) Price is increasing in each of the parameters, except strike price, which is decreasing.
(c) $\sigma^2 \approx 0.211$.

Solutions for Chapter 9

9.1 The distribution is normal, with

$$E\left(\int_0^t sB_s\, ds\right) = \int_0^t sE(B_s)\, ds = 0,$$

and

$$\begin{aligned}
Var\left(\int_0^t sB_s\, ds\right) &= E\left(\left(\int_0^t sB_s\, ds\right)^2\right) = \int_{x=0}^t \int_{y=0}^t E(xB_x yB_y)\, dy\, dx \\
&= \int_{x=0}^t \int_{y=0}^x xyE(B_xB_y)\, dy\, dx + \int_{x=0}^t \int_{y=x}^t xyE(B_xB_y)\, dy\, dx \\
&= \int_{x=0}^t \int_{y=0}^x xy^2\, dy\, dx + \int_{x=0}^t \int_{y=x}^t x^2y\, dy\, dx \\
&= \int_{x=0}^t \frac{x^4}{3}\, dx + \int_{x=0}^t x^2\left(\frac{t^2}{2} - \frac{x^2}{2}\right)\, dx \\
&= \frac{t^5}{15} + \frac{t^5}{6} - \frac{t^5}{10} = \frac{2t^5}{15}.
\end{aligned}$$

9.3 By Ito's Lemma, with $g(t, x) = x^4$,

$$d(B_t^4) = 6B_t^2 dt + 4B_t^3\, dB_t,$$

which gives

$$B_t^4 = 6\int_0^t B_s^2\, ds + 4\int_0^t B_s^3\, dB_s,$$

and

$$E(B_t^4) = 6\int_0^t E(B_s^2)\, ds = 6\int_0^t s\, ds = 3t^2.$$

9.5 The desired martingale is $B_t^4 - 6tB_t^2 + 3t^2$.

9.9 (b) $E(X_3) = 4$; $Var(X_3) = 30$; $P(X_3 < 5) = 0.7314$.