Math 236 Algebra 2 Assignment 6

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Problem 1a.

- $T(1,0,0) = (1,3) = 1*(1,0) + 3*(0,1)$
- $T(0,1,0) = (1,0) = 1*(1,0) + 0*(0,1)$
- $T(0,0,1) = (1,-2) = 1*(1,0) - 2*(0,1)$

$$\implies M(T) = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \end{pmatrix}$$

Problem 1b.

- $T(1,0,0) = (1,3) = -1*(1,1) + 2*(1,2)$
- $T(0,1,0) = (1,0) = 2*(1,1) - 1*(1,2)$
- $T(0,0,1) = (1,-2) = 4*(1,1) - 3*(1,2)$

$$\implies M(T) = \begin{pmatrix} -1 & 2 & 4 \\ 2 & -1 & -3 \end{pmatrix}$$

Problem 2. For a vector $v \in ker f$, consider the conditions on $f(v)$:

$$3x_3 - 2x_4 = 0 \implies x_4 = \frac{3}{2}x_3$$

$$x_1 - 2x_3 + x_4 = 0 \implies x_4 = 2x_3 - x_1$$

$$\implies x_3 = 2x_1$$

From these conditions it can be seen that v is of the form $(a, b, 2a, 3a)$ where $a, b \in \mathbb{F}$. Consider the set of vectors,

$$\{(1,0,2,3), (1,1,2,3)\}$$

Let $a, b \in \mathbb{F}$.

$$a(1,0,2,3) + b(1,1,2,3) = (a+b, b, 2a+2b, 3a+3b) = (a+b, b, 2(a+b), 3(a+b))$$

Therefore $\{(1,0,2,3),(1,1,2,3)\}$ spans the vector space. Solve $a(1,0,2,3)+b(1,1,2,3)=0$

$$a(1,0,2,3)+b(1,1,2,3)=(a+b,b,2(a+b),3(a+b))$$
$$\implies b=0$$
$$\implies a=0$$

Therefore $\{(1,0,2,3),(1,1,2,3)\}$ are linearly independent. It follows that $\{(1,0,2,3),(1,1,2,3)\}$ is a basis for $ker\ f$. We conclude $dim\ kerf=2$ From the fundamental theorem of linear maps:

$$dim\mathbb{F}^4 = dim\ kerf + dim\ rangef$$
$$4 = 2 + dim\ rangef \implies dim\ rangef = 2$$

Since f maps to $\mathbb{F}^2$ and $\mathbb{F}^2$ has dimension 2, it follows that the standard basis is a basis for$rangef$. In other words $\{(1,0),(0,1)\}$ is a basis for $rangef$.

Problem 3. Assume that $ker(T) = range(T)$, therefore $dim\ \ ker(T) = dim\ \ range(T)$. From the fundamental theorem of linear maps,

$$dimV = dim\ ker(T) + dim\ range(T) \Leftrightarrow dimV = 2\,dim\ ker(T)$$

Since the dimension of V is odd this is a contradiction. Therefore there does not exist a linear map such that $ker(T) = range(T)$.

Problem 4. Consider $\{1, 2x, 3x^2\}$, this is a basis of $\mathcal{P}_2(\mathbb{R})$ and $\{x, x^2, x^3, 1\}$, this is a basis of $\mathcal{P}_3(\mathbb{R})$. It follows,

$$D(x) = 1 = 1(1) + 0(2x) + 0(3x^2)$$
$$D(x^2) = 2x = 0(1) + 1(2x) + 0(3x^2)$$
$$D(x^3) = 3x^2 = 0(1) + 0(2x) + 1(3x^2)$$
$$D(1) = 0 = 0(1) + 0(2x) + 0(3x^2)$$

$$M(D) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Problem 5a.

- $0 \in E$
- Suppose $T, S \in E$. $(T+S)v = Tv + Sv = 0 + 0 = 0$. Therefore $T + S \in E$
- Suppose $\lambda \in \mathbb{F}, T \in E$. $(\lambda T)v = \lambda(Tv) = \lambda(0) = 0$. Therefore $\lambda T \in E$

Therefore E is subspace.