Math 316 Assignment 1

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September 20, 2018

Problem 1. Suppose $|z| = 1$. Let $y = \frac{z-1}{z+1}$,

$$\overline{y} = \overline{\frac{z-1}{z+1}} = \frac{\overline{z-1}}{\overline{z+1}} = \frac{\overline{z}-1}{\overline{z}+1}$$

Now,

$$\begin{aligned} y + \overline{y} &= \frac{z-1}{z+1} + \frac{\overline{z}-1}{\overline{z}+1} \\ &= \frac{z-1}{z+1} \cdot \frac{\overline{z}+1}{\overline{z}+1} + \frac{\overline{z}-1}{\overline{z}+1} \cdot \frac{z+1}{z+1} \\ &= \frac{z\bar{z} - \bar{z} + z - 1 + z\bar{z} - z + \bar{z} - 1}{(z+1)(\bar{z}+1)} \\ &= \frac{2(|z|^2 - 1)}{(z+1)(\bar{z}+1)} \end{aligned}$$

Using $|z| = 1$,

$$\begin{aligned} &= \frac{2(1^2 - 1)}{(z+1)(\bar{z}+1)} \\ &= 0 \end{aligned}$$

Since $y + \overline{y} = 2Re(y)$, we have $2Re(y) = 0$ and therefore we conclude that $y = \frac{z-1}{z+1}$ is purely imaginary.

Now, suppose $\frac{z-1}{z+1}$ is purely imaginary. Let $y = \frac{z-1}{z+1}$. From above we have,

$$y + \overline{y} = \frac{2(|z|^2 - 1)}{(z+1)(\bar{z}+1)} = 2Re(y)$$

Since y is purely imaginary, $Re(y) = 0$

$$\implies \frac{2(|z|^2 - 1)}{(z+1)(\bar{z}+1)} = 0$$

$$\implies |z|^2 - 1 = 0$$

Since $|\cdot| \geq 0$

$$\implies |z| = 1$$

Therefore $|z| = 1$ if and only if $\frac{z-1}{z+1}$ is purely imaginary

Problem 2. Clearly $z \neq 0$, therefore,

$$\left(\frac{z-1}{z}\right)^n = 1$$

Further we have,

$$1 = cos(n\theta) + isin(n\theta) \quad \text{when } n\theta = 0 \bmod 2\pi$$

Therefore when $k \in \{0, 1, ..., n-1\}$ we have,

$$1 - \frac{1}{z} = (\cos(2k\pi) + i\sin(2k\pi))^{1/n}$$

$$= \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})$$

It follows,

$$z = \frac{1}{1 - \cos(\frac{2k\pi}{n}) - i\sin(\frac{2k\pi}{n})}$$

$$= \frac{1}{1 - \cos(\frac{2k\pi}{n}) - i\sin(\frac{2k\pi}{n})} \cdot \frac{1 - \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})}{1 - \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})}$$

$$= \frac{1 - \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})}{\left(1 - \cos(\frac{2k\pi}{n})\right)^2 + \sin^2(\frac{2k\pi}{n})}$$

$$= \frac{1 - \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})}{1 - 2\cos(\frac{2k\pi}{n}) + \cos^2(\frac{2k\pi}{n}) \ + sin^2(\frac{2k\pi}{n})}$$

$$= \frac{1 - \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})}{2(1 - \cos(\frac{2k\pi}{n}))}$$

$$= \frac{1}{2} + i\frac{\sin(\frac{2k\pi}{n})}{2(1 - \cos(\frac{2k\pi}{n}))}$$

Problem 3.

$$\overline{\phi} = \overline{\frac{w-z}{1-\overline{w}z}} = \frac{\overline{w-z}}{\overline{1-\overline{w}z}} = \frac{\overline{w}-\overline{z}}{1-w\overline{z}}$$

Now consider,

$$\phi \cdot \overline{\phi} = \frac{w-z}{1-\overline{w}z} \cdot \frac{\overline{w}-\overline{z}}{1-w\overline{z}}$$

$$= \frac{w\overline{w} - w\overline{z} - \overline{w}z + z\overline{z}}{1 - w\overline{z} - \overline{w}z + w\overline{w}z\overline{z}}$$

$$= \frac{|w|^2 + |z|^2 - w\overline{z} - \overline{w}z}{1 + |w|^2|z|^2 - w\overline{z} - \overline{w}z}$$

if $|z| = 1$ we have,

$$\phi \cdot \overline{\phi} = \frac{1 + |w|^2 - w\overline{z} - \overline{w}z}{1 + |w|^2 - w\overline{z} - \overline{w}z} = 1$$

now since $\phi \cdot \overline{\phi} = |\phi|^2$,

$$\implies |\phi|^2 = 1$$

Since $|\cdot| \geq 0$

$$\implies |\phi| = 1$$

Therefore $|\phi| = 1$ whenever $|z| = 1$.

Now if $|z| < 1$

$$\implies |z|^2 < 1$$

We know that $|w| < 1 \implies |w|^2 < 1$ and therefore $|w|^2 - 1 < 0$, it follows

$$|w|^2 - 1 < |z|^2(|w|^2 - 1)$$

$$\implies |w|^2 - 1 < |w|^2|z|^2 - |z|^2$$

$$\implies |w|^2 + |z|^2 < 1 + |w|^2|z|^2$$

$$|w|^2 + |z|^2 - w\overline{z} - \overline{w}z < 1 + |w|^2|z|^2 - w\overline{z} - \overline{w}z$$

$$\implies \phi \cdot \overline{\phi} = \frac{|w|^2 + |z|^2 - w\overline{z} - \overline{w}z}{1 + |w|^2|z|^2 - w\overline{z} - \overline{w}z} < 1$$

now since $\phi \cdot \overline{\phi} = |\phi|^2$,

$$\implies |\phi|^2 < 1$$

Since $|\cdot| \geq 0$

$$\implies |\phi| < 1$$

Therefore $|\phi| < 1$ whenever $|z| < 1$.

Problem 4.

$$\sum_{n \geq 0} \frac{cos(n\theta)}{2^n} = Re\Big(\sum_{n \geq 0} \frac{e^{ni\theta}}{2^n}\Big)$$

Considering the new sequence,

$$\sum_{n \geq 0} \frac{e^{ni\theta}}{2^n} = \sum_{n \geq 0} \left(\frac{e^{i\theta}}{2}\right)^n$$

Assuming $|\frac{e^{i\theta}}{2}| < 1$ (otherwise the series diverges),

$$= \frac{1}{1 - \frac{e^{i\theta}}{2}}$$

$$= \frac{2}{2 - e^{i\theta}}$$

$$= \frac{2}{2 - cos\theta - isin\theta}$$
$$= \frac{4 - 2cos\theta + 2isin\theta}{5 - 4cos\theta}$$

Therefore,

$$Re\Big(\sum_{n\geq 0} \frac{e^{ni\theta}}{2^n}\Big) = \frac{4 - 2cos\theta}{5 - 4cos\theta}$$
$$\implies \sum_{n\geq 0} \frac{cos(n\theta)}{2^n} = \frac{4 - 2cos\theta}{5 - 4cos\theta}$$

Problem 5i. Base case:

$$n = 0 : T_0(x) = 1,\ \ deg(1) = 0$$
$$n = 1 : T_1(x) = x,\ \ deg(x) = 1$$

Therefore $T_0(x)$ has degree 0 and $T_1(x)$ has degree 1

Induction step: Assume $T_{n-1}(x)$ and $T_{n-2}(x)$ have degree $n-1$ and $n-2$ respectively.

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$$

Therefore,

$$deg(T_n(x)) = deg(2xT_{n-1}(x) - T_{n-2}(x))$$

Since $deg(2x) = 1$ and $deg(T_{n-1}(x)) = n-1$, we have

$$deg(2xT_{n-1}(x)) = n$$
$$\implies deg(2xT_{n-1}(x) - T_{n-2}(x)) = n$$

Therefore $T_n(x)$ has degree n.

Problem 5ii. Base case:

$$n = 0 : T_0(cos\theta) = 1 = cos(0) = cos(0 \cdot \theta)$$
$$n = 1 : T_1(cos\theta) = cos\theta = cos(1 \cdot \theta)$$

Induction step: Assume $T_{n-1}(cos\theta) = cos(n-1)\theta$ and $T_{n-2}(cos\theta) = cos(n-2)\theta$.

$$T_n(cos\theta) = 2cos\theta \cdot cos(n-1)\theta - cos(n-2)\theta$$
$$= 2\Big[\frac{1}{2}(e^{i\theta} + e^{-i\theta})\Big]\Big[\frac{1}{2}(e^{i(n-1)\theta} + e^{-i(n-1)\theta})\Big] - \Big[\frac{1}{2}(e^{i(n-2)\theta} + e^{-i(n-2)\theta})\Big]$$
$$\frac{1}{2}\Big[e^{in\theta} + e^{i(n-2)\theta} + e^{-i(n-2)\theta} + e^{-in\theta}\Big] - \frac{1}{2}\Big[e^{i(n-2)\theta} + e^{-i(n-2)\theta}\Big]$$
$$= \frac{1}{2}\Big[e^{in\theta} + e^{-in\theta}\Big]$$
$$= cos(n\theta)$$

Therefore $T_n(cos\theta) = cos(n\theta)$.

Problem 5iii. For $x \in [-1, 1]$ $\exists \theta$ such that $x = cos\theta$. Therefore let $x = cos\theta$.

$$\begin{aligned} T_m(T_n(x)) &= T_m(T_n(cos\theta)) \\ &= T_m(cos(n\theta)) \\ &= cos(mn\theta) \\ &= T_{mn}(cos\theta) \\ &= T_{mn}(x) \end{aligned}$$

Therefore for $x \in [-1, 1]$, $T_m(T_n(x)) = T_{mn}(x)$. Since $[-1, 1]$ is an infinite set, $T_m(T_n(x)) = T_{mn}(x)$ is satisfied everywhere.

Problem 6i.
Let $U_1, U_2, ...$ be any number of open subsets of $\mathbb{C}$. Let $z \in \mathbb{C}$ and suppose $z \in \bigcup_{i\geq 1} U_i$. Therefore, $\exists i \in \{1, 2, ...\}$ such that $z \in U_i$. Since U_i is open $\exists r > 0$ such that $D(z, r) \subseteq U_i$.

$$\implies D(z, r) \subseteq U_i \subseteq \bigcup_{i\geq 1} U_i$$

Since z is an arbitrary point in the union, we conclude that the union of arbitrarily many open subsets is an open subset.

Problem 6ii.
Let $U_1, U_2, ..., U_n$ be a finite number of open subsets of $\mathbb{C}$. Let $z \in \mathbb{C}$ and suppose $z \in \bigcap_{i=1}^{n} U_i$

$$\implies z \in U_i \quad \forall i \in \{1, 2, ..., n\}$$

Therefore since U_i is open, $\exists r_i > 0$ such that $D(z, r_i) \subseteq U_i$. Let $r_{min} = min(r_1, r_2, ..., r_n)$.

$$\implies D(z, r_{min}) \subseteq D(z, r_i) \subseteq U_i \quad \forall i \in \{1, 2, ..., n\}$$

$$\implies D(z, r_{min}) \subseteq \bigcap_{i=1}^{n} U_i$$

Since z in an arbitrary point in the intersection, we conclude that the intersection of finitely many open subsets is an open subset.

Problem 6iii.
Let $U_1, U_2, ..., U_n$ be a finite number of closed subsets of $\mathbb{C}$. Now consider $\left(\bigcup_{i=1}^{n} U_i\right)^c$. Using De Morgan's Laws (for sets) we get,

$$\left(\bigcup_{i=1}^{n} U_i\right)^c = \bigcap_{i=1}^{n} U_i^c$$

Where U_i^c is the compliment of the closed subset U_i and is therefore an open subset of $\mathbb{C}$. Therefore, using part ii we get that $\bigcap_{i=1}^{n} U_i^c$ is an open subset and therefore $\left(\bigcup_{i=1}^{n} U_i\right)^c$ is also an open subset. Finally since $\bigcup_{i=1}^{n} U_i$ is the compliment of the open subset $\left(\bigcup_{i=1}^{n} U_i\right)^c$ we conclude that $\bigcup_{i=1}^{n} U_i$ is a closed subset.

Therefore the union of finitely many closed subsets is a closed subset.

Problem 6iv.

Let $U_1, U_2, ...$ be any number of closed subsets of $\mathbb{C}$. Now consider $\left(\bigcap_{i\geq 1} U_i\right)^c$. Using De Morgan's Laws (for sets) we get,

$$\left(\bigcap_{i\geq 1} U_i\right)^c = \bigcup_{i\geq 1} U_i^c$$

Where U_i^c is the compliment of the closed subset U_i and is therefore an open subset of $\mathbb{C}$. Therefore, using part i we get that $\bigcup_{i\geq 1} U_i^c$ is an open subset and therefore $\left(\bigcap_{i\geq 1} U_i\right)^c$ is also an open subset. Finally since $\bigcap_{i\geq 1} U_i$ is the compliment of the open subset $\left(\bigcap_{i\geq 1} U_i\right)^c$ we conclude that $\bigcap_{i\geq 1} U_i$ is a closed subset.

Therefore the intersection of arbitrarily many closed subsets is a closed subset.