Math 256 Winter 2017

Due Friday, February 3

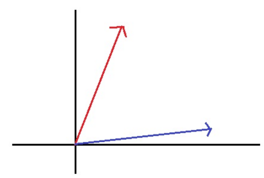
1. Consider the vector space $\mathbb{R}^3$ over $\mathbb{R}$. Let $B = ((1,0,1),(0,1,1),(1,1,0))$ and $C = ((4,3,3),(1,2,1),(-3,-1,5))$ be two bases.

   (a) Let $v = (1,0,1)$, expressed in the standard basis for $\mathbb{R}^3$. Find $[v]_B$ and $[v]_C$.

   (b) Find the change-of-basis matrix ${}_BM_C$.

   (c) Verify that $[v]_B = {}_B M_C[v]_C$.

2. Consider two linear transformations T and L from in $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ with the property that $T(v_1) = L(v_1)$ and $T(v_2) = L(v_2)$ for the vectors v_1 and v_2 sketched below. Prove that $L(v) = T(v)$ for all vectors $v \in \mathbb{R}^2$.



3. Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x,y,z) = (5x - 2y + 3z + b, 9y + cxz).$$

   Show that T is linear if and only if $b = c = 0$. (Be sure to prove both directions!)

4. (Ax 3.A.4) Suppose $T \in \mathcal{L}(V,W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

5. (Ax 3.B.2) Suppose V is vector space and $S, T \in \mathcal{L}(V,V)$ are such that

$$\text{range}(S) \subset \ker(T).$$

   Prove $(ST)^2 = 0$.

6. Suppose that $T \in \mathcal{L}(V,W)$ is injective and $v_1, \dots, v_n$ in linearly independent in V. Prove that $Tv_1, \dots, Tv_n$ is linearly independent in W.

7. Let V be a finite-dimensional vector space. A linear map $P : V \to V$ is called *idempotent* if $P \circ P = P$. (In other words, $P(P(v)) = P(v)$.) Prove that if P is idempotent,

$$V = \ker(P) \oplus \text{range}(P).$$

   (Hint: One way to do this is to first show that $\ker(P) \cap \text{range}(P) = \{0\}$. Then compare the dim-sum formula to the Fundamental Theorem of Linear Maps to conclude $\ker(P) + \text{range}(P) = V$.

8. **This question is optional. You do not have to hand it in.** Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.