Assignment 5: Linear Systems
Math 327/397 Winter 2019

No Due Date

Instructions

This is a problem sheet covering a portion of the final exam. It is not to be submitted nor graded.

Question 1: LU Decomposition

Let
$$A = \begin{bmatrix} 4 & 2 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & -1 & 2 & 1 \\ 0 & 0.6 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 16 \\ 16 \\ 8 \\ 4 \end{bmatrix}.$$

(a) Find matrices L_1, L_2, L_3 such that $L_3 L_2 L_1 A = U$, where U is upper triangular.

(b) State the matrix L which is lower triangular such that $A = LU$.

Question 2: Forward and Backward Substitution

For the matrix A and vector b in Question 1:

(a) Solve $Ly = b$ to find y by forward substitution.

(b) Solve $Uz = y$ by backward substitution to find z.

(c) What is the solution x of $Ax = b$?

Question 3:Symmetric Positive Definite Matrices

For a symmetric $A \in \mathbb{R}^{n\times n}$, we define A to be positive definite if $x^T Ax > 0$ for every $x \in \mathbb{R}^n$, $x \neq 0$. Show the following.

(a) A real symmetric matrix A is positive definite if and only if all its eigenvalues are positive.

(b) For any $B \in \mathbb{R}^{m\times n}$, if B has full column rank then $B^T B$ is symmetric positive definite. What happens is B is not full column rank?

Question 4: Cholesky Decomposition

For the following matrices either find (without using a computer) the Cholesky factorization $A = R^T R$ of A, if A is positive definite, or show that A is not positive definite by demonstrating that the Cholesky factorization algorithm fails.

$$(a) \qquad A = \begin{bmatrix} 9 & 3 & -3 \\ 3 & 5 & 1 \\ -3 & 1 & 11 \end{bmatrix}, \qquad (b) \qquad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Question 5: Sums of Polynomials

Consider the problem of finding a formula for the sum of squares and suppose we know the form of the solution:

$$a_0 + a_1 n + a_2 n^2 + a_3 n^3 = \sum_{j=0}^{n} j^2$$

where the a_j are unknown.

(a) Using the column vector $[a_0, a_1, a_2, a_3]^T$ to represent to the polynomial on the left, write the equations for the above for $n = 0, 1, 2, 3$ into a linear system of the form $Ax = b$.

(b) Solve the system. What is the formula for summing the squares?

(c) Can you generalize this procedure to polynomials of arbitrary degree?

Question 6 (Math 397): Sums of Polynomials

In Question 5 we assumed the form of the solution. Here, show that if p is a polynomial of degree k, then there exists a polynomial q of degree $k + 1$ such that

$$q(n) = \sum_{j=0}^{n} p(j).$$