Math 243 Analysis 2 Assignment 4

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Problem 2. Let $\epsilon > 0$. Define α and ω on the interval [0,1] as follows,

$$\alpha(x) = 0$$

$$\omega(x) = \begin{cases} 1, & \text{if } x \in [0, \epsilon] \\ f(x), & \text{if } x \in (\epsilon, 1] \end{cases}$$

Note that $\alpha \leq f \leq \omega \ \forall x \in [0, 1]$. α is a step function and therefore Riemann Integrable. ω is a function of two parts, the first being when $\omega = 1$ for $x \in [0, \epsilon]$, therefore the first section of ω is Riemann Integrable. The second part is when $\omega = f$ for $x \in (\epsilon, 1]$. By the Archimedean Property there is a natural number N such that $\frac{1}{N} < e$. Therefore there are a finite number of points where $x = \frac{1}{q}$ and $1 \leq q \leq N$. In other words the second part of ω is constant at 0 except for finitely many points where $x = \frac{1}{q}$, therefore this section of the function is Riemann Integrable and its integral is equal to that of a function that is constantly equal to 0. By the additivity property of Riemann Integrals ω is Riemann Integrable on [0,1] and,

$$\int_0^1 \omega = \int_0^\epsilon \omega + \int_\epsilon^1 \omega = 1 * \epsilon + 0 * (1 - \epsilon) = \epsilon$$

As well,

$$\int_0^1 \alpha = 0 * 1 = 0$$

Therefore,

$$\int_0^1 \omega - \alpha = \int_0^1 \omega - \int_0^1 \alpha = \epsilon - 0 = \epsilon$$

We conclude that f is Riemann Integrable on [0,1]. Next for all $\epsilon > 0$,

$$0 \leq \int_0^1 f \leq \epsilon$$

$$\implies \int_0^1 f = 0.$$

Problem 3b. Let $\epsilon > 0$. Divide [0,1] into n subintervals of equal width $= \frac{1}{n}$ where $n > \frac{2}{\epsilon}$. Define α and ω as follows,

$$\alpha(x) = \begin{cases} \frac{(i-1)^2}{n^2}, & \text{if } x \in [\frac{i-1}{n}, \frac{i}{n}] \\ 1, & \text{if x=1} \end{cases}$$

and,

$$\omega(x) = \begin{cases} \frac{i^2}{n^2}, & \text{if } x \in [\frac{i-1}{n}, \frac{i}{n}] \\ 1, & \text{if x=1} \end{cases}$$

α and ω are step functions and thus Riemann Integrable on [0,1] and $\alpha \leq f \leq \omega$ $\forall x \in [0,1]$.

$$\int_0^1 \omega - \alpha = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - i^2 + 2i - 1}{n^2} = \frac{2i-1}{n^2} \leq \frac{2n-1}{n^2} \leq \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

Therefore f is Riemann Integrable on [0,1]. Further,

$$\int_0^1 \alpha = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} * \frac{1}{n} = \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 - 2i + 1 = \frac{1}{n^3} \Big[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\Big]$$

$$= \frac{1}{6n^2}\Big[(n+1)(2n+1) - 6(n+1) + 6\Big] = \frac{1}{6n^2}\Big[2n^2 + 3n + 1 - 6n - 6 + 6\Big] = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$$\int_0^1 \omega = \sum_{i=1}^{n} \frac{i^2}{n^2} * \frac{1}{n} = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 = \frac{1}{n^3} \Big[\frac{n(n+1)(2n+1)}{6}\Big] = \frac{1}{6n^2}\Big[(n+1)(2n+1)\Big]$$

$$= \frac{1}{6n^2}\Big[2n^2 + 3n + 1\Big] = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

It follows $\forall n \in \mathbb{N}$,

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \leq \int_0^1 f \leq \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Therefore by the squeeze theorem,

$$\int_0^1 f = \frac{1}{3}$$