Math 236 Algebra 2 Assignment 3

Jonathan Pearce, 260672004

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Problem 1. $span(v_1, v_2, v_3, v_4) = \{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_i \in \mathbb{F}\ \forall 1 \leq i \leq 4\}$
Consider $span(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$

$$= \{b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 : b_i \in \mathbb{F}\ \forall 1 \leq i \leq 4\}$$

$$= \{b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 : b_i \in \mathbb{F}\ \forall 1 \leq i \leq 4\}$$

Now suppose,

$$b_1 := a_1,\ b_2 := a_2 + a_1,\ b_3 := a_3 + a_2 + a_1,\ b_4 := a_4 + a_3 + a_2 + a_1$$

Then,

$$span(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) = \{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_1, a_2, a_3, a_4 \in \mathbb{F}\}$$

$$\implies span(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4) = span(v_1, v_2, v_3, v_4)$$

$$\implies v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\ \ spans\ \ V$$

Problem 2. Consider $3x + y = 2$ and $x + 5y = 6$ Then,

$$y = 2 - 3x$$

$$x + 5(2 - 3x) = 6 \implies x = \frac{2}{7}$$

$$3x + y = 2 \Leftrightarrow 3\frac{2}{7} + y = 2 \implies y = \frac{8}{7}$$

$$t = 4\frac{2}{7} + 9\frac{8}{7} = \frac{80}{7}$$

Therefore if $t = \frac{80}{7}$ then $(3, 1, 4), (1, 5, 9), (2, 6, \frac{80}{7})$ are not linearly independent in $\mathbb{R}^3$.
Why is the set not linearly independent?

$$\frac{2}{7}(3, 1, 4) + \frac{8}{7}(1, 5, 9) + (-1)(2, 6, \frac{80}{7}) = (0, 0, 0)$$

Since the coefficients are non zero, this implies that $(3, 1, 4), (1, 5, 9), (2, 6, \frac{80}{7})$ are not linearly independent in $\mathbb{R}^3$.

Problem 3a. Prove $(1+i, 1-i)$ is linearly independent. Suppose $a, b \in \mathbb{R}$, now solve

$$a(1+i) + b(1-i) = 0$$

$$a(1+i) + b(1-i) = a + ai + b - bi = (a+b) + i(a-b) = 0$$

$$\implies a + b = 0 \, and \, a - b = 0 \implies a = 0, b = 0$$

Therefore $(1+i, 1-i)$ is linearly independent.

Problem 3b. Prove $(1+i, 1-i)$ is linearly dependent. Suppose $a+bi, c+di \in \mathbb{C}$, now solve

$$(a+bi)(1+i) + (c+di)(1-i) = 0$$

$$(a+bi)(1+i)+(c+di)(1-i) = a+ai+bi-b+c-ci+di+d = (a-b+c+d)+i(a+b-c+d) = 0$$

Suppose $a = 0, d = 0, c = 1, b = 1$ These values satisfy the above expression. Therefore $(1+i, 1-i)$ is linearly dependent.

Problem 4. Take $a_1, a_2, a_3, a_4 \in \mathbb{F}$. Suppose $a_1(v_1-v_2)+a_2(v_2-v_3)+a_3(v_3-v_4)+a_4v_4 = 0$. Then $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ are linearly independent in V if and only if $a_1 = a_2 = a_3 = a_4 = 0$ provides the only solution to the above equation.

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = a_1v_1 - a_1v_2 + a_2v_2 - a_2v_3 + a_3v_3 - a_3v_4 + a_4v_4$$

$$= a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0$$

Since v_1, v_2, v_3, v_4 are linearly independent in V

$$\implies a_1 = 0, a_2 - a_1 = 0, a_3 - a_2 = 0, a_4 - a_3 = 0$$

$$\implies a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$$

Therefore $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent in V.

Problem 5. Take $a_1, a_2, ..., a_m \in \mathbb{F}$. Suppose $a_1\lambda v_1 + a_2\lambda v_2 + ... + a_m\lambda v_m = 0$. Then $\lambda v_1, \lambda v_2, ..., \lambda v_m$ are linearly independent in V if and only if $a_1 = a_2 = ... = a_m = 0$ provides the only solution to the above equation.

$$a_1\lambda v_1 + a_2\lambda v_2 + ... + a_m\lambda v_m = (a_1\lambda)v_1 + (a_2\lambda)v_2 + ... + (a_m\lambda)v_m$$

Since $v_1, v_2, ..., v_m$ are linearly independent in V

$$\implies a_1\lambda = a_2\lambda = ... = a_m\lambda = 0$$

Since $\lambda \neq 0$

$$\implies a_1 = a_2 = ... = a_m = 0$$

Therefore $\lambda v_1, \lambda v_2, ..., \lambda v_m$ are linearly independent in V.

Problem 6 and 7 omitted on purpose.