Assignment 4: SVD, EVD
Math 327/397 Winter 2019

Due: Monday April 8, 2019

Instructions

Submit a complete paper copy of your solutions in class or to the math department by 4:30pm on the due date. Questions 1 – 5 are for everyone. Question 6 is for Math 397 only.

Question 1: Geometry of the SVD

(a) A k-dimensional ellipse, surface and interior, with axes along the standard coordinates is algebraically defined as the set of points $z = (z_1, z_2, \ldots, z_k)^T$ satisfying $\left(\frac{z_1}{\alpha_1}\right)^2 + \ldots + \left(\frac{z_k}{\alpha_k}\right)^2 \leq 1$. Note that we can have a k-dimensional ellipse embedded inside $\mathbb{R}^n$ even in the case $n > k$ by allowing some of the z_j to be identically zero. Using these definitions, show that the matrix $\Sigma = diag(\sigma_1, \ldots, \sigma_{\min(m,n)}) \in \mathbb{R}^{m \times n}$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$, maps the unit sphere $\{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$, surface and interior, to an ellipse. Under what conditions is the surface of the unit sphere mapped to the surface of the ellipse? (Suggestion: Consider the cases $m \geq n$ and $n > m$ separately. Also, some of the axes of the ellipse may be zero, so it may be convenient to introduce $r \leq \min(m, n)$ such that $\sigma_1 \geq \ldots \geq \sigma_r > 0$.)

(b) Write a matlab function *plotMatrixImage(A,N,n)* which does the following:

- partitions the unit 2-D unit circle into N equally spaced angles, computes the x and y values for these and plots the circle.
- plots the image of that circle under multiplication by the 2-by-2 matrix A.
- generates n random points drawn from the normal distribution using *randn(2,n)*, plots them and also plots their image under multiplication by A.

Show output for *plotMatrixImage(A,20,100)*, *plotMatrixImage(A,20,1000)*, and *plotMatrixImage(A,20,10000)* for the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. This should give you three different figures where each time the function is run you plot all elements on a single figure, using the same scaling for the x and y axes.

Question 2: Compression with SVD - Application

Download the 256×256 image matrix *A.mat* from MyCourses, and read it into Matlab using *load A*. To see that it is an image execute (on the same line) $imagesc(A)$; $colormap(gray)$; We'll use the SVD to compute low rank approximations to A.

(a) Using Matlab's command $[U, S, V] = svd(A)$, find the SVD of A and then compute the best (in the 2-norm) rank k approximations A_k to A for $k = 1, 2, 4, 16, 64$. Plot them and compare visually with the original image.

(b) Using Matlab's command *norm* compute the error of A_k relative to A, i.e, $E_k := \|A - A_k\|_2 / \|A\|_2$, and plot a graph of E_k against k. Also plot the ratio σ_i/σ_1 of A against i for $i = 1, \ldots 256$. Comment on the two graphs. How large should k be to ensure that $E_k = \|A_k - A\|_2/\|A\|_2 \leq 0.05$?

(c) Let $A = \sum_{j=1}^{\min(m,n)} \sigma_j u_j v_j^T \in \mathbb{R}^{m \times n}$ represent the sum-of-rank-one matrices SVD of A, where the u_j are the left singular vectors, v_j are the right singular vectors and σ_j are ranked in decreasing order. Assuming that $rank(A) \geq k$, show that the best rank k approximation of A in the matrix 2-norm is given by $A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^T$.

Question 3: Eigenvalue Theory 1

(a) Let $A \in \mathbb{C}^{n \times n}$, and let $(\lambda_1, x_1), \ldots (\lambda_k, x_k)$ be eigenpairs where all λ_i are distinct. Show that the corresponding eigenvectors $x_1, \ldots x_k$ are linearly independent.

(b) Let $A, B \in \mathbb{C}^{n \times n}$ be similar. Show that A and B have the same characteristic polynomial, same eigenvalues including algebraic and geometric multiplicities, same determinant, and same trace.

(c) Do A and B from (b) share the same singular values? Justify.

Question 4: Eigenvalue Theory 2

Let $A \in \mathbb{C}^{n \times n}$. For each of the following statements show that it is true or give a counterexample to show that it is false.

(a) If λ is an eigenvalue of A, and $\mu \in \mathbb{C}^n$ then $\lambda - \mu$ is an eigenvalue of $A - \mu I$.

(b) If A is real and λ is an eigenvalue of A then so is $-\lambda$.

(c) If A is real and λ is an eigenvalue of A, then so is $\bar{\lambda}$.

(d) If A is nonsingular and λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}.

(e) If all the eigenvalues of A are zero then $A = 0$.

(f) If A is Hermitian and λ is an eigenvalue then $|\lambda|$ is a singular value of A.

(g) If A is diagonalizable and all its eigenvalues are equal then A is diagonal.

Question 5: Basic QR algorithm, Power Method

Let $A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 3 & 1 & 2 & 4 \end{bmatrix}$.

(a) Implement *[Q,D] = myQRbasic(A,N)* in Matlab, the basic version of QR algorithm where N is the number of iterations in your QR step. Run it for $N = 10$ steps on A above and show the output.

(b) Implement *[lambda,v] = myPowerMethod(A,N)*, the power method in Matlab. Run it for $N = 10$ steps on A above and (log) plot the convergence $|\lambda^{(k)} - \lambda|$ where $\lambda^{(k)}$ is the k-th iterate of your algorithm and λ is the nearly exact largest eigenvalue computed using Matlab's *eig* function.

Question 6 (Math 397): Normal Matrices

Let $A \in \mathbb{C}^{n \times n}$. Show that A is normal if and only if $A^H A = AA^H$.