Econ 546 Assignment 4

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Problem 1. Subgame 1: Initial move is (C,C)

$$2 + 2\delta + 2\delta^2 + 2\delta^3 + ... \geq 2 + 3\delta + 0\delta^2 + 3\delta^3 + ...$$

$$2\delta[1 + \delta + \delta^2 + ...] \geq 3\delta[1 + \delta^2 + \delta^4 + ...]$$

$$\frac{2\delta}{1-\delta} \geq \frac{3\delta}{1-\delta^2}$$

$$2(1+\delta) \geq 3$$

$$\delta \geq \frac{1}{2}$$

Subgame 2: Initial move is (D,C)

$$3 + 0\delta + 3\delta^2 + 0\delta^3 + ... \geq 3 + 1\delta + 1\delta^2 + 1\delta^3 + ...$$

$$3\delta^2[1 + \delta^2 + \delta^4 + ...] \geq \delta[1 + \delta + \delta^2 + ...]$$

$$\frac{3\delta^2}{1-\delta^2} \geq \frac{\delta}{1-\delta}$$

$$3\delta \geq 1 + \delta$$

$$\delta \geq \frac{1}{2}$$

Subgame 3: Initial move is (C,D)

$$0 + 3\delta + 0\delta^2 + 3\delta^3 + ... \geq 0 + 2\delta + 2\delta^2 + 2\delta^3 + ...$$

$$3\delta[1 + \delta^2 + \delta^4 + ...] \geq 2\delta[1 + \delta + \delta^2 + ...]$$

$$\frac{3\delta}{1-\delta^2} \geq \frac{2\delta}{1-\delta}$$

$$3 \geq 2 + 2\delta$$

$$\frac{1}{2} \geq \delta$$

Subgame 4: Initial move is (D,D)

$$1 + 1\delta + 1\delta^2 + 1\delta^3 + ... \geq 1 + 0\delta + 3\delta^2 + 0\delta^3 + ...$$

$$\delta[1 + \delta + \delta^2 + ...] \geq 3\delta^2[1 + \delta^2 + \delta^4 + ...]$$

$$\frac{\delta}{1-\delta} \geq \frac{3\delta^2}{1-\delta^2}$$
$$1+\delta \geq 3\delta$$
$$\frac{1}{2} \geq \delta$$

From these 4 subgames we have,

$$\frac{1}{2} \geq \delta \geq \frac{1}{2}$$
$$\Rightarrow \delta = \frac{1}{2}$$

Therefore $\delta = \frac{1}{2}$ is the only discount factor that sustains the tit-for-tat strategy.

Problem 2. We can complete figure 275.1 to outline the payoffs for both players.

	(B,B)	(B,S)	(S,B)	(S,S)
B	$(2, \frac{1}{2})$	$(1, \frac{3}{2})$	$(1, 0)$	$(0, 1)$
S	$(0, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, \frac{3}{2})$	$(1, 1)$

From this table we can see if Player 1 plays S, Type 1 of Player 2 will play S and Type 2 will play B, this strategy yields a payoff of $\frac{1}{2}$ for Player 1 and $\frac{3}{2}$ for Player 2. However, if Player 1 knows that Player 2 will play (Type 1, Type 2) = (S,B), then they would switch and play B to raise their payoff to 1. Thus there is no Nash equilibrium in which player 1 chooses S.

Denote p the probability that player 1 plays B

$$EU_2(\text{Type } 1 = B) = \frac{1}{2}p + \frac{1}{2}(1-p) + \frac{3}{2}p + 0(1-p) = \frac{1}{2} + \frac{3p}{2}$$
$$EU_2(\text{Type } 1 = S) = 0p + \frac{3}{2}(1-p) + 1p + 1(1-p) = \frac{5}{2} - \frac{3p}{2}$$
$$EU_2(\text{Type } 1 = B) = EU_2(\text{Type } 1 = S) \Rightarrow p = \frac{2}{3}$$

Therefore Player 1 plays according to $(B, S) = (\frac{2}{3}, \frac{1}{3})$,

$$EU_2(\text{Type } 2 = B) = \frac{1}{2} * \frac{2}{3} + \frac{1}{2} * \frac{1}{3} + 0 * \frac{2}{3} + \frac{3}{2} * \frac{1}{3} = 1$$
$$EU_2(\text{Type } 2 = S) = \frac{3}{2} * \frac{2}{3} + 0 * \frac{1}{3} + 1 * \frac{2}{3} + 1 * \frac{1}{3} = 2$$

Therefore Type 2 of Player 2 plays according to $(B, S) = (0, 1)$. Denote q the probability that Type 1 of player 2 plays B. Using these two strategies of player 2,

$$EU_1(B) = \frac{1}{2} * 0 + \frac{1}{2} * 2q = q$$
$$EU_1(S) = \frac{1}{2} * (1-q) + \frac{1}{2} * 1 = 1 - \frac{q}{2}$$
$$\Rightarrow q = \frac{2}{3}$$

Therefore Type 2 of Player 2 plays according to $(B, S) = (\frac{2}{3}, \frac{1}{3})$.

We conclude that the first mixed strategy Nash Equilibrium consists of Player 1 playing according to $(\frac{2}{3}, \frac{1}{3})$, Type 1 of Player 2 playing according to $(\frac{2}{3}, \frac{1}{3})$ and Type 2 of Player 2 playing according to $(0, 1)$.

Denote p the probability that player 1 plays B

$$EU_2(\text{Type } 2 = B) = \frac{1}{2}p + \frac{1}{2}(1-p) + 0 * p + \frac{3}{2}(1-p) = 2 - \frac{3p}{2}$$

$$EU_2(\text{Type } 2 = S) = \frac{3}{2}p + 0(1-p) + 1p + 1(1-p) = 1 + \frac{3p}{2}$$

$$\Rightarrow p = \frac{1}{3}$$

Therefore Player 1 plays according to $(B, S) = (\frac{1}{3}, \frac{2}{3})$,

$$EU_2(\text{Type } 1 = B) = \frac{1}{2} * \frac{1}{3} + \frac{1}{2} * \frac{2}{3} + \frac{3}{2} * \frac{1}{3} + 0 * \frac{2}{3} = 1$$

$$EU_2(\text{Type } 1 = S) = 0 * \frac{1}{3} + \frac{3}{2} * \frac{2}{3} + 1 * \frac{1}{3} + 1 * \frac{2}{3} = 2$$

Therefore Type 1 of Player 2 plays according to $(B, S) = (0, 1)$. Denote q the probability that Type 2 of player 2 plays B. Using these two strategies of player 2,

$$EU_1(B) = \frac{1}{2} * 0 + \frac{1}{2} * 2q = q$$

$$EU_1(S) = \frac{1}{2} * (1-q) + \frac{1}{2} * 1 = 1 - \frac{q}{2}$$

$$\Rightarrow q = \frac{2}{3}$$

Therefore Type 1 of Player 2 plays according to $(B, S) = (\frac{2}{3}, \frac{1}{3})$.

We conclude that the second mixed strategy Nash Equilibrium consists of Player 1 playing according to $(\frac{1}{3}, \frac{2}{3})$, Type 1 of Player 2 playing according to $(0, 1)$ and Type 2 of Player 2 playing according to $(\frac{2}{3}, \frac{1}{3})$.

Problem 3.

- Players: Player 1, Player 2
- States: Strong, Weak
- Actions: Fight, Yield (for each player)
- Signals:
  - Player 1's signal: Same signal in each state
  - Player 2's signal: Different signals depending on state
- Beliefs:

- Player 1's belief of Player 2's strength: (Strong, Weak) = (α,1-α)
- Player 2's belief of Player 1's strength: (Strong, Weak) = (1,0) or (0,1) depending on their signal

- Payoffs: Outlined in Tables below

	F	Y
F	(-1,$\underline{1}$)	(1,0)
Y	(0,$\underline{1}$)	(0,0)

Table 1: State = Strong

	F	Y
F	(1,-1)	(1,$\underline{0}$)
Y	(0,$\underline{1}$)	(0,0)

Table 2: State = Weak

In the payoff tables we have underlined Player 2's best responses. Using these we can calculate the best action of Player 1.

$$EU_1(F) = -1 * \alpha + 1 * (1 - \alpha) = 1 - 2\alpha$$

$$EU_1(Y) = 0 * \alpha + 0 * (1 - \alpha) = 0$$

$$EU_1(F) > EU_1(Y) \Rightarrow \alpha < \frac{1}{2}$$

Therefore we conclude when $\alpha < \frac{1}{2}$, Player 1's best action is Fight, and when $\alpha > \frac{1}{2}$, Player 1's best action is Yield. Therefore for $\alpha < \frac{1}{2}$ this game has a unique Nash equilibrium: Player 1 chooses Fight and Player 2 chooses Fight if they are Strong and Yield if they are weak. For $\alpha > \frac{1}{2}$ the game has a unique Nash equilibrium, in which Player 1 chooses Yield and Player 2 chooses Fight regardless of whether they are strong or weak.

Problem 4. Player 1's best response function:

$$b_1(q_L, q_H) = \begin{cases} \frac{1}{2}(\alpha - c_1 - (\theta q_L + (1-\theta)q_H)), & \text{if } \theta q_L + (1-\theta)q_H \leq \alpha - c_1 \\ 0, & \text{else} \end{cases}$$

Player 2's best response functions:

$$b_L(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_L - q_1), & \text{if } q_1 \leq \alpha - c_L \\ 0, & \text{else} \end{cases}$$

$$b_H(q_1) = \begin{cases} \frac{1}{2}(\alpha - c_H - q_1), & \text{if } q_1 \leq \alpha - c_H \\ 0, & \text{else} \end{cases}$$

From the question we know all the outputs are positive. Solving for q_1^*:

$$q_1^* = \frac{1}{2}(\alpha - c_1 - (\theta \frac{1}{2}(\alpha - c_L - q_1^*) + (1-\theta)\frac{1}{2}(\alpha - c_H - q_1^*)))$$

$$q_1^* = \frac{1}{2}(\alpha - c_1 - (\frac{1}{2}\theta(\alpha - c_L - q_1^*) + \frac{1}{2}(\alpha - c_H - q_1^*) - \frac{1}{2}\theta(\alpha - c_H - q_1^*)))$$

$$q_1^* = \frac{1}{2}(\alpha - c_1 - \frac{1}{2}(\theta(\alpha - c_L - q_1^*) + (\alpha - c_H - q_1^*) - \theta(\alpha - c_H - q_1^*)))$$

$$q_1^* = \frac{1}{2}(\alpha - c_1 - \frac{1}{2}(\theta\alpha - \theta c_L - \theta q_1^* + \alpha - c_H - q_1^* - \theta\alpha + \theta c_H + \theta q_1^*))$$

$$q_1^* = \frac{1}{2}(\alpha - c_1 - \frac{1}{2}(-\theta c_L + \alpha - c_H - q_1^* + \theta c_H))$$

$$q_1^* = \frac{1}{4}(2\alpha - 2c_1 + \theta c_L - \alpha + c_H + q_1^* - \theta c_H)$$

$$q_1^* = \frac{1}{4}(\alpha - 2c_1 + \theta c_L + c_H + q_1^* - \theta c_H)$$

$$\frac{3}{4}q_1^* = \frac{1}{4}(\alpha - 2c_1 + \theta c_L + (1-\theta)c_H)$$

$$q_1^* = \frac{1}{3}(\alpha - 2c_1 + \theta c_L + (1-\theta)c_H)$$

Solving for q_L^*:

$$q_L^* = \frac{1}{2}(\alpha - c_L - \frac{1}{3}(\alpha - 2c_1 + \theta c_L + (1-\theta)c_H))$$

$$q_L^* = \frac{1}{6}(3\alpha - 3c_L - \alpha + 2c_1 - \theta c_L - (1-\theta)c_H))$$

$$q_L^* = \frac{1}{6}(2\alpha - 4c_L + 2c_1 + c_L - \theta c_L - (1-\theta)c_H))$$

$$q_L^* = \frac{1}{6}(2\alpha - 4c_L + 2c_1 + (1-\theta)c_L - (1-\theta)c_H))$$

$$q_L^* = \frac{1}{6}(2\alpha - 4c_L + 2c_1 - (1-\theta)(c_H - c_L))$$

Solving for q_H^* (By symmetry):

$$q_H^* = \frac{1}{6}(2\alpha - 4c_H + 2c_1 + \theta(c_H - c_L))$$

Therefore the imperfect information equilibrium is,

$$\begin{cases} q_1^* = \frac{1}{3}(\alpha - 2c_1 + \theta c_L + (1-\theta)c_H) \\ q_L^* = \frac{1}{6}(2\alpha - 4c_L + 2c_1 - (1-\theta)(c_H - c_L)) \\ q_H^* = \frac{1}{6}(2\alpha - 4c_H + 2c_1 + \theta(c_H - c_L)) \end{cases}$$

If Player 1 knows that Player 2's unit cost is c_L, in other words $\theta = 1$, the equilibrium becomes

$$\begin{cases} q_1^* = \frac{1}{3}(\alpha - 2c_1 + c_2) \\ q_2^* = \frac{1}{3}(\alpha - 2c_2 + c_1) \end{cases}$$

And in fact the equilibrium for when Player 1 knows that Player 2's unit cost is c_H, in other words $\theta = 0$ is the same. Therefore this is the equilibrium for perfect information.

From the imperfect information equilibrium we see that if Player 2's cost is in fact c_L, this results in a lower output compared to its output in the perfect information equilibrium, because $-(1-\theta)(c_H - c_L)) \leq 0$. Similarly if Player 2's cost is in fact c_H, this results in a greater output compared to its output in the perfect information equilibrium, because $\theta(c_H - c_L)) \geq 0$.