6

POISSON PROCESS

The count has arrived …

–Thomas Carlyle, *Count Cagliostro*

6.1 INTRODUCTION

Text messages arrive on your cell phone at irregular times throughout the day. Accidents occur on the highway in a seemingly random distribution of time and place. Babies are born at chance moments on a maternity ward.

All of these phenomena are well modeled by the Poisson process, a stochastic process used to model the occurrence, or arrival, of events over a continuous interval. Typically, the interval represents time.

A Poisson process is a special type of *counting process*. Given a stream of events that arrive at random times starting at $t = 0$, let N_t denote the number of arrivals that occur by time t, that is, the number of events in $[0, t]$. For instance, N_t might be the number of text messages received up to time t.

For each $t \geq 0$, N_t is a random variable. The collection of random variables $(N_t)_{t\geq 0}$ is a continuous-time, integer-valued stochastic process, called a counting process. Since N_t counts events in $[0, t]$, as t increases, the number of events N_t increases.

Introduction to Stochastic Processes with R, First Edition. Robert P. Dobrow.

Counting Process

A *counting process* $(N_t)_{t\geq 0}$ is a collection of non-negative, integer-valued random variables such that if $0 \leq s \leq t$, then $N_s \leq N_t$.

Unlike a Markov chain, which is a *sequence* of random variables, a counting process forms an uncountable collection, since it is indexed over a continuous time interval.

Figure 6.1 shows the path of a counting process in which events occur at times t_1, t_2, t_3, t_4, t_5. As shown, the path of a counting process is a right-continuous step function. If $0 \leq s < t$, then $N_t - N_s$ is the number of events in the interval $(s, t]$.

Note that in the discussion of the Poisson process, we use the word *event* in a loose, generic sense, and not in the rigorous sense of probability theory where an event is a subset of the sample space.

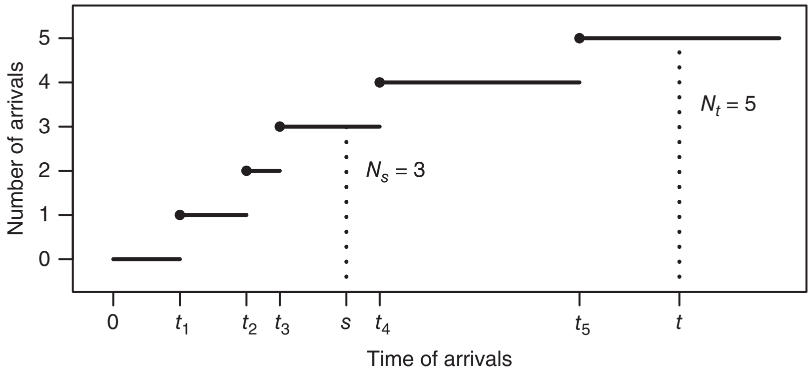

Figure 6.1 Counting process.

There are several ways to characterize the Poisson process. One can focus on (i) the *number* of events that occur in fixed intervals, (ii) *when* events occur, and the times between those events, or (iii) the probabilistic behavior of individual events on infinitesimal intervals. This leads to three equivalent definitions of a Poisson process, each of which gives special insights into the stochastic model.

Poisson Process—Definition 1

A *Poisson process with parameter* λ is a counting process $(N_t)_{t\geq 0}$ with the following properties:

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with parameter λt.

3. (Stationary increments) For all $s, t > 0$, $N_{t+s} - N_s$ has the same distribution as N_t. That is,

$$P(N_{t+s} - N_s = k) = P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \text{ for } k = 0, 1, \ldots$$

4. (Independent increments) For $0 \le q < r \le s < t$, $N_t - N_s$ and $N_r - N_q$ are independent random variables.

The stationary increments property says that the distribution of the number of arrivals in an interval only depends on the *length* of the interval.

The independent increments property says that the number of arrivals on disjoint intervals are independent random variables.

Since N_t has a Poisson distribution, $E(N_t) = \lambda t$. That is, we expect about λt arrivals in t time units. Thus, the *rate* of arrivals is $E(N_t)/t = \lambda$.

Example 6.1 Starting at 6 a.m., customers arrive at Martha's bakery according to a Poisson process at the rate of 30 customers per hour. Find the probability that more than 65 customers arrive between 9 and 11 a.m.

Solution Let $t = 0$ represent 6 a.m. Then, the desired probability is $P(N_5 - N_3 > 65)$. By stationary increments,

$$\begin{aligned} P(N_5 - N_3 > 65) &= P(N_2 > 65) = 1 - P(N_2 \le 65) \\ &= 1 - \sum_{k=0}^{65} P(N_2 = k) \\ &= 1 - \sum_{k=0}^{65} \frac{e^{-30(2)}(30(2))^k}{k!} = 0.2355. \end{aligned}$$

In R , the result is obtained by typing

```
> 1-ppois(65,2*30)
[1] 0.2355065
```

■

Example 6.2 Joe receives text messages starting at 10 a.m. at the rate of 10 texts per hour according to a Poisson process. Find the probability that he will receive exactly 18 texts by noon and 70 texts by 5 p.m.

Solution The desired probability is $P(N_2 = 18, N_7 = 70)$, with time as hours. If 18 texts arrive in $[0, 2]$ and 70 texts arrive in $[0, 7]$, then there are $70 - 18 = 52$ texts in

(2, 7]. That is,

$$\{N_2 = 18, N_7 = 70\} = \{N_2 = 18, N_7 - N_2 = 52\}.$$

The intervals [0, 2] and (2, 7] are disjoint, which gives

$$\begin{aligned}
P(N_2 = 18, N_7 = 70) &= P(N_2 = 18, N_7 - N_2 = 52) \\
&= P(N_2 = 18)\, P(N_7 - N_2 = 52) \\
&= P(N_2 = 18)\, P(N_5 = 52) \\
&= \left(\frac{e^{-2(10)}(2(10))^{18}}{18!}\right)\left(\frac{e^{-5(10)}(5(10))^{52}}{52!}\right) \\
&= 0.0045,
\end{aligned}$$

where the second equality is because of independent increments, and the third equality is because of stationary increments. The final calculation in R is

```
> dpois(18,2*10)*dpois(52,5*10)
[1] 0.004481021
```

Warning: It would be incorrect to write

$$P(N_2 = 18, N_7 - N_2 = 52) = P(N_2 = 18, N_5 = 52).$$

It is not true that $N_7 - N_2 = N_5$. The number of arrivals in (2, 7] is not necessarily equal to the number of arrivals in (0, 5]. What is true is that the *distribution* of $N_7 - N_2$ is equal to the distribution of N_5. Note that while $N_7 - N_2$ is independent of N_2, the random variable N_5 is not independent of N_2. Indeed, $N_5 \geq N_2$. ■

Translated Poisson Process

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. For fixed time $s > 0$, consider the *translated process* $(N_{t+s} - N_s)_{t\geq 0}$. The translated process is probabilistically equivalent to the original process.

Translated Poisson Process is a Poisson Process

Proposition 6.1. *Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. For $s > 0$, let*

$$\widetilde{N}_t = N_{t+s} - N_s, \quad for \ \ t \geq 0.$$

Then, $(\widetilde{N}_t)_{t\geq 0}$ is a Poisson process with parameter λ.

We have that $\left(\widetilde{N}_t\right)_{t\geq 0}$ is a counting process with $\widetilde{N}_0 = N_s - N_s = 0$. By stationary increments, $\widetilde{N}_t$ has the same distribution as N_t. And the new process inherits stationary and independent increments from the original. It follows that if $N_s = k$, the distribution of $N_{t+s} - k$ is equal to the distribution of N_t.

Example 6.3 On election day, people arrive at a voting center according to a Poisson process. On average, 100 voters arrive every hour. If 150 people arrive during the first hour, what is the probability that at most 350 people arrive before the third hour?

Solution Let N_t denote the number of arrivals in the first t hours. Then, $(N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda = 100$. Given $N_1 = 150$, the distribution of $N_3 - N_1 = N_3 - 150$ is equal to the distribution of N_2. This gives

$$\begin{aligned} P(N_3 \leq 350 | N_1 = 150) &= P(N_3 - 150 \leq 200 | N_1 = 150) \\ &= P(N_2 \leq 200) \\ &= \sum_{k=0}^{200} \frac{e^{-100\,(2)}(100(2))^k}{k!} \\ &= 0.519. \end{aligned}$$

■

6.2 ARRIVAL, INTERARRIVAL TIMES

For a Poisson process with parameter λ, let X denote the time of the first arrival. Then, $X > t$ if and only if there are no arrivals in $[0, t]$. Thus,

$$P(X > t) = P(N_t = 0) = e^{-\lambda t}, \quad \text{for} \quad t > 0.$$

Hence, X has an exponential distribution with parameter λ.

The exponential distribution plays a central role in the Poisson process. What is true for the time of the first arrival is also true for the time between the first and second arrival, and for all *interarrival times*. A Poisson process is a counting process for which interarrival times are independent and identically distributed exponential random variables.

Poisson Process—Definition 2

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. exponential random variables with parameter λ. For $t > 0$, let

$$N_t = \max\{n : X_1 + \cdots + X_n \leq t\},$$

with $N_0 = 0$. Then, $(N_t)_{t\geq 0}$ defines a Poisson process with parameter λ.

Let

$$S_n = X_1 + \cdots + X_n, \quad \text{for } n = 1, 2, \ldots$$

We call $S_1, S_2, \ldots$ the *arrival times* of the process, where S_k is the time of the kth arrival. Furthermore,

$$X_k = S_k - S_{k-1}, \quad \text{for } k = 1, 2, \ldots$$

is the *interarrival time* between the $(k-1)$th and kth arrival, with $S_0 = 0$.

The relationship between the interarrival and arrival times for a Poisson process is illustrated in Figure 6.2.

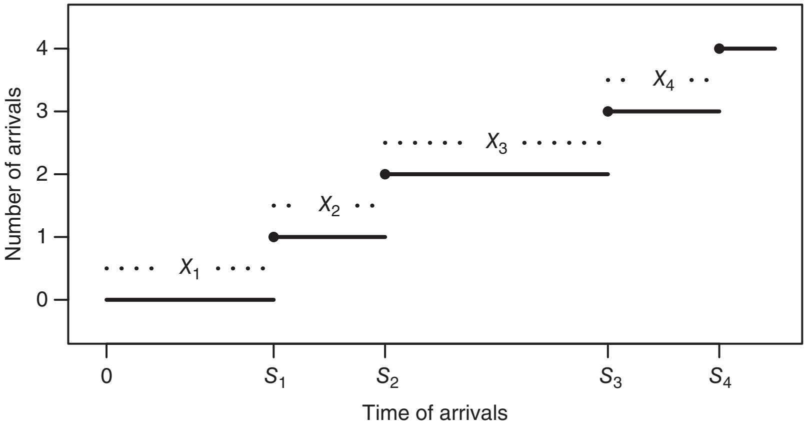

Figure 6.2 Arrival times $S_1, S_2, \ldots$, and interarrival times $X_1, X_2, \ldots$

In the next section, we show that Definitions 1 and 2 are equivalent. A benefit of Definition 2 is that it leads to a direct method for constructing, and simulating, a Poisson process:

1. Let $S_0 = 0$.
2. Generate i.i.d. exponential random variables $X_1, X_2, \ldots$
3. Let $S_n = X_1 + \cdots + X_n$, for $n = 1, 2, \ldots$
4. For each $k = 0, 1, \ldots$, let $N_t = k$, for $S_k \leq t < S_{k+1}$.

Two realizations of a Poisson process with $\lambda = 0.1$ obtained by this method on the interval [0, 100] are shown in Figure 6.3.

The importance of the exponential distribution to the Poisson process lies in its unique memoryless property, a topic from probability that merits review.

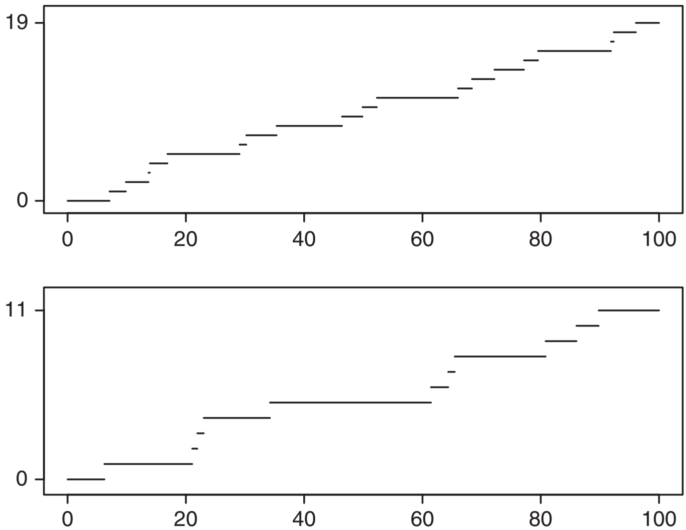

Figure 6.3 Realizations of a Poisson process with $\lambda = 0.1$.

Memorylessness

To illustrate, assume that Amy and Zach each want to take a bus. Buses arrive at a bus stop according to a Poisson process with parameter $\lambda = 1/30$. That is, the times between buses have an exponential distribution, and buses arrive, on average, once every 30 minutes. Unlucky Amy gets to the bus stop just as a bus pulls out of the station. Her waiting time for the next bus is about 30 minutes. Zach arrives at the bus stop 10 minutes after Amy. Remarkably, the time that Zach waits for a bus also has an exponential distribution with parameter $\lambda = 1/30$. Memorylessness means that their waiting time distributions are the same, and they will both wait, on average, the same amount of time!

To prove it true, observe that Zach waits more than t minutes if and only if Amy waits more than $t + 10$ minutes, given that a bus does not come in the first 10 minutes. Let A and Z denote Amy and Zach's waiting times, respectively. Amy's waiting time is exponentially distributed. Hence,

$$P(Z > t) = P(A > t + 10|A > 10) = \frac{P(A > t + 10)}{P(A > 10)}$$
$$= \frac{e^{-(t+10)/30}}{e^{-10/30}} = e^{-t/30} = P(A > t),$$

from which it follows that A and Z have the same distribution. See the `R` code and Figure 6.4 for the results of a simulation.

Of course, there is nothing special about $t = 10$. Memorylessness means that regardless of how long you have waited, the distribution of the time you still have to wait is the same as the original waiting time.

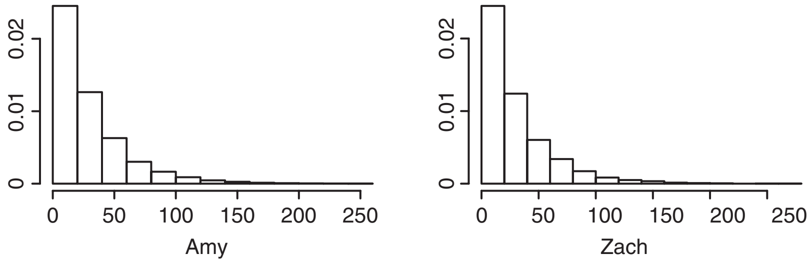

Figure 6.4 Waiting time distributions for Amy and Zach. Zach arrives 10 minutes after Amy. By memorylessness, the distributions are the same.

R : Bus Waiting Times

```
# buswaiting.R
> trials <- 5000
> amy <- numeric(trials)
> zach <- numeric(trials)
> for (i in 1:trials) {
 + bus <- rexp(1,1/30)
 + amy[i] <- bus
 + while (bus < 10) { bus <- bus + rexp(1,1/30) }
 + zach[i] <- bus-10 }
> mean(amy)
[1] 29.8043
> mean(zach)
[1] 30.39833
> hist(amy,xlab="Amy",prob=T,ylab="",main="")
> hist(zach,xlab="Zach",prob=T,ylab="",main="")
```

The exponential distribution is the only continuous distribution that is memoryless. (The geometric distribution has the honors for the discrete case.) Here is the general statement of the property.

Memorylessness

A random variable X is *memoryless* if, for all $s, t > 0$,

$$P(X > s + t | X > s) = P(X > t).$$

Results for the *minimum* of independent exponential random variables are particularly useful when working with the Poisson process. We highlight two properties that arise in many settings.

Minimum of Independent Exponential Random Variables

Let $X_1, \dots, X_n$ be independent exponential random variables with respective parameters $\lambda_1, \dots, \lambda_n$. Let $M = \min(X_1, \dots, X_n)$.

1. For $t > 0$,
$$P(M > t) = e^{-t(\lambda_1 + \cdots + \lambda_n)}. \tag{6.1}$$
That is, M has an exponential distribution with parameter $\lambda_1 + \cdots + \lambda_n$.
2. For $k = 1, \dots, n$,
$$P(M = X_k) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}. \tag{6.2}$$

Proof.

1. For $t > 0$,
$$\begin{aligned} P(M > t) &= P(X_1 > t, \dots, X_n > t) = P(X_1 > t) \cdots P(X_n > t) \\ &= e^{-\lambda_1 t} \cdots e^{-\lambda_n t} = e^{-t(\lambda_1 + \cdots + \lambda_n)}. \end{aligned}$$
2. For $1 \le k \le n$, conditioning on X_k gives
$$\begin{aligned} P(M = X_k) &= P(\min(X_1, \dots, X_n) = X_k) \\ &= P(X_1 \ge X_k, \dots, X_n \ge X_k) \\ &= \int_0^\infty P(X_1 \ge t, \dots, X_n \ge t | X_k = t) \lambda_k e^{-\lambda_k t}\, dt \\ &= \int_0^\infty P(X_1 \ge t, \dots, X_{k-1} \ge t, X_{k+1} \ge t, \dots, X_n \ge t) \lambda_k e^{-\lambda_k t}\, dt \\ &= \int_0^\infty P(X_1 \ge t) \cdots P(X_{k-1} \ge t) P(X_{k+1} \ge t) \cdots P(X_n \ge t) \lambda_k e^{-\lambda_k t}\, dt \\ &= \lambda_k \int_0^\infty e^{-t(\lambda_1 + \cdots + \lambda_n)}\, dt = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}. \end{aligned}$$

■

Example 6.4 A Boston subway station services the red, green, and orange lines. Subways on each line arrive at the station according to three independent Poisson processes. On average, there is one red train every 10 minutes, one green train every 15 minutes, and one orange train every 20 minutes.

(i) When you arrive at the station what is the probability that the first subway that arrives is for the green line?

(ii) How long will you wait, on average, before some train arrives?

(iii) You have been waiting 20 minutes for a red train and have watched three orange trains go by. What is the expected additional time you will wait for your subway?

Solution

(i) Let X_G, X_R, and X_O denote, respectively, the times of the first green, red, and orange subways that arrive at the station. The event that the first subway is green is the event that X_G is the minimum of the three independent random variables. The desired probability is

$$P(\min(X_G, X_R, X_O) = X_G) = \frac{1/15}{1/10 + 1/15 + 1/20} = \frac{4}{13} = 0.31.$$

(ii) The time of the first train arrival is the minimum of X_G, X_R, and X_O, which has an exponential distribution with parameter

$$\frac{1}{10} + \frac{1}{15} + \frac{1}{20} = \frac{13}{60}.$$

Thus, you will wait, on average $60/13 = 4.615$ minutes. A *quick* simulation gives

```
> sim <- replicate(10000,
+   min(rexp(1,1/10),rexp(1,1/15),rexp(1,1/20)))
> mean(sim)
[1] 4.588456
```

(iii) Your waiting time is independent of the orange arrivals. By memorylessness of interarrival times, the additional waiting time for the red line has the same distribution as the original waiting time. You will wait, on average, 10 more minutes. ■

For a Poisson process, each arrival time S_n is a sum of n i.i.d. exponential interarrival times. A sum of i.i.d. exponential variables has a gamma distribution.

Arrival Times and Gamma Distribution

For $n = 1, 2, \ldots$, let S_n be the time of the nth arrival in a Poisson process with parameter λ. Then, S_n has a gamma distribution with parameters n and λ. The density function of S_n is

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad \text{for } t > 0.$$

Mean and variance are

$$E(S_n) = \frac{n}{\lambda} \quad \text{and} \quad Var(S_n) = \frac{n}{\lambda^2}.$$

For a general gamma distribution, the parameter n does not have to be an integer. When it is, the distribution is sometimes called an *Erlang distribution*. Observe that if $n = 1$, the gamma distribution reduces to the exponential distribution with parameter λ.

Two ways to derive the gamma distribution for sums of i.i.d. exponentials are (i) by induction on n, using the fact that $S_n = S_{n-1} + X_n$, and (ii) by moment-generating functions. We leave both derivations to the exercises.

Example 6.5 The times when goals are scored in hockey are modeled as a Poisson process in Morrison (1976). For such a process, assume that the average time between goals is 15 minutes.

(i) In a 60-minute game, find the probability that a fourth goal occurs in the last 5 minutes of the game.

(ii) Assume that at least three goals are scored in a game. What is the mean time of the third goal?

Solution The parameter of the hockey Poisson process is $\lambda = 1/15$.

(i) The desired probability is

$$P(55 < S_4 \leq 60) = \frac{1}{6}\int_{55}^{60} (1/15)^4 t^3 e^{-t/15}\, dt = 0.068.$$

In R , the probability is found by typing

```
> pgamma(60,4,1/15)-pgamma(55,4,1/15)
[1] 0.06766216
```

(ii) The desired expectation is

$$\begin{aligned} E(S_3|S_3 < 60) &= \frac{1}{P(S_3 < 60)}\int_0^{60} t f_{S_3}(t)\, dt \\ &= \frac{1}{P(S_3 < 60)}\int_0^{60} t\frac{(1/15)^3 t^2 e^{-t/15}}{2}\, dt \\ &= \frac{25.4938}{0.7619} = 33.461 \text{ minutes.} \end{aligned}$$

■

6.3 INFINITESIMAL PROBABILITIES

A third way to define the Poisson process is based on an infinitesimal description of the distribution of points (e.g., events) in small intervals.

To state the new definition, we use *little-oh* notation. Write $f(h) = \mathrm{o}(h)$ to mean that

$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

More generally, say that a function f is *little-oh of* g, and write $f(h) = \mathrm{o}(g(h))$, to mean that

$$\lim_{h\to 0} \frac{f(h)}{g(h)} = 0.$$

Little-oh notation is often used when making order of magnitude statements about a function, or in referencing the remainder term of an approximation.

For example, the Taylor series expansion of e^h with remainder term is

$$e^h = 1 + h + \frac{h^2}{2} + \frac{h^3}{6} + \cdots = 1 + h + R(h),$$

where $R(h) = e^z h^2/2$, for some $z \in (-h, h)$. Since $R(h)/h = e^z h/2 \to 0$, as $h \to 0$, we can write

$$e^h = 1 + h + \mathrm{o}(h).$$

Note that if two functions f and g are little-oh of h, then $f(h) + g(h) = \mathrm{o}(h)$, since $(f(h) + g(h))/h \to 0$, as $h \to 0$. Similarly, if $f(h) = \mathrm{o}(h)$, then $cf(h) = \mathrm{o}(h)$, for any constant c. If $f(h) = o(1)$, then $f(h) \to 0$, as $h \to 0$.

Poisson Process—Definition 3

A *Poisson process with parameter* λ is a counting process $(N_t)_{t\ge 0}$ with the following properties:

1. $N_0 = 0$.
2. The process has stationary and independent increments.
3. $P(N_h = 0) = 1 - \lambda h + \mathrm{o}(h)$.
4. $P(N_h = 1) = \lambda h + \mathrm{o}(h)$.
5. $P(N_h > 1) = \mathrm{o}(h)$.

Properties 3–5 essentially ensure that there cannot be infinitely many arrivals in a finite interval, and that in an infinitesimal interval there may occur at most one event.

It is straightforward to show that Definition 3 is a consequence of Definition 1. If N_h has a Poisson distribution with parameter λh, then

$$P(N_h = 0) = e^{-\lambda h} = 1 - \lambda h + \mathrm{o}(h),$$
$$P(N_h = 1) = e^{-\lambda h}\lambda h = (1 - \lambda h + \mathrm{o}(h))\lambda h = \lambda h + \mathrm{o}(h),$$

and

$$\begin{aligned} P(N_h > 1) &= 1 - P(N_h = 0) - P(N_h = 1) \\ &= 1 - (1 - \lambda h + \mathrm{o}(h)) - (\lambda h + \mathrm{o}(h)) \\ &= \mathrm{o}(h). \end{aligned}$$

The converse, that Definition 3 implies Definition 1, is often shown by deriving and solving a differential equation. We will forego the rigorous proof, but give a heuristic explanation, which offers insight into the nature of Poisson arrivals.

Assume that Definition 3 holds. We need to show that N_t has a Poisson distribution with parameter λt.

Consider N_t, the number of points in the interval $[0, t]$. Partition $[0, t]$ into n subintervals each of length t/n. Properties 3–5 imply that for sufficiently large n, each subinterval will contain either 0 or 1 point with high probability. The chance that a small subinterval contains 2 or more points is negligible. See Figure 6.5.

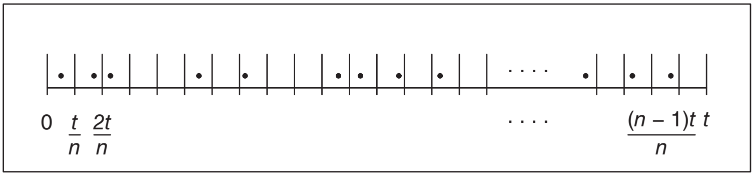

Figure 6.5 A partition of $[0, t]$ where each subinterval contains 0 or 1 point.

A subinterval has the form $((k-1)t/n, kt/n]$ for some $k = 1, \ldots, n$. By stationary increments,

$$P(N_{kt/n} - N_{(k-1)t/n} = 1) = P(N_{t/n} = 1) = \frac{\lambda t}{n} + \mathrm{o}\left(\frac{t}{n}\right), \quad \text{for all } k.$$

Furthermore, by independent increments, whether or not a point is contained in a particular subinterval is independent of the points in any other subinterval.

Hence, for large n, the outcomes in each subinterval can be considered a sequence of n i.i.d. Bernoulli trials, where the probability p_n that a point is contained in a subinterval is

$$p_n = \frac{\lambda t}{n} + \mathrm{o}\left(\frac{t}{n}\right).$$

Thus, the number of points in $[0, t]$, being the sum of n i.i.d. Bernoulli trials, has a binomial distribution with parameters n and p_n.

The result that N_t has a Poisson distribution is obtained by letting n tend to infinity and appealing to the Poisson approximation of the binomial distribution. Since

$$np_n = n\left[\frac{\lambda t}{n} + \mathrm{o}\left(\frac{t}{n}\right)\right] = \lambda t + n\left[\mathrm{o}\left(\frac{t}{n}\right)\right] \to \lambda t > 0, \quad \text{as } n \to \infty,$$

the approximation holds, which gives that N_t has a Poisson distribution with parameter λt. Note that for the $\mathrm{o}(t/n)$ term, since

$$\lim_{1/n\to 0} \frac{\mathrm{o}(t/n)}{t/n} = 0, \quad \text{then equivalently} \quad \lim_{n\to\infty} n[\mathrm{o}(t/n)] = 0.$$

A statement and proof of the Poisson approximation of the binomial distribution is given in Appendix B, Section B.4.

Equivalence of Poisson Definitions

We have shown that Definitions 1 and 3 are equivalent. Here, we show that Definitions 1 and 2 are equivalent.

Assume Definition 2. That is, let $X_1, X_2, \ldots$ be an i.i.d. sequence of exponential random variables with parameter λ. For each n, let $S_n = X_1 + \cdots + X_n$, with $S_0 = 0$, and let $N_t = \max\{n : S_n \le t\}$. We show N_t has a Poisson distribution with parameter λt.

Observe that for $k \ge 0$, $N_t = k$ if and only if the kth arrival occurs by time t and the $(k+1)$th arrival occurs after t. That is, $S_k \le t < S_k + X_{k+1}$. Since S_k is a function of $X_1, \ldots, X_k$, S_k and X_{k+1} are independent random variables, and their joint density is the product of their marginal densities. This gives

$$f_{S_k,X_{k+1}}(s,x) = f_{S_k}(s) f_{X_{k+1}}(x) = \left(\frac{\lambda^k s^{k-1} e^{-\lambda s}}{(k-1)!}\right) \lambda e^{-\lambda x}, \quad \text{for } s, x > 0.$$

For $k \ge 0$,

$$\begin{aligned}
P(N_t = k) &= P(S_k \le t \le S_k + X_{k+1}) \\
&= P(S_k \le t, X_{k+1} \ge t - S_k) \\
&= \int_0^t \int_{t-s}^{\infty} f_{S_k,X_{k+1}}(s,x)\, dx\, ds \\
&= \int_0^t \int_{t-s}^{\infty} \left(\frac{\lambda^k s^{k-1} e^{-\lambda s}}{(k-1)!}\right) \lambda e^{-\lambda x}\, dx\, ds \\
&= \frac{\lambda^k}{(k-1)!} \int_0^t \left(s^{k-1} e^{-\lambda s}\right)\, e^{-\lambda(t-s)}\, ds \\
&= \frac{e^{-\lambda t}\lambda^k}{(k-1)!} \int_0^t s^{k-1}\, ds = \frac{e^{-\lambda t}(\lambda t)^k}{k!},
\end{aligned}$$

which gives the desired Poisson distribution.

The fact that the interarrival times of a Poisson process are memoryless means that the pattern of arrivals from an arbitrary time s onward behaves the same as the pattern of arrivals from time 0 onward. It follows that the number of arrivals in the interval $(s, s+t]$ has the same distribution as the number of arrivals in $(0, t]$. Stationary, as well as independent, increments are direct consequences.

Conversely, assume Definition 1. Consider the distribution of the first interarrival time X_1. As shown at the beginning of Section 6.2 the distribution is exponential with parameter λ.

For X_2, consider $P(X_2 > t|X_1 = s)$. If the first arrival occurs at time s, and the second arrival occurs more than t time units later, then there are no arrivals in the interval $(s, s+t]$. Conversely, if there are no arrivals in $(s, s+t]$ and $X_1 = s$, then $X_2 > t$. Thus,

$$\begin{aligned} P(X_2 > t|X_1 = s) &= P(N_{s+t} - N_s = 0|X_1 = s) \\ &= P(N_{s+t} - N_s = 0) \\ &= P(N_t = 0) = e^{-\lambda t}, \quad \text{for } t > 0. \end{aligned}$$

The second equality is because of independent increments. It follows that X_1 and X_2 are independent, and the distribution of X_2 is exponential with parameter λ.

For the general case, consider $P(X_{k+1} > t|X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$. If the kth arrival occurs at time $x_1 + \cdots + x_k$ and $X_{k+1} > t$, then there are no arrivals in the interval $(x_1 + \cdots + x_k, x_1 + \cdots + x_k + t]$. See Figure 6.6.

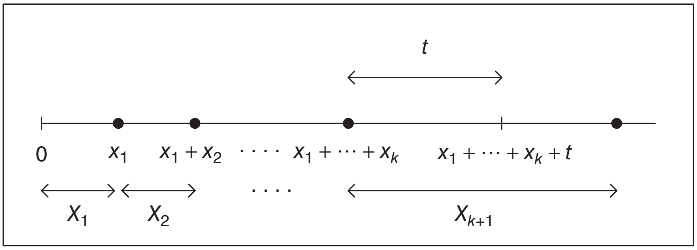

Figure 6.6 With $X_{k+1} > t$, there are no arrivals between $x_1 + \cdots + x_k$ and $x_1 + \cdots + x_{k+t}$.

We have that

$$\begin{aligned} &P(X_{k+1} > t|X_1 = x_1, \ldots, X_k = x_k) \\ &\qquad = P(N_{x_1+\cdots+x_k+t} - N_{x_1+\cdots+x_k} = 0|X_1 = x_1, \ldots, X_k = x_k) \\ &\qquad = P(N_{x_1+\cdots+x_k+t} - N_{x_1+\cdots+x_k} = 0) \\ &\qquad = P(N_t = 0) = e^{-\lambda t}, \quad \text{for } t > 0. \end{aligned}$$

Hence, X_{k+1} is independent of $X_1, \ldots, X_k$, and the distribution of X_{k+1} is exponential with parameter λ.

It follows that $X_1, X_2, \ldots$ is an i.i.d. sequence of exponential random variables with parameter λ, which gives Definition 2.

The equivalence of all three definitions has been established.

6.4 THINNING, SUPERPOSITION

According to the United Nations Population Division, the worldwide sex ratio at birth is 108 boys to 100 girls. Thus, the probability that any birth is a boy is

$$p = \frac{108}{108 + 100} = 0.519.$$

That this probability is greater than one-half is said to be nature's way of balancing the fact that boys have a slightly higher risk than girls of not surviving birth.

Assume that babies are born on a maternity ward according to a Poisson process $(N_t)_{t\geq 0}$ with parameter λ. How can the number of male births and the number of female births be described?

Babies' sex is independent of each other. We can think of a male birth as the result of a coin flip whose heads probability is p. Assume that there are n births by time t. Then, the number of male births by time t is the number of heads in n i.i.d. coin flips, which has a binomial distribution with parameters n and p. Similarly the number of female births in $[0, t]$ has a binomial distribution with parameters n and $1 - p$.

Let M_t denote the number of male births by time t. Similarly define the number of female births F_t. Thus, $M_t + F_t = N_t$. The joint probability mass function of (M_t, F_t) is

$$\begin{aligned}
P(M_t = m, F_t = f) &= P(M_t = m, F_t = f, N_t = m + f) \\
&= P(M_t = m, F_t = f | N_t = m + f) P(N_t = m + f) \\
&= P(M_t = m | N_t = m + f) P(N_t = m + f) \\
&= \frac{(m+f)!}{m!\,f!} p^m (1-p)^f \frac{e^{-\lambda t}(\lambda t)^{m+f}}{(m+f)!} \\
&= \frac{p^m (1-p)^f e^{-\lambda t(p+(1-p))}(\lambda t)^{m+f}}{m!\,f!} \\
&= \left(\frac{e^{-\lambda p t}(\lambda p t)^m}{m!}\right)\left(\frac{e^{-\lambda(1-p)\,t}(\lambda(1-p)t)^f}{f!}\right),
\end{aligned}$$

for $m, f = 0, 1, \ldots$ This shows that M_t and F_t are independent Poisson random variables with parameters $\lambda p t$ and $\lambda(1-p)t$, respectively. In fact, each process $(M_t)_{t\geq 0}$ and $(F_t)_{t\geq 0}$ is a Poisson process, called a *thinned process*. It is not hard to show that both processes inherit stationary and independent increments from the original Poisson birth process.

The birth example with two thinned processes illustrates a general result.

Thinned Poisson Process

Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ. Assume that each arrival, independent of other arrivals, is marked as a *type k* event with probability p_k, for $k = 1, \ldots, n$, where $p_1 + \cdots + p_n = 1$. Let $N_t^{(k)}$ be the number of type k events in $[0, t]$. Then, $\left(N_t^{(k)}\right)_{t\geq 0}$ is a Poisson process with parameter λp_k. Furthermore, the processes

$$\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$$

are independent. Each process is called a *thinned Poisson process*.

Example 6.6 Consider the male and female birth processes. Assume that births occur on a maternity ward at the average rate of 2 births per hour.

(i) On an 8-hour shift, what is the expectation and standard deviation of the number of female births?

(ii) Find the probability that only girls were born between 2 and 5 p.m.

(iii) Assume that five babies were born on the ward yesterday. Find the probability that two are boys.

Solution Let $(N_t)_{t\geq 0}$, $(M_t)_{t\geq 0}$, and $(F_t)_{t\geq 0}$ denote the overall birth, male, and female processes, respectively.

(i) Female births form a Poisson process with parameter

$$\lambda(1 - p) = 2(0.481) = 0.962.$$

The number of female births on an 8-hour shift F_8 has a Poisson distribution with expectation

$$E(F_8) = \lambda(1 - p)8 = 2(0.481)8 = 7.696,$$

and standard deviation

$$SD(F_8) = \sqrt{7.696} = 2.774.$$

(ii) The desired probability is $P(M_3 = 0, F_3 > 0)$. By independence,

$$\begin{aligned} P(M_3 = 0, F_3 > 0) &= P(M_3 = 0)P(F_3 > 0) \\ &= e^{-2(0.519)3}\left(1 - e^{-2(0.481)3}\right) \\ &= e^{-3.114}\left(1 - e^{-2.886}\right) = 0.042. \end{aligned}$$

(iii) Conditional on there being five births in a given interval, the number of boys in that interval has a binomial distribution with parameters $n = 5$ and $p = 0.519$. The desired probability is

$$\frac{5!}{2!3!}(0.519)^2(0.481)^3 = 0.30.$$

■

Related to the thinned process is the *superposition* process obtained by merging, or adding, independent Poisson processes. We state the following intuitive result without proof:

Superposition Process

Assume that $\left(N_t^{(1)}\right)_{t\geq 0}, \ldots, \left(N_t^{(n)}\right)_{t\geq 0}$ are n independent Poisson processes with respective parameters $\lambda_1, \ldots, \lambda_n$. Let $N_t = N_t^{(1)} + \cdots + N_t^{(n)}$, for $t \geq 0$. Then, $(N_t)_{t\geq 0}$ is a Poisson process with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.

See Figure 6.7 to visualize the superposition of three independent Poisson processes.

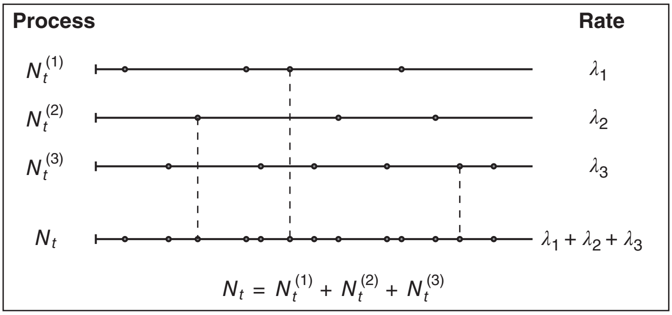

Figure 6.7 The N_t process is the superposition of $N_t^{(1)}$, $N_t^{(2)}$, and $N_t^{(3)}$.

Example 6.7 (Oh my!) In the land of Oz, sightings of lions, tigers, and bears each follow a Poisson process with respective parameters, $\lambda_L, \lambda_T, \lambda_B$, where the time unit is hours. Sightings of the three species are independent of each other.

(i) Find the probability that Dorothy will not see any animal in the first 24 hours from when she arrives in Oz.

(ii) Dorothy saw three animals one day. Find the probability that each species was seen.

Solution

(i) The process of animal sightings $(N_t)_{t\geq 0}$ is the superposition of three independent Poisson processes. Thus, it is a Poisson process with parameter $\lambda_L + \lambda_T + \lambda_B$. The desired probability is

$$P(N_{24} = 0) = e^{-24(\lambda_L+\lambda_T+\lambda_B)}.$$

(ii) Let L_t, T_t, and B_t be the numbers of lions, tigers, and bears, respectively, seen by time t. The desired probability is

$$\begin{aligned}
&P(L_{24} = 1, B_{24} = 1, T_{24} = 1 | N_{24} = 3)\\
&\quad = \frac{P(L_{24} = 1, B_{24} = 1, T_{24} = 1, N_{24} = 3)}{P(N_{24} = 3)}\\
&\quad = \frac{P(L_{24} = 1, B_{24} = 1, T_{24} = 1)}{P(N_{24} = 3)}\\
&\quad = \frac{P(L_{24} = 1)P(B_{24} = 1)P(T_{24} = 1)}{P(N_{24} = 3)}\\
&\quad = \frac{(e^{-24\lambda_L} 24\lambda_L)(e^{-24\lambda_T} 24\lambda_T)(e^{-24\lambda_B} 24\lambda_B)}{e^{-24(\lambda_L+\lambda_T+\lambda_B}(24(\lambda_L + \lambda_T + \lambda_B))^3/3!}\\
&\quad = \frac{6\lambda_L\lambda_B\lambda_T}{(\lambda_L + \lambda_B + \lambda_T)^3}.
\end{aligned}$$

■

Embedding and the Birthday Problem

Sometimes discrete problems can be solved by *embedding* them in continuous ones. The methods in this section were popularized in Blom and Holst (1991) for solving discrete balls-and-urn models, which involve sampling with and without replacement. We illustrate the method on the famous birthday problem. (If your probability class did not cover the birthday problem, you should ask for your money back.)

The classic birthday problem asks, "How many people must be in a room before the probability that some share a birthday, ignoring year and leap days, is at least 50%?"

The probability that two people have the same birthday is 1 minus the probability that no one shares a birthday, which is

$$p_k = 1 - \prod_{i=1}^{k} \frac{366 - i}{365} = 1 - \frac{365!}{(365 - k)!365^k}.$$

One finds that $p_{22} = 0.476$ and $p_{23} = 0.507$. Thus, 23 people are needed.

Consider this variant of the birthday problem, assuming a random person's birthday is uniformly distributed on the 365 days of the year. People enter a room one by one. How many people are in the room the first time that two people share the same birthday? Let K be the desired number. We show how to find the mean and standard deviation of K by embedding.

Consider a continuous-time version of the previous question. People enter a room according to a Poisson process $(N_t)_{t\geq 0}$ with rate $\lambda = 1$. Each person is independently *marked* with one of 365 birthdays, where all birthdays are equally likely. The procedure creates 365 thinned Poisson processes, one for each birthday. Each of the 365 processes are independent, and their superposition gives the process of people entering the room. See Figure 6.8.

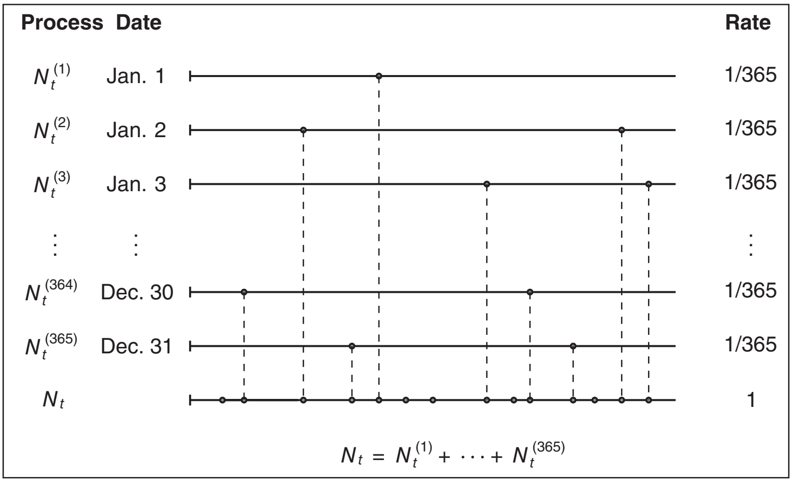

Figure 6.8 Embedding the birthday problem in a superposition of Poisson processes.

Let $X_1, X_2, \ldots,$ be the interarrival sequence for the process of people entering the room. The X_i are i.i.d. exponential random variables with mean 1. Let T be the first time when two people in the room share the same birthday. Then,

$$T = \sum_{i=1}^{K} X_i. \tag{6.3}$$

Equation (6.3) relates the interarrival times $X_1, X_2, \ldots,$ the continuous time T, and the discrete count K.

The X_i are independent of K. The random variable T is represented as a random sum of random variables. By results for such sums (see Example 1.27),

$$E[T] = E[K]\ E[X_1] = E[K].$$

For each $k = 1, \ldots, 365$, let Z_k be the time when the second person marked with birthday k enters the room. Then, the first time two people in the room have the same birthday is $T = \min_{1\leq k\leq 365} Z_k$. Each Z_k, being the arrival time of the second event of a Poisson process, has a gamma distribution with parameters $n = 2$ and $\lambda = 1/365$, with density

$$f(t) = \frac{te^{-t/365}}{365^2}, \quad \text{for } t > 0,$$

and cumulative distribution function

$$P(Z_1 \leq t) = \int_0^t \frac{se^{-s/365}}{365^2}\, ds = 1 - \frac{e^{-t/365}(365 + t)}{365}.$$

This gives

$$\begin{aligned} P(T > t) &= P(\min(Z_1, \dots, Z_{365}) > t) \\ &= P(Z_1 > t, \dots, Z_{365} > t) \\ &= P(Z_1 > t)^{365} = \left(1 + \frac{t}{365}\right)^{365} e^{-t}, \quad \text{for } t > 0. \end{aligned}$$

The desired birthday expectation is

$$E(K) = E(T) = \int_0^\infty P(T > t)\, dt = \int_0^\infty \left(1 + \frac{t}{365}\right)^{365} e^{-t}\, dt. \tag{6.4}$$

The second equality makes use of a general result for the expectation of a positive, continuous random variable. (For reference, see Exercise 1.21.) The last integral of Equation (6.4) is difficult to solve exactly. It can be estimated using a Taylor series approximation. A numerical software package finds $E(K) = 24.617$ and $Var\,(K) = 779.23$, with standard deviation 27.91.

We invite the reader to use embedding to find the expected number of people needed for *three* people to share the same birthday. See Exercise 6.19.

6.5 UNIFORM DISTRIBUTION

It is common to think of a Poisson process as modeling a *completely random* distribution of events, or points, on the positive number line. Although it is not possible to have a uniform distribution on $[0, \infty)$, or any unbounded interval, there is nevertheless a strong connection between a Poisson process and the uniform distribution.

If a Poisson process contains exactly n events in an interval $[0, t]$, then the unordered *locations*, or times, of those events are uniformly distributed on the interval.

For the case of one event, the assertion is easily shown. Consider the distribution of the time of the first arrival, conditional on there being one arrival by time t. For $0 \leq s \leq t$,

$$\begin{aligned} P(S_1 \leq s | N_t = 1) &= \frac{P(S_1 \leq s, N_t = 1)}{P(N_t = 1)} = \frac{P(N_s = 1, N_t = 1)}{P(N_t = 1)} \\ &= \frac{P(N_s = 1, N_t - N_s = 0)}{P(N_t = 1)} \\ &= \frac{P(N_s = 1)P(N_{t-s} = 0)}{P(N_t = 1)} \\ &= \frac{e^{-\lambda s}\lambda s e^{-\lambda(t-s)}}{e^{-\lambda t}\lambda t} = \frac{s}{t}, \end{aligned}$$

which is the cumulative distribution function of the uniform distribution on $[0, t]$.

To discuss the case of more than one arrival in $[0, t]$, we introduce the topic of *order statistics*. Let $U_1, \ldots, U_n$ be an i.i.d. sequence of random variables uniformly distributed on $[0, t]$. Their joint density function is

$$f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) = \frac{1}{t^n}, \text{ for } 0 \leq u_1, \ldots, u_n \leq t.$$

Arrange the U_i in increasing order $U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(n)}$, where $U_{(k)}$ is the kth smallest of the U_i. The ordered sequence $(U_{(1)}, \ldots, U_{(n)})$ is called the *order statistics* of the original sequence.

The joint density function of the order statistics is

$$f_{U_{(1)},\ldots,U_{(n)}}(u_1, \ldots, u_n) = \frac{n!}{t^n}, \text{for } 0 \leq u_1 < \cdots < u_n \leq t.$$

We will not prove this rigorously, but give an intuitive argument. Assume that

$$U_{(1)} = u_1, \ldots, U_{(n)} = u_n, \text{ for } 0 < u_1 < \cdots < u_n < t.$$

Consider a sample of n independent uniform random variables on $[0, t]$. There are $n!$ such samples that would give rise to these order statistic values, as there are $n!$ orderings of the n distinct numbers $u_1, \ldots, u_n$. The value of the joint density for each of these uniform samples is equal to $1/t^n$. Hence, the infinitesimal probability

$$\begin{aligned} f_{U_{(1)},\ldots,U_{(n)}}(u_1, \ldots, u_n)du_1 \cdots du_n &= n! f_{U_1,\ldots,U_n}(u_1, \ldots, u_n)du_1 \cdots du_n \\ &= \frac{n!}{t^n} du_1 \cdots du_n, \end{aligned}$$

which establishes the claim.

We can now describe the joint distribution of the arrival times in a Poisson process, conditional on the number of arrivals.

Arrival Times and Uniform Distribution

Let $S_1, S_2, \ldots,$ be the arrival times of a Poisson process with parameter λ. Conditional on $N_t = n$, the joint distribution of $(S_1, \ldots, S_n)$ is the distribution of the order statistics of n i.i.d. uniform random variables on $[0, t]$. That is, the joint density function of $S_1, \ldots, S_n$ is

$$f(s_1, \ldots, s_n) = \frac{n!}{t^n}, \text{ for } \quad 0 < s_1 < \cdots < s_n < t. \tag{6.5}$$

Equivalently, let $U_1, \ldots, U_n$ be an i.i.d. sequence of random variables uniformly distributed on $[0, t]$. Then, conditional on $N_t = n$,

$$(S_1, \ldots, S_n) \text{ and } (U_{(1)}, \ldots, U_{(n)})$$

have the same distribution.

Proof. For jointly distributed random variables $S_1, \ldots, S_n$ with joint density f,

$$f(s_1, \ldots, s_n) = \lim_{\epsilon_1 \to 0} \cdots \lim_{\epsilon_n \to 0} \frac{P(s_1 \le S_1 \le s_1 + \epsilon_1, \ldots, s_n \le S_n \le s_n + \epsilon_n)}{\epsilon_1 \cdots \epsilon_n}.$$

To establish Equation (6.5), assume that $0 < s_1 < s_2 < \cdots < s_n < t$ and consider the event

$$\{s_1 \le S_1 \le s_1 + \epsilon_1, \ldots, s_n \le S_n \le s_n + \epsilon_n\},$$

given that there are exactly n arrivals in $[0, t]$. For $\epsilon_1, \ldots, \epsilon_n$ sufficiently small, this is the event that each of the intervals $(s_i, s_i + \epsilon_i]$ contains exactly one arrival, and no arrivals occur elsewhere in $[0, t]$. By stationary and independent increments,

$$\begin{aligned}
P(s_1 \le & S_1 \le s_1 + \epsilon_1, \ldots, s_n \le S_n \le s_n + \epsilon_n | N_t = n) \\
&= \frac{P(N_{s_1+\epsilon_1} - N_{s_1} = 1, \ldots, N_{s_n+\epsilon_n} - N_{s_n} = 1, N_t = n)}{P(N_t = n)} \\
&= \frac{\lambda\epsilon_1 e^{-\lambda\epsilon_1} \cdots \lambda\epsilon_n e^{-\lambda\epsilon_n} e^{-\lambda(t-\epsilon_1-\cdots-\epsilon_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\
&= \frac{n!\epsilon_1 \cdots \epsilon_n}{t^n}.
\end{aligned}$$

Dividing by $\epsilon_1 \cdots \epsilon_n$, and letting each $\epsilon_i \to 0$, gives the result. ■

There is a lot of information contained in Equation (6.5). From this, one can obtain joint distributions of any subset of the arrival times, including the marginal distributions. For instance, the *last (first)* arrival time, conditional on there being n arrivals in $[0, t]$, has the same distribution as the maximum (minimum) of n independent uniform random variables on $(0, t)$.

Example 6.8 Starting at time $t = 0$, patrons arrive at a restaurant according to a Poisson process with rate 20 customers per hour.

(i) Find the probability that the 60th customer arrives in the interval $[2.9, 3]$.

(ii) If 60 people arrive at the restaurant by time $t = 3$, find the probability that the 60th customer arrives in the interval $[2.9, 3]$.

Solution

(i) The time of the 60th arrival S_{60} has a gamma distribution with parameters $n = 60$ and $\lambda = 20$. The desired probability is $P(2.9 < S_{60} < 3)$. In R, type

```
> pgamma(3,60,20)-pgamma(2.9,60,20)
[1] 0.1034368
```

(ii) Given $N_3 = 60$, the arrival time of the 60th customer has the same distribution as the maximum M of 60 i.i.d. random variables uniformly distributed on $(0, 3)$. The desired probability is

$$\begin{aligned}
P(2.9 < S_{60} < 3|N_3 = 60) &= P(2.9 < M < 3) = 1 - P(M \le 2.9)\\
&= 1 - P(U_1 \le 2.9, \dots , U_{60} \le 2.9)\\
&= 1 - P(U_1 \le 2.9)^{60}\\
&= 1 - \left(\frac{2.9}{3}\right)^{60}\\
&= 1 - 0.131 = 0.869.
\end{aligned}$$

■

Example 6.9 Concert-goers arrive at a show according to a Poisson process with parameter λ. The band starts playing at time t. The kth person to arrive in $[0, t]$ waits $t - S_k$ time units for the start of the concert, where S_k is the kth arrival time. Find the expected total waiting time of concert-goers who arrive before the band starts.

Solution The desired expectation is $E\left(\sum_{k=1}^{N_t}(t - S_k)\right)$. Conditioning on N_t,

$$\begin{aligned}
E\left(\sum_{k=1}^{N_t}(t - S_k)\right) &= \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n}(t - S_k)|N_t = n\right) P(N_t = n)\\
&= \sum_{n=1}^{\infty} E\left(tn - \sum_{k=1}^{n} S_k|N_t = n\right) P(N_t = n)\\
&= \sum_{n=1}^{\infty} \left(tn - E\left(\sum_{k=1}^{n} S_k|N_t = n\right)\right) P(N_t = n)\\
&= \lambda t^2 - \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} S_k|N_t = n\right) P(N_t = n).
\end{aligned}$$

Conditional on n arrivals in $[0, t]$, $S_1 + \cdots + S_n$ has the same distribution as the sum of the uniform order statistics. Furthermore, $\sum_{k=1}^{n} U_{(k)} = \sum_{k=1}^{n} U_k$. This gives

$$\begin{aligned}
\sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} S_k | N_t = n\right) P(N_t = n) &= \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} U_{(k)}\right) P(N_t = n) \\
&= \sum_{n=1}^{\infty} E\left(\sum_{k=1}^{n} U_k\right) P(N_t = n) \\
&= \sum_{n=1}^{\infty} \frac{nt}{2} P(N_t = n) \\
&= \frac{\lambda t^2}{2}.
\end{aligned}$$

The desired expectation is

$$E\left(\sum_{k=1}^{N_t} (t - S_k)\right) = \lambda t^2 - \frac{\lambda t^2}{2} = \frac{\lambda t^2}{2}.$$

■

Example 6.10 Students enter a campus building according to a Poisson process $(N_t)_{t \geq 0}$ with parameter λ. The times spent by each student in the building are i.i.d. random variables with continuous cumulative distribution function $F(t)$. Find the probability mass function of the number of students in the building at time t, assuming there are no students in the building at time 0.

Solution Let B_t denote the number of students in the building at time t. Conditioning on N_t,

$$\begin{aligned}
P(B_t = k) &= \sum_{n=k}^{\infty} P(B_t = k | N_t = n) P(N_t = n) \\
&= \sum_{n=k}^{\infty} P(B_t = k | N_t = n) \frac{e^{-\lambda t} (\lambda t)^n}{n!}.
\end{aligned}$$

Assume that n students enter the building by time t, with arrival times $S_1, \ldots, S_n$. Let Z_k be the length of time spent in the building by the kth student, for $1 \leq k \leq n$. Then, $Z_1, \ldots, Z_n$ are i.i.d. random variables with cdf F, and students leave the building at times $S_1 + Z_1, \ldots, S_n + Z_n$. There are k students in the building at time t if and only if k of the departure times $S_1 + Z_1, \dots, S_n + Z_n$ exceed t. This gives

$$\begin{aligned} P(B_t = k|N_t = n) &= P(k \text{ of the } S_1 + Z_1, \dots, S_n + Z_n \text{ exceed } t|N_t = n) \\ &= P(k \text{ of the } U_{(1)} + Z_1, \dots, U_{(n)} + Z_n \text{ exceed } t) \\ &= P(k \text{ of the } U_1 + Z_1, \dots, U_n + Z_n \text{ exceed } t) \\ &= \binom{n}{k} p^k (1-p)^{n-k}, \end{aligned}$$

where

$$p = P(U_1 + Z_1 > t) = \frac{1}{t}\int_0^t P(Z_1 > t - x)\, dx = \frac{1}{t}\int_0^t [1 - F(x)]\, dx.$$

This gives

$$\begin{aligned} P(B_t = k) &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= \frac{p^k (\lambda t)^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} (\lambda t)^{n-k}}{(n-k)!} \\ &= \frac{p^k (\lambda t)^k}{k!} e^{\lambda(1-p)t} \\ &= \frac{e^{-\lambda p t} (\lambda p t)^k}{k!}, \quad \text{for } k = 0, 1, \dots \end{aligned}$$

That is, B_t has a Poisson distribution with parameter λp, where

$$p = \int_0^t [1 - F(x)]\, dx.$$

■

Simulation

Results for arrival times and the uniform distribution offer a new method for simulating a Poisson process with parameter λ on an interval $[0, t]$:

1. Simulate the number of arrivals N in $[0, t]$ from a Poisson distribution with parameter λt.
2. Generate N i.i.d. random variables uniformly distributed on $(0, t)$.
3. Sort the variables in increasing order to give the Poisson arrival times.

R: Simulating a Poisson Process on $[0, t]$

Following is a simulation of the arrival times of a Poisson process with parameter $\lambda = 1/2$ on $[0, 30]$.

```
# poissonsim.R
> t <- 30
> lambda <- 1/2
> N <- rpois(1,lambda*t)
> unifs <- runif(N,0,t))
> arrivals <- sort(unifs)
> arrivals
 [1]  8.943  9.835 11.478 12.039 16.009 17.064 17.568
 [8] 17.696 18.663 22.961 24.082 24.440 28.250
```

6.6 SPATIAL POISSON PROCESS

The spatial Poisson process is a model for the distribution of events, or *points*, in two- or higher-dimensional space. Such processes have been used to model the location of trees in a forest, galaxies in the night sky, and cancer clusters across the United States. For $d \geq 1$ and $A \subseteq \mathbb{R}^d$, let N_A denote the number of points in the set A. Write $|A|$ for the *size* of A (e.g., area in $\mathbb{R}^2$, volume in $\mathbb{R}^3$).

Spatial Poisson Process

A collection of random variables $(N_A)_{A\subseteq\mathbb{R}^d}$ is a spatial Poisson process with parameter λ if

1. for each bounded set $A \subseteq \mathbb{R}^d$, N_A has a Poisson distribution with parameter $\lambda|A|$.
2. whenever A and B are disjoint sets, N_A and N_B are independent random variables.

Observe how the spatial Poisson process generalizes the regular one-dimensional Poisson process. Property 1 gives the analogue of stationary increments, where the *size* of an interval is the length of the interval. Property 2 is the counterpart of independent increments.

Example 6.11 A spatial Poisson process in the plane has parameter $\lambda = 1/2$. Find the probability that a disk of radius 2 centered at $(3, 4)$ contains exactly 5 points.

Solution Let C denote the disk. Then, $|C| = \pi r^2 = 4\pi$. The desired probability is

$$P(N_C = 5) = \frac{e^{-\lambda|C|}(\lambda|C|)^5}{5!} = \frac{e^{-2\pi}(2\pi)^5}{5!} = 0.152.$$

∎

The uniform distribution arises for the spatial process in a similar way to how it does for the one-dimensional Poisson process. Given a bounded set $A \subseteq \mathbb{R}^d$, then conditional on there being n points in A, the locations of the points are uniformly distributed in A. For this reason, a spatial Poisson process is sometimes called a model of *complete spatial randomness*.

To simulate a spatial Poisson process with parameter λ on a bounded set A, first simulate the number of points N in A according to a Poisson distribution with parameter $\lambda|A|$. Then, generate N points uniformly distributed in A.

Four realizations of a spatial Poisson process with parameter $\lambda = 100$ on the square $[0, 1] \times [0, 1]$ are shown in Figure 6.9. The circle C inside the square is centered at $(0.7, 0.7)$ with radius $r = 0.2$. The simulation was repeated 100,000 times, counting the number of points in the circle at each iteration. Table 6.1 shows the numerical results. The distribution of counts is seen to be close to the expected counts for a Poisson distribution with mean $\lambda|C| = 100\pi(0.2)^2 = 12.567$.

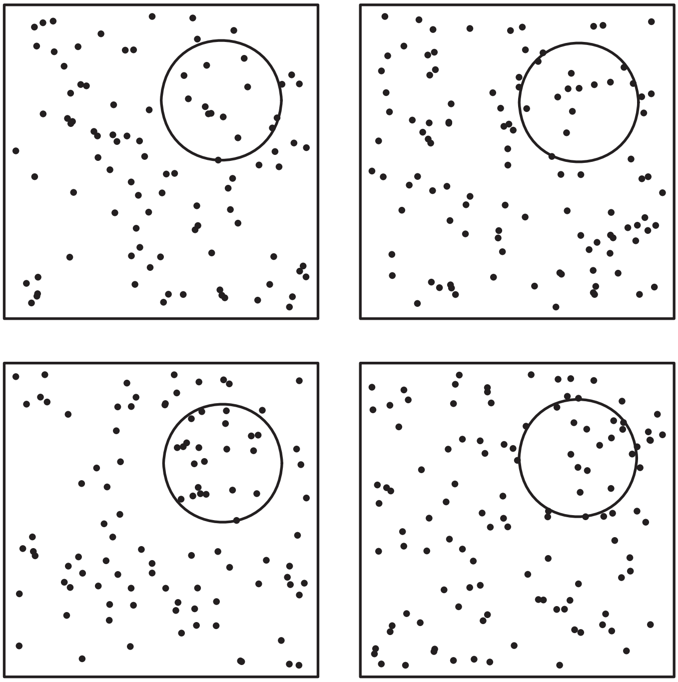

Figure 6.9 Samples of a spatial Poisson process.

TABLE 6.1 Number of Points in a Circle of Radius $r = 0.2$ for a Spatial Poisson Process with $\lambda = 100$.

Counts	0–4	5–9	10–14	15–19	20–24	25–29
Observed	522	19200	52058	24975	3135	106
Expected	510.0	19130.0	522215.6	24941.5	3075.00	125.9

Simulation is based on 100,000 trials.

R: Simulating a Spatial Poisson Process

```
# spatialPoisson.R
> lambda <- 100
> squarearea <- 1
> trials <- 100000
> simlist <- numeric(trials)
> for (i in 1:trials) {
+  N <- rpois(1,lambda*squarearea)
+  xpoints <- runif(N,0,1)
+  ypoints <- runif(N,0,1)
+  ct <- sum(((xpoints-0.7)^2+(ypoints-0.7)^2)<=0.2^2)
+  simlist[i] <- ct } # number of points in circle
> mean(simlist)
[1] 12.57771
> var(simlist)
[1] 12.57435
> # Compare with theoretical mean and variance
> lambda*pi*(0.2)^2
[1] 12.56637
```

A spatial Poisson process is a special case of a *point process*, which is a general model for the distribution of points in space. There is an abundance of applications of point processes, which include models that incorporate clustering, attraction, repulsion, time and space dependence, and so on. Often one wants to measure how close or far a given point pattern is from complete spatial randomness, that is, from a spatial Poisson process. A common measure is the *nearest-neighbor distance* defined to be the distance between an arbitrary point and the point of the process closest to it.

Consider a spatial Poisson process in $\mathbb{R}^2$ with parameter λ. Let x denote a fixed point in the plane. Let D be the distance from x to its nearest neighbor. The event $\{D > t\}$ occurs if and only if there are no points in the circle centered at x of radius t. Let C_x denote such a circle. Then,

$$P(D > t) = P\left(N_{C_x} = 0\right) = e^{-\lambda|C_x|} = e^{-\lambda\pi t^2}, \text{ for } t > 0.$$

Differentiating gives the density function for the nearest-neighbor distance

$$f_D(t) = e^{-\lambda\pi t^2} 2\lambda\pi t, \quad \text{for } t > 0,$$

with mean and variance

$$E(D) = \frac{1}{2\sqrt{\lambda}} \quad \text{and} \quad Var(D) = \frac{4-\pi}{4\pi\lambda}.$$

Example 6.12 (South Carolina swamp forest) Jones et al. (1994) introduce data for the locations of 630 trees, including 91 cypress trees, in a swamp hardwood forest in South Carolina for the purpose of studying tree population dynamics. See Figure 6.10. The data are contained in a 200×50 m^2 area. The average nearest-neighbor distance for all tree locations is 1.990 m. For the cypress trees it is 5.08 m.

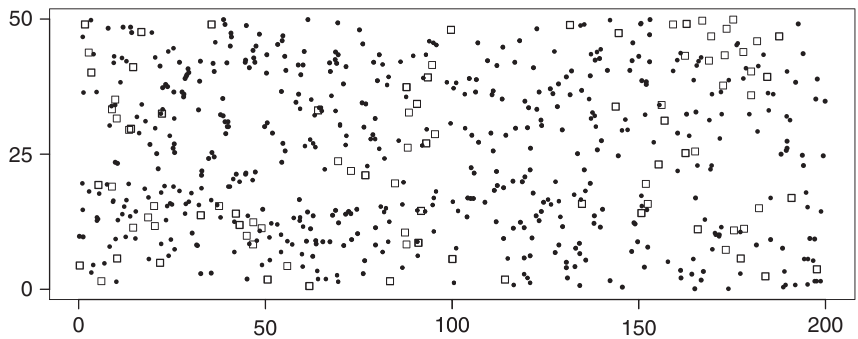

Figure 6.10 Plot of tree locations in a hardwood swamp in South Carolina. Squares are locations of cypress trees and dots locations of any other species. *Source:* Dixon (2012).

Researchers are interested in whether there is evidence of clustering. In a spatial Poisson process, with points distributed at the rate of 630 per 10,000 m^2, the expected nearest-neighbor distance is

$$E(D) = \frac{1}{2\sqrt{630/10000}} = 1.992 \text{ m},$$

with standard error

$$SD(D)/\sqrt{n} = \sqrt{\frac{4-\pi}{4\pi 630/10000}} \Big/ \sqrt{630} = 0.041.$$

For the 91 cypress trees, a model of complete spatial randomness would yield

$$E(D) = \frac{1}{2\sqrt{91/100000}} = 5.241 \text{ m},$$

with standard error

$$SD(D)/\sqrt{n} = \sqrt{\frac{4-\pi}{4\pi 630/10000}}\Big/\sqrt{91} = 1.041 \text{ m}.$$

As measured by nearest-neighbor distance, the data do not show evidence of clustering. We note that researchers were able to detect some small evidence of clustering for these data by using more sophisticated spatial statistic tools. ■

6.7 NONHOMOGENEOUS POISSON PROCESS

In a Poisson process, arrivals occur at a constant rate, independent of time. However, for many applications this is an unrealistic assumption. Consider lunch time at a college cafeteria. The doors open at 11 a.m. Students arrive at an increasing rate until the noon peak hour. Then, the rate stays constant for 2 hours, after which it declines until 3 p.m., when the cafeteria closes.

Such activity can be modeled by a nonhomogeneous Poisson process with rate $\lambda = \lambda(t)$, which depends on t. Such a rate *function* is called the *intensity function*.

Nonhomogeneous Poisson Process

A counting process $(N_t)_{t\geq 0}$ is a *nonhomogeneous Poisson process with intensity function* $\lambda(t)$, if

1. $N_0 = 0$.
2. For all $t > 0$, N_t has a Poisson distribution with mean

$$E(N_t) = \int_0^t \lambda(x)\, dx.$$

3. For $0 \leq q < r \leq s < t$, $N_r - N_q$ and $N_t - N_s$ are independent random variables.

A nonhomogeneous Poisson process has independent increments, but not necessarily stationary increments. It can be shown that for $0 < s < t$, $N_t - N_s$ has a Poisson distribution with parameter $\int_s^t \lambda(x)\, dx$. If $\lambda(t) = \lambda$ is constant, we obtain the regular Poisson process with parameter λ.

Example 6.13 Students arrive at the cafeteria for lunch according to a nonhomogeneous Poisson process. The arrival rate increases linearly from 100 to 200 students per hour between 11 a.m. and noon. The rate stays constant for the next 2 hours, and then decreases linearly down to 100 from 2 to 3 p.m. Find the probability that there are at least 400 people in the cafeteria between 11:30 a.m. and 1:30 p.m.

Solution The intensity function is

$$\lambda(t) = \begin{cases} 100 + 100t, & 0 \le t \le 1, \\ 200, & 1 < t \le 3, \\ 500 - 100t, & 3 \le t < 4, \end{cases}$$

where t represents hours past 11 a.m. See Figure 6.11.

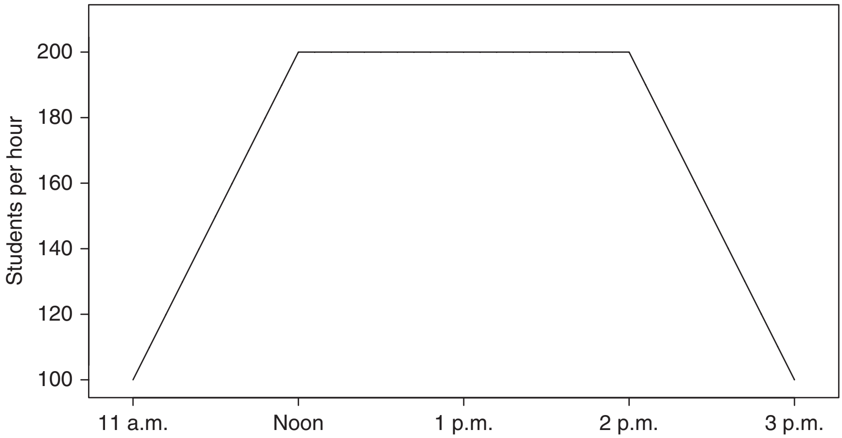

Figure 6.11 Intensity function.

The desired probability is $P(N_{2.5} - N_{0.5} \ge 400)$, where $N_{2.5} - N_{0.5}$ has a Poisson distribution with mean

$$E(N_{2.5} - N_{0.5}) = \int_{0.5}^{2.5} \lambda(t)\, dt = \int_{0.5}^{1} (100 + 100t)\, dt + \int_{1}^{2.5} 200\, dt = 387.5.$$

Then,

$$P(N_{2.5} - N_{0.5} \ge 400) = 1 - \sum_{k=0}^{399} \frac{e^{-387.5}(387.5)^k}{k!} = 0.269.$$

■

Example 6.14 In reliability engineering one is concerned with the probability that a system is working during an interval of time. A common model for failure times is a nonhomogeneous Poisson process with intensity function of the form

$$\lambda(t) = \alpha\beta t^{\beta-1},$$

where $\alpha, \beta > 0$ are parameters, and t represents the age of the system. At $\beta = 1$, the model reduces to a homogeneous Poisson process with parameter α. If the system starts at $t = 0$, the expected number of failures after t time units is

$$E(N_t) = \int_0^t \lambda(x)\, dx = \int_0^t \alpha\beta x^{\beta-1}\, dx = \alpha t^{\beta}.$$

Because of the power law form of the mean failure time, the process is sometimes called a *power law Poisson process*.

Of interest, is the *reliability* $R(t)$, defined as the probability that a system, which starts at time t, is operational up through time $t + c$, for some constant c, that is, the probability of no failures in the interval $(t, t + c]$. This gives

$$R(t) = P(N_{t+c} - N_t = 0) = e^{-\int_t^{t+c} \lambda(x)\, dx} = e^{-\int_t^{t+c} \alpha\beta x^{\beta-1}\, dx} = e^{-\alpha((t+c)^{\beta}-t^{\beta})}.$$

■

6.8 PARTING PARADOX

> How wonderful that we have met with a paradox. Now we have some hope of making progress.
>
> –Niels Bohr

The following classic is based on Feller (1968). Buses arrive at a bus stop according to a Poisson process. The time between buses, on average, is 10 minutes. Lisa gets to the bus stop at time t. How long can she expect to wait for a bus?

Here are two possible answers:

(i) By memorylessness, the time until the next bus is exponentially distributed with mean 10 minutes. Lisa will wait, on average, 10 minutes.

(ii) Lisa arrives at some time between two consecutive buses. The expected time between consecutive buses is 10 minutes. By symmetry, her expected waiting time should be half that, or 5 minutes.

Paradoxically, *both* answers have some truth to them! On the one hand, the time until the next bus will be shown to have an exponential distribution with mean 10 minutes. But the *backwards* time to the previous bus is almost exponential as well, with mean close to 10 minutes. Thus, the time when Lisa arrives at the bus stop is a point in an interval whose length is about 20 minutes. And the argument in (ii) essentially holds. By symmetry, her expected waiting time should be half that, or 10 minutes.

The surprising result is that the interarrival time of the buses before and after Lisa's arrival is about 20 minutes. And yet the expected interarrival time for buses is 10 minutes!

R : Waiting Time Paradox

A Poisson process with parameter $\lambda = 1/10$ is generated on $[0, 200]$. Lisa arrives at the bus stop at time $t = 50$. Simulation shows her average wait time is about 10 minutes.

```
# waitingparadox.R
> mytime <- 50
> lambda <- 1/10
> trials <- 10000
> simlist <- numeric(trials)
> for (i in 1:trials) {
+  N <- rpois(1,300*lambda)
+  arrivals <- sort(runif(N,0,300))
+  wait <- arrivals[arrivals > mytime][1] - mytime
+  simlist[i] <- wait }
> mean(simlist)
[1] 10.04728
```

To explain the paradox, consider the process of bus arrivals. The rate of one arrival per 10 minutes is an average. The time between buses is random, and buses may arrive one right after the other, or there may be a long time between consecutive buses. When Lisa gets to the bus stop, she is more likely to get there during a longer interval between buses than a shorter interval.

To illustrate the idea, pick a random number between 1 and 200. Do it now before reading on. Now look at Figure 6.12, which gives arrival times for a Poisson process with parameter $\lambda = 1/10$ on $[0, 200]$. Find your number. Is your number in a short interval (length less than 10) or a long interval (length greater than 10)? Most readers will find their number in a long interval.

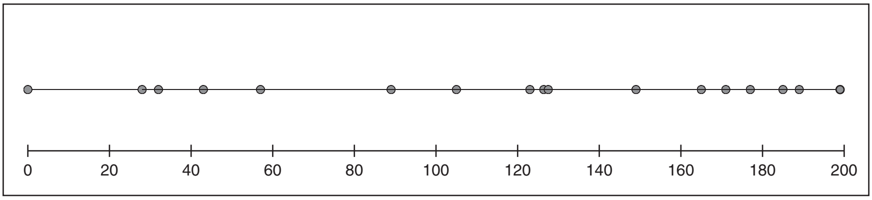

Figure 6.12 Pick a number from 0 to 200. Is your number in a short or long interval?

This example illustrates the phenomenon of *length-biased* or *size-biased* sampling. If you reach into a bag containing pieces of string of different lengths and pick a string *at random*, you tend to pick a longer rather than a shorter piece. Bus interarrival times are analogous to lengths of string.

Here is another example of size-biased sampling. Suppose you want to estimate how much time people spend exercising at the gym. If you go to the gym to survey people at random, you are likely to get a biased estimate, as you are more likely to sample people who work out a lot. Those who rarely go to the gym are not likely to be there when you go!

For the bus waiting problem, the expected length of an interarrival time, *which contains a fixed time* t, is larger—about twice as large—than the average interval length between buses.

Here is an exact analysis. Consider a fixed $t > 0$. The time of the last bus before t is S_{N_t}. The time of the next bus after t is S_{N_t+1}. The expected length of the interval containing t is

$$E\left(S_{N_t+1} - S_{N_t}\right) = E\left(S_{N_t+1}\right) - E\left(S_{N_t}\right).$$

For $E\left(S_{N_t+1}\right)$, condition on N_t. Consider

$$E\left(S_{N_t+1}|N_t = k\right) = E(S_{k+1}|N_t = k) = E(S_{k+1}) = \frac{k+1}{\lambda}.$$

The second equality is because the $(k + 1)$th arrival occurs after time t, and is thus independent of N_t. It follows that $E\left(S_{N_t+1}|N_t\right) = (N_t + 1)/\lambda$. By the law of total expectation,

$$E\left(S_{N_t+1}\right) = E\left(E\left(S_{N_t+1}|N_t\right)\right) = E\left(\frac{N_t+1}{\lambda}\right) = \frac{\lambda t+1}{\lambda} = t + \frac{1}{\lambda}. \tag{6.6}$$

For $E\left(S_{N_t}\right)$, we have that $E\left(S_{N_t}|N_t = k\right) = E(S_k|N_t = k)$. Conditional on $N_t = k$, the kth arrival time has the same distribution as the maximum of k i.i.d. uniform random variables distributed on $(0, t)$. We leave it to the reader to show that this expectation is equal to $tk/(k + 1)$. That is, $E\left(S_{N_t}|N_t = k\right) = tk/(k + 1)$ and thus

$$E\left(S_{N_t}|N_t\right) = tN_t/(N_t + 1) = t - t/(N_t + 1).$$

This gives

$$E\left(S_{N_t}\right) = E\left(E\left(S_{N_t}|N_t\right)\right) = E\left(t - \frac{t}{N_t+1}\right) = t - tE\left(\frac{1}{N_t+1}\right). \tag{6.7}$$

We find

$$\begin{aligned}
E\left(\frac{1}{N_t+1}\right) &= \sum_{k=0}^{\infty}\left(\frac{1}{k+1}\right)\frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
&= \frac{e^{-\lambda t}}{\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^{k+1}}{(k+1)!} = \frac{e^{-\lambda t}}{\lambda t}\sum_{k=1}^{\infty}\frac{(\lambda t)^k}{k!} \\
&= \frac{e^{-\lambda t}}{\lambda t}(e^{\lambda t} - 1) = \frac{1 - e^{-\lambda t}}{\lambda t}.
\end{aligned}$$

Together with Equation (6.7),

$$E\left(S_{N_t}\right) = t - \frac{1}{\lambda} + \frac{e^{-\lambda t}}{\lambda}.$$

Finally with Equation (6.6), the expected length of the interval that contains t is

$$E\left(S_{N_t+1} - S_{N_t}\right) = \left(t + \frac{1}{\lambda}\right) - \left(t - \frac{1}{\lambda} + \frac{e^{-\lambda t}}{\lambda}\right) = \frac{2 - e^{-\lambda t}}{\lambda} \approx \frac{2}{\lambda},$$

for large (or even moderate) t.

EXERCISES

6.1 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 1.5$. Find the following:

(a) $P(N_1 = 2, N_4 = 6)$

(b) $P(N_4 = 6|N_1 = 2)$

(c) $P(N_1 = 2|N_4 = 6)$

6.2 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 2$. Find the following:

(a) $E(N_3N_4)$

(b) $E(X_3X_4)$

(c) $E(S_3S_4)$

6.3 Calls are received at a company call center according to a Poisson process at the rate of five calls per minute.

(a) Find the probability that no call occurs over a 30-second period.

(b) Find the probability that exactly four calls occur in the first minute, and six calls occur in the second minute.

(c) Find the probability that 25 calls are received in the first 5 minutes and six of those calls occur in the first minute.

6.4 Starting at 9 a.m., patients arrive at a doctor's office according to a Poisson process. On average, three patients arrive every hour.

(a) Find the probability that at least two patients arrive by 9:30 a.m.

(b) Find the probability that 10 patients arrive by noon and eight of them come to the office before 11 a.m.

(c) If six patients arrive by 10 a.m., find the probability that only one patient arrives by 9:15 a.m.

6.5 Let $(N_t)_{t\geq 0}$ be a Poisson process. Explain what is wrong with the following *proof* that N_3 is a constant.

$$\begin{aligned} E\left((N_3)^2\right) &= E(N_3 N_3) = E(N_3(N_6 - N_3)) \\ &= E(N_3)E(N_6 - N_3) = E(N_3)E(N_3) \\ &= E(N_3)^2. \end{aligned}$$

Thus, $\text{Var}(N_3) = E\left((N_3)^2\right) - E(N_3)^2 = 0$, which gives that N_3 is a constant with probability 1.

6.6 Occurrences of landfalling hurricanes during an El Niño event are modeled as a Poisson process in Bove et al. (1998). The authors assert that "During an El Niño year, the probability of two or more hurricanes making landfall in the United States is 28%." Find the rate of the Poisson process.

6.7 Ben, Max, and Yolanda are at the front of three separate lines in the cafeteria waiting to be served. The serving times for the three lines follow independent Poisson processes with respective parameters 1, 2, and 3.

(a) Find the probability that Yolanda is served first.

(b) Find the probability that Ben is served before Yolanda.

(c) Find the expected waiting time for the first person served.

6.8 Starting at 6 a.m., cars, buses, and motorcycles arrive at a highway toll booth according to independent Poisson processes. Cars arrive about once every 5 minutes. Buses arrive about once every 10 minutes. Motorcycles arrive about once every 30 minutes.

(a) Find the probability that in the first 20 minutes, exactly three vehicles—two cars and one motorcycle—arrive at the booth.

(b) At the toll booth, the chance that a driver has exact change is 1/4, independent of vehicle. Find the probability that no vehicle has exact change in the first 10 minutes.

(c) Find the probability that the seventh motorcycle arrives within 45 minutes of the third motorcycle.

(d) Find the probability that at least one other vehicle arrives at the toll booth between the third and fourth car arrival.

6.9 Show that the geometric distribution is memoryless.

6.10 Assume that $X_1, X_2, \ldots$ is an i.i.d. sequence of exponential random variables with parameter λ. Let $S_n = X_1 + \cdots + X_n$. Show that S_n has a gamma distribution with parameters n and λ

(a) by moment-generating functions,

(b) by induction on n.

6.11 Show that a continuous probability distribution that is memoryless must be exponential. Hint: For $g(t) = P(X > t)$, show that $g(t) = (g(1))^t$ for all positive, rational t.

6.12 Starting at noon, diners arrive at a restaurant according to a Poisson process at the rate of five customers per minute. The time each customer spends eating at the restaurant has an exponential distribution with mean 40 minutes, independent of other customers and independent of arrival times. Find the distribution, as well as the mean and variance, of the number of diners in the restaurant at 2 p.m.

6.13 Assume that $(N_t)_{t\geq 0}$ is a Poisson process with parameter λ. Find the conditional distribution of N_s given $N_t = n$, for

(a) $s < t$,

(b) $s > t$.

6.14 Red cars arrive at an intersection according to a Poisson process with parameter r. Blue cars arrive, independently of red cars, according to a Poisson process with parameter b. Let X be the number of blue cars which arrive between two successive red cars. Show that X has a geometric distribution.

6.15 Failures occur for a mechanical process according to a Poisson process. Failures are classified as either major or minor. Major failures occur at the rate of 1.5 failures per hour. Minor failures occur at the rate of 3.0 failures per hour.

(a) Find the probability that two failures occur in 1 hour.

(b) Find the probability that in half an hour, no major failures occur.

(c) Find the probability that in 2 hours, at least two major failures occur or at least two minor failures occur.

6.16 Accidents occur at a busy intersection according to a Poisson process at the rate of two accidents per week. Three out of four accidents involve the use of alcohol.

(a) What is the probability that three accidents involving alcohol will occur next week?

(b) What is the probability that at least one accident occurs tomorrow?

(c) If six accidents occur in February (four weeks), what is the probability that less than half of them involve alcohol?

6.17 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter λ and arrival times $S_1, S_2, \ldots$ Evaluate the expected sum of squares of the arrival times before time t,

$$E\left(\sum_{n=1}^{N_t} S_n^2\right).$$

6.18 The planets of the Galactic Empire are distributed in space according to a spatial Poisson process at an approximate density of one planet per cubic parsec. From the Death Star, let X be the distance to the nearest planet.

(a) Find the probability density function of X.

(b) Find the mean distance from the Death Star to the nearest planet.

6.19 Members of a large audience are asked to state their birthdays, one at a time. How many people will be asked before three persons are found to have the same birthday? Use embedding to estimate the expected number. You will need to use numerical methods to evaluate the resulting integral.

6.20 Consider a spatial point process in $\mathbb{R}^2$ with parameter λ. Assume that A is a bounded set in $\mathbb{R}^2$ which contains exactly one point of the process. Given $B \subseteq A$, find the probability that B contains one point.

6.21 For a Poisson process with parameter λ show that for $s < t$, the correlation between N_s and N_t is

$$\text{Corr}(N_s, N_t) = \sqrt{\frac{s}{t}}.$$

6.22 At the Poisson Casino, two dice are rolled at random times according to a Poisson process with parameter λ. Find the probability that in $[0, t]$ every pair of dice rolled comes up 7.

6.23 Oak and maple trees are each located in the arboretum according to independent spatial Poisson processes with parameters λ_O and λ_M, respectively.

(a) In a region of x square meters, find the probability that both species of trees are present.

(b) In the arboretum, there is a circular pond of radius 100 m. Find the probability that within 20 m of the pond there are only oaks.

6.24 Tom is bird watching in the arboretum. The times when he sights a meadowlark occur in accordance with a Poisson process with parameter λ. The times when he sights a sparrow occur as a Poisson process with parameter μ. Assume that meadowlark and sparrow sightings are independent.

(a) Find the probability that a meadowlark is seen first.

(b) Find the probability the one bird is seen by time $t = 1$.

(c) Find the probability that one sparrow and two meadowlarks are seen by time $t = 2$.

6.25 Computers in the lab fail, on average, twice a day, according to a Poisson process. Last week, 10 computers failed. Find the expected time of the last failure, and give an approximate time of day when the last failure occurred.

6.26 If $(N_t)_{t\geq 0}$ is a Poisson process with parameter λ, find the probability generating function of N_t.

6.27 In a small parliamentary election, votes are counted according to a Poisson process at the rate of 60 votes per minute. There are six political parties, whose popularity among the electorate is shown by this distribution.

A	B	C	D	E	F
0.05	0.30	0.10	0.10	0.25	0.20

(a) In the first 2 minutes of the vote tally, 40 people had voted for parties E and F. Find the probability that more than 100 votes were counted in the first 2 minutes.

(b) If vote counting starts at 3 p.m., find the probability that the first vote is counted by the first second after 3 p.m.

(c) Find the probability that the first vote for party C is counted before a vote for B or D.

6.28 Tornadoes hit a region according to a Poisson process with $\lambda = 2$. The number of insurance claims filed after any tornado has a Poisson distribution with mean 30. The number of tornadoes is independent of the number of insurance claims. Find the expectation and standard deviation of the total number of claims filed by time t.

6.29 Job offers for a recent college graduate arrive according to a Poisson process with mean two per month. A job offer is acceptable if the salary offered is at least \$35,000. Salary offers follow an exponential distribution with mean \$25,000. Find the probability that an acceptable job offer will be received within 3 months.

6.30 See Example 6.9. Find the variance of total waiting time of concert-goers who arrive before the band starts.

6.31 See Example 6.9. Find the expected *average* waiting time of concert-goers who arrive before the band starts.

6.32 Investors purchase \$1,000 bonds at the random times of a Poisson process with parameter λ. If the interest rate is r, then the *present value* of an investment purchased at time t is $1000e^{-rt}$. Show that the expected total present value of the bonds purchased by time t is $1000\lambda(1 - e^{-rt})/r$.

6.33 Let $S_1, S_2, \ldots$ be the arrival times of a Poisson process with parameter λ. Given the time of the nth arrival, find the expected time $E(S_1|S_n)$ of the first arrival.

6.34 Describe in words the random variable S_{N_t}. Find the distribution of S_{N_t} by giving the cumulative distribution function.

6.35 See the definitions for the spatial and nonhomogeneous Poisson processes. Define a nonhomogeneous, spatial Poisson process in $\mathbb{R}^2$. Consider such a process $(N_A)_{A\subseteq\mathbb{R}^2}$ with intensity function

$$\lambda(x, y) = e^{-(x^2+y^2)}, \text{for } -\infty < x, y < \infty.$$

Let C denote the unit circle, that is, the circle of radius 1 centered at the origin. Find $P(N_C = 0)$.

6.36 Starting at 9 a.m., customers arrive at a store according to a nonhomogeneous Poisson process with intensity function $\lambda(t) = t^2$, for $t > 0$, where the time unit is hours. Find the probability mass function of the number of customers who enter the store by noon.

6.37 A *compound Poisson process* $(C_t)_{t\geq 0}$ is defined as

$$C_t = \sum_{k=1}^{N_t} X_k,$$

where $(N_t)_{t\geq 0}$ is a Poisson process, and $X_1, X_2, \ldots$ is an i.i.d. sequence of random variables that are independent of $(N_t)_{t\geq 0}$.

Assume that automobile accidents at a dangerous intersection occur according to a Poisson process at the rate of 3 accidents per week. Furthermore, the number of people seriously injured in an accident has a Poisson distribution with mean 2. Show that the process of serious injuries is a compound Poisson process, and find the mean and standard deviation of the number of serious injuries over 1 year's time.

6.38 A *mixed Poisson process*, also called a *Cox process* or *doubly stochastic process*, arises from a Poisson process where the parameter Λ is itself a random variable. If $(N_t)_{t\geq 0}$ is a mixed Poisson process, then the conditional distribution of N_t given $\Lambda = \lambda$ is Poisson with parameter λt. Assume that for such a process Λ has an exponential distribution with parameter μ. Find the probability mass function of N_t.

6.39 See Exercise 6.38. Assume that $(N_t)_{t\geq 0}$ is a mixed Poisson process with rate parameter uniformly distributed on (0, 1). Find $P(N_1 = 1)$.

6.40 Assume that $(N_t)_{t\geq 0}$ is a mixed Poisson process whose rate Λ has a gamma distribution with parameters n and λ. Show that

$$P(N_t = k) = \binom{n+k-1}{k}\left(\frac{\lambda}{\lambda+t}\right)^n\left(\frac{t}{\lambda+t}\right)^k, \quad \text{for } k = 0, 1, \ldots$$

6.41 `R` : Goals occur in a soccer game according to a Poisson process. The average total number of goals scored for a 90-minute match is 2.68. Assume that two teams are evenly matched. Use simulation to estimate the probability both teams will score the same number of goals. Compare with the theoretical result.

6.42 `R` : Simulate the restaurant results of Exercise 6.12.

6.43 `R` : Simulate a spatial Poisson process with $\lambda = 10$ on the box of volume 8 with vertices at the 8 points $(\pm 1, \pm 1, \pm 1)$. Estimate the mean and variance of the number of points in the ball centered at the origin of radius 1. Compare with the exact values.

6.44 `R` : See Exercise 6.32. Simulate the expected total present value of the bonds if the interest rate is 4%, the Poisson parameter is $\lambda = 50$, and $t = 10$. Compare with the exact value.

6.45 `R` : Simulate the birthday problem of Exercise 6.19.