Math 243 Analysis 2 Assignment 2

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Problem 2a. Let $\epsilon > 0$, $\delta := \epsilon$ and let $\dot{\mathcal{P}}$ be an arbitrary tagged partition of [a,b] such that $||\dot{\mathcal{P}}|| < \delta$.

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$S(g;\dot{\mathcal{P}}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1})$$

$$S(g;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} g(t_i)(x_i - x_{i-1}) - f(t_i)(x_i - x_{i-1})$$

$$\sum_{i=1}^{n}(x_i - x_{i-1})(g(t_i) - f(t_i)) \geq 0$$

Since $(x_i - x_{i-1}) > 0$ and $g(t_i) - f(t_i) \geq 0$

$$\implies S(f;\dot{\mathcal{P}}) \leq S(g;\dot{\mathcal{P}})$$

Since f and g are Riemann integrable,

$$|S(f;\dot{\mathcal{P}}) - \int_a^b f| < \epsilon \ and \ |S(g;\dot{\mathcal{P}}) - \int_a^b g| < \epsilon$$

Therefore,

$$-\epsilon < S(f;\dot{\mathcal{P}}) - \int_a^b f < \epsilon \implies -\epsilon + \int_a^b f < S(f;\dot{\mathcal{P}}) < \epsilon + \int_a^b f$$

$$-\epsilon < S(g;\dot{\mathcal{P}}) - \int_a^b g < \epsilon \implies -\epsilon + \int_a^b g < S(g;\dot{\mathcal{P}}) < \epsilon + \int_a^b g$$

Since $S(f;\dot{\mathcal{P}}) \leq S(g;\dot{\mathcal{P}})$,

$$-\epsilon + \int_a^b f \leq \epsilon + \int_a^b g \implies \int_a^b f \leq \int_a^b g + 2\epsilon$$

Since the choice of ϵ is arbitrary, then

$$\int_a^b f \leq \int_a^b g$$

Problem 2b. Let $\epsilon > 0$, $\delta := \epsilon$ and let $\dot{\mathcal{P}}$ be any tagged partition of [a,b] such that $||\dot{\mathcal{P}}||< \delta$.

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} |f(t_i)|(x_i - x_{i-1}) \leq \sum_{i=1}^{n} M(x_i - x_{i-1})$$

$$= M\sum_{i=1}^{n}(x_i - x_{i-1}) = M(x_n - x_0) = M(b-a)$$

Therefore,

$$S(f;\dot{\mathcal{P}}) \leq M(b-a)$$

Now,

$$S(f;\dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \geq \sum_{i=1}^{n} -|f(t_i)|(x_i - x_{i-1}) \geq \sum_{i=1}^{n} -M(x_i - x_{i-1})$$

$$= -M\sum_{i=1}^{n}(x_i - x_{i-1}) = -M(x_n - x_0) = -M(b-a)$$

Therefore,

$$S(f;\dot{\mathcal{P}}) \geq -M(b-a)$$

Since f is Riemann integrable,

$$|S(f;\dot{\mathcal{P}}) - \int_a^b f| < \epsilon \Leftrightarrow |\int_a^b f - S(f;\dot{\mathcal{P}})| < \epsilon$$

$$\implies -\epsilon < \int_a^b f - S(f;\dot{\mathcal{P}}) < \epsilon \implies -\epsilon + S(f;\dot{\mathcal{P}}) < \int_a^b f < \epsilon + S(f;\dot{\mathcal{P}})$$

Therefore,

$$-\epsilon - M(b-a) \leq -\epsilon + S(f;\dot{\mathcal{P}}) < \int_a^b f < \epsilon + S(f;\dot{\mathcal{P}}) \leq \epsilon + M(b-a)$$

$$\implies -\epsilon - M(b-a) \leq \int_a^b f \leq \epsilon + M(b-a)$$

$$\implies |\int_a^b f| \leq M(b-a) + \epsilon$$

Since the choice of ϵ is arbitrary, then

$$|\int_a^b f| \leq M(b-a)$$

Problem 5a. Let $\epsilon > 0$, $\delta := \epsilon$ and let $\dot{\mathcal{P}}$ be any tagged partition of [-1,1] such that $||\dot{\mathcal{P}}||< \delta$. Let $\dot{\mathcal{P}}_1$ be the subset of $\dot{\mathcal{P}}$ having its tags in [-1,0) and let $\dot{\mathcal{P}}_2$ be the subset of $\dot{\mathcal{P}}$ having its tags in [0,1].

$$\implies S(f;\dot{\mathcal{P}}) = S(f;\dot{\mathcal{P}}_1) + S(f;\dot{\mathcal{P}}_2)$$

Let U_1 denote the union of the subintervals in $\dot{\mathcal{P}}_1$, then

$$[-1,-\delta] \subset U_1 \subset [-1,\delta]$$

Since $f(t_k) = 0$ for any tag in $\dot{\mathcal{P}}_1$,

$$0(1-\delta) \leq S(f;\dot{\mathcal{P}}_1) \leq 0(1+\delta) \implies 0 \leq S(f;\dot{\mathcal{P}}_1) \leq 0 \implies S(f;\dot{\mathcal{P}}_1) = 0$$

Let U_2 denote the union of the subintervals in $\dot{\mathcal{P}}_2$, then

$$[\delta, 1] \subset U_1 \subset [-\delta, 1]$$

Since $f(t_k) = 1$ for any tag in $\dot{\mathcal{P}}_2$,

$$1(1-\delta) \leq S(f;\dot{\mathcal{P}}_2) \leq 1(1+\delta) \implies (1-\delta) \leq S(f;\dot{\mathcal{P}}_2) \leq (1+\delta)$$

Adding the expression for $S(f;\dot{\mathcal{P}}_1)$ and the inequality for $S(f;\dot{\mathcal{P}}_2)$ we get,

$$0 + (1-\delta) \leq S(f;\dot{\mathcal{P}}_1) + S(f;\dot{\mathcal{P}}_2) \leq 0 + (1+\delta)$$

$$\implies 1-\delta \leq S(f;\dot{\mathcal{P}}) \leq 1+\delta$$

$$\implies -\delta \leq S(f;\dot{\mathcal{P}}) - 1 \leq \delta$$

$$\implies |S(f;\dot{\mathcal{P}}) - 1| \leq \delta = \epsilon$$

Therefore,

$$\int_{-1}^{1} f_1 = 1$$