Lecture 14

Robust Price-of-Anarchy Bounds in Smooth Games

The preceding lecture introduced several relaxations of the pure Nash equilibrium concept. The benefit of enlarging the set of equilibria is increased plausibility and computational tractability. The drawback is that price-of-anarchy bounds, which concern the worst equilibrium of a game, can only degrade as the set of equilibria grows. This lecture introduces "smooth games," in which POA bounds for PNE extend without degradation to several relaxed equilibrium concepts, including coarse correlated equilibria.

Section 14.1 outlines a four-step recipe for proving POA bounds for PNE, inspired by our results for atomic selfish routing networks. Section 14.2 introduces a class of location games and uses the four-step recipe to prove a good POA bound for the PNE of such games. Section 14.3 defines smooth games, and Section 14.4 proves that POA bounds in such games extend to several relaxations of PNE.

*14.1 A Recipe for POA Bounds

Theorem 12.3 shows that the POA in every atomic selfish routing network with affine cost functions is at most $\frac{5}{2}$. To review, the proof has the following high-level steps.

1. Given an arbitrary PNE $\mathbf{s}$, the equilibrium hypothesis is invoked once per agent i with the hypothetical deviation s_i^*, where $\mathbf{s}^*$ is an optimal outcome, to derive the inequality $C_i(\mathbf{s}) \leq C_i(s_i^*, \mathbf{s}_{-i})$ for each i. Importantly, the deviations $\mathbf{s}^*$ are independent of the choice of the PNE $\mathbf{s}$. This is the only time that the PNE hypothesis is invoked in the entire proof.

2. The k inequalities that bound individuals' equilibrium costs are summed over the agents. The left-hand side of the resulting

inequality (12.9) is the cost of the PNE $\mathbf{s}$; the right-hand side is a strange entangled function of $\mathbf{s}$ and $\mathbf{s}^*$.

3. The hardest step is to relate the entangled term $\sum_{i=1}^{k} C_i(s_i^*, \mathbf{s}_{-i})$ generated by the previous step to the only two quantities that we care about, the costs of $\mathbf{s}$ and $\mathbf{s}^*$. Specifically, inequality (12.10) proves an upper bound of $\frac{5}{3}\sum_{i=1}^{k} C_i(\mathbf{s}^*) + \frac{1}{3}\sum_{i=1}^{k} C_i(\mathbf{s})$. This step is just algebra, and is agnostic to our choices of $\mathbf{s}$ and $\mathbf{s}^*$ as a PNE and an optimal outcome, respectively.
4. The final step is to solve for the POA. Subtracting $\frac{1}{3}\sum_{i=1}^{k} C_i(\mathbf{s})$ from both sides and multiplying through by $\frac{3}{2}$ proves that the POA is at most $\frac{5}{2}$.

This proof is canonical, in that POA proofs for many other classes of games follow the same four-step recipe. The main point of this lecture is that this recipe generates "robust" POA bounds that apply to all of the equilibrium concepts defined in Lecture 13.

*14.2 A Location Game

Before proceeding to the general theory, it is helpful to have another concrete example under our belt.

14.2.1 The Model

Consider a *location game* with the following ingredients:

- A set L of possible locations. These could represent servers capable of hosting a Web cache, gentrifying neighborhoods ready for an artisanal chocolate shop, and so on.
- A set of k agents. Each agent i chooses one location from a set $L_i \subseteq L$ from which to provide a service. All agents provide the same service, and differ only in where they are located. There is no limit on the number of markets that an agent can provide service to.
- A set M of markets. Each market $j \in M$ has a value v_j that is known to all agents. This is the market's maximum willingness-to-pay for receiving the service.

- For each location $\ell \in L$ and market $j \in M$, there is a cost $c_{\ell j}$ of serving j from ℓ. This could represent physical distance, the degree of incompatibility between two technologies, and so on.

Given a location choice by each agent, each agent tries to capture as many markets as possible, at the highest prices possible. To define the payoffs precisely, we start with an example. Figure 14.1 shows a location game with $L = \{\ell_1, \ell_2, \ell_3\}$ and $M = \{m_1, m_2\}$. There are two agents, with $L_1 = \{\ell_1, \ell_2\}$ and $L_2 = \{\ell_2, \ell_3\}$. Both markets have value 3. The cost between location ℓ_2 and either market is 2. Locations ℓ_1 and ℓ_3 have cost 1 to markets m_1 and m_2, respectively, and infinite cost to the other market.

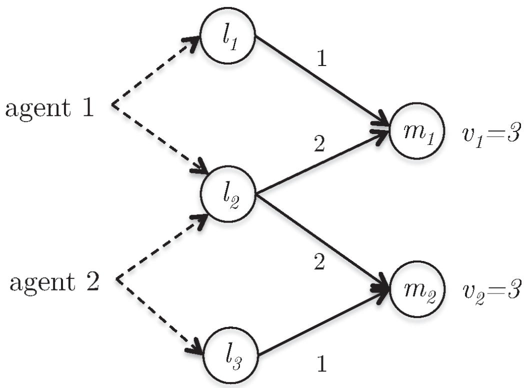

Figure 14.1: A location game with two agents (1 and 2), three locations ($L = \{\ell_1, \ell_2, \ell_3\}$), and two markets ($M = \{m_1, m_2\}$).

Continuing the example, suppose the first agent chooses location ℓ_1 and the second agent chooses location ℓ_3. Then, each agent has a monopoly in the market that they entered. The only thing restricting the price charged is the maximum willingness-to-pay of each market. Thus, each agent can charge 3 for her service to her market. Since the cost of service is 1 in both cases, both agents have a payoff of $3 - 1 = 2$.

Alternatively, suppose the first agent switches to location ℓ_2, while the second agent remains at location ℓ_3. Agent 1 still has a monopoly in market m_1, and thus can still charge 3. Her service cost has jumped to 2, however, so her payoff from that market has dropped to 1. In market m_2, agent 2 can no longer charge a price of 3 without consequence—at any price strictly bigger than 2, agent 1 can profitably undercut the price and take the market. Thus, agent 2 will charge the highest price she can without losing the market to the competition, which is 2. Since her cost of serving the market is 1, agent 2's payoff is $2 - 1 = 1$.

In general, in a strategy profile $\mathbf{s}$ of a location game in which T is the set of chosen locations and agent i chooses $\ell \in T$, agent i's payoff from a market $j \in M$ is

$$\pi_{ij}(\mathbf{s}) = \begin{cases} 0 & \text{if } c_{\ell j} \geq v_j \text{ or } \ell \text{ is not the closest} \\ & \quad \text{location of } T \text{ to } j \\ d_j^{(2)}(\mathbf{s}) - c_{\ell j} & \text{otherwise,} \end{cases} \tag{14.1}$$

where $d_j^{(2)}(\mathbf{s})$ is the highest price that agent i can get away with, namely the minimum of v_j and the second-smallest cost between a location of T and j. The definition in (14.1) assumes that each market is served by the potential provider with the lowest service cost, at the highest competitive price. The payoff $\pi_{ij}(\mathbf{s})$ is thus the "competitive advantage" that i has over the other agents for market j, up to a cap of v_j minus the service cost.

Agent i's total payoff is then

$$\pi_i(\mathbf{s}) = \sum_{j \in M} \pi_{ij}(\mathbf{s}).$$

Location games are examples of payoff-maximization games (Remark 13.1).

The objective function in a location game is to maximize the social welfare, which for a strategy profile $\mathbf{s}$ is defined as

$$W(\mathbf{s}) = \sum_{j \in M} \left(v_j - d_j(\mathbf{s})\right), \tag{14.2}$$

where $d_j(\mathbf{s})$ is the minimum of v_j and the smallest cost between a chosen location and j. The definition (14.2) assumes that each market j is served by the chosen location with the smallest cost of serving j, or not at all if this cost is at least v_j.

The welfare $W(\mathbf{s})$ depends on the strategy profile $\mathbf{s}$ only through the set of locations chosen by some agent in $\mathbf{s}$. The definition (14.2) makes sense more generally for any subset of chosen locations T, and we sometimes write $W(T)$ for this quantity.

Every location game has at least one PNE (Exercise 14.1). We next work toward a proof that every PNE of every location game has social welfare at least 50% times the maximum possible.[1]

Theorem 14.1 (POA Bound for Location Games) *The POA of every location game is at least* $\frac{1}{2}$.

The bound of $\frac{1}{2}$ is tight in the worst case (Exercise 14.2).

14.2.2 Properties of Location Games

We next identify three properties possessed by every location game. Our proof of Theorem 14.1 relies only on these properties.

(P1) For every strategy profile $\mathbf{s}$, the sum $\sum_{i=1}^{k} \pi_i(\mathbf{s})$ of agents' payoffs is at most the social welfare $W(\mathbf{s})$.

This property follows from the facts that each market $j \in M$ contributes $v_j - d_j(\mathbf{s})$ to the social welfare and $d_j^{(2)}(\mathbf{s}) - d_j(\mathbf{s})$ to the payoff of the closest location, and that $d_j^{(2)}(\mathbf{s}) \leq v_j$ by definition.

(P2) For every strategy profile $\mathbf{s}$, $\pi_i(\mathbf{s}) = W(\mathbf{s}) - W(\mathbf{s}_{-i})$. That is, an agent's payoff is exactly the extra welfare created by the presence of her location.

To see this property, write the contribution of a market j to $W(\mathbf{s}) - W(\mathbf{s}_{-i})$ as $\min\{v_j, d_j(\mathbf{s}_{-i})\} - \min\{v_j, d_j(\mathbf{s})\}$. When the upper bound v_j is not binding, this is the extent to which the closest chosen location to j is closer in $\mathbf{s}$ than in $\mathbf{s}_{-i}$. This quantity is zero unless agent i's location is the closest to j in $\mathbf{s}$, in which case it is

$$\min\{v_j, d_j^{(2)}(\mathbf{s})\} - \min\{v_j, d_j(\mathbf{s})\}. \tag{14.3}$$

Either way, this is precisely market j's contribution $\pi_{ij}(\mathbf{s})$ to agent i's payoff in $\mathbf{s}$. Summing over all $j \in M$ proves the property.

[1] With a maximization objective, the POA is always at most 1, the closer to 1 the better.

(P3) The social welfare W is *monotone* and *submodular* as a function of the set of chosen locations. Monotonicity just means that $W(T_1) \leq W(T_2)$ whenever $T_1 \subseteq T_2 \subseteq L$; this property is evident from (14.2). Submodularity is a set-theoretic version of diminishing returns, defined formally as

$$W(T_2 \cup \{\ell\}) - W(T_2) \leq W(T_1 \cup \{\ell\}) - W(T_1) \qquad (14.4)$$

for every location $\ell \in L$ and subsets $T_1 \subseteq T_2 \subseteq L$ of locations (Figure 14.2). This property follows from our expression (14.3) for the welfare increase caused by one new location ℓ; Exercise 14.3 asks you to provide the details.

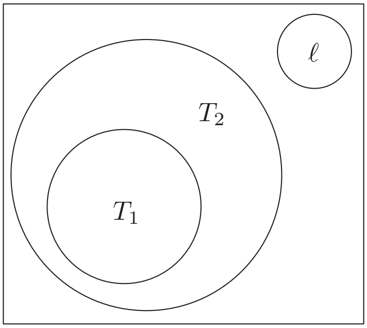

Figure 14.2: Definition of submodularity. Adding ℓ to the bigger set T_2 yields a smaller increase in social welfare than adding ℓ to the smaller set T_1.

14.2.3 Proof of Theorem 14.1

We follow the four-step recipe in Section 14.1. Let $\mathbf{s}$ denote an arbitrary PNE and $\mathbf{s}^*$ a social welfare-maximizing outcome. In the first step, we invoke the PNE hypothesis once per agent, with the outcome $\mathbf{s}^*$ providing hypothetical deviations. That is, since $\mathbf{s}$ is a PNE,

$$\pi_i(\mathbf{s}) \geq \pi_i(s_i^*, \mathbf{s}_{-i}) \qquad (14.5)$$

for every agent i. This is the only step of the proof that uses the assumption that $\mathbf{s}$ is a PNE.

The second step is to sum (14.5) over all of the agents, yielding

$$W(\mathbf{s}) \geq \sum_{i=1}^{k} \pi_i(\mathbf{s}) \geq \sum_{i=1}^{k} \pi_i(s_i^*, \mathbf{s}_{-i}), \tag{14.6}$$

where the first inequality is property (P1) of location games.

The third step is to disentangle the final term of (14.6) and relate it to the only two quantities that we care about, $W(\mathbf{s})$ and $W(\mathbf{s}^*)$. By property (P2) of location games, we have

$$\sum_{i=1}^{k} \pi_i(s_i^*, \mathbf{s}_{-i}) = \sum_{i=1}^{k} \left[W(s_i^*, \mathbf{s}_{-i}) - W(\mathbf{s}_{-i})\right]. \tag{14.7}$$

To massage the right-hand side into a telescoping sum, we add the extra locations $s_1^*, \ldots, s_{i-1}^*$ to the ith term.[2] By submodularity of W (property (P3)), we have

$$W(s_i^*, \mathbf{s}_{-i}) - W(\mathbf{s}_{-i}) \geq W(s_1^*, \ldots, s_i^*, \mathbf{s}) - W(s_1^*, \ldots, s_{i-1}^*, \mathbf{s})$$

for each $i = 1, 2, \ldots, k$. Thus, the right-hand side of (14.7) can be bounded below by

$$\sum_{i=1}^{k} \left[W(s_1^*, \ldots, s_i^*, \mathbf{s}) - W(s_1^*, \ldots, s_{i-1}^*, \mathbf{s})\right],$$

which simplifies to

$$W(s_1^*, \ldots, s_k^*, s_1, \ldots, s_k) - W(\mathbf{s}) \geq W(\mathbf{s}^*) - W(\mathbf{s}),$$

where the inequality follows from the monotonicity of W (property (P3)). Summarizing, we have

$$\sum_{i=1}^{k} \pi_i(s_i^*, \mathbf{s}_{-i}) \geq W(\mathbf{s}^*) - W(\mathbf{s}), \tag{14.8}$$

completing the third step of the proof.

[2]Some of the locations in $\mathbf{s}$ and $\mathbf{s}^*$ may coincide, but this does not affect the proof.

The fourth and final step is to solve for the POA. Inequalities (14.6) and (14.8) imply that

$$W(\mathbf{s}) \geq W(\mathbf{s}^*) - W(\mathbf{s}),$$

and so

$$\frac{W(\mathbf{s})}{W(\mathbf{s}^*)} \geq \frac{1}{2}$$

and the POA is at least $\frac{1}{2}$. This completes the proof of Theorem 14.1.

*14.3 Smooth Games

The following definition is an abstract version of the third "disentanglement" step in the proofs of the POA bounds for atomic selfish routing games (Theorem 12.3) and location games (Theorem 14.1). The goal is not generalization for its own sake; POA bounds established via this condition are automatically robust in several senses.

Definition 14.2 (Smooth Games)

(a) A cost-minimization game is (λ, μ)-*smooth* if

$$\sum_{i=1}^{k} C_i(s_i^*, \mathbf{s}_{-i}) \leq \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s}) \tag{14.9}$$

for all strategy profiles $\mathbf{s}, \mathbf{s}^*$. Here $\text{cost}(\cdot)$ is an objective function that satisfies $\text{cost}(\mathbf{s}) \leq \sum_{i=1}^{k} C_i(\mathbf{s})$ for every strategy profile $\mathbf{s}$.

(b) A payoff-maximization game is (λ, μ)-*smooth* if

$$\sum_{i=1}^{k} \pi_i(s_i^*, \mathbf{s}_{-i}) \geq \lambda \cdot W(\mathbf{s}^*) - \mu \cdot W(\mathbf{s}) \tag{14.10}$$

for all strategy profiles $\mathbf{s}, \mathbf{s}^*$. Here $W(\cdot)$ is an objective function that satisfies $W(\mathbf{s}) \geq \sum_{i=1}^{k} \pi_i(\mathbf{s})$ for every strategy profile $\mathbf{s}$.

Every game is (λ, μ)-smooth for suitable choices of λ and μ, but good POA bounds require that neither λ nor μ is too large (see Section 14.4).

The smoothness condition controls the effect of a set of "one-dimensional perturbations" of an outcome, as a function of both the initial outcome $\mathbf{s}$ and the perturbations $\mathbf{s}^*$. Intuitively, in a (λ, μ)-smooth game with small values of λ and μ, the externality imposed by one agent on the others is bounded.

Atomic selfish routing networks are $(\frac{5}{3}, \frac{1}{3})$-smooth cost-minimization games. This follows from our proof that the right-hand side of (12.6) is bounded above by the right-hand side of (12.10), and the choice of the objective function (11.3) as $\text{cost}(\mathbf{s}) = \sum_{i=1}^{k} C_i(\mathbf{s})$. Location games are $(1, 1)$-smooth payoff-maximization games, as witnessed by property (P1) of Section 14.2.2 and inequality (14.8).[3]

Remark 14.3 (Smoothness with Respect to a Profile)
A game is (λ, μ)-*smooth with respect to the strategy profile* $\mathbf{s}^*$ if the inequality (14.9) or (14.10) holds for the specific strategy profile $\mathbf{s}^*$ and all strategy profiles $\mathbf{s}$. All known consequences of Definition 14.2, including those in Section 14.4, only require smoothness with respect to some optimal outcome $\mathbf{s}^*$. See Problems 14.1–14.3 for applications of this relaxed condition.

*14.4 Robust POA Bounds in Smooth Games

This section shows that POA bounds for smooth games apply to several relaxations of PNE. In general, the POA of an equilibrium concept is defined as the ratio (13.3), with the "MNE" in the numerator replaced by the present concept.

14.4.1 POA Bounds for PNE

In a (λ, μ)-smooth cost-minimization game with $\mu < 1$, every PNE $\mathbf{s}$ has cost at most $\frac{\lambda}{1-\mu}$ times that of an optimal outcome $\mathbf{s}^*$. To see this, use the assumption that the objective function satisfies $\text{cost}(\mathbf{s}) \leq$

[3]When we proved the "disentanglement" inequalities for atomic selfish routing and location games, we had in mind the case where $\mathbf{s}$ and $\mathbf{s}^*$ are a PNE and an optimal outcome, respectively. Our proofs do not use these facts, however, and apply more generally to all pairs of strategy profiles.

$\sum_{i=1}^{k} C_i(\mathbf{s})$, the PNE condition (once per agent), and the smoothness assumption to derive

$$\begin{aligned}\text{cost}(\mathbf{s}) &\le \sum_{i=1}^{k} C_i(\mathbf{s}) \\ &\le \sum_{i=1}^{k} C_i(s_i^*, \mathbf{s}_{-i}) \\ &\le \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s}).\end{aligned}$$

Rearranging terms establishes the bound of $\frac{\lambda}{1-\mu}$.

Similarly, every PNE of a (λ, μ)-smooth payoff-maximization game has objective function value at least $\frac{\lambda}{1+\mu}$ times that of an optimal outcome. These observations generalize our POA bounds of $\frac{5}{2}$ and $\frac{1}{2}$ for atomic selfish routing networks with affine cost functions and location games, respectively.

14.4.2 POA Bounds for CCE

We next describe the first sense in which the POA bound of $\frac{\lambda}{1-\mu}$ or $\frac{\lambda}{1+\mu}$ for a (λ, μ)-smooth game is robust: it applies to all coarse correlated equilibria (CCE) of the game (Definition 13.5).

Theorem 14.4 (POA of CCE in Smooth Games) *In every (λ, μ)-smooth cost-minimization game with $\mu < 1$, the POA of CCE is at most $\frac{\lambda}{1-\mu}$.*

That is, the exact same POA bound that we derived in in the previous section for PNE holds more generally for all CCE. CCE are therefore a "sweet spot" equilibrium concept in smooth games—permissive enough to be highly tractable (see Lecture 17), yet stringent enough to allow good worst-case bounds.

Given the definitions, we can prove Theorem 14.4 just by following our nose.

Proof of Theorem 14.4: Consider a (λ, μ)-smooth cost-minimization game, a coarse correlated equilibrium σ, and an optimal outcome $\mathbf{s}^*$.

We can write

$$
\begin{aligned}
\mathbf{E}_{\mathbf{s}\sim\sigma}[\text{cost}(\mathbf{s})] &\le \mathbf{E}_{\mathbf{s}\sim\sigma}\left[\sum_{i=1}^{k} C_i(\mathbf{s})\right] && (14.11)\\
&= \sum_{i=1}^{k} \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] && (14.12)\\
&\le \sum_{i=1}^{k} \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(s_i^*, \mathbf{s}_{-i})] && (14.13)\\
&= \mathbf{E}_{\mathbf{s}\sim\sigma}\left[\sum_{i=1}^{k} C_i(s_i^*, \mathbf{s}_{-i})\right] && (14.14)\\
&\le \mathbf{E}_{\mathbf{s}\sim\sigma}[\lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s})] && (14.15)\\
&= \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \mathbf{E}_{\mathbf{s}\sim\sigma}[\text{cost}(\mathbf{s})]\,, && (14.16)
\end{aligned}
$$

where inequality (14.11) follows from the assumption on the objective function, equations (14.12), (14.14), and (14.16) follow from linearity of expectation, inequality (14.13) follows from the definition (13.5) of a coarse correlated equilibrium (applied once per agent i, with the hypothetical deviation s_i^*), and inequality (14.15) follows from (λ, μ)-smoothness. Rearranging terms completes the proof. ■

Similarly, in (λ, μ)-smooth payoff-maximization games, the POA bound of $\frac{\lambda}{1+\mu}$ applies to all CCE (Exercise 14.4).

Our POA bounds of $\frac{5}{2}$ and $\frac{1}{2}$ for atomic selfish routing games (Theorem 12.3) and location games (Theorem 14.1) may initially seem specific to PNE, but since the proofs establish the stronger smoothness condition (Definition 14.2), Theorem 14.4 implies that they hold for all CCE.

14.4.3 POA Bounds for Approximate PNE

Smooth games have a number of other nice properties, as well. For example, the POA bound of $\frac{\lambda}{1-\mu}$ or $\frac{\lambda}{1+\mu}$ for a (λ, μ)-smooth game applies automatically to approximate equilibria, with the POA bound degrading gracefully as a function of the approximation parameter.

Definition 14.5 (ϵ-Pure Nash Equilibrium) For $\epsilon \ge 0$, an outcome $\mathbf{s}$ of a cost-minimization game is an *ϵ-pure Nash equilibrium*

(ϵ-PNE) if, for every agent i and deviation $s'_i \in S_i$,

$$C_i(\mathbf{s}) \leq (1+\epsilon) \cdot C_i(s'_i, \mathbf{s}_{-i}). \tag{14.17}$$

This is, in an ϵ-PNE, no agent can decrease her cost by more than a $1+\epsilon$ factor via a unilateral deviation. The following guarantee holds (Exercise 14.5).

Theorem 14.6 (POA of ϵ-PNE in Smooth Games) *In every (λ, μ)-smooth cost-minimization game with $\mu < 1$, for every $\epsilon < \frac{1}{\mu} - 1$, the POA of ϵ-PNE is at most*

$$\frac{(1+\epsilon)\lambda}{1-\mu(1+\epsilon)}.$$

Similar results hold for (λ, μ)-smooth payoff-maximization games, and for approximate versions of other equilibrium concepts.

For example, in atomic selfish routing networks with affine cost functions, which are $(\frac{5}{3}, \frac{1}{3})$-smooth, the POA of ϵ-PNE with $\epsilon < 2$ is at most $\frac{5+5\epsilon}{2-\epsilon}$.

The Upshot

☆ A four-step recipe for proving POA bounds is: (1) invoke the equilibrium condition once per agent, using an optimal outcome to define hypothetical deviations, to bound agents' equilibrium costs; (2) add up the resulting inequalities to bound the total equilibrium cost; (3) relate this entangled bound back to the equilibrium and optimal costs; (4) solve for the POA.

☆ The POA bound for atomic selfish routing networks with affine cost functions follows from this four-step recipe.

☆ In a location game where agents choose locations from which to provide a service and compete for several markets, this four-step recipe proves that the POA is at least $\frac{1}{2}$.

☆ The definition of a smooth game is an abstract version of the third "disentanglement" step in this recipe.

☆ The POA bound implied by the smoothness condition extends to all coarse correlated equilibria.

☆ The POA bound implied by the smoothness condition extends to all approximate equilibria, with the POA bound degrading gracefully as a function of the approximation parameter.

Notes

The definition of location games and Theorem 14.1 are due to Vetta (2002). The importance of POA bounds that apply beyond Nash equilibria is articulated in Mirrokni and Vetta (2004). The POA of CCE is first studied in Blum et al. (2008). Definition 14.2 and Theorems 14.4 and 14.6 are from Roughgarden (2015). The term "smooth" is meant to succinctly suggest an analogy between Definition 14.2 and a Lipschitz-type condition. Problems 14.1–14.3 are from Hoeksma and Uetz (2011), Caragiannis et al. (2015), and Christodoulou et al. (2008), respectively. For more on the POA in (non-DSIC) auctions and mechanisms, see the survey by Roughgarden et al. (2016).

Exercises

Exercise 14.1 *(H)* Prove that every location game is a potential game (Section 13.3) and hence has at least one PNE.

Exercise 14.2 Prove that Theorem 14.1 is tight, in that there is a location game in which the POA of PNE is $\frac{1}{2}$.

Exercise 14.3 Prove that the social welfare function of a location game is a submodular function of the set of chosen locations.

Exercise 14.4 Prove that the POA of CCE of a (λ, μ)-smooth payoff-maximization game is at least $\frac{\lambda}{1+\mu}$.

Exercise 14.5 *(H)* Prove Theorem 14.6.

Problems

Problem 14.1 This problem studies a scenario with k agents, where agent j has a processing time p_j. There are m identical machines. Each agent chooses a machine, and the agents on each machine are processed serially from shortest to longest. (You can assume that the p_j's are distinct.) For example, if agents with processing times 1, 3, and 5 are scheduled on a common machine, then they will complete at times 1, 4, and 9, respectively. The following questions concern the cost-minimization game in which agents choose machines to minimize their completion times, and the objective function of minimizing the sum $\sum_{j=1}^{k} C_j$ of the agents' completion times.

(a) Define the *rank* R_j of agent j in a schedule as the number of agents on j's machine with processing time at least p_j, including j itself. For example, if agents with processing times 1, 3, and 5 are scheduled on a common machine, then they have ranks 3, 2, and 1, respectively.

Prove that the objective function value $\sum_{j=1}^{k} C_j$ of an outcome can also be written as $\sum_{j=1}^{k} p_j R_j$.

(b) Prove that the following algorithm produces an optimal outcome: (1) sort the agents from largest to smallest; (2) for $j = 1, 2, \ldots, k$, assign the jth agent in this ordering to machine $j \bmod m$ (where machine 0 means machine m).

(c) *(H)* Prove that in every such scheduling game, the POA of CCE is at most 2.

Problem 14.2 The Generalized Second Price sponsored search auction described in Problem 3.1 induces a payoff-maximization game, where bidder i strives to maximize her utility $\alpha_{j(i)}(v_i - p_{j(i)})$, where v_i is her value-per-click, $j(i)$ is her assigned slot, and $p_{j(i)}$ and $\alpha_{j(i)}$

are the price-per-click and click-through rate of this slot. (If i is not assigned a slot, then $\alpha_{j(i)} = p_{j(i)} = 0$.)

(a) Assume that each bidder can bid any nonnegative number. Show that even with one slot and two bidders, the POA of PNE can be 0.

(b) *(H)* Now assume that each bidder i always bids a number between 0 and v_i. Prove that the POA of CCE is at least $\frac{1}{4}$.

Problem 14.3 This problem concerns combinatorial auctions (Example 7.2) where each bidder i has a unit-demand valuation v_i (Exercise 7.5). This means that there are values $v_{i1}, \ldots, v_{im}$ such that $v_i(S) = \max_{j \in S} v_{ij}$ for every subset S of items.

Consider a payoff-maximization game in which each bidder i submits one bid b_{ij} for each item j and each item is sold separately using a second-price single-item auction. Similarly to Problem 14.2(b), assume that each bid b_{ij} lies between 0 and v_{ij}. The utility of a bidder is her value for the items won less her total payment. For example, if bidder i has values v_{i1} and v_{i2} for two items, and wins both items when the second-highest bids are p_1 and p_2, then her utility is $\max\{v_{i1}, v_{i2}\} - (p_1 + p_2)$.

(a) *(H)* Prove that the POA of PNE in such a game can be at most $\frac{1}{2}$.

(b) *(H)* Prove that the POA of CCE in every such game is at least $\frac{1}{2}$.