McGill University
Department of Mathematics and Statistics
MATH 243 Analysis 2, Winter 2017
Assignment 3

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 2 and 5(a).**

This assignment is due **Tuesday, January 31, at 2:30am** in class. **Late assignments will not be accepted!**

1. Let $c \in [a, b]$ and let
$$f : [a, b] \to \mathbb{R},\ f(x) := \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases}$$
Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f = 0$.

2. (a) Let f and g be Riemann integrable on $[a, b]$ such that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f \leq \int_a^b g$.
Hint: Prove first that $S(f; \dot{\mathcal{P}}) \leq S(g; \dot{\mathcal{P}})$ for all tagged partitions $\dot{\mathcal{P}}$ of $[a, b]$.
(b) Let f be Riemann integrable on $[a, b]$ and let $M \in \mathbb{R}$ be a constant such that $|f(x)| \leq M$ for all $x \in [a, b]$. Prove that $\left| \int_a^b f \right| \leq M(b - a)$.

3. Use induction to prove that if $f_1, \dots, f_n$ are Riemann integrable on $[a, b]$ and $k_1, \dots, k_n \in \mathbb{R}$, then the linear combination $f := k_1 f_1 + \dots k_n f_n$ is Riemann integrable on $[a, b]$ and
$$\int_a^b f = k_1 \int_a^b f_1 + \dots + k_n \int_a^b f_n$$

4. (a) Let $c_1, \dots, c_n \in [a, b]$ and let $f : [a, b] \to \mathbb{R}$ be a function such that $f(x) = 0$ for all $x \in [a, b] \setminus \{c_1, \dots, c_n\}$. Prove that f is Riemann integrable on $[a, b]$ and that $\int_a^b f = 0$.
(b) Let $c_1, \dots, c_n \in [a, b]$ and let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions such that $f(x) = g(x)$ for all $x \in [a, b] \setminus \{c_1, \dots, c_n\}$. Prove that if f is Riemann integrable on $[a, b]$ then g is Riemann integrable on $[a, b]$ and $\int_a^b f = \int_a^b g$.

—**Please turn over!**—

5. Let

   (a)

$$f_1(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

   Prove that f_1 is Riemann integrable on $[-1, 1]$ and compute $\displaystyle\int_{-1}^{1} f_1$.

   (b) Let

$$f_2(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 2 & \text{if } x > 0 \end{cases}$$

   Prove that f_2 is Riemann integrable on $[-1, 1]$ and compute $\displaystyle\int_{-1}^{1} f_2$.

   Hint: Use questions 3 or 4.

6. (a) Let f be Riemann integrable on $[a, b]$ and let $(\dot{P}_n)$ be any sequence of tagged partitions of $[a, b]$ with $\lim \left(||\dot{P}_n||\right) = 0$. Prove that $\left(S(f; \dot{P}_n)\right)$ converges and that $\displaystyle\int_a^b f = \lim \left(S(f; \dot{P}_n)\right)$.

   (b) Let $f(x) := 0$ if $x \in [0, 1]$ is rational and $f(x) := 1/x$ if $x \in [0, 1]$ is irrational.

      (i) Prove that f is not Riemann integrable on $[0, 1]$.

      (ii) However, prove that there exists a sequence $(\dot{P}_n)$ of tagged partitions of $[0, 1]$ such that $\lim \left(||\dot{P}_n||\right) = 0$ and $\lim \left(S(f; \dot{P}_n)\right)$ exists. The converse of part (a) thus does not hold.