Lecture 11

Selfish Routing and the Price of Anarchy

This lecture commences the second part of the course. In many settings, there is no option to design a game from scratch. Unlike all of our carefully crafted DSIC mechanisms, games "in the wild" generally have no dominant strategies. Predicting the outcome of such a game requires a notion of "equilibrium." As the outcome of self-interested behavior, there is no reason to expect equilibria to be socially desirable outcomes. Happily, in many interesting models, equilibria are near-optimal under relatively weak assumptions. This lecture and the next consider "selfish routing," a canonical such model.

Section 11.1 uses three examples to provide a gentle introduction to selfish routing. Sections 11.2 and 11.3 state and interpret the main result of this lecture, that the worst-case price of anarchy of selfish routing is always realized in extremely simple networks, and consequently equilibria are near-optimal in networks without "highly nonlinear" cost functions. After Section 11.4 formally defines equilibrium flows and explains some of their properties, Section 11.5 proves the main result.

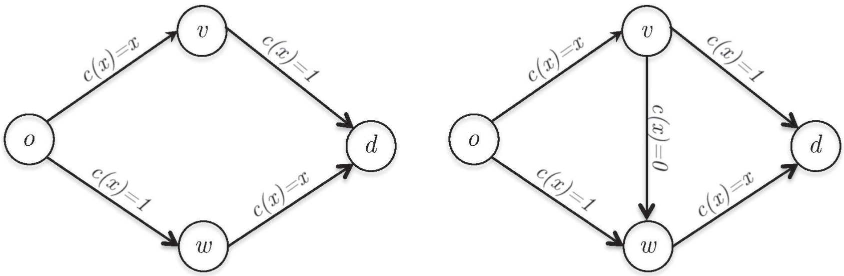
11.1 Selfish Routing: Examples

Before formally defining our model of selfish routing, we develop intuition and motivate the main results through a sequence of examples.

11.1.1 Braess's Paradox

Lecture 1 introduced Braess's paradox (Section 1.2). To recap, one unit of traffic, perhaps representing rush-hour drivers, leaves an origin o for a destination d. In the network in Figure 11.1(a), by symmetry, at equilibrium half of the traffic uses each route and the common travel time is $\frac{3}{2}$. After installing a teleportation device that allows drivers to travel instantly from v to w (Figure 11.1(b)), the new route

$o \to v \to w \to d$ is a dominant strategy for every driver. The common travel time in this new equilibrium is 2. The minimum-possible travel time in the new network is $\frac{3}{2}$, as there is no profitable way to use the teleporter. The *price of anarchy (POA)* of this selfish routing network, defined as the ratio between the travel time in an equilibrium and the minimum-possible average travel time, is $2/\frac{3}{2} = \frac{4}{3}$.[1]



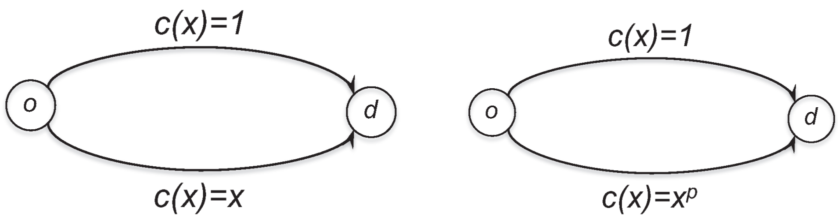
(a) Initial network (b) Augmented network

Figure 11.1: Braess's paradox revisited. Edges are labeled with their cost functions, which describe the travel time of an edge as a function of the amount x of traffic that uses it. In (b), the price of anarchy is 4/3.

11.1.2 Pigou's Example

Pigou's example, shown in Figure 11.2(a), is an even simpler selfish routing network in which the POA is $\frac{4}{3}$. Even when the lower edge of this network carries all of the traffic, it is no worse than the alternative. Thus, the lower edge is a dominant strategy for every driver, and in equilibrium all drivers use it and experience travel time 1. An altruistic dictator would minimize the average travel time by splitting the traffic equally between the two edges. This results in an average travel time of $\frac{3}{4}$, showing that the POA in Pigou's example is $1/\frac{3}{4} = \frac{4}{3}$.

[1] This definition makes sense because the selfish routing networks considered in this lecture always have at least one equilibrium, and because the average travel time is the same in every equilibrium (see Lecture 13). Lecture 12 extends the definition of the POA to games with multiple equilibria.



(a) Pigou's example (b) A nonlinear variant

Figure 11.2: Pigou's example and a nonlinear variant.

11.1.3 Pigou's Example: A Nonlinear Variant

The POA is $\frac{4}{3}$ in both Braess's paradox and Pigou's example, which is quite reasonable for completely unregulated behavior. The story is not so rosy in all networks, however. In the nonlinear variant of Pigou's example (Figure 11.2(b)), the cost function of the lower edge is $c(x) = x^p$ rather than $c(x) = x$, where p is large. The lower edge remains a dominant strategy, and the equilibrium travel time remains 1. The optimal solution, meanwhile, is now much better. If the traffic is again equally split between the two edges, then the average travel time tends to $\frac{1}{2}$ as $p \to \infty$, with the traffic on the bottom edge arriving at d nearly instantaneously. Even better is to assign a $1-\epsilon$ fraction of the traffic to the bottom edge, where ϵ tends to 0 as p tends to infinity. Then, almost all of the traffic gets to d with travel time $(1-\epsilon)^p$, which is close to 0 when p is sufficiently large, and the ϵ fraction of martyrs on the upper edge contribute little to the average travel time. We conclude that the POA in the nonlinear variant of Pigou's example is unbounded as $p \to \infty$.

11.2 Main Result: Informal Statement

The POA of selfish routing can be large (Section 11.1.3) or small (Sections 11.1.1 and 11.1.2). The goal of this lecture is to provide a thorough understanding of when the POA of selfish routing is close to 1. Looking at our three examples, we see that "highly nonlinear" cost functions can prevent a selfish routing network from having a POA close to 1, while our two examples with linear cost functions have a small POA. The coolest statement that might be true is that highly nonlinear cost functions are the *only* obstacle to a small POA—that every selfish routing network with not-too-nonlinear cost functions, no matter how complex, has POA close to 1. The main result of this lecture formulates and proves this conjecture.

We study the following model. There is a directed graph $G = (V, E)$, with vertex set V and directed edge set E, with an origin vertex o and a destination vertex d. There are r units of traffic (or *flow*) destined for d from o.[2] We treat G as a flow network, in the sense of the classical maximum- and minimum-cost flow problems. Each edge e of the network has a *cost function*, describing the travel time (per unit of traffic) as a function of the amount of traffic using the edge. Edges do not have explicit capacities. In this lecture and the next, we always assume that every cost function is nonnegative, continuous, and nondecreasing. These are very mild assumptions in most relevant applications, like road or communication networks.

We first state an informal version of this lecture's main result and explain how to interpret and use it. We give a formal statement in Section 11.3 and a proof in Sections 11.4 and 11.5. Importantly, the theorem is parameterized by a set $\mathcal{C}$ of permissible cost functions. This reflects our intuition that the POA of selfish routing seems to depend on the "degree of nonlinearity" of the network's cost functions. The result is already interesting for simple classes $\mathcal{C}$ of cost functions, such as the set $\{c(x) = ax + b : a, b \geq 0\}$ of affine functions with nonnegative coefficients.

Theorem 11.1 (Tight POA Bounds for Selfish Routing)
Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.

Section 11.3 makes the term "Pigou-like networks" precise. The point of Theorem 11.1 is that worst-case examples are always simple. The principal culprit for inefficiency in selfish routing is nonlinear cost functions, not complex network structure.

For a particular cost function class $\mathcal{C}$ of interest, Theorem 11.1 reduces the problem of computing the worst-case POA to a back-of-

[2] To minimize notation, we state and prove the main result only for "single-commodity networks," where there is one origin and one destination. The main result and its proof extend to networks with multiple origins and destinations (Exercise 11.5).

the-envelope calculation. Without Theorem 11.1, one would effectively have to search through all networks with cost functions in $\mathcal{C}$ to find the one with the largest POA. Theorem 11.1 guarantees that the much simpler search through Pigou-like networks is sufficient.

For example, when $\mathcal{C}$ is the set of affine cost functions with nonnegative coefficients, Theorem 11.1 implies that Pigou's example (Section 11.1.2) maximizes the POA. Thus, the POA is always at most $\frac{4}{3}$ in selfish routing networks with such cost functions. When $\mathcal{C}$ is the set of polynomials with nonnegative coefficients and degree at most p, Theorem 11.1 implies that the worst example is the nonlinear variant of Pigou's example (Section 11.1.3). Computing the POA of this worst-case example yields an upper bound on the POA of every selfish routing network with such cost functions. See Table 11.1 for several examples, which demonstrate the point that the POA of selfish routing is large only in networks with "highly nonlinear" cost functions. For example, quartic functions have been proposed as a reasonable model of road traffic in some situations, and the worst-case POA with respect to such functions is slightly larger than 2. Lecture 12 discusses cost functions germane to communication networks.

Table 11.1: The worst-case POA in selfish routing networks with cost functions that are polynomials with nonnegative coefficients and degree at most p.

Description	Typical Representative	Price of Anarchy
Linear	$ax+b$	$4/3$
Quadratic	ax^2+bx+c	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	ax^3+bx^2+cx+d	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4+bx^3+cx^2+dx+e$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Degree $\leq p$	$\sum_{i=0}^{p} a_i x^i$	$\frac{(p+1)\sqrt[p]{p+1}}{(p+1)\sqrt[p]{p+1}-p} \approx \frac{p}{\ln p}$

11.3 Main Result: Formal Statement

To formalize the statement of Theorem 11.1, we need to define the "Pigou-like networks" for a class $\mathcal{C}$ of cost functions. We then formulate a lower bound on the POA based solely on these trivial instances.

Theorem 11.2 states a matching upper bound on the POA of every selfish routing network with cost functions in $\mathcal{C}$.

Ingredients of a Pigou-Like Network

1. Two vertices, o and d.
2. Two edges from o to d, an "upper" edge and a "lower" edge.
3. A nonnegative traffic rate r.
4. A cost function $c(\cdot)$ on the lower edge.
5. The cost function everywhere equal to $c(r)$ on the upper edge.

See also Figure 11.3. There are two free parameters in the description of a Pigou-like network, the traffic rate r and the cost function $c(\cdot)$ of the lower edge.

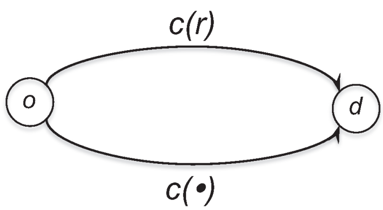


Figure 11.3: A Pigou-like network.

The POA of a Pigou-like network is easy to compute. By construction, the lower edge is a dominant strategy for all traffic—it is no less attractive than the alternative (with constant cost $c(r)$), even when it is fully congested. Thus, in the equilibrium all traffic travels on the lower edge, and the total travel time is $r \cdot c(r)$—the amount of traffic times the common per-unit travel time experienced by all of the traffic. We can write the minimum-possible total travel time as

$$\inf_{0 \le x \le r} \{x \cdot c(x) + (r - x) \cdot c(r)\}, \tag{11.1}$$

where x is the amount of traffic routed on the lower edge.[3] For later convenience, we allow x to range over all nonnegative reals, not just over $[0, r]$. Since cost functions are nondecreasing, this larger range does not change the quantity in (11.1)—there is always an optimal choice of x in $[0, r]$. We conclude that the POA in a Pigou-like network with traffic rate $r > 0$ and lower edge cost function $c(\cdot)$ is

$$\sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}.$$

Let $\mathcal{C}$ be an arbitrary set of nonnegative, continuous, and nondecreasing cost functions. Define the *Pigou bound* $\alpha(\mathcal{C})$ as the largest POA in a Pigou-like network in which the lower edge's cost function belongs to $\mathcal{C}$. Formally,

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}. \tag{11.2}$$

The first two suprema search over all choices of the two free parameters $c \in \mathcal{C}$ and $r \geq 0$ in a Pigou-like network; the third computes the best-possible outcome in the chosen Pigou-like network.[4]

The Pigou bound can be evaluated explicitly for many sets $\mathcal{C}$ of interest. For example, if $\mathcal{C}$ is the set of affine (or even concave) nonnegative and nondecreasing functions, then $\alpha(\mathcal{C}) = \frac{4}{3}$ (Exercises 11.1 and 11.2). The expressions in Table 11.1 are precisely the Pigou bounds for sets of polynomials with nonnegative coefficients and bounded degree (Exercise 11.3). The Pigou bound is achieved for these sets of cost functions by the nonlinear variant of Pigou's example (Section 11.1.3).

Suppose a set $\mathcal{C}$ contains all of the constant functions. Then the Pigou-like networks that define $\alpha(\mathcal{C})$ use only functions from $\mathcal{C}$, and $\alpha(\mathcal{C})$ is a lower bound on the worst-case POA of selfish routing networks with cost functions in $\mathcal{C}$.[5]

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[3] Continuity of the cost function c implies that this infimum is attained, but we won't need this fact.

[4] Whenever the ratio reads $\frac{0}{0}$, we interpret it as 1.

[5] As long as $\mathcal{C}$ contains at least one function c with $c(0) > 0$, the Pigou bound is a lower bound on the POA of selfish routing networks with cost functions in $\mathcal{C}$. The reason is that, under this weaker assumption, Pigou-like networks can be simulated by slightly more complex networks with cost functions only in $\mathcal{C}$ (Exercise 11.4).

The formal version of Theorem 11.1 is that the Pigou bound $\alpha(\mathcal{C})$ is an upper bound on the POA of *every* selfish routing network with cost functions in $\mathcal{C}$, whether Pigou-like or not.

Theorem 11.2 (Tight POA Bounds for Selfish Routing) *For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the POA is at most $\alpha(\mathcal{C})$.*

11.4 Technical Preliminaries

Before proving Theorem 11.2, we review some flow network preliminaries. While notions like flow and equilibria are easy to define in Pigou-like networks, defining them in general networks requires a little care.

Let $G = (V, E)$ be a selfish routing network, with r units of traffic traveling from o to d. Let $\mathcal{P}$ denote the set of o-d paths of G, which we assume is nonempty. A *flow* describes how traffic is split over the o-d paths, and is a nonnegative vector $\{f_P\}_{P\in\mathcal{P}}$ with $\sum_{P\in\mathcal{P}} f_P = r$. For example, in Figure 11.4, half of the traffic takes the zig-zag path $o \to v \to w \to d$, while the other half is split equally between the two two-hop paths.

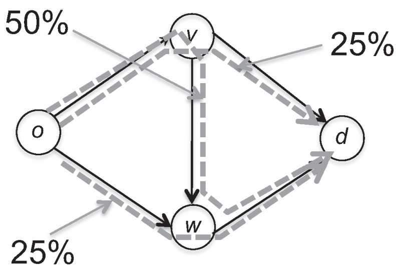

Figure 11.4: Example of a flow, with 25% of the traffic routed on each of the paths $o \to v \to t$ and $o \to w \to t$, and the remaining 50% on the path $o \to v \to w \to d$.

For an edge $e \in E$ and a flow f, we write $f_e = \sum_{P\in\mathcal{P}\,:\,e\in P} f_P$ for the amount of traffic that uses a path that includes e. For example, in Figure 11.4, $f_{(o,v)} = f_{(w,d)} = \frac{3}{4}$, $f_{(o,w)} = f_{(v,d)} = \frac{1}{4}$, and $f_{(v,w)} = \frac{1}{2}$.

In an equilibrium flow, traffic travels only on the shortest o-d paths, where "shortest" is defined using the travel times induced by the flow.

Definition 11.3 (Equilibrium Flow) A flow f is an *equilibrium* if $f_{\widehat{P}} > 0$ only when

$$\widehat{P} \in \operatorname{argmin}_{P \in \mathcal{P}} \left\{ \sum_{e \in P} c_e(f_e) \right\}.$$

For example, with cost functions as in Braess's paradox (Figure 11.1(b)), the flow in Figure 11.4 is not an equilibrium because the only shortest path is the zig-zag path, and some of the traffic doesn't use it.

We denote our objective function, the total travel time incurred by traffic in a flow f, by $C(f)$. We sometimes call the total travel time the *cost* of a flow. This objective function can be computed in two different ways, and both ways are useful. First, we can define

$$c_P(f) = \sum_{e \in E} c_e(f_e)$$

as the travel time along a path and tally the total travel time path-by-path:

$$C(f) = \sum_{P \in \mathcal{P}} f_P \cdot c_P(f). \tag{11.3}$$

Alternatively, we can tally it edge-by-edge:

$$C(f) = \sum_{e \in E} f_e \cdot c_e(f_e). \tag{11.4}$$

Recalling that $f_e = \sum_{P \in \mathcal{P} \,:\, e \in P} f_P$, a simple reversal of sums verifies the equivalence of (11.3) and (11.4).

*11.5 Proof of Theorem 11.2

We now prove Theorem 11.2. Fix a selfish routing network $G = (V, E)$ with cost functions in $\mathcal{C}$ and traffic rate r. Let f and f^* denote equilibrium and minimum-cost flows in the network, respectively. The proof has two parts.

The first part of the proof shows that after "freezing" the cost of every edge e at its equilibrium value $c_e(f_e)$, the equilibrium flow f is optimal. This makes sense, since an equilibrium flow routes all traffic on shortest paths with respect to the edge costs it induces.

Formally, since f is an equilibrium flow, if $f_{\widehat{P}} > 0$, then $c_{\widehat{P}}(f) \le c_P(f)$ for all $P \in \mathcal{P}$ (Definition 11.3). In particular, all paths $\widehat{P}$ used by the equilibrium flow have a common cost $c_{\widehat{P}}(f)$, call it L, and $c_P(f) \ge L$ for every path $P \in \mathcal{P}$. Thus,

$$\sum_{P\in\mathcal{P}} \underbrace{f_P}_{\text{sums to } r} \cdot \underbrace{c_P(f)}_{= L \text{ if } f_P > 0} = r \cdot L \tag{11.5}$$

while

$$\sum_{P\in\mathcal{P}} \underbrace{f^*_P}_{\text{sums to } r} \cdot \underbrace{c_P(f)}_{\ge L} \ge r \cdot L. \tag{11.6}$$

Expanding $c_P(f) = \sum_{e\in E} c_e(f_e)$ and reversing the order of summation as in (11.3)–(11.4), we can write the left-hand sides of (11.5) and (11.6) as sums over edges and derive

$$\sum_{e\in E} f_e \cdot c_e(f_e) = r \cdot L \tag{11.7}$$

and

$$\sum_{e\in E} f^*_e \cdot c_e(f_e) \ge r \cdot L. \tag{11.8}$$

Subtracting (11.7) from (11.8) yields

$$\sum_{e\in E} (f^*_e - f_e) c_e(f_e) \ge 0. \tag{11.9}$$

The inequality (11.9) is stating something very intuitive: since the equilibrium flow f routes all traffic on shortest paths, no other flow f^* can be better if we keep all edge costs fixed at $\{c_e(f_e)\}_{e\in E}$.

The second part of the proof quantifies the extent to which the optimal flow f^* can be better than f. The rough idea is to show that, edge by edge, the gap in costs between f and f^* is no worse than the Pigou bound. This statement only holds up to an error term for each edge, but we can control the sum of the error terms using the inequality (11.9) from the first part of the proof.

Formally, for each edge $e \in E$, instantiate the right-hand side of the Pigou bound (11.2) using c_e for c, f_e for r, and f_e^* for x. Since $\alpha(\mathcal{C})$ is the supremum over all possible choices of c, r, and x, we have

$$\alpha(\mathcal{C}) \geq \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)}.$$

The definition of the Pigou bound accommodates both the cases $f_e^* < f_e$ and $f_e^* \geq f_e$. Rearranging,

$$f_e^* \cdot c_e(f_e^*) \geq \frac{1}{\alpha(\mathcal{C})} \cdot f_e \cdot c_e(f_e) + (f_e^* - f_e)c_e(f_e). \qquad (11.10)$$

Summing (11.10) over all edges $e \in E$ gives

$$C(f^*) \geq \frac{1}{\alpha(\mathcal{C})} \cdot C(f) + \underbrace{\sum_{e \in E} (f_e^* - f_e)c_e(f_e)}_{\geq 0 \text{ by (11.9)}} \geq \frac{C(f)}{\alpha(\mathcal{C})}.$$

Thus the POA $C(f)/C(f^*)$ is at most $\alpha(\mathcal{C})$, and the proof of Theorem 11.2 is complete.

The Upshot

☆ In an equilibrium flow of a selfish routing network, all traffic travels from the origin to the destination on shortest paths.

☆ The price of anarchy (POA) of a selfish routing network is the ratio between the total travel time in an equilibrium flow and the minimum-possible total travel time.

☆ The POA is $\frac{4}{3}$ in Braess's paradox and Pigou's example, and is unbounded in the nonlinear variant of Pigou's example.

☆ The POA of a selfish routing network is large only if it has "highly nonlinear" cost functions.

☆ The Pigou bound for a set $\mathcal{C}$ of edge cost functions is the largest POA arising in a two-vertex, two-edge network with one cost function in $\mathcal{C}$ and one constant cost function.

☆ The Pigou bound for $\mathcal{C}$ is an upper bound on the POA of every selfish routing network with cost functions in $\mathcal{C}$.

Notes

Additional background on flow networks is in, for example, Cook et al. (1998). Pigou's example is described qualitatively by Pigou (1920). Selfish routing in general networks is proposed and studied in Wardrop (1952) and Beckmann et al. (1956). Braess's paradox is from Braess (1968). The price of anarchy of selfish routing is first considered in Roughgarden and Tardos (2002), who also proved that the POA is at most $\frac{4}{3}$ in every (multicommodity) network with affine cost functions. Theorem 11.2 is due to Roughgarden (2003) and Correa et al. (2004). Sheffi (1985) discusses the use of quartic cost functions for modeling road traffic. Problem 11.3 is from Roughgarden (2006). Roughgarden (2005) contains much more material on the price of anarchy of selfish routing.

Exercises

Exercise 11.1 *(H)* Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Pigou bound $\alpha(\mathcal{C})$ is $\frac{4}{3}$.

Exercise 11.2 *(H)* Prove that if $\mathcal{C}$ is the set of nonnegative, nondecreasing, and concave cost functions, then $\alpha(\mathcal{C}) = \frac{4}{3}$.

Exercise 11.3 For a positive integer p, let $\mathcal{C}_p$ denote the set of polynomials with nonnegative coefficients and degree at most p: $\mathcal{C}_p = \{\sum_{i=0}^{p} a_i x^i \,:\, a_0, \ldots, a_p \geq 0\}$.

(a) Prove that the Pigou bound of the singleton set $\{x^p\}$ is increasing in p.

(b) Prove that the Pigou bound of the set $\{ax^i \,:\, a \geq 0, i \in \{0, 1, 2 \ldots, p\}\}$ is the same as that of the set $\{x^p\}$.

(c) *(H)* Prove that Pigou bound of $\mathcal{C}_p$ is the same as that of the set $\{x^p\}$.

Exercise 11.4 Let $\mathcal{C}$ be a set of nonnegative, continuous, and nondecreasing cost functions.

(a) *(H)* Prove that if $\mathcal{C}$ includes functions c with $c(0) = \beta$ for all $\beta > 0$, then there are selfish routing networks with cost functions in $\mathcal{C}$ and POA arbitrarily close to the Pigou bound $\alpha(\mathcal{C})$.

(b) *(H)* Prove that if $\mathcal{C}$ includes a function c with $c(0) > 0$, then there are selfish routing networks with cost functions in $\mathcal{C}$ and POA arbitrarily close to the Pigou bound $\alpha(\mathcal{C})$.

Exercise 11.5 Consider a *multicommodity* network $G = (V, E)$, where for each $i = 1, 2, \ldots, k$, r_i units of traffic travel from an origin $o_i \in V$ to a destination $d_i \in V$.

(a) Extend the definitions of a flow and of an equilibrium flow (Definition 11.3) to multicommodity networks.

(b) Extend the two expressions (11.3) and (11.4) for the total travel time to multicommodity networks.

(c) Prove that Theorem 11.2 continues to hold for multicommodity networks.

Problems

Problem 11.1 In Pigou's example (Section 11.1.2), the optimal flow routes some traffic on a path with cost twice that of a shortest path. Prove that, in every selfish routing network with affine cost functions, an optimal flow f^* routes all traffic on paths with cost at most twice that of a shortest path (according to the travel times induced by f^*).

Problem 11.2 In this problem we consider an alternative objective function, that of minimizing the *maximum* travel time

$$\max_{P \in \mathcal{P} \,:\, f_P > 0} \sum_{e \in P} c_e(f_e)$$

of a flow f. The price of anarchy (POA) with respect to this objective is then defined as the ratio between the maximum cost of an equilibrium flow and that of a flow with minimum-possible maximum cost.[6]

We assume throughout this problem that there is one origin, one destination, one unit of traffic, and affine cost functions (of the form $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$).

(a) Prove that in networks with only two vertices o and d, and any number of parallel edges, the POA with respect to the maximum cost objective is 1.

(b) *(H)* Prove that the POA with respect to the maximum cost objective can be as large as 4/3.

(c) *(H)* Prove that the POA with respect to the maximum cost objective is never larger than 4/3.

Problem 11.3 This problem considers Braess's paradox in selfish routing networks with nonlinear cost functions.

(a) Modify Braess's paradox (Section 11.1.1) to show that adding an edge to a network with nonlinear cost functions can double the travel time of traffic in an equilibrium flow.

(b) *(H)* Show that adding edges to a network with nonlinear cost functions can increase the equilibrium travel time by strictly more than a factor of 2.

[6] For an equilibrium flow, the maximum cost is just the common cost incurred by all of the traffic.