Math 316: Complex Variables **Fall 2018**

Assignment 1 Due: September 20th, 11:30 (in class)

Exercise 1. Let $z \in \mathbb{C} \setminus \{-1\}$. Show that $|z| = 1$ if and only if $\frac{z-1}{z+1}$ is purely imaginary.

Exercise 2. Let n be a positive integer. Describe the complex solutions of the equation $(z-1)^n = z^n$ in standard form.

Exercise 3. Let $w \in \mathbb{C}$ with $|w| = 1$. Show that the map

$$\phi(z) = \frac{w - z}{1 - \overline{w}z}$$

satisfies $|\phi(z)| < 1$ whenever $|z| < 1$, and $|\phi(z)| = 1$ whenever $|z| = 1$.

Exercise 4. Compute

$$\sum_{n \geq 0} \frac{\cos(n\theta)}{2^n}$$

as a function of the real parameter θ.

Exercise 5. Consider the sequence of (real) polynomials $T_n(x)$ defined by the recurrence relation

$$T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$$

where $T_0(x) = 1$ and $T_1(x) = x$.

(i) Show that $T_n(x)$ has degree n.
(ii) Show that $T_n(\cos\theta) = \cos n\theta$.
(iii) Show that $T_m(T_n(x)) = T_{mn}(x)$.

Exercise 6. Show that, in $\mathbb{C}$, the following hold:

- the union of arbitrarily many open subsets is an open subset;
- the intersection of finitely many open subsets is an open subset;

and

- the union of finitely many closed subsets is a closed subset;
- the intersection of arbitrarily many closed subsets is a closed subset.