Math 447 Assignment 2

Jonathan Pearce, 260672004

February 18, 2018

Problem 3.19. Assume the transition probability from each state is distributed uniformly. To simplify the problem we remove states b and c and add a transition from e to e, thus the transition probability matrix is as follows

$$\begin{array}{c c} & \begin{array}{cccc} a & d & e & f \end{array} \\ \begin{array}{c} a \\ d \\ e \\ f \end{array} & \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Let e_x be the expected time to hit d for the walk started in x, then we have the following

$$\begin{cases} e_a = \frac{1}{2}(1+e_f) + \frac{1}{2}(1+e_e) \\ e_e = \frac{1}{4} + \frac{1}{4}(1+e_a) + \frac{1}{4}(1+e_e) + \frac{1}{4}(1+e_f) \\ e_f = \frac{1}{2}(1+e_a) + \frac{1}{2}(1+e_e) \end{cases}$$

Solving this system,

$$\begin{cases} e_a = 10 \\ e_e = 8 \\ e_f = 10 \end{cases}$$

Thus the expected time to hit d for the walk started in a is 10.

Problem 3.27a.
Irreducibility: Let i,j be states in Figure 3.16. Without loss of generality suppose $i >= j$.
Case 1: Show j is accessible from i.

$$P_{ij}^{j+1} = P_{i0}P_{0j}^{j}$$

$$= P_{i0}\Big(\prod_{k=0}^{j-1} P_{k(k+1)}\Big)$$

$$= \frac{1}{i+1}\Big(\prod_{k=1}^{j-1} \frac{k}{k+1}\Big)$$

$$= \frac{1}{i+1}\Big(\frac{1}{2} * \frac{2}{3} * \frac{3}{4} * ... * \frac{j-2}{j-1} * \frac{j-1}{j}\Big)$$
$$= \frac{1}{i+1}\Big(\frac{1}{j}\Big)$$
$$= \frac{1}{j(i+1)} > 0$$

Therefore for $n = j + 1$ we have $P_{ij}^n > 0$

Case 2: Show i is accessible from j.

$$P_{ji}^{i-j} = \prod_{k=j}^{i-1} P_{k(k+1)}$$

$$= \prod_{k=j}^{i-1} \frac{k}{k+1}$$
$$= \frac{j}{j+1} * \frac{j+1}{j+2} * ... * \frac{i-2}{i-1} * \frac{i-1}{i}$$
$$= \frac{j}{i} > 0$$

Therefore for $n = i - j$ we have $P_{ij}^n > 0$

We have shown that states i and j communicate with each other for any i and j and therefore this Markov chain is irreducible.

Aperiodicity: Suppose we are at state 0.

$$P_{00}^2 = P_{01}P_{10}$$
$$= 1 * \frac{1}{2} = \frac{1}{2} > 0$$

$$P_{00}^3 = P_{01}P_{12}P_{20}$$
$$= 1 * \frac{1}{2} * \frac{1}{3} = \frac{1}{6} > 0$$

We have found return times for state 0 of 2 and 3 steps. Since 2 and 3 are both primes, their greatest common divisor is 1, therefore state 0 is aperiodic. By Lemma 3.7 (page 108) in the textbook we conclude that the entire Markov chain is aperiodic.

Problem 3.27b. Consider the probability of our first return to state 0 for the chain started at 0.

$$
\begin{aligned}
f_0 &= \sum_{i=1}^{n} P_{00}^{i} \\
&= 0 + \Big(1 * \frac{1}{2}\Big) + \Big(1 * \frac{1}{2} * \frac{1}{3}\Big) + \Big(1 * \frac{1}{2} * \frac{2}{3} * \frac{1}{4}\Big) + \Big(1 * \frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \frac{1}{5}\Big)... \\
&= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + ... \\
&= \sum_{i=1}^{n} \frac{1}{i(i+1)}
\end{aligned}
$$

To calculate the probability of returning to 0, we can compute f_0 as $n \to \infty$,

$$
\lim_{n\to\infty} f_0 = \lim_{n\to\infty} \sum_{i=1}^{n} P_{00}^{i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} = 1
$$

Therefore $f_0 = 1$, we are guaranteed to return to state 0 when starting a chain at state 0. We conclude state 0 is a recurrent state. By part (a) of this question and Theorem 3.3 (page 99) of the textbook, we conclude the chain is recurrent.

Problem 3.27c. We calculate the expected return time to state 0. Define e_0 to be the expected return time to state 0.

$$
e_0 = \sum_{l=1}^{\infty} l\Big(P_{0(l-1)}^{l-1} * P_{(l-1)0}\Big)
$$

$$
e_0 = 1(0) + 2\Big(1 * \frac{1}{2}\Big) + 3\Big(1 * \frac{1}{2} * \frac{1}{3}\Big) + 4\Big(1 * \frac{1}{2} * \frac{2}{3} * \frac{1}{4}\Big) + 5\Big(1 * \frac{1}{2} * \frac{2}{3} * \frac{3}{4} * \frac{1}{5}\Big)...
$$

$$
e_0 = 0 + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}....
$$

$$
e_0 = \sum_{l=1}^{\infty} \frac{1}{l} = \infty
$$

Therefore State 0 is null recurrent. By Lemma 3.12 (page 138) in the textbook we conclude the chain is null recurrent.

Problem 3.34. Assume that the Markov Chain is finite. Now, consider states i, j,

- $\tilde{P}_{ij} = pP_{ij}$ if $i \neq j$
- $\tilde{P}_{ij} = pP_{ij} + (1-p) = 1 + p(P_{ij} - 1)$ if $i = j$

Since P is a stochastic matrix we have for any row i,

$$P_{i0} + P_{i1} + P_{i2} + ... + P_{ii} + ... + P_{in} = 1$$

$$\Rightarrow P_{i0} + P_{i1} + P_{i2} + ... + P_{in} = 1 - P_{ii}$$

Now consider any row i in $\tilde{P}$

$$\tilde{P}_{i0} + \tilde{P}_{i1} + \tilde{P}_{i2} + ... + \tilde{P}_{ii} + ... + \tilde{P}_{in}$$

$$= pP_{i0} + pP_{i1} + pP_{i2} + ... + (1 + p(P_{ii} - 1)) + ... + pP_{in}$$
$$= p(P_{i0} + P_{i1} + P_{i2} + ... + P_{in}) + 1 - p(1 - P_{ii})$$
$$= p(1 - P_{ii}) + 1 - p(1 - P_{ii})$$
$$= 1$$

Therefore $\tilde{P}$ is a stochastic matrix.

Consider states i and j. Since P is irreducible the Markov Chain contains no self absorbing states. Therefore there must exist N_0 such that,

$$P_{ij}^{N_0} > 0$$

and the path taken from i to j does not loop to the same state twice in a row (i.e. always moves to a new state). It follows that,

$$\tilde{P}_{ij}^{N_0} = p^{N_0} P_{ij}^{N_0} > 0$$

Similarly, there must exist N_1 such that,

$$P_{ji}^{N_1} > 0$$

and the path taken from j to i does not loop to the same state twice in a row. It follows that,

$$\tilde{P}_{ji}^{N_1} = p^{N_1} P_{ji}^{N_1} > 0$$

Therefore $\tilde{P}$ is irreducible.

Since P is irreducible and therefore has no self absorbing states, all diagonal entries of P are less than 1. Therefore for all i,

$$0 \leq P_{ii} < 1$$

$$\Rightarrow -1 \leq P_{ii} - 1 < 0$$

$$\Rightarrow -p \leq p(P_{ii} - 1) < 0$$

$$\Rightarrow 1 - p \leq 1 + p(P_{ii} - 1) < 1$$

$$\Rightarrow 1 - p \leq \tilde{P}_{ii} < 1$$

Since $0 < p < 1$

$$\Rightarrow 0 < \tilde{P}_{ii} < 1$$

We conclude that all diagonal entries of $\tilde{P}$ have positive probability not equal to 1. Therefore in $\tilde{P}$ any state i has a possible return time of 1, and therefore is aperiodic. Since $\tilde{P}$ is irreducible, we conclude that $\tilde{P}$ is aperiodic. Finally we conclude that $\tilde{P}$ is a stochastic matrix for an ergodic Markov chain.

Let π be the stationary distribution of P, it follows $\pi P = \pi$

$$\begin{aligned}
\pi\tilde{P} &= \pi\Big(pP + (1-p)I\Big) \\
&= \pi pP + \pi(1-p)I \\
&= p\pi P + (1-p)\pi I \\
&= p\pi + (1-p)\pi \\
&= \pi
\end{aligned}$$

Therefore P and $\tilde{P}$ have the same stationary distribution

Chains in the Markov chain associated with $\tilde{P}$ will tend to stay at one state for longer periods of time than chains derived from the Markov chain associated with P. Further, each column of $\tilde{P}$ evolves towards the corresponding stationery distribution value at a slower rate than the columns of P. When the value of p is close to 1 the difference is very small, where as if p is close to 0, $\tilde{P}$ will take much longer than P before its value begin to approach the stationary distribution values.

Problem 3.38a.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix}
(\frac{5}{6})^5 & 5(\frac{5}{6})^4(\frac{1}{6}) & 10(\frac{5}{6})^3(\frac{1}{6})^2 & 10(\frac{5}{6})^2(\frac{1}{6})^3 & 5(\frac{5}{6})(\frac{1}{6})^4 & (\frac{1}{6})^5 \\
0 & (\frac{5}{6})^4 & 4(\frac{5}{6})^3(\frac{1}{6}) & 6(\frac{5}{6})^2(\frac{1}{6})^2 & 4(\frac{5}{6})(\frac{1}{6})^3 & (\frac{1}{6})^4 \\
0 & 0 & (\frac{5}{6})^3 & 3(\frac{5}{6})^2\frac{1}{6} & 3(\frac{5}{6})(\frac{1}{6})^2 & (\frac{1}{6})^3 \\
0 & 0 & 0 & (\frac{5}{6})^2 & 2(\frac{5}{6})(\frac{1}{6}) & (\frac{1}{6})^2 \\
0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \end{array}$$

Problem 3.38b.

$$e_5 = 0$$

$$6 * e_4 = 5(e_4 + 1) + 1 \Rightarrow e_4 = 6$$

$$6^2 * e_3 = 5^2(e_3 + 1) + 2 * 5(e_4 + 1) + 1 \Rightarrow e_3 = \frac{96}{11}$$

$$6^3 * e_2 = 5^3(e_2 + 1) + 3 * 5^2(e_3 + 1) + 3 * 5(e_4 + 1) + 1 \Rightarrow e_2 = \frac{10566}{1001}$$

$$6^4 * e_1 = 5^4(e_1+1) + 4*5^3(e_2+1) + 6*5^2(e_3+1) + 4*5(e_4+1) + 1 \Rightarrow e_1 = \frac{728256}{61061}$$

$$6^5 * e_0 = 5^5(e_0+1) + 5*5^4(e_1+1) + 10*5^3(e_2+1) + 10*5^2(e_3+1) + 6*5(e_4+1) + 1 \Rightarrow e_0 = \frac{3700788121}{283994711}$$

$$\Rightarrow e_0 \approx 13.03$$

Problem 3.42.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ ... \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & ... \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & ... \\ p & 0 & 1-p & 0 & 0 & ... \\ 0 & p & 0 & 1-p & 0 & ... \\ 0 & 0 & p & 0 & 1-p & ... \\ ... & ... & ... & ... & ... & ... \end{pmatrix} \end{array}$$

We observe that P is a banded matrix with the transition from State 0 to State 1 with probability 1. Let $X = (1, x_2, x_3 ...)$, now solve $XP = X$

$$\begin{cases} px_2 = 1 \\ 1 + px_3 = x_2 \\ (1-p)x_2 + px_4 = x_3 \\ (1-p)x_3 + px_5 = x_4 \\ ... \end{cases}$$

Solving,

$$\Rightarrow \begin{cases} x_2 = \frac{1}{p} \\ x_3 = \frac{1-p}{p^2} \\ x_4 = \frac{(1-p)^2}{p^3} \\ x_5 = \frac{(1-p)^3}{p^4} \\ ... \\ x_n = \frac{(1-p)^{n-2}}{p^{n-1}} \end{cases}$$

Summing over the vector X, we get,

$$= 1 + \sum_{i=2}^{\infty} \frac{(1-p)^{i-2}}{p^{i-1}}$$

$$= 1 + \frac{1}{p} \sum_{i=2}^{\infty} \left(\frac{1-p}{p}\right)^{i-2}$$

$$= 1 + \frac{1}{p} \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^{i}$$

The summation is a geometric series and in order for this sum to converge we require the following,

$$\frac{1-p}{p} < 1$$
$$\Rightarrow p > 1/2$$

It follows that when $p \in (\frac{1}{2}, 1)$,

$$\sum_{i=0}^{\infty} \Big(\frac{1-p}{p}\Big)^i = \frac{1}{1 - \frac{1-p}{p}} = \frac{p}{2p-1}$$

Finally we find the sum of the elements of the vector X whenever $p \in (\frac{1}{2}, 1)$ to be,

$$= 1 + \frac{1}{p}\frac{p}{(2p-1)} = \frac{2p-1+1}{2p-1} = \frac{2p}{2p-1}$$

Otherwise the sum diverges to infinity.

It follows that the stationary distribution when $p \in (\frac{1}{2}, 1)$ is,

$$\frac{1}{\frac{2p}{2p-1}}\Big(1, \frac{1}{p}, \frac{1-p}{p^2}, \frac{(1-p)^2}{p^3}, ...\Big)$$

$$= \Big(\frac{2p-1}{2p}, \frac{2p-1}{2p^2}, \frac{(2p-1)(1-p)}{2p^3}, \frac{(2p-1)(1-p)^2}{2p^4}, ...\Big)$$

Now that we have the stationary distribution and the transition probability matrix we can find what values of p make the chain time reversible by solving the detailed balance equations. Because of the symmetry of the graph each non zero entry is only in 1 detailed balance equation. Therefore we only need to check 2 equations, the first being the equation with transitions between 0 and 1, the second being the general case with transitions between i and $i+1$ ($i \neq 0$).

$$\pi_0 P_{01} = \Big(\frac{2p-1}{2p}\Big)(1) = \Big(\frac{2p-1}{2p^2}\Big)(p) = \pi_1 P_{01}$$

$$\pi_i P_{i(i+1)} = \Big(\frac{(2p-1)(1-p)^{i-1}}{2p^{i+1}}\Big)(1-p) = \Big(\frac{(2p-1)(1-p)^i}{2p^{i+2}}\Big)(p) = \pi_{i+1} P_{(i+1)i}$$

Therefore the detailed balance equations are satisfied. We conclude that the chain is time reversible whenever $p \in (\frac{1}{2}, 1)$. With these values of p the stationary distribution π is,

$$\pi = \Big(\frac{2p-1}{2p}, \frac{2p-1}{2p^2}, \frac{(2p-1)(1-p)}{2p^3}, \frac{(2p-1)(1-p)^2}{2p^4}, ...\Big)$$

Problem 3.44. Let $X = (1, x_2, x_3, x_4)$, now solve $XP = X$

$$\begin{cases} (1-p-2r) + x_2p + x_3q + x_4q = 1 \\ p + x_2(1-p-2r) + x_3q + x_4q = x_2 \\ r + x_2r + x_3(1-p-2q) + x_4p = x_3 \\ r + x_2r + x_3p + x_4(1-p-2q) = x_4 \end{cases}$$

solving this system we get,

$$X = (1, 1, \frac{r}{q}, \frac{r}{q})$$

It follows that the stationary distribution is,

$$\pi = \frac{1}{1 + x_2 + x_3 + x_4} X$$

$$= \frac{1}{1 + 1 + \frac{r}{q} + \frac{r}{q}} (1, 1, \frac{r}{q}, \frac{r}{q})$$

$$= \frac{q}{2r + 2q} (1, 1, \frac{r}{q}, \frac{r}{q})$$

$$= \left(\frac{q}{2(r+q)}, \frac{q}{2(r+q)}, \frac{r}{2(r+q)}, \frac{r}{2(r+q)} \right)$$

$$= \frac{1}{2(r+q)} (q, q, r, r)$$

Let R to be a non stochastic matrix where $R_{ij} = \pi_i P_{ij}$

$$R = \frac{1}{2(r+q)} \begin{array}{c} \\ a \\ c \\ g \\ t \end{array} \begin{array}{c} \begin{array}{cccc} a & c & g & t \end{array} \\ \begin{pmatrix} q(1-p-2r) & qp & qr & qr \\ qp & q(1-p-2r) & qr & qr \\ rq & rq & r(1-p-2r) & rp \\ rq & rq & rp & r(1-p-2r) \end{pmatrix} \end{array}$$

We observe that R is symmetric and therefore the vector $\frac{1}{2(r+q)}(q, q, r, r)$ (i.e. π) satisfies the detailed balance equations and the chain is reversible.

Case: $p = 0.1$, $q = 0.2$, and $r = 0.3$

$$P = \begin{array}{c} \\ a \\ c \\ g \\ t \end{array} \begin{array}{c} \begin{array}{cccc} a & c & g & t \end{array} \\ \begin{pmatrix} 0.3 & 0.1 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.5 \end{pmatrix} \end{array}$$

$$\pi = (0.2, 0.2, 0.3, 0.3)$$

$$R = \begin{array}{c} \\ a \\ c \\ g \\ t \end{array} \begin{array}{c} \begin{array}{cccc} a & c & g & t \end{array} \\ \begin{pmatrix} 0.06 & 0.02 & 0.06 & 0.06 \\ 0.02 & 0.06 & 0.06 & 0.06 \\ 0.06 & 0.06 & 0.15 & 0.03 \\ 0.06 & 0.06 & 0.03 & 0.15 \end{pmatrix} \end{array}$$

For the given values R is symmetric. Therefore the vector $\pi = (0.2, 0.2, 0.3, 0.3)$ satisfies the detailed balance equations and the chain is reversible.

Math 447 Assignment 2 Continued

Jonathan Pearce

February 12, 2018

3.34

Rough Work to see how chains evolve

```
##### Matrix powers ###############################
# matrixpower(mat,k) mat^k
#
matrixpower <- function(mat,k) {
if (k == 0) return (diag(dim(mat)[1]))
if (k == 1) return(mat)
if (k > 1) return( mat %*% matrixpower(mat, k-1))
}

stationary <- function(mat) {
x = eigen(t(mat))$vectors[,1]
as.double(x/sum(x))
}

P = matrix(c(1/9,1/9,7/9, 1/9,7/9,1/9, 2/9,2/9,5/9),ncol=3,byrow=T)
P_hat = matrix(c(892/900,1/900,7/900,1/900,898/900,1/900,2/900,2/900,896/900),ncol=3,byrow=T)

stationary(P)
```

```
## [1] 0.1538462 0.4615385 0.3846154
```

```
stationary(P_hat)
```

```
## [1] 0.1538462 0.4615385 0.3846154
```

```
matrixpower(P,10)
```

```
##           [,1]      [,2]      [,3]
## [1,] 0.1542111 0.4592893 0.3864995
## [2,] 0.1535028 0.4636546 0.3828427
## [3,] 0.1541122 0.4598988 0.3859890
```

```
matrixpower(P_hat,10)
```

```
##            [,1]       [,2]       [,3]
## [1,] 0.91537184 0.01131807 0.07331010
## [2,] 0.01067846 0.97816338 0.01115816
## [3,] 0.02103711 0.02167672 0.95728616
```

3.44

Confirm results for $p = 0.1$, $q = 0.2$, and $r = 0.3$.

```
P_DNA = matrix(c(3/10,1/10,3/10,3/10,1/10,3/10,3/10,3/10,2/10,2/10,5/10,1/10,2/10,2/10,1/10,5/10),ncol=
```

```r
pi = stationary(P_DNA)

#stationary distribtution
pi
```

```
## [1] 0.2 0.2 0.3 0.3
```

```r
#Detailed balance equations
for (i in 1:4){
  for (j in 1:4){
    #print the two sides of a detailed balance equation, check they're equal by inspection
    cat(sprintf("\"%f\" \"%f\"\n", pi[i]*P_DNA[i,j], pi[j]*P_DNA[j,i]))
  }

}
```

```
## "0.060000" "0.060000"
## "0.020000" "0.020000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.020000" "0.020000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.150000" "0.150000"
## "0.030000" "0.030000"
## "0.060000" "0.060000"
## "0.060000" "0.060000"
## "0.030000" "0.030000"
## "0.150000" "0.150000"
```

Therefore we have confirmed via computation that our results are correct and work with the case $p = 0.1$, $q = 0.2$, and $r = 0.3$.

3.54

I is an absorbing state. Therefore we remove it from the matrix and form a new matrix Q. We then compute F the fundamental matrix for absorbing classes.

```r
Q = matrix(c(0,1/2,0,1/2,0,0,0,0,1/3,0,1/3,0,1/3,0,0,0,0,1/2,0,0,0,1/2,0,0,1/2,0,0,0,0,0,1/2,0,0,1/3,0,

F = solve(diag(8)-Q)
F[1,1]
```

```
## [1] 7.5
```

```r
rowSums(F)
```

```
## [1] 44.5 39.5 39.0 47.5 32.0 36.5 48.5 17.0
```

a

The sum of the first row of F is the expected number of rooms the mouse will visit before it finds the cheese. Therefore we expect the mouse to visit 44.5 rooms on average before finding the cheese

b

F_{11} is the the expected number of times the mouse will visit room A before it finds the cheese. Therefore we expect the mouse to visit room A 7.5 times on average before finding the cheese (we count our initial state of room A as 1 visit)

3.66

We consider d an absorbing state in this case and we form a new matrix Q. We then compute F the fundamental matrix for absorbing classes.

```
Q = matrix(c(0,1/4.0,1/4.0,1/4.0,1/4.0,1/2.0,0,1/2.0,0,0,1/4.0,1/4.0,0,1/4.0,0,1/4.0,0,1/4.0,0,1/4.0,1/2

F = solve(diag(5)-Q)
F
```

```
##          [,1]      [,2]     [,3]     [,4]      [,5]
## [1,] 3.333333 1.3333333 2.000000 2.000000 1.3333333
## [2,] 2.666667 2.2222222 2.222222 1.777778 1.1111111
## [3,] 2.000000 1.1111111 2.444444 1.555556 0.8888889
## [4,] 2.000000 0.8888889 1.555556 2.444444 1.1111111
## [5,] 2.666667 1.1111111 1.777778 2.222222 2.2222222
```

```
rowSums(F)
```

```
## [1] 10 10  8  8 10
```

The sum of the first row of F is the expected time to hit d for the walk started in a. Therefore we expect the walk from a to d to take 10 steps. This agrees with our answer in 3.19.