INTRODUCTION TO STOCHASTIC PROCESSES WITH R

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To my family

CONTENTS

PREFACE

> The last thing one discovers in composing a work is what to put first.
>
> —Blaise Pascal

The intended audience for this book are students who like probability. With that prerequisite, I am confident that you will love stochastic processes.

Stochastic, or random, processes is the dynamic side of probability. What differential equations is to calculus, stochastic processes is to probability. The material appeals to those who like applications and to those who like theory. It is both excellent preparation for future study, as well as a *terminal* course, in the sense that we do not have to tell students to wait until the next class or the next year before seeing the good stuff. This *is* the good stuff! Stochastic processes, as a branch of probability, speaks in the language of rolling dice, flipping coins, and gambling games, but in the service of applications as varied as the spread of infectious diseases, the evolution of genetic sequences, models for climate change, and the growth of the World Wide Web.

The book assumes that the reader has taken a calculus-based probability course and is familiar with matrix algebra. Conditional probability and conditional expectation, which are essential tools, are offered in the introductory chapter, but may be skimmed over depending upon students' background. Some topics assume a greater knowledge of linear algebra than basic matrices (such as eigenvalues and eigenvectors) but these are optional, and relevant sections are starred. The book does not assume background in combinatorics, differential equations, or real analysis. Necessary mathematics is introduced as needed.

A focus of this book is the use of simulation. I have chosen to use the popular statistical freeware `R`, which is an accessible interactive computing environment. The use of simulation, important in its own right for applied work and mathematical research, is a powerful pedagogical tool for making theoretical concepts come alive with practical, hands-on demonstrations. It is not necessary to use `R` in order to use this book; code and script files are supplemental. However, the software is easy—and fun—to learn, and there is a tutorial and exercises in an appendix for bringing students up to speed.

The book contains more than enough material for a standard one-semester course. Several topics may lend themselves to individual or group projects, such as card shuffling, perfect sampling (coupling from the past), queueing theory, stochastic calculus, martingales, and stochastic differential equations. Such specialized material is contained in starred sections.

An undergraduate textbook poses many challenges. I have struggled with trying to find the right balance between theory and application, between conceptual understanding and formal proof. There are, of course, some things that cannot be said. Continuous-time processes, in particular, require advanced mathematics based on measure theory to be made precise. Where these subjects are presented I have emphasized intuition over rigor.

Following is a synopsis of the book's nine chapters.

Chapter 1 introduces stochastic and deterministic models, the generic features of stochastic processes, and simulation. This is essential material. The second part of the chapter treats conditional probability and conditional expectation, which can be reviewed at a fast pace.

The main features of discrete-time Markov chains are covered in Chapters 2 and 3. Many examples of Markov chains are introduced, and some of them are referenced throughout the book. Numerical and simulation-based methods motivate the discussion of limiting behavior. In addition to basic computations, topics include stationary distributions, ergodic and absorbing chains, time reversibility, and the strong Markov property. Several important limit theorems are discussed in detail, with proofs given at the end of the chapter. Instructors may choose to limit how much time is spent on proofs.

Branching processes are the topic of Chapter 4. Although branching processes are Markov chains, the methods of analysis are different enough to warrant a separate chapter. Probability-generating functions are introduced, and do not assume prior exposure.

The focus of Chapter 5 is Markov chain Monte Carlo, a relatively new topic but one with exponentially growing application. Instructors will find many subjects to pick and choose. Several case studies make for excellent classroom material, in particular (i) a randomized method for decoding text, from Diaconis (2009), and (ii) an application that combines ecology and counting matrices with fixed row and column totals, based on Cobb and Chen (2003). Other topics include coupling from the past, card shuffling, and rates of convergence of Markov chains.

Chapter 6 is devoted to the Poisson process. The approach emphasizes three alternate definitions and characterizations, based on the (i) counting process, (ii) arrival process, and (iii) infinitesimal description. Additional topics are spatial processes, nonhomogeneous Poisson processes, embedding, and arrival time paradoxes.

Continuous-time Markov chains are discussed in Chapter 7. For continuous-time stochastic processes, here and in Chapter 8, there is an emphasis on intuition, examples, and applications. In addition to basic material, there are sections on queueing theory (with Little's formula), absorbing processes, and Poisson subordination.

Brownian motion is the topic of Chapter 8. The material is more challenging. Topics include the invariance principle, transformations, Gaussian processes, martingales, and the optional stopping theorem. Examples include scoring in basketball and animal tracking. The Black–Scholes options pricing formula is derived.

Chapter 9 is a gentle introduction to stochastic calculus. *Gentle* means no measure theory, sigma fields, or filtrations, but an emphasis on examples and applications. I decided to include this material because of its growing popularity and application. Stochastic differential equations are introduced. Simulation and numerical methods help make the topic accessible.

Book appendices include (i) getting started with `R`, with exercises, (ii) probability review, with short sections on the main discrete and continuous probability distributions, (iii) summary table of common probability distributions, and (iv) matrix algebra review. Resources for students include a suite of `R` functions and script files for generating many of the processes from the book.

The book contains more than 200 examples, and about 600 end-of-chapter exercises. Short solutions to most odd-numbered exercises are given at the end of the book. A web site www.people.carleton.edu/rdobrow/stochbook is established. It contains errata and relevant files. All the `R` code and script files used in the book are available at this site. A solutions manual with detailed solutions to all exercises is available for instructors.

Much of this book is a reflection of my experience teaching the course over the past 10 years. Here is a suggested one-semester syllabus, which I have used.

1. Introduction and review—1.1, 1.2, 1.3 (quickly skim 1.4 and 1.5)
2. One-day introduction to `R`—Appendix A
3. Markov chains—All of chapters 2 and 3
4. Branching processes—Chapter 4
5. MCMC—5.1, 5.2
6. Poisson process—6.1, 6.2, 6.4, 6.5, 6.8
7. Continuous-time Markov chains—7.1, 7.2, 7.3, 7.4
8. Brownian motion—8.1, 8.2, 8.3, 8.4, 8.5, 8.7

If instructors have questions on syllabus, homework assignments, exams, or projects, I am happy to share resources and experiences teaching this most rewarding course.

Stochastic Processes is a great mathematical adventure. Bon voyage!

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LIST OF SYMBOLS AND NOTATION

> Such is the advantage of a well constructed language that its simplified notation often becomes the source of profound theories.
>
> —Pierre-Simon Laplace

$\emptyset$	empty set
$A \cup B$	union of sets A and B
$A \cap B$	intersection of sets A and B
$A \subseteq B$	A is a subset of B
$x \in A$	x is an element of set A
$\binom{n}{k}$	binomial coefficient $\dfrac{n!}{k!(n-k)!}$
$\lfloor x \rfloor$	floor of x
$\lceil x \rceil$	ceiling of x
$\Vert \boldsymbol{x} \Vert$	length of vector $\boldsymbol{x}$
$\boldsymbol{A}^{-1}$	matrix inverse of $\boldsymbol{A}$
$\boldsymbol{A}^T$	matrix transpose of $\boldsymbol{A}$
$e^{\boldsymbol{A}}$	matrix exponential of $\boldsymbol{A}$
B_t	standard Brownian motion
$\mathrm{Corr}(X, Y)$	correlation of X and Y
$\mathrm{Cov}(X, Y)$	covariance of X and Y
$d(i)$	period of state i
$\deg(v)$	degree of vertex v in a graph
e	extinction probability (for branching process)

$E(X)$	expectation of X
$E(X^k)$	kth moment of X
$E(X\|Y)$	conditional expectation of X given Y
f_j	probability of eventual return to j
$f_X(x)$	probability density function of X
$f(x, y)$	joint probability density function of X and Y
$f_{X\|Y}(x\|y)$	conditional density function of X given $Y = y$
$G_X(s)$	probability generating function of X
$\boldsymbol{I}$	identity matrix
I_A	indicator function of the event A
$k(s, t)$	covariance function
$K_t(x, y)$	Markov transition kernel
λ_*	second largest eigenvalue
$m_X(t)$	moment generating function of X
N_t	Poisson process
N_A	spatial Poisson process ($A \subseteq \mathbb{R}^d$)
ω	simple outcome of a sample space
Ω	sample space
$\boldsymbol{\pi}$	stationary distribution of Markov chain
$\boldsymbol{P}$	transition matrix for Markov chain
$\boldsymbol{P}(t)$	transition function for continuous-time Markov chain
$\boldsymbol{Q}$	generator matrix
$\mathbb{R}$	real numbers
$\mathbb{R}^2$	two-dimensional Euclidean space
$\mathbb{R}^n$	n-dimensional Euclidean space
$\mathcal{S}$	state space for Markov chain
$SD(X)$	standard deviation of X
T_a	first hitting time of a
$U_{(1)}, \ldots, U_{(n)}$	order statistics
$v(t)$	total variation distance
$Var(X)$	variance of X
$Var(X\|Y)$	conditional variance of X given Y

NOTATION CONVENTIONS

Matrices are represented by bold, capital letters, for example, $\boldsymbol{M}$, $\boldsymbol{P}$.

Vectors are represented by bold, lowercase letters, for example, $\boldsymbol{\alpha}$, $\boldsymbol{\lambda}$, $\boldsymbol{\pi}$. When vectors are used to represent discrete probability distributions they are row vectors.

ABBREVIATIONS

cdf	cumulative distribution function
gcd	greatest common divisor
i.i.d.	independent and identically distributed
MCMC	Markov chain Monte Carlo
mgf	moment-generating function
pdf	probability density function
pgf	probability generating function
pmf	probability mass function
sde	stochastic differential equation
slln	strong law of large numbers
wlln	weak law of large numbers

ABOUT THE COMPANION WEBSITE

This book is accompanied by a companion website:
http://www.people.carleton.edu/~rdobrow/stochbook/

The website includes:

- Solutions manual available to instructors.
- R script files
- Errata