Hints to Selected Exercises and Problems

Problem 2.1(c): Shoot for $c = \frac{1}{4}$. Use the first half of the bidders to get calibrated.

Problem 3.1(b): Adopt bidder i's perspective and "target" slot j.

Problem 3.1(d): First prove that, in a locally envy-free bid profile, the bidders must be sorted in nonincreasing order of values-per-click.

Problem 3.1(e): Use (3.8). What bids would yield these payments in a GSP auction? Use part (d) to argue that these bids form an equilibrium.

Problem 3.2(b): This boils down to checking that the payment rule of the Revenue Target Auction satisfies Myerson's payment formula.

Exercise 4.7: For example, what do auction houses such as Christie's and Sotheby's use?

Problem 4.2(b): If S^* is the optimal solution (with item values $\mathbf{v}$), and S is the computed solution (optimal for item values $\tilde{\mathbf{v}}$), then $\sum_{i \in S} v_i \geq m \sum_{i \in S} \tilde{v}_i \geq m \sum_{i \in S^*} \tilde{v}_i \geq \sum_{i \in S^*} (v_i - m)$.

Problem 4.2(e): Try many different values of m and use part (c). Under what conditions does taking the better of two monotone allocation rules yield another monotone allocation rule?

Problem 4.3(a): Reduce from the problem of computing the largest independent set of a graph (see, e.g., Garey and Johnson (1979)).

Problem 4.3(c): When the greedy algorithm makes a mistake by selecting some bidder, how many other bidders can it "block"?

Exercise 5.6: The distribution has infinite expectation, violating the assumptions of Section 5.1.3.

Problem 5.2: Use Problem 5.1(c).

Problem 5.3(c): First extend part (a), with $b_i(v_i)$ set to the expected value of the second-highest valuation, conditioned on the event that v_i is the highest valuation.

Exercise 6.1(b): Two bidders with valuations drawn from different uniform distributions suffice.

Exercise 6.2: Define t such that $\Pr[\pi_i > t \text{ for all } i] \leq \frac{1}{2} \leq \Pr[\pi_i \geq t \text{ for all } i]$. Show that at least one of the two corresponding strategies—either taking the first prize with value at least t, or the first with value exceeding t—satisfies the requirement.

Exercise 6.4: Use the Bulow-Klemperer theorem. Use Theorem 5.2 to bound the amount by which the optimal expected revenue can decrease when one bidder is removed.

Problem 6.1(a): Take $n = 2$.

Problem 6.2(b): Use downward-closure to reason about the outcome selected by $\mathcal{M}^*$.

Problem 6.2(c): Use part (a).

Problem 6.3(b): Given posted prices $p_1, \ldots, p_n$, consider a single-item auction that applies a reserve price of p_i to each bidder i and then awards the item to the remaining bidder (if any) with the largest value of $v_i - p_i$.

Problem 6.3(c): Identify posted prices $p_1, \ldots, p_n$ as in the proof of Theorem 6.4.[15] Show that only less expected revenue is earned by the single-item auction that applies a reserve price of p_i to each bidder i and then awards the item to the remaining bidder (if any) with the smallest value of p_i. Use the prophet inequality (Theorem 6.1 and Remark 6.2) to lower bound the expected virtual welfare, and hence expected revenue, of this auction.

Problem 6.4(b): Instantiate Theorem 6.5 with $n = 1$ to deduce that, with one bidder and one item, the expected revenue earned

[15] Warning: as a non-single-parameter setting, you cannot assume that expected revenue equals expected virtual welfare (cf., Theorem 5.2).

by a posted price p drawn randomly from F is at least half that by a monopoly price p^* for F. Use regularity to argue that, for every $t \geq 0$, this guarantee continues to hold for the prices $\max\{p, t\}$ and $\max\{p^*, t\}$. How much expected revenue does a bidder $i \neq j$ contribute to the optimal and given mechanisms?

Exercise 7.5: Use the fact that a maximum-weight matching of a bipartite graph can be computed in polynomial time.

Problem 7.1(b): Sum up the VCG payments (7.2) and simplify to obtain a multiple of the left-hand side of (7.4) and bid-independent terms.

Problem 7.3(b): Use subadditivity.

Problem 7.3(c): Use Problem 7.2. Exercise 7.5 is also relevant.

Exercise 8.1: First show that the sum of bidders' utilities (at prices $\mathbf{p}$) is maximized, then cancel out the price terms.

Exercise 8.2: Use the same example that illustrates the exposure problem.

Problem 8.3: For the reader familiar with linear programming duality, the allocation corresponds to a maximum-weight bipartite matching, the prices to an optimal dual solution, and the equilibrium conditions to complementary slackness conditions. Alternatively, use the payments of the VCG mechanism to define the item prices, and the structure of optimal matchings to verify the equilibrium conditions.

Exercise 9.3: Construct an example where one bidder can delay reporting a demand decrease to cause a different bidder to pay extra, resulting in lower prices for future items.

Exercise 9.4(c): Consider the realistic setting in which each B_i/v_i is modestly large but still far smaller than m.

Problem 9.1(b): First prove that, for every such deterministic DSIC auction, there is a simple probability distribution over valuation profiles such that the expected social welfare of the auction is at most c/n times the expected highest valuation. Explain why this implies the desired lower bound for both deterministic and randomized auctions.

Problem 9.3(c): Generalize the mechanism in (b) in two different ways. The less obvious way is to supplement the reported peaks with additional "dummy peaks."

Exercise 10.1: Adding back an edge of $E_i \setminus F_i$ either has no effect on which vertices before i get matched, or else guarantees that i is matched.

Exercise 10.6: If a hospital w prefers its match v in the applicant-optimal stable matching to its match v' in some other stable matching M', then (v, w) form a blocking pair for M'.

Problem 10.1: First consider a misreport that differs from the true preference list only in the order of two consecutive hospitals. Use induction to extend to arbitrary misreports.

Exercise 11.1: Prove that there is no loss of generality restricting to Pigou-like networks with $a = r = 1$. The POA in such networks is decreasing in b.

Exercise 11.2: Proceed by direct computation, or alternatively show how to replace the concave cost functions of a network by affine cost functions so that the POA can only increase.

Exercise 11.3(c): Transform a network with polynomial cost functions into one with the same POA and monomial cost functions.

Exercise 11.4(a): Starting from a Pigou-like example, simulate the edge with constant cost function $c(x) = \beta$ by many parallel edges, each with a cost function c satisfying $c(0) = \beta$.

Exercise 11.4(b): Let $\overline{\mathcal{C}}$ denote the set of all nonnegative scalar multiples of cost functions in $\mathcal{C}$. Apply part (a) to $\overline{\mathcal{C}}$ and simulate scalar multiples using paths of multiple edges.

Problem 11.2(b): Braess's paradox.

Problem 11.2(c): This is a relatively straightforward consequence of Theorem 11.2 and Exercise 11.1.

Problem 11.3(b): Add two edges to a network with six vertices.

Exercise 12.2(c): Follow the proof of Theorem 11.2. In (11.10), invoke the β-over-provisioned assumption to justify using α_β in place of $\alpha(\mathcal{C})$.

Exercise 12.6: Check all cases where y and z are both small. What happens as y or z grows large?

Problem 12.1: Prove that, with an affine cost function, the inequality (12.4) holds even with an extra factor of $\frac{1}{4}$ on the right-hand side.

Problem 12.3(a): Two useful lower bounds on the minimum-possible makespan are $\max_{i=1}^k w_i$ and $\sum_{i=1}^k w_i/m$.

Exercise 13.4: Proceed edge-by-edge.

Problem 13.1: Consider the special case of $k = m$ and $w_i = 1$ for all agents i. Invoke well-known properties of occupancy (i.e., "balls into bins") problems that are discussed in standard texts like Mitzenmacher and Upfal (2005) and Motwani and Raghavan (1996).

Problem 13.3: For the "only if" direction, set $C_1^t, \ldots, C_k^t$ equal to the potential function.

Problem 13.4(a): The resources E correspond to the outcomes of the team game. Map each strategy s_i of agent i in the team game to the subset of E corresponding to outcomes where i chooses s_i. The cost of each resource is zero except when used by all of the agents.

Problem 13.4(b): The resources E correspond to choices of an agent i and strategies $\mathbf{s}_{-i}$ of the others. Map each strategy s_i of agent i in the dummy game to the set of resources of the form $\mathbf{s}_{-i}$ or $\mathbf{s}_{-j}$ with i playing a strategy other than s_i. The cost of each resource is zero except when used by a single agent. (Such cost functions may be decreasing, as permitted in congestion games.)

Exercise 14.1: Use property (P2).

Exercise 14.5: Follow the derivation in Section 14.4.1.

Problem 14.1(c): Prove that every such game is $(2, 0)$-smooth with respect to the optimal outcome in (b) (see Remark 14.3).

Problem 14.2(b): Prove that every such game is $(\frac{1}{2}, 1)$-smooth with respect to the optimal outcome in which each bidder bids half her value.

Problem 14.3(a): Consider two bidders and two items, with $v_{11} = v_{22} = 2$ and $v_{12} = v_{21} = 1$.

Problem 14.3(b): Fix an optimal outcome in which each bidder i receives at most one item $j(i)$. Prove that every such game is $(1, 1)$-smooth with respect to the optimal outcome in which each bidder i bids $v_{ij(i)}$ on item $j(i)$ and zero on all other items.

Exercise 15.6: Prove a stronger version of (15.4).

Problem 15.3: Generalize Exercise 13.4 and proceed as in the proof of Theorem 15.1.

Exercise 16.1: Two agents with three strategies each suffice.

Exercise 16.4: Create a directed graph as in Figure 16.1 and topologically sort the vertices.

Problem 16.2: Reprove Lemma 16.5, again using that agent i was chosen over j and that agent j has the option of deviating to s_i'.

Problem 16.3(b): Three agents suffice.

Problem 16.3(c): Proceed by induction on the number of agents. After adding a new agent to an inductively defined PNE, show that best-response dynamics converges to a PNE in at most k iterations.

Exercise 17.1: Reduce the problem to the special case of costs in $[-1, 1]$.

Exercise 17.2: Restart the algorithm with a new "guess" for T each time it reaches a time step t that is a power of 2.

Exercise 17.3: Use a time-averaged version of (16.11).

Problem 17.2(a): For the upper bound, follow the advice of the majority of the remaining potentially omniscient experts.

Problem 17.2(c): Follow the advice of one of the remaining potentially omniscient experts, chosen uniformly at random.

Problem 17.3: Pre-program σ into the algorithms $\mathcal{A}_1, \ldots, \mathcal{A}_k$. To make sure that each $\mathcal{A}_i$ is a no-regret algorithm, switch to the multiplicative weights algorithm if some other agent j fails to use the agreed-upon algorithm $\mathcal{A}_j$.

Problem 17.4(a): Sample the X_a's gradually by flipping coins only as needed, pausing once the action a^* with smallest perturbed cumulative cost is identified. Resuming, only X_{a^*} is not yet fully determined. What can you say if the next coin flip comes up "tails?"

Problem 17.4(b): Consider first the special case where $X_a = 0$ for all a. Iteratively transform the action sequence that always selects the best action in hindsight to the sequence chosen by the proposed algorithm. Work backward from time T, showing that the cost only decreases with each step of the transformation.

Problem 17.4(d): By (a), at each time step, the FTPL algorithm chooses the same action as the algorithm in (b) except with probability η.

Problem 17.4(e): Use a new perturbation at each time step.

Exercise 18.1: Look to Rock-Paper-Scissors for inspiration.

Exercise 18.4: Use Exercise 18.3.

Exercise 18.6: Given an arbitrary two-player game, add a "dummy player" to make it zero-sum.

Problem 18.1: Two agents with two strategies each suffice.

Problem 18.3(a): Each inequality has the form $\sum_{\mathbf{s} \in O} z_{\mathbf{s}} \cdot C_i(\mathbf{s}) \leq \sum_{\mathbf{s} \in O} z_{\mathbf{s}} \cdot C_i(s_i', \mathbf{s}_{-i})$ for an agent i and a strategy $s_i' \in S_i$.

Problem 18.4: Use Exercise 18.3 and characterize the minimax pairs instead. To compute a strategy for the row player, solve for a mixed strategy $\mathbf{x}$ and the largest real number ζ such that, for every pure (and hence mixed) strategy that the column player might play, the row player's expected payoff when playing $\mathbf{x}$ is at least ζ.

Exercise 19.2: No. Describing a congestion game with k agents and m edges requires only km parameters, while the linear program in Problem 18.3 has size exponential in k.

Exercise 19.3: Use the reduction in the proof of Theorem 19.4.

Problem 19.1(a): Reduce the problem of computing a global minimizer of the potential function (13.6) to the minimum-cost flow problem (see, e.g., Cook et al. (1998)).

Problem 19.1(b): Proceed directly or use the fact that minimum-cost flows are characterized by the nonexistence of improving cycles in the "residual graph."

Exercise 20.1: Consider some MNE of $(\mathbf{A}, \mathbf{B})$, and suppose the row and column players place positive probability only on the rows R and columns C, respectively. Solve a system of linear equations to recover the probabilities of a MNE where the row and column players randomize only over R and C, respectively.

Exercise 20.2: Given only the descriptions of $\mathcal{A}_1$ and $\mathcal{A}_2$, how can you be sure there is always such a witness? If there isn't one, how do you solve the problem in $\mathcal{NP} \cap co\mathcal{NP}$?

Problem 20.1: Use the solution to Exercise 20.1.

Problem 20.2(a): Use Chernoff-Hoeffding bounds, as presented in standard texts like Mitzenmacher and Upfal (2005) and Motwani and Raghavan (1996), to prove that that the expected payoff of every pure strategy is almost the same in $(\mathbf{x}^*, \mathbf{y}^*)$ and in $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$.

Problem 20.2(b): Adapt the solution to Problem 20.1. How many components of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are nonzero?

Problem 20.3(a): Given a bimatrix game $(\mathbf{A}, \mathbf{B})$, have the players play twice in parallel, once in either role. That is, after translating the payoffs, use the payoff matrix $\begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{B}^\top & \mathbf{0} \end{pmatrix}$ and its transpose.

Problem 20.3(c): Prove that the symmetric games generated by the reduction in (a) are guaranteed to possess asymmetric MNE.