Math 243 Analysis Assignment 5

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Problem 1a. Suppose $x_1 \in [0,1]$ such that $x_1 \in (\frac{1}{n+1}, \frac{1}{n}]$, therefore $f(x_1) = \frac{1}{n}$. Take $x_2 \in [0,1]$ such that $x_2 > x_1$. There are two cases. The first being that $x_2 \in (\frac{1}{n+1}, \frac{1}{n}]$ and therefore $f(x_2) = \frac{1}{n} = f(x_1)$. The second case is that $x_2 \in (\frac{1}{n}, \frac{1}{n-1}]$ and therefore $f(x_2) = \frac{1}{n-1} > f(x_1)$. This proves that $f(x_2) \geq f(x_1)$ and thus f is increasing on [0,1]. Therefore f is Riemann Integrable on [0,1].

Problem 1b. Let $\epsilon > 0$.

$$(f-g)(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0, & \text{elsewhere} \end{cases}$$

Define α and ω on the interval [0,1] as follows,

$$\alpha(x) = 0$$

$$\omega(x) = \begin{cases} 1, & \text{if } x \in [0, \epsilon] \\ (f-g)(x), & \text{if } x \in (\epsilon, 1] \end{cases}$$

Note that $\alpha \leq f \leq \omega \ \forall x \in [0,1]$ and α is a step function and therefore Riemann Integrable.

$$\int_0^1 \alpha = 0 * 1 = 0$$

ω is a function of two parts, the first being when $\omega = 1$ for $x \in [0, \epsilon]$, therefore the first section of ω is Riemann Integrable. The second part is when $\omega = f - g$ for $x \in (\epsilon, 1]$. By the Archimedean Property there is $N \in \mathbb{N}$ such that $\frac{1}{N} < e$. Therefore there are a finite number of points where $x = \frac{1}{n}$ and $1 \leq n \leq N$. In other words the second part of ω is constant at 0 except for finitely many points where $x = \frac{1}{n}$, therefore this section of the function is Riemann Integrable and its integral is equal to that of a function that is constantly equal to 0. By the additivity property of Riemann Integrals ω is Riemann Integrable on [0,1] and,

$$\int_0^1 \omega = \int_0^\epsilon \omega + \int_\epsilon^1 \omega = 1 * \epsilon + 0 * (1 - \epsilon) = \epsilon$$

Therefore,

$$\int_0^1 \omega - \alpha = \int_0^1 \omega - \int_0^1 \alpha = \epsilon - 0 = \epsilon$$

We conclude that $f-g$ is Riemann Integrable on [0,1]. Next for all $\epsilon>0$,

$$0 \leq \int_0^1 f-g \leq \epsilon$$

$$\implies \int_0^1 f-g = \int_0^1 f - \int_0^1 g = 0$$

$$\implies \int_0^1 f = \int_0^1 g$$

We conclude that g is Riemann Integrable.

Problem 2a. There are 5 cases to consider:
Case $a<0, b<0$: g is constant at -1 on $[a,b]$. Therefore g is Riemann Integrable.
Case $a<0, b=0$: g is constant at -1 on $[a,b]$ except for finitely many points. Therefore g is Riemann Integrable.
Case $a>0, b>0$: g is constant at 1 on $[a,b]$. Therefore g is Riemann Integrable.
Case $a=0, b>0$: g is constant at 1 on $[a,b]$ except for finitely many points. Therefore g is Riemann Integrable.
Case $a<0, b>0$: Divide the interval $[a,b]$ into two subintervals $I_1=[a,0]$ and $I_2=(0,b)$. Consider g on I_1, this is simply case 2. Similarly consider g on I_2, this is a situation that is equivalent to case 3. By the additivity theorem, g is Riemann Integrable on $I_1 \cup I_2 = [a,b]$. Therefore g is Riemann integrable on any interval $[a,b]$ with $a<b$.

Problem 2b. Let $\mathcal{P}$ be any partition of $[0,1]$ and let $\epsilon = \frac{1}{2}$
Consider the tagged partition $\dot{\mathcal{P}}_1 = \{t_1, t_2, ..., t_n\}$ such that $t_i \in \mathbb{R}\backslash\mathbb{Q}$ $\forall 1 \leq i \leq n$ and $||\dot{\mathcal{P}}_1|| < \delta$. Then since t_i is irrational.

$$S(g \circ f; \dot{\mathcal{P}}_1) = \sum_{i=1}^{n} g(f(t_i))(x_i - x_{i-1}) = \sum_{i=1}^{n} g(0)(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1})$$

$$\implies S(g \circ f; \dot{\mathcal{P}}_1) = 0$$

Now consider the tagged partition $\dot{\mathcal{P}}_2 = \{s_1, s_2, ..., s_n\}$ such that $s_i \in \mathbb{Q}$ $\forall 1 \leq i \leq n$ and $||\dot{\mathcal{P}}_2|| < \delta$. Then since s_i is rational $\exists q \in \mathbb{N}$ such that,

$$S(g \circ f; \dot{\mathcal{P}}_2) = \sum_{i=1}^{n} g(f(s_i))(x_i - x_{i-1}) = \sum_{i=1}^{n} g(\frac{1}{q})(x_i - x_{i-1}) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = x_n - x_0$$

$$\implies S(g \circ f; \dot{\mathcal{P}}_2) = 1$$

$$|S(g \circ f; \dot{\mathcal{P}}_1) - S(g \circ f; \dot{\mathcal{P}}_2)| = 1 > \epsilon$$

Therefore by the Cauchy Criterion $g \circ f$ is not Riemann Integrable.