Math 236 Winter 2017

Due Friday, February 17

1. In this exercise, you will work out the details of the proof of the following lemma from Professor Goren's notes.

   Lemma: *Let T be a nilpotent operator on an n-dimensional vector space V. Then $T^n = 0$, where $0 \in \mathcal{L}(V)$ is the zero map.*

   (a) First, show that since T is nilpotent, $\dim\ker T > 0$.

   (b) Next, show that either T is the zero map or $\dim\ker T^2 > \dim\ker T$.

   (c) Show in general that for $k \in \{2, 3, \ldots\}$ either T^{k-1} is the zero map or $\dim\ker T^k > \dim\ker T^{k-1}$.

   (d) Using parts (a)-(c), why must $T^n = 0$?

2. (Ax 3.C.4) Suppose $B = \{v_1, \ldots, v_m\}$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $C = \{w_1, \ldots, w_n\}$ of W such that all entries in the first column of $\mathcal{M}(T; B, C) =_C [T]_B$ are 0 except for possibly a 1 in the first row of the first column.

3. Consider the matrix A shown below.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 3 \end{pmatrix}$$

   (a) Using cycle notation, write down all permutations in S_3 and indicate the sign of each permutation.

   (b) Circle the entries of A that show up in the term of the determinant corresponding to the permutation $(1, 2, 3)$.

   (c) Using the permutation definition of determinant, compute the determinant of A.

4. Prove that the function det:$M_n(\mathbb{F}) \to \mathbb{F}$ defined by

$$\det(a_{ij}) = \sum_{\sigma \in S_n} \mathbf{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

   satisfies property (2) of Theorem 5.3.1 in Professor Goren's notes.

5. Prove that the definition of determinant in the previous problem satisfies property (4) of Theorem 5.3.1 in Professor Goren's notes.

6. Consider the $n \times n$ upper triangular matrix M shown below. Using the permutation definition of the determinant, prove that $\det(M) = M_{11}M_{22}M_{33} \cdots M_{nn}$.

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} & \ldots & M_{1(n-2)} & M_{1n} \\ 0 & M_{22} & M_{23} & \ldots & M_{2(n-1)} & M_{2n} \\ 0 & 0 & M_{33} & \ldots & M_{3(n-1)} & M_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & M_{nn} \end{pmatrix}$$

7. Prove or give a counterexmple: for any matrices $S, T \in M_n(\mathbb{F})$, $\det(S + T) = \det(S) + \det(T)$.