Math 256 Winter 2017

Due Friday, January 27

1. Find a basis for the following vector spaces. In each case, prove it is a basis and state the dimension of the vector space.

   (a) $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \mid x_1 + 3x_2 - x_4 = x_3 + x_4 = 0\}$

   (b) $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\right\}$, where $\cdot$ denotes matrix multiplication

2. Let X be any set, and let $V = \{f : X \to \mathbb{F}\}$ be the vector space of functions on X with pointwise addition, i.e. $(f + g)(x) = f(x) + g(x)$ for all $f, g \in V$.

   (a) For each $x \in X$, let $f_x \in V$ be the vector defined by

   $$f_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

   For any finite subset $\{x_1, x_2, \dots, x_n\} \subset X$, show that $\{f_{x_1}, f_{x_2}, \dots, f_{x_n}\}$ is linearly independent.

   (b) When is V finite-dimensional? When it is finite-dimensional, what is its dimension? Your answers should be stated in terms of X.

3. (Ax 2.B.8) Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

   $$u_1, \dots, u_m, w_1, \dots, w_n$$

   is a basis of V.

4. (Ax 2.C.2) Show that the subspaces of $\mathbb{R}^2$ are precisely $\{0\}$, $\mathbb{R}^2$, and all lines in $\mathbb{R}^2$ through the origin.

5. (Ax 2.C.5) Let $\mathcal{P}_4(\mathbb{R}) = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \mid a_i \in \mathbb{F}\}$

   (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) \mid p''(6) = 0\}$. Find a basis of U.

   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4(\mathbb{R})$.

   (c) Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

6. (Ax 2.C.12) Suppose U and W are both five-dimensional subspaces of $\mathbb{R}^9$. Prove that $U \cap W \neq \{0\}$.

7. (Ax 2.C.15) Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \dots, U_n$ of V such that

   $$V = U_1 \oplus \cdots \oplus U_n.$$