Assignment 2: Orthogonalization
Math 327/397 Winter 2019

Due: Friday Feb 22, 2019

Instructions

Submit a complete paper copy of your solutions in class or to the math department by 4pm on the due date. Questions 1 – 5 are for everyone. Question 6 is for Math 397 only.

Question 1: Orthogonal Matrices

(a) Let $u, v \in \mathbb{R}^n$, $< u, v >= u^T v$ denote the standard inner product, and $\| \cdot \|$ denote the induced (2-)norm. If $Q \in \mathbb{R}^{n\times n}$ is orthogonal, show that Q preserves inner products, norms, distances, and angles under matrix-vector multiplication.

(b) Show that the orthogonal projection of u onto v, is the vector in $span(v)$ closest to u. Extend the result to subspaces, that is, if $W \subseteq \mathbb{R}^n$ is a subspace, show that the orthogonal projection of u onto W is the vector inside W closest to u. Is the closest vector unique? Justify.

Question 2: Classical Gram-Schmidt

Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

In constructing the QR decomposition, ensure that R has non-negative diagonal elements. For by-hand computations keep at least four significant digits in your intermediate calculations.

(a) Compute by hand the reduced QR decomposition of A using the Classical Gram-Schmidt method. Show your work.

(b) Assuming the input is full rank, write the pseudo-code for the Classical Gram-Schmidt (you may refer to the class notes).

(c) Implement the Classical Gram-Schmidt into a Matlab m-file.

(d) Run your algorithm on A and paste the output.

(e) How many floating point operations does the algorithm require for an $m \times n$ input matrix? How much memory storage counted in floating point numbers? Justify, aiming to get at least the order and the first coefficient correct.

Question 3: Modified Gram-Schmidt

Repeat question 2, (a)–(d) only, with the Modified Gram-Schmidt.

Question 4: Givens Rotations

Repeat question 2, (a)–(d) only, with the Givens Rotations method. In this case, construct the full QR decomposition instead of the reduced one.

Question 5: Householder Reflections

Repeat question 2, (a)–(d) only, with the Householder Reflections method. Again, construct the full QR decomposition.

Question 6 (Math 397): Loss of Orthogonality

Define $B = rand(100, 80)$ in Matlab.

(a) In your m-files for Classical Gram-Schmidt, add lines of code to track the loss of orthogonality using the following metric: $LCGS_k := \|Q_k^T Q_k - I_k\|_F$ at each step. Here $\| \cdot \|_F$ refers to the Frobenius Norm. Do the same for Modified Gram-Schmidt defining the loss of orthogonality as $LMGS_k$. Run both algorithms on B and plot $LCGS_k$ and $LMGS_k$ on the same chart against the iteration index k. (You might find it useful to explore different plotting options for better visualization, e.g. logplot.) Which method loses orthogonality faster?

(b) Run your Givens rotations and Householder methods on B. At completion compute the corresponding loss of orthogonality quantities LG_n, LH_n. Here, only consider the relevant portion of Q, whose column space is the same as the one of B. Comparing $LCGS_n$, $LMGS_n$, LG_n, and LH_n, which of the four methods gives the most orthogonal Q according to this metric?