Math 256 Winter 2017

Due Friday, January 13

1. Decide whether or not U is a subspace of V in the following.
   (a) $V = \mathbb{C}^2$ and $U = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$
   (b) $V = \mathbb{R}^4$ and $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 \geq x_2\}$
   (c) $V = \mathbb{R}^2$ and $U = \{(x, \cos y) \mid x, y \in \mathbb{R}\}$

2. (Ax. 1.C.7) Give an example of a nonempty subset U of $\mathbb{R}^2$ that is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbb{R}^2$.

3. (Ax 1.C.12) Let V be a vector space. Prove that the union of two subspaces of V is a subspace if and only if one subspace contains the other.

4. (Ax 1.C.18) Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have an additive inverse?

5. (Ax 1.C.21) Suppose $U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 \mid x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^5$ such that $\mathbb{F}^5 = U \oplus W$.

6. (Ax 1.C.23) Let V be a vector space and U_1, U_2 be subspaces. Prove or provide a counterexamle: If there exists a subspace W of V such that
$$U_1 \oplus W = U_2 \oplus W$$
then $U_1 = U_2$.

7. Let
$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$
This is a vector space with operations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix} \text{ and } \lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$
Define the *transpose operator* $^t : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$
Observe that $(A^t)^t = A$ and $(aA + bB)^t = aA^t + bB^t$, for any $A, B \in M_2(\mathbb{R})$ and $a, b \in \mathbb{R}$. (You should verify these properties yourself as well as the fact that $M_2(\mathbb{R})$ is a vector space, but you do not need to hand it in.)
   (a) Prove that $U := \{A \in M_2(\mathbb{R}) \mid A = A^t\}$ and $V = \{A \in M_2(\mathbb{R}) \mid A = -A^t\}$ are subspaces of $M_2(\mathbb{R})$.
   (b) Prove that
$$M_2(\mathbb{R}) = U \oplus V.$$