**McGill University**
**Department of Mathematics and Statistics**
**MATH 243 Analysis 2, Winter 2017**
**Assignment 4**

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 2 and 3(b).**

This assignment is due **Tuesday, February 7, at 2:30pm** in class. **Late assignments will not be accepted!**

1. Let $a$, $b$, $c$, $d$ be real numbers with $a \le c \le d \le b$. Show that the function

$$f : [a,b] \to \mathbb{R}, \quad f(x) := \begin{cases} 1 & \text{if } x \in [c,d] \\ 0 & \text{if } x \notin [c,d] \end{cases}$$

(called an *elementary step function*) is Riemann integrable on $[a,b]$ and that $\int_a^b f = d - c$.

2. Let

$$f(x) := \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Prove that $f$ is Riemann integrable on $[0,1]$ and compute $\int_0^1 f$.

Hint: Use the squeeze theorem.

3. (a) Use induction to prove the summation formula $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

   (b) Show that $x^2$ is Riemann integrable on the interval $[0,1]$ and compute $\int_0^1 x^2\,dx$.

   Hint: Consider a partition $\mathcal{P}$ of $[0,1]$ into $n$ intervals of equal width and define step functions $\alpha$ and $\omega$ with respect to this partition such that $\alpha(x) \le x^2 \le \omega(x)$ for all $x \in [0,1]$. Then compute $\int_0^1 \alpha$ and $\int_0^1 \omega$.

4. (Hard) Let

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that $f$ is Riemann integrable on $[0,1]$.

Hint: Use the squeeze theorem. Note that you are *not* expected to compute $\int_0^1 f$.

5. (a) Suppose that $f$ is continuous on $[a,b]$, that $f(x) \ge 0$ for all $x \in [a,b]$ and that $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

   (b) Show by providing a concrete counterexample that the continuity condition in part (a) cannot be dropped.