Math 236 Algebra 2 Assignment 7

Jonathan Pearce, 260672004

February 17, 2017

Problem 1a. Suppose $dim\, kerT = 0$. Therefore $kerT$ only contains the zero vector. Suppose $V := \{0, v\}$, since the kernel is trivial we have $T(0) = 0$ and $T(v) = v$. Therefore $v \in T^n$ and it follows that $T^n \neq 0$. We conclude that $dimkerT > 0$.

Problem 1b. If T is the zero map, then $T^n = 0$ and thus T is nilpotent. Suppose T is not the zero map, therefore $\exists v \in V$ such that $Tv \neq 0 \implies dimrangeT > 0$. Since 0 maps to 0 for any linear operator we have, $dim\, kerT^2 \geq dim\, kerT$. Assume $dim\, kerT^2 = dim\, kerT$, from the fundamental theorem of linear maps we get $0 < dim\, rangeT = dim\, rangeT^2$, it follows that $dim\, rangeT^n > 0$ therefore $T^n \neq 0$. We conclude that $dim\, kerT^2 > dim\, kerT$.

Problem 1c. Let $k \in \mathbb{N}$ and $3 \leq k < n$. If T^{k-1} is the zero map, then $T^n = 0$ and thus T is nilpotent. Suppose T^{k-1} is not the zero map, therefore $\exists v \in V$ such that $T^{k-1}v \neq 0 \implies dimrangeT^{k-1} > 0$. Since 0 maps to 0 for any linear operator we have, $dim\, kerT^k \geq dim\, kerT^{k-1}$. Assume $dim\, kerT^k = dim\, kerT^{k-1}$, from the fundamental theorem of linear maps we get $0 < dim\, rangeT^{k-1} = dim\, rangeT^k$, it follows that $dim\, rangeT^n > 0$ therefore $T^n \neq 0$. We conclude that $dim\, kerT^k > dim\, kerT^{k-1}$.

Problem 1d. Since V is a n-dimensional vector space (i.e. finite dimensional) this process outlined in b and c can be repeated at most n times before T^j becomes the zero map for some $j \in \mathbb{N}$ and $1 \leq j \leq n$

Problem 2.

Problem 3a.

$$S_3[sgn] = (1,2,3)[1], (1,3,2)[1], (1,2)[-1], (2,3)[-1], (1,3)[-1], 1[1]$$

Problem 3b. $\begin{pmatrix} & & 0 \\ 0 & & \\ & 1 & \end{pmatrix}$, are the entries in the term of the determinant corresponding to (1, 2, 3).

Problem 3c.

$$det(A) = \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3}$$

$$= sgn((1,2,3))a_{21}a_{32}a_{13} + sgn((1,3,2))a_{31}a_{12}a_{23} + sgn((1,2))a_{21}a_{12}a_{33} + sgn((2,3))a_{11}a_{32}a_{23}$$
$$+ sgn((1,3))a_{31}a_{22}a_{13} + sgn(1)a_{11}a_{22}a_{33}$$
$$= 1(0)(1)(0) + 1(1)(2)(-1) - 1(0)(2)(3) - 1(1)(1)(-1) - 1(1)(1)(0) + 1(1)(1)(3)$$
$$= 0 - 2 + 0 + 1 + 0 + 3 = 2$$

Therefore $det(A) = 2$.

Problem 4.

$$det(v_1 v_2 ... v_i ... v_n) + det(v_1 v_2 ... v_i{}' ... v_n)$$
$$= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(i)i} \cdots a_{\sigma(n)n} + \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a'_{\sigma(i)i} \cdots a_{\sigma(n)n}$$
$$= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(i)i} \cdots a_{\sigma(n)n} + sgn(\sigma) a_{\sigma(1)1} \cdots a'_{\sigma(i)i} \cdots a_{\sigma(n)n}$$
$$= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} (a_{\sigma(i)i} + a'_{\sigma(i)i})$$
$$= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots (a_{\sigma(i)i} + a'_{\sigma(i)i}) \cdots a_{\sigma(n)n}$$
$$det(v_1 v_2 ... (v_i + v_i{}') ... v_n)$$

Problem 5.

$$det(e_1 e_2 ... e_n)$$
$$= \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

Let $\sigma_1 \in S_n$ and $\sigma_1 := 1$

$$\implies sgn(1) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} = sgn(1) a_{11} a_{22} \cdots a_{nn} = 1$$

Let $\sigma_2 \in S_n$ and $\sigma_2 \neq \sigma_1$. Therefore there exists $a_{\sigma(j)j}$ such that $\sigma(j) \neq j$ and it follows $a_{\sigma(j)j} = 0$

$$\implies sgn(\sigma_2) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(j)j} \cdots a_{\sigma(n)n} = 0$$

Therefore,

$$det(e_1 e_2 ... e_n) = \sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = 0 + 0 + ... + 1 = 1$$

Problem 6. Prove by induction.
Base case n=1: M is a 1x1 matrix then by definition

$$det(M) = M_{11}$$

Induction step: Assume $det(M) = M_{11}M_{22}\cdots M_{nn}$. Add a row and column to M such that it is an $(n+1)\times(n+1)$ upper triangular matrix, define it as $\hat{M}$. Using Laplace's Theorem expand about the $n+1^{\text{th}}$ row.

$$det(\hat{M}) = 0 * M^{(n+1)1} + 0 * M^{(n+1)2} + ... + 0 * M^{(n+1)n} + M_{(n+1)(n+1)} * M^{(n+1)(n+1)}$$

Deleting the $n+1^{\text{th}}$ row and $n+1^{\text{th}}$ column gives us the matrix M. Therefore from the induction hypothesis we get the following,

$$M^{(n+1)(n+1)} = -1^{(n+1)+(n+1)}det(M) = M_{11}M_{22}\cdots M_{nn}$$

It follows

$$det(\hat{M}) = 0 + 0 + ... + 0 + M_{(n+1)(n+1)}M_{11}M_{22}\cdots M_{nn} = M_{11}M_{22}\cdots M_{nn}M_{(n+1)(n+1)}$$

Therefore we conclude that the determinant of an upper triangular matrix is the product of its diagonal entries.

Problem 7. Define S and T.

$$S = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore,

$$S + T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$det(S) = 0\cdot 1 - (-1)\cdot 0 = 0$$

$$det(T) = 1\cdot 0 - 0\cdot 1 = 0$$

$$det(S+T) = 1\cdot 1 - (-1)\cdot 1 = 2$$

It follows,

$$det(S) + det(T) \neq det(S+T)$$