Math 243 Analysis 2 Assignment 1

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Problem 1. Prove the product rule. Show $(f \cdot g)(x)$ is differentiable at c

$$\lim_{x\to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c} = \lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x\to c} \frac{(f(c) + \varphi(x)(x - c))(g(c) + \psi(x)(x - c)) - f(c)g(c)}{x - c}$$

$$= \lim_{x\to c} \frac{f(c)g(c) + f(c)\psi(x)(x - c) + g(c)\varphi(x)(x - c) + \varphi(x)(x - c)\psi(x)(x - c) - f(c)g(c)}{x - c}$$

$$= \lim_{x\to c} \frac{f(c)\psi(x)(x - c) + g(c)\varphi(x)(x - c) + \varphi(x)(x - c)\psi(x)(x - c))}{x - c}$$

$$= \lim_{x\to c}(f(c)\psi(x) + g(c)\varphi(x) + \varphi(x)\psi(x)(x - c))$$

From the limit laws,

$$= \lim_{x\to c} f(c)\psi(x) + \lim_{x\to c} g(c)\varphi(x) + \lim_{x\to c} \varphi(x)\psi(x)(x - c)$$

$$= f(c)\psi(c) + g(c)\varphi(c) + 0$$

$$= f(c)g'(c) + g(c)f'(c)$$

Therefore, $(f \cdot g)(x)$ is differentiable at c and $(f \cdot g)'(c) = f(c)g'(c) + g(c)f'(c)$

Problem 4a. Differentiate f at 0

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{x^2 sin(\frac{1}{x^2})}{x} = \lim_{x\to 0} x sin(\frac{1}{x^2})$$

Since $\mid sin(\frac{1}{x^2}) \mid \leq 1$. It follows,

$$\mid x sin(\frac{1}{x^2}) \mid \ \leq \ \mid x \mid$$

hence,

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x sin(\frac{1}{x^2}) \leq \lim_{x\to 0} x$$

By the Squeeze Theorem,

$$\lim_{x\to 0} -x = 0 = \lim_{x\to 0} x \implies \lim_{x\to 0} x sin(\frac{1}{x^2}) = 0$$

$$\implies f'(0) = 0$$

Differentiate f at $\mathbb{R}\backslash\{0\}$. There exists $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ such that $(g \cdot h)(x) = f(x)$. Suppose,

$$g(x) = x^2 \ and \ h(x) = \begin{cases} sin(\frac{1}{x^2}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Therefore by the product rule, $f'(x) = (g \cdot h)'(x) = g(x)h'(x) + h(x)g'(x)$
Differentiate g at $c \in \mathbb{R}$:

$$\lim_{x\to c} \frac{x^2 - c^2}{x - c} = \lim_{x\to c} \frac{(x-c)(x+c)}{x-c} = \lim_{x\to c} x + c = 2c$$

Therefore $g'(x) = 2x$, $\forall x \in \mathbb{R}$

h can be expressed as the composition of two functions $\varphi \circ \psi$ where $\varphi = sin(x)$ and $\psi = \frac{1}{x^2}$
Therefore $h(x) = (\varphi \circ \psi)(x) = \varphi(\psi(x))$. From the chain rule,

$$h'(x) = (\varphi \circ \psi)'(x) = \varphi'(\psi(x)) \cdot \psi'(x).$$

By definition $\varphi'(x) = cos(x)$.
Differentiate ψ at $c \in \mathbb{R}\backslash\{0\}$:

$$\lim_{x\to c} \frac{\frac{1}{x^2} - \frac{1}{c^2}}{x - c} = \lim_{x\to c} \frac{\frac{c^2 - x^2}{x^2c^2}}{x - c} = \lim_{x\to c} \frac{\frac{(c-x)(c+x)}{x^2c^2}}{x - c} = \lim_{x\to c} \frac{-(c+x)}{x^2c^2} = \frac{-2c}{c^4} = \frac{-2}{c^3}$$

Therefore $\psi'(x) = -2x^{-3}$, $\forall x \in \mathbb{R}\backslash\{0\}$.
Therefore $h'(x) = (\varphi \circ \psi)'(x) = \varphi'(\psi(x)) \cdot \psi'(x) = cos(\frac{1}{x^2})(\frac{-2}{x^3})\forall x \in \mathbb{R}\backslash\{0\}$
From the Product Rule,

$$f'(x) = (g \cdot h)'(x) = g(x)h'(x) + h(x)g'(x) = x^2cos(\frac{1}{x^2})(\frac{-2}{x^3}) + sin(\frac{1}{x^2})2x, \forall x \in \mathbb{R}\backslash\{0\}$$

$$\implies f'(x) = \begin{cases} 2xsin(\frac{1}{x^2}) - (\frac{2}{x})cos(\frac{1}{x^2}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Therefore f' is differentiable on $\mathbb{R}$.

Problem 4b. Suppose f is bounded on $[-1,1]$. Then,

$$\exists M \in \mathbb{R}, M > 0 \ such \ that \ |f'(x)| < M, \forall x \in [-1,1]$$

Now let $x_0 \in \mathbb{R}$ such that $x_0 > M$ and $x_0 > 1$. Let $x_0 := \sqrt{n\pi}$ for some $n \in \mathbb{N}$
Since $x_0 > 1 \implies 1 > \frac{1}{x_0} > 0 \iff \frac{1}{x_0} \in (0,1) \subseteq [-1,1]$. Now calculate,

$$\left|f'(\frac{1}{x_0})\right| = \left|2(\frac{1}{x_0})sin(\frac{1}{(\frac{1}{x_0})^2}) - (\frac{2}{(\frac{1}{x_0})})cos(\frac{1}{(\frac{1}{x_0})^2})\right| = \left|\frac{2}{x_0}sin(x_0^2) - (2x_0)cos(x_0^2)\right|$$

$$= \left|\frac{2}{\sqrt{n\pi}}sin(n\pi) - (2\sqrt{n\pi})cos(n\pi)\right| = \left|-(2\sqrt{n\pi})cos(n\pi)\right| = 2\sqrt{n\pi} = 2x_0 > M$$

This is a contradiction to the assumption that f is bounded by M, and thus f is unbounded on $[-1,1]$.

Problem 4c. Prove f' is discontinuous at 0. Define a sequence,

$$x_n := \sqrt{\frac{1}{2\pi n}}$$

$$\lim_{n\to\infty}(x_n) = 0$$

$$f(x_n) = 2x_n sin(\frac{1}{x_n^2}) - \frac{2}{x_n}cos(\frac{1}{x_n^2}) = 2\sqrt{\frac{1}{2\pi n}}sin(2\pi n) - 2\sqrt{2\pi n}\,cos(2\pi n) = -2\sqrt{2\pi n}$$

$$\lim_{n\to\infty}(f(x_n)) = -\infty \neq f'(0)$$

Therefore f' is discontinuous at 0.