Lecture 13

Equilibria: Definitions, Examples, and Existence

Equilibrium flows in atomic selfish routing networks (Definition 12.2) are a form of "pure" Nash equilibria, in that the agents do not randomize over paths. The Rock-Paper-Scissors game (Section 1.3) shows that some games have no pure Nash equilibria. When are pure Nash equilibria guaranteed to exist? How do we analyze games without any pure Nash equilibria?

Section 13.1 introduces three relaxations of pure Nash equilibria, each more permissive and computationally tractable than the previous one. All three of these relaxed equilibrium concepts are guaranteed to exist in all finite games. Section 13.2 proves that every routing game has at least one pure Nash equilibrium. Section 13.3 generalizes the argument and defines the class of potential games.

13.1 A Hierarchy of Equilibrium Concepts

Many games have no pure Nash equilibria. In addition to Rock-Paper-Scissors, another example is the generalization of atomic selfish routing networks to agents with different sizes (Exercise 13.5). For a meaningful equilibrium analysis of such games, such as a price-of-anarchy analysis, we need to enlarge the set of equilibria to recover guaranteed existence. Figure 13.1 illustrates the hierarchy of equilibrium concepts defined in this section. Lecture 14 proves worst-case performance guarantees for all of these equilibrium concepts in several games of interest.

13.1.1 Cost-Minimization Games

A *cost-minimization game* has the following ingredients:

- a finite number k of agents;

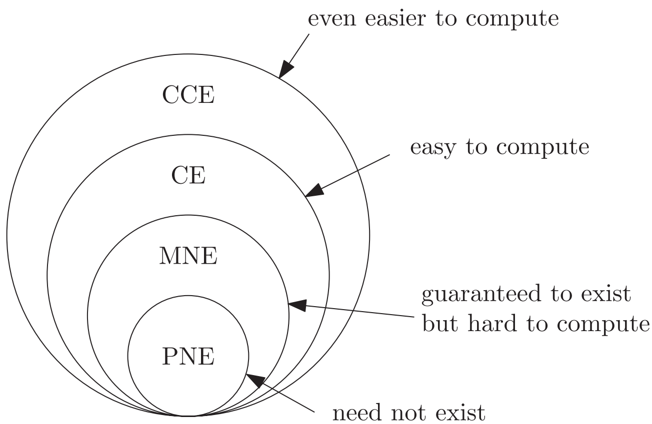

Figure 13.1: A hierarchy of equilibrium concepts: pure Nash equilibria (PNE), mixed Nash equilibria (MNE), correlated equilibria (CE), and coarse correlated equilibria (CCE).

- a finite set S_i of *pure strategies*, or simply *strategies*, for each agent i;
- a nonnegative cost function $C_i(\mathbf{s})$ for each agent i, where $\mathbf{s} \in S_1 \times \cdots \times S_k$ denotes a *strategy profile* or *outcome*.

For example, every atomic selfish routing network corresponds to a cost-minimization game, with $C_i(\mathbf{s})$ denoting i's travel time on her chosen path, given the paths chosen by the other agents.

Remark 13.1 (Payoff-Maximization Games) In a *payoff-maximization game*, the cost function C_i of each agent i is replaced by a payoff function π_i. This is the more conventional way to define games, as in the Rock-Paper-Scissors game in Section 1.3. The following equilibrium concepts are defined analogously in payoff-maximization games, except with all of the inequalities reversed. The formalisms of cost-minimization and payoff-maximization games are equivalent, but in most applications one is more natural than the other.

13.1.2 Pure Nash Equilibria (PNE)

A pure Nash equilibrium is an outcome in which a unilateral deviation by an agent can only increase the agent's cost.

Definition 13.2 (Pure Nash Equilibrium (PNE)) A strategy profile $\mathbf{s}$ of a cost-minimization game is a *pure Nash equilibrium (PNE)* if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$C_i(\mathbf{s}) \leq C_i(s_i{}', \mathbf{s}_{-i}). \tag{13.1}$$

By $\mathbf{s}_{-i}$ we mean the vector $\mathbf{s}$ of all strategies, with the ith component removed. Equivalently, in a PNE $\mathbf{s}$, every agent i's strategy s_i is a *best response* to $\mathbf{s}_{-i}$, meaning that it minimizes $C_i(s_i', \mathbf{s}_{-i})$ over $s_i' \in S_i$. PNE are easy to interpret but, as discussed above, do not exist in many games of interest.

13.1.3 Mixed Nash Equilibria (MNE)

Lecture 1 introduced the idea of an agent randomizing over her strategies via a *mixed strategy*. In a mixed Nash equilibrium, agents randomize independently and unilateral deviations can only increase an agent's expected cost.

Definition 13.3 (Mixed Nash Equilibrium (MNE)) Distributions $\sigma_1, \ldots, \sigma_k$ over strategy sets $S_1, \ldots, S_k$ of a cost-minimization game constitute a *mixed Nash equilibrium (MNE)* if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}\left[C_i(s_i{}', \mathbf{s}_{-i})\right], \tag{13.2}$$

where σ denotes the product distribution $\sigma_1 \times \cdots \times \sigma_k$.

Definition 13.3 considers only unilateral deviations to pure strategies, but allowing deviations to mixed strategies does not change the definition (Exercise 13.1).

Every PNE is a MNE in which every agent plays deterministically. The Rock-Paper-Scissors game shows that a game can have MNE that are not PNE.

Two facts discussed at length in Lecture 20 are relevant here. First, every cost-minimization game has at least one MNE. We can therefore define the *POA of MNE* of a cost-minimization game, with respect to an objective function defined on the game's outcomes, as the ratio

$$\frac{\text{expected objective function value of worst MNE}}{\text{objective function value of best outcome}}. \tag{13.3}$$

Second, computing a MNE appears to be a computationally intractable problem, even when there are only two agents.[1] This raises the concern that POA bounds for MNE need not be meaningful. In games where we don't expect the agents to quickly reach an equilibrium, why should we care about performance guarantees for equilibria? This objection motivates the search for more permissive and computationally tractable equilibrium concepts.

13.1.4 Correlated Equilibria (CE)

Our next equilibrium notion takes some getting used to. We define it, then explain the standard semantics, and then offer an example.

Definition 13.4 (Correlated Equilibrium (CE)) A distribution σ on the set $S_1 \times \cdots \times S_k$ of outcomes of a cost-minimization game is a *correlated equilibrium (CE)* if for every agent $i \in \{1, 2, \ldots, k\}$, strategy $s_i \in S_i$, and deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s} \sim \sigma}[C_i(\mathbf{s}) \mid s_i] \leq \mathbf{E}_{\mathbf{s} \sim \sigma}\left[C_i({s_i}', \mathbf{s}_{-i}) \mid s_i\right]. \tag{13.4}$$

Importantly, the distribution σ in Definition 13.4 need not be a product distribution; in this sense, the strategies chosen by the agents are correlated. The MNE of a game correspond to the CE that are product distributions (Exercise 13.2). Since MNE are guaranteed to exist, so are CE. CE also have a useful equivalent definition in terms of "swapping functions" (Exercise 13.3).

The usual interpretation of a correlated equilibrium involves a trusted third party. The distribution σ over outcomes is publicly known. The trusted third party samples an outcome $\mathbf{s}$ according to σ. For each agent $i = 1, 2, \ldots, k$, the trusted third party privately suggests the strategy s_i to i. The agent i can follow the suggestion s_i, or not. At the time of decision making, an agent i knows the distribution σ and one component s_i of the realization $\mathbf{s}$, and accordingly has a posterior distribution on others' suggested strategies $\mathbf{s}_{-i}$. With these semantics, the correlated equilibrium condition (13.4) requires that every agent minimizes her expected cost by playing the suggested

[1] The precise statement uses an analog of $\mathcal{NP}$-completeness suitable for equilibrium computation problems (see Lecture 20).

strategy s_i. The expectation is conditioned on i's information—σ and s_i—and assumes that other agents play their recommended strategies $\mathbf{s}_{-i}$.

Believe it or not, a traffic light is a perfect example of a CE that is not a MNE. Consider the following two-agent game, with each matrix entry listing the costs of the row and column agents in the corresponding outcome:

	Stop	Go
Stop	1,1	1,0
Go	0,1	5,5

There is a modest cost (1) for waiting and a large cost (5) for getting into an accident. This game has two PNE, the outcomes (Stop, Go) and (Go, Stop). Define σ by randomizing uniformly between these two PNE. This is not a product distribution over the game's four outcomes, so it cannot correspond to a MNE of the game. It is, however, a CE. For example, consider the row agent. If the trusted third party (i.e., the stoplight) recommends the strategy "Go" (i.e., is green), then the row agent knows that the column agent was recommended "Stop" (i.e., has a red light). Assuming the column agent plays her recommended strategy and stops at the red light, the best strategy for the row agent is to follow her recommendation and to go. Similarly, when the row agent is told to stop, she assumes that the column agent will go, and under this assumption stopping is the best strategy.

Lecture 18 proves that, unlike MNE, CE are computationally tractable. There are even distributed learning algorithms that quickly guide the history of joint play to the set of CE. Thus bounding the *POA of CE*, defined as the ratio (13.3) with "MNE" replaced by "CE," provides a meaningful equilibrium performance guarantee.

13.1.5 Coarse Correlated Equilibria (CCE)

We should already be quite pleased with positive results, such as good POA bounds, that apply to the computationally tractable set of CE. But if we can get away with it, we'd be happy to enlarge the set of equilibria even further, to an "even more tractable" concept.

Definition 13.5 (Coarse Correlated Equilibrium (CCE))
A distribution σ on the set $S_1 \times \cdots \times S_k$ of outcomes of a cost-minimization game is a *coarse correlated equilibrium (CCE)* if for every agent $i \in \{1, 2, \ldots, k\}$ and every unilateral deviation $s_i' \in S_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{\mathbf{s}\sim\sigma}\left[C_i({s_i}', \mathbf{s}_{-i})\right]. \tag{13.5}$$

The condition (13.5) is the same as that for MNE (13.2), except without the restriction that σ is a product distribution. In this condition, when an agent i contemplates a deviation ${s_i}'$, she knows only the distribution σ and *not* the component s_i of the realization. Put differently, a CCE only protects against unconditional unilateral deviations, as opposed to the unilateral deviations conditioned on s_i that are addressed in Definition 13.4. It follows that every CE is a CCE, and so CCE are guaranteed to exist and are computationally tractable. Lecture 17 demonstrates that the distributed learning algorithms that quickly guide the history of joint play to the set of CCE are even simpler and more natural than those for the set of CE.

13.1.6 An Example

We next increase intuition for the four equilibrium concepts in Figure 13.1 with a concrete example. Consider an atomic selfish routing network (Section 12.4) with four agents. The network is simply a common origin vertex o, a common destination vertex d, and an edge set $E = \{0, 1, 2, 3, 4, 5\}$ consisting of 6 parallel o-d edges. Each edge has the cost function $c(x) = x$.

The pure Nash equilibria are the outcomes in which each agent chooses a distinct edge. Every agent suffers only one unit of cost in such an equilibrium. One mixed Nash equilibrium that is obviously not pure has each agent independently choosing an edge uniformly at random. Every agent suffers expected cost $\frac{3}{2}$ in this equilibrium. The uniform distribution over all outcomes in which there is one edge with two agents and two edges with one agent each is a (non-product) correlated equilibrium, since both sides of (13.4) read $\frac{3}{2}$ for every i, s_i, and s_i'. The uniform distribution over the subset of these outcomes in which the set of chosen edges is either $\{0, 2, 4\}$ or $\{1, 3, 5\}$ is a coarse correlated equilibrium, since both sides of (13.5) read $\frac{3}{2}$ for every i and s_i'. It is not a correlated equilibrium, since an agent i that is recommended the edge s_i can reduce her conditional expected cost to 1 by choosing the deviation s_i' to the successive edge (modulo 6).

13.2 Existence of Pure Nash Equilibria

This section proves that equilibrium flows exist in atomic selfish routing networks (Section 13.2.1) and are also essentially unique in nonatomic selfish routing networks (Section 13.2.2), and introduces the class of congestion games (Section 13.2.3).

13.2.1 Existence of Equilibrium Flows

Section 12.4 asserts that atomic selfish routing networks are special games, in that a (pure) equilibrium flow always exists. We now prove this fact.

Theorem 13.6 (Existence of PNE in Routing Games) *Every atomic selfish routing network has at least one equilibrium flow.*

Proof: Define a function on the flows of an atomic selfish routing network by

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i), \tag{13.6}$$

where f_e is the number of agents that choose a path in f that includes the edge e. The inner sum in (13.6) is the "area under the curve" of the cost function c_e; see Figure 13.2. By contrast, the corresponding term $f_e \cdot c_e(f_e)$ of the cost objective function (11.4) corresponds to the shaded bounding box in Figure 13.2.[2]

Consider a flow f, an agent i using the o_i-d_i path P_i in f, and a deviation to some other o_i-d_i path $\hat{P}_i$. Let $\hat{f}$ denote the flow after i's deviation from P_i to $\hat{P}_i$. We claim that

$$\Phi(\hat{f}) - \Phi(f) = \sum_{e \in \hat{P}_i} c_e(\hat{f}_e) - \sum_{e \in P_i} c_e(f_e). \tag{13.7}$$

In words, the change in Φ under a unilateral deviation is exactly the same as the change in the deviator's individual cost. Thus, the single

[2]This similarity between the function Φ and the cost objective function is useful; see Lecture 15.

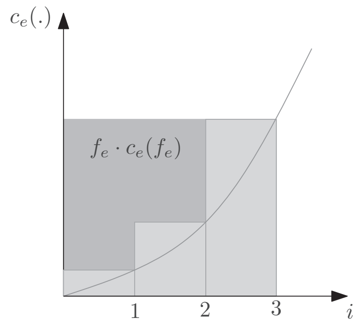

Figure 13.2: Edge e's contribution to the potential function Φ and to the cost objective function.

function Φ simultaneously tracks the effect of deviations by each of the agents.

To prove this claim, consider how Φ changes when i switches her path from P_i to $\hat{P}_i$. The inner sum of the potential function corresponding to edge e picks up an extra term $c_e(f_e + 1)$ whenever e is in $\hat{P}_i$ but not P_i, and sheds its final term $c_e(f_e)$ whenever e is in P_i but not $\hat{P}_i$. Thus, the left-hand side of (13.7) is

$$\sum_{e \in \hat{P}_i \setminus P_i} c_e(f_e + 1) - \sum_{e \in P_i \setminus \hat{P}_i} c_e(f_e),$$

which is exactly the same as the right-hand side of (13.7).

To complete the proof of the theorem, consider a flow f that minimizes Φ. There are only finitely many flows, so such a flow exists. No unilateral deviation by any agent can decrease Φ. By (13.7), no agent can decrease her cost by a unilateral deviation, and f is an equilibrium flow. ■

13.2.2 Uniqueness of Nonatomic Equilibrium Flows

This section sketches the analog of Theorem 13.6 for the nonatomic selfish routing networks introduced in Lecture 11. Since agents have negligible size in such networks, we replace the inner sum in (13.6) by an integral:

$$\Phi(f) = \sum_{e \in E} \int_0^{f_e} c_e(x)dx, \qquad (13.8)$$

where f_e is the amount of traffic routed on edge e by the flow f. Because edge cost functions are continuous and nondecreasing, the function Φ is continuously differentiable and convex. The first-order optimality conditions of Φ are precisely the equilibrium flow conditions (Definition 11.3), and so the local minima of Φ correspond to equilibrium flows. Since Φ is continuous and the space of all flows is compact, Φ has a global minimum, and this flow is an equilibrium. Moreover, the convexity of Φ implies that its only local minima are its global minima. When Φ is strictly convex, there is only one global minimum and hence only one equilibrium flow. When Φ has multiple global minima, these correspond to equilibrium flows that all have the same cost.

13.2.3 Congestion Games

The proof of Theorem 13.6 never uses the network structure of an atomic selfish routing game. The argument remains valid for *congestion games*, where there is an abstract set E of resources (previously, the edges), each with a cost function, and each agent i has an arbitrary collection $S_i \subseteq 2^E$ of strategies (previously, the o_i-d_i paths), each a subset of resources. Congestion games play an important role in Lecture 19.

The proof of Theorem 13.6 also does not use the assumption that edge cost functions are nondecreasing. The generalization of Theorem 13.6 to networks with decreasing cost functions is useful in Lecture 15.

13.3 Potential Games

A *potential game* is one for which there exists a *potential function* Φ with the property that, for every unilateral deviation by some agent, the change in the potential function value equals the change in the deviator's cost. Formally,

$$\Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}) = C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s}) \tag{13.9}$$

for every outcome $\mathbf{s}$, agent i, and unilateral deviation $s_i' \in S_i$. Intuitively, the agents of a potential game are inadvertently and collectively striving to optimize Φ. We consider only finite potential games in these lectures.

The identity (13.7) in the proof of Theorem 13.6 shows that every atomic selfish routing game, and more generally every congestion game, is a potential game. Lectures 14 and 15 furnish additional examples.

The final paragraph of the proof of Theorem 13.6 implies the following result.

Theorem 13.7 (Existence of PNE in Potential Games)
Every potential game has at least one PNE.

Potential functions are one of the only general tools for proving the existence of PNE.

The Upshot

☆ Pure Nash equilibria (PNE) do not exist in many games, motivating relaxations of the equilibrium concept.

☆ Mixed Nash equilibria (MNE), where each agent randomizes independently over her strategies, are guaranteed to exist in all finite games.

☆ In a correlated equilibrium (CE), a trusted third party chooses an outcome $\mathbf{s}$ from a public distribution σ, and every agent i, knowing σ and s_i, prefers strategy s_i to every unilateral deviation $s_i{}'$.

☆ Unlike MNE, CE are computationally tractable.

☆ A coarse correlated equilibrium (CCE) is a relaxation of a correlated equilibrium in which an agent's unilateral deviation must be independent of s_i.

☆ CCE are even easier to learn than CE.

☆ A potential game is one with a potential function such that, for every unilateral deviation by some agent, the change in the potential function value equals the change in the deviator's cost.

☆ Every potential game has at least one PNE.

☆ Every atomic selfish routing game is a potential game.

Notes

Nash (1950) proves that every finite game has at least one mixed Nash equilibrium. The correlated equilibrium concept is due to Aumann (1974). Coarse correlated equilibria are implicit in Hannan (1957) and explicit in Moulin and Vial (1978). The existence and uniqueness of equilibrium flows in nonatomic selfish routing networks (Section 13.2.2) is proved in Beckmann et al. (1956). Theorem 13.6 and the definition of congestion games are from Rosenthal (1973). Theorem 13.7 and the definition of potential games are from Monderer and Shapley (1996).

The example in Exercise 13.5 is from Goemans et al. (2005), while Exercise 13.6 is due to Fotakis et al. (2005). Problem 13.1 is discussed by Koutsoupias and Papadimitriou (1999). The results in Problems 13.2 and 13.4 are due to Monderer and Shapley (1996), although the suggested proof of the latter follows Voorneveld et al. (1999). Problem 13.3 is from Facchini et al. (1997).

Exercises

Exercise 13.1 Prove that the mixed Nash equilibria of a cost-minimization game are precisely the mixed strategy profiles $\sigma_1, \ldots, \sigma_k$ that satisfy

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \leq \mathbf{E}_{s_i'\sim\sigma_i', \mathbf{s}_{-i}\sim\sigma_{-i}}\left[C_i(s_i{}', \mathbf{s}_{-i})\right]$$

for every agent i and mixed strategy σ_i' of i.

Exercise 13.2 Consider a cost-minimization game and a product distribution $\sigma = \sigma_1 \times \cdots \times \sigma_k$ over the game's outcomes, where σ_i is a mixed strategy for agent i. Prove that σ is a correlated equilibrium of the game if and only if $\sigma_1, \ldots, \sigma_k$ form a mixed Nash equilibrium of the game.

Exercise 13.3 Prove that a distribution σ over the outcomes $S_1 \times \cdots \times S_k$ of a cost-minimization game is a correlated equilibrium if and only if it has the following property: for every agent i and swapping function $\delta : S_i \to S_i$,

$$\mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\mathbf{s})] \le \mathbf{E}_{\mathbf{s}\sim\sigma}[C_i(\delta(s_i), \mathbf{s}_{-i})] \,.$$

Exercise 13.4 *(H)* Consider an atomic selfish routing network where each edge e has an affine cost function $c_e(x) = a_e x + b_e$ with $a_e, b_e \ge 0$. Let $C(f)$ denote the cost (11.4) of a flow f and $\Phi(f)$ the potential function value (13.6). Prove that

$$\frac{1}{2}C(f) \le \Phi(f) \le C(f)$$

for every flow f.

Exercise 13.5 In a *weighted* atomic selfish routing network, each agent i has a positive weight w_i and chooses a single o_i-d_i path on which to route all of her traffic. Consider the network shown in Figure 13.3, and suppose there are two agents with weights 1 and 2, both with the origin vertex o and the destination vertex d. Each edge is labeled with its cost function, which is a function of the total amount of traffic routed on the edge. For example, if agents 1 and 2 choose the paths $o \to v \to w \to d$ and $o \to w \to d$, then they incur costs of 48 and 74 per-unit of traffic, respectively.

Prove that there is no (pure) equilibrium flow in this network.

Exercise 13.6 Consider a weighted atomic selfish routing network (Exercise 13.5) where each edge has an affine cost function. Use the potential function

$$\Phi(f) = \sum_{e\in E}\left(c_e(f_e)f_e + \sum_{i\in S_e} c_e(w_i)w_i\right),$$

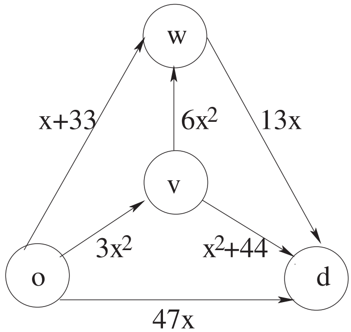

Figure 13.3: Exercise 13.5. A weighted atomic selfish routing network with no equilibrium.

where S_e denotes the set of agents that use edge e in the flow f, to prove that there is at least one (pure) equilibrium flow.

Problems

Problem 13.1 *(H)* Recall the class of cost-minimization games introduced in Problem 12.3, where each agent $i = 1, 2, \ldots, k$ has a positive weight w_i and chooses one of m identical machines to minimize her load. We again consider the objective of minimizing the makespan, defined as the maximum load of a machine. Prove that, as k and m tend to infinity, the worst-case POA of mixed Nash equilibria (13.3) in such games is not upper bounded by any constant.

Problem 13.2 This problem and the next two problems develop further theory about potential games (Section 13.3). Recall that a *potential function* is defined on the outcomes of a cost-minimization game and satisfies

$$\Phi(s_i', \mathbf{s}_{-i}) - \Phi(\mathbf{s}) = C_i(s_i', \mathbf{s}_{-i}) - C_i(\mathbf{s})$$

for every outcome $\mathbf{s}$, agent i, and deviation s_i' by i.

(a) Prove that if a cost-minimization game admits two potential functions Φ_1 and Φ_2, then there is a constant $b \in \mathbb{R}$ such that $\Phi_1(\mathbf{s}) = \Phi_2(\mathbf{s}) + b$ for every outcome $\mathbf{s}$ of the game.

(b) Prove that a cost-minimization game is a potential game if and only if for every two outcomes $\mathbf{s}^1$ and $\mathbf{s}^2$ that differ in the strategies of exactly two agents i and j,

$$\left[C_i(s_i^2, \mathbf{s}_{-i}^1) - C_i(\mathbf{s}^1)\right] + \left[C_j(\mathbf{s}^2) - C_j(s_i^2, \mathbf{s}_{-i}^1)\right] = \\ \left[C_j(s_j^2, \mathbf{s}_{-j}^1) - C_j(\mathbf{s}^1)\right] + \left[C_i(\mathbf{s}^2) - C_i(s_j^2, \mathbf{s}_{-j}^1)\right].$$

Problem 13.3 *(H)* A *team game* is a cost-minimization game in which all agents have the same cost function: $C_1(\mathbf{s}) = \cdots = C_k(\mathbf{s})$ for every outcome $\mathbf{s}$. In a *dummy game*, the cost of every agent i is independent of her strategy: $C_i(s_i, \mathbf{s}_{-i}) = C_i(s_i', \mathbf{s}_{-i})$ for every $\mathbf{s}_{-i}$ and every $s_i, s_i' \in S_i$.

Prove that a cost-minimization game with agent cost functions $C_1, \ldots, C_k$ is a potential game if and only if

$$C_i(\mathbf{s}) = C_i^t(\mathbf{s}) + C_i^d(\mathbf{s})$$

for every i and $\mathbf{s}$, where $C_1^t, \ldots, C_k^t$ is a team game and $C_1^d, \ldots, C_k^d$ is a dummy game.

Problem 13.4 Section 13.2.3 defines congestion games and notes that every such game is a potential game, even when cost functions need not be nondecreasing. This problem proves the converse, that every potential game is a congestion game in disguise. Call two games $\mathcal{G}_1$ and $\mathcal{G}_2$ *isomorphic* if: (1) they have the same number k of agents; (2) for each agent i, there is a bijection f_i from the strategies S_i of i in $\mathcal{G}_1$ to the strategies T_i of i in $\mathcal{G}_2$; and (3) these bijections preserve costs, so that $C_i^1(s_1, \ldots, s_k) = C_i^2(f_1(s_1), \ldots, f_k(s_k))$ for every agent i and outcome $s_1, \ldots, s_k$ of $\mathcal{G}_1$. (Here C^1 and C^2 denote agents' cost functions in $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively.)

(a) *(H)* Prove that every team game, as defined in the previous problem, is isomorphic to a congestion game.

(b) *(H)* Prove that every dummy game, as defined in the previous problem, is isomorphic to a congestion game.

(c) Prove that every potential game is isomorphic to a congestion game.