Math 236 Algebra 2 Assignment 5

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Problem 1a.

$$[v]_B = (1, 0, 0)$$

Find $[v]_C$:

$$\begin{cases} 4a + b - 3c = 1 \\ 3a + 2b - c = 0 \\ 3a + b + 5c = 1 \end{cases} \implies a = \frac{16}{35}, b = \frac{-23}{35}, c = \frac{2}{35}$$

Therefore

$$[v]_C = (\frac{16}{35}, \frac{-23}{35}, \frac{2}{35})$$

Problem 1b. Find ${}_BM_C$

$$(4, 3, 3) = 2(1, 0, 1) + 1(0, 1, 1) + 2(1, 1, 0)$$

$$(1, 2, 1) = 0(1, 0, 1) + 1(0, 1, 1) + 1(1, 1, 0)$$

$$(-3, -1, 5) = \frac{3}{2}(1, 0, 1) + \frac{7}{2}(0, 1, 1) + -\frac{9}{2}(1, 1, 0)$$

Therefore,

$${}_BM_C = \begin{pmatrix} 2 & 0 & \frac{3}{2} \\ 1 & 1 & \frac{7}{2} \\ 2 & 1 & -\frac{9}{2} \end{pmatrix}$$

Problem 1c.

$${}_BM_C[v]_C = \begin{pmatrix} 2 & 0 & \frac{3}{2} \\ 1 & 1 & \frac{7}{2} \\ 2 & 1 & -\frac{9}{2} \end{pmatrix} (\frac{16}{35}, \frac{-23}{35}, \frac{2}{35}) = (1, 0, 0) = [v]_B$$

Problem 2. From the diagram on the assignment it appears that $v_1 \neq \lambda v_2$ where $\lambda \in \mathbb{R}$. Therefore v_1, v_2 are linearly independent. It follows that since the dimension of $\mathbb{R}^2$ is 2 that $\{v_1, v_2\}$ is a basis of $\mathbb{R}^2$. Therefore any vector $v \in \mathbb{R}^2$ can be expressed as $v = av_1 + bv_2$ where $a, b \in \mathbb{R}$. Since T, L are linear transformations it follows:

$$T(v) = T(av_1 + bv_2) = T(av_1) + T(bv_2) = aT(v_1) + bT(v_2) = aL(v_1) + bT(v_2)$$

$$= L(av_1) + L(bv_2) = L(av_1 + bv_2) = L(v)$$

Therefore $T(v) = L(v)$ for all vectors $v \in \mathbb{R}^2$

Problem 3. $\Rightarrow$. Suppose $b = c = 0$. Therefore $T(x, y, z) = (5x - 2y + 3z, 9y)$. Check two properties of linear maps.

- Take $T(a_1, b_1, c_1), T(a_2, b_2, c_2)$
  $T(a_1, b_1, c_1) = (5a_1 - 2b_1 + 3c_1, 9b_1)$ and $T(a_2, b_2, c_2) = (5a_2 - 2b_2 + 3c_2, 9b_2)$
  $T(a_1, b_1, c_1) + T(a_2, b_2, c_2) = (5a_1 - 2b_1 + 3c_1, 9b_1) + (5a_2 - 2b_2 + 3c_2, 9b_2)$
  $= (5a_1 - 2b_1 + 3c_1 + 5a_2 - 2b_2 + 3c_2, 9b_1 + 9b_2) = (5a_1 + 5a_2 - 2b_1 - 2b_2 + 3c_1 + 3c_2, 9b_1 + 9b_2)$
  $= (5(a_1 + a_2) - 2(b_1 + b_2) + 3(c_1 + c_2), 9(b_1 + b_2)) = T(a_1 + a_2, b_1 + b_2, c_1 + c_2)$

- Take $T(a, b, c)$ and $\lambda \in \mathbb{R}$
  $T(\lambda(a, b, c)) = T(\lambda a, \lambda b, \lambda c) = (5\lambda a - 2\lambda b + 3\lambda c, 9\lambda b) = \lambda(5a - 2b + 3c, 9b) = \lambda T(a, b, c)$

Therefore T is a linear map.
$\Leftarrow$ Now suppose T is a linear map. Therefore T has the homogeneity property and take $T(x, y, z)$ and $\lambda \in \mathbb{R}$

$$T(\lambda(x, y, z)) = T(\lambda x, \lambda y, \lambda z) = (5\lambda x - 2\lambda y + 3\lambda z + b, 9\lambda y + c\lambda x\lambda z) = v_1$$

$$\lambda T(x, y, z) = \lambda(5x - 2y + 3z + b, 9y + cxz) = (5\lambda x - 2\lambda y + 3\lambda z + \lambda b, 9\lambda y + \lambda cxz) = v_2$$

In order for T to be a linear map, v_1 must equal v_2

$$(5\lambda x - 2\lambda y + 3\lambda z + b, 9\lambda y + c\lambda x\lambda z) = (5\lambda x - 2\lambda y + 3\lambda z + \lambda b, 9\lambda y + \lambda cxz)$$

$$\implies b = \lambda b \implies b = 0$$

Since $v_1 = v_2$ for all $\lambda \in \mathbb{R}$

$$\implies \lambda^2 cxz = \lambda cxz \implies \lambda cxz = cxz \implies c = 0$$

Since $v_1 = v_2$ for all $\lambda \in \mathbb{R}$ and for all $(x, y, z) \in \mathbb{R}^3$ Therefore if T is a linear map then b and c must be 0.

Problem 4. Suppose $a_1, ..., a_n \in \mathbb{F}$ and,

$$0 = a_1 v_1 + ... + a_n v_n$$

Therefore,

$$T(0) = T(a_1 v_1 + ... + a_n v_n) = T(a_1 v_1) + ... + T(a_n v_n) = a_1 T v_1 + ... + a_n T v_n$$

Since $T(0) = 0$,

$$0 = a_1 T(v_1) + ... + a_n T(v_n)$$

$Tv_1, ..., Tv_n$ are linearly independent, therefore $a_1 = ... = a_n = 0$ and thus $v_1, ...v_n$ are linearly independent.

Problem 5. $RangeS \subset kerT$. Therefore $\forall v \in V$, $TSv = 0$. Now take $u \in V$ such that $Tu = v$.

$$(ST)^2 u = S[TS(Tu)] = S[TS(v)] = S[0] = 0$$

Problem 6. Suppose $a_1, ..., a_n \in \mathbb{F}$ and,

$$0 = a_1 Tv_1 + ... + a_n Tv_n = T(a_1 v_1 + ... + a_n v_n)$$

Since T is injective $kerT = \{0\}$. Therefore $T(0) = 0$,

$$T(0) = T(a_1 v_1 + ... + a_n v_n)$$

$$\implies 0 = a_1 v_1 + ... + a_n v_n$$

$v_1, ..., v_n$ are linearly independent, therefore $a_1 = ... = a_n = 0$ and thus $Tv_1, ...Tv_n$ are linearly independent.

Problem 7. Let $v \in V$, then,

$$P(P(v)) = P(v) \implies P(v) - P(P(v)) = 0 \implies P(v - P(v)) = 0$$

Let $v - P(v) = k$.

$$\implies v = P(v) + k \implies V = range(P) + ker(P)$$

Since $P(v) \in range(P)$ and $k \in ker(p)$.

Let $w \in ker(P) \cap range(P)$. Therefore $w = P(x)$ for some $x \in V$ since $w \in range(P)$. It follows that $P(w) = P(P(x))$. Now since $w \in ker(P)$,

$$0 = P(w) = P(P(x)) = P(x) = w$$

Therefore $ker(P) \cap range(P) = \{0\}$. This proves that $V = ker(P) \oplus range(P)$.