Lecture 8

Spectrum Auctions

This lecture is a case study on the practical implementation of combinatorial auctions for wireless spectrum, an important and challenging multi-parameter mechanism design problem. While our sponsored search case studies (Sections 2.6 and 5.3) involve billions of small-stakes auctions, spectrum auction design concerns a single auction with billions of dollars of potential revenue.

Section 8.1 explains the practical benefits of indirect mechanisms. Section 8.2 discusses the prospects for selling multiple items via separate single-item auctions. Section 8.3 describes simultaneous ascending auctions, the primary workhorse in wireless spectrum auctions, while Section 8.4 weighs the pros and cons of packing bidding. Section 8.5 outlines the cutting edge of spectrum auction design, the 2016 FCC Incentive Auction.

8.1 Indirect Mechanisms

In a combinatorial auction (Example 7.2) there are n bidders, m items, and each bidder i's valuation specifies her value $v_i(S)$ for each bundle S of items that she might receive. In principle, the VCG mechanism provides a DSIC and welfare-maximizing combinatorial auction (Theorem 7.3). This mechanism is potentially practical if bidders' valuations are sufficiently simple (Exercise 7.5), but not otherwise (Section 7.3). For example, the number of parameters that each bidder reports in the VCG mechanism, or any other direct-revelation mechanism, grows exponentially with the number of items m.

The utter absurdity of direct-revelation combinatorial auctions motivates *indirect* mechanisms, which learn information about bidders' preferences only on a "need-to-know" basis. The canonical indirect auction is the ascending English auction; see also Exercise 2.7. This auction format is familiar from the movies: an auctioneer keeps

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track of the current price and tentative winner, and the auction stops when only one interested bidder remains.[1] Each bidder has a dominant strategy, which is to stay in the auction as long as the current price is below her valuation (the bidder might win for positive utility) and to drop out once the current price reaches her valuation (after which winning can only lead to negative utility). If all bidders play these strategies, then the outcome of the English auction is the same as that of a second-price (sealed-bid) auction. The second-price auction is the result of applying the revelation principle (Theorem 4.3) to the English auction.

Indirect mechanisms that elicit only a modest amount of information about bidders' valuations are unavoidable for all but the simplest combinatorial auction design problems.[2] This entails giving up on both the DSIC guarantee and full welfare maximization; we will miss these properties, but have no alternative.

8.2 Selling Items Separately

What's a natural indirect auction format for combinatorial auctions that avoids eliciting valuations for every possible bundle from each bidder? The simplest mechanisms to try are those that sell the items separately, using some type of single-item auction for each. Such a mechanism requires only one bid per bidder per item, and this is arguably the minimum number imaginable. Before pinning down the precise single-item auction format, we consider a basic question:

[1]There are a few variants. The movies, and auction houses like Christie's and Sotheby's, use an "open outcry" auction in which bidders can drop out and return, and can make "jump bids" to aggressively raise the current price. When doing mathematical analysis, the "Japanese" variant is usually more convenient: the auction begins at some opening price, which is publicly displayed and increases at a steady rate. Each bidder either chooses "in" or "out" at the current price, and once a bidder drops out, she cannot return. The winner is the last bidder in, and the sale price is the price at which the second-to-last bidder dropped out.

[2]Indirect mechanisms can also be useful in single-parameter settings like single-item auctions. Empirical studies show that bidders are more likely to play their dominant strategy in an English auction than in a sealed-bid second-price auction, where some bidders inexplicably overbid. Second, ascending auctions leak less valuation information to the seller. In a second-price auction, the seller learns the highest bid; in an English auction, the seller only learns a lower bound on the highest bid, namely the final selling price.

could selling items separately conceivably lead to allocations with high social welfare, even in principle?

There is a fundamental dichotomy between combinatorial auctions in which items are *substitutes* and those in which items can also be *complements*. The former are far easier than the latter, in both theory and practice. Roughly speaking, items are substitutes if they provide diminishing returns—having one item only makes others less valuable. For two items A and B, for example, the substitutes condition means that $v(AB) \leq v(A) + v(B)$. In a spectrum auction context, two licenses for the same area with equal-sized frequency ranges are usually substitute items. Theory indicates that selling items separately can work well when items are (mostly) substitutes. For starters, welfare maximization is a computationally tractable problem when items are substitutes and the true valuations are known. Also, the undesirable incentive and revenue properties of the VCG mechanism (Section 7.3 and Exercises 7.3 and 7.4) evaporate when items are substitutes, generalizing the reassuring properties of second-price single-item auctions. But even though substitute items constitute the "easy" case, we'll see that it is easy to screw up when trying to sell them separately.

Items are complements if there are synergies between them, so that possessing one makes others more valuable. With two items A and B, this translates to the property $v(AB) > v(A) + v(B)$. Complements arise naturally in wireless spectrum auctions, as some bidders want a collection of licenses that are adjacent, either in their geographic areas or in their frequency ranges. When items can be complements, welfare maximization is a computationally intractable problem, even without incentive constraints (Problem 4.3). We cannot expect a simple auction format like separate single-item auctions to perform well in these cases.

The items in spectrum auctions, and most real-world combinatorial auctions, are a mixture of substitutes and complements. If the problem is "mostly substitutes," then separate single-item auctions can perform well, if properly implemented. If not, then more complex auction formats are needed to achieve allocations with high social welfare (see Section 8.4).

8.3 Case Study: Simultaneous Ascending Auctions

8.3.1 Two Rookie Mistakes

There are numerous ways to organize separate single-item auctions. This section discusses two design decisions that seem to matter a lot in practice.

Rookie Mistake #1

Hold the single-item auctions sequentially, one at a time.

To see why holding auctions sequentially can be a bad idea, consider the especially easy case of k-unit auctions (Example 3.2), where the items are identical and each bidder only wants one of them. There is a simple DSIC and welfare-maximizing auction in this case (Exercise 2.3). Suppose we instead hold a sequence of single-item auctions—say, two identical items, sold via back-to-back second-price auctions. Now imagine that you are a bidder with a very high valuation—you expect to win any auction that you participate in. What should you do? First, suppose that every other bidder participates and bids her true valuation (until she wins an item). If you participate in the first auction, you would win and pay the second-highest valuation. If you skip it, the bidder with the second-highest valuation would win the first auction and disappear, leaving you to win the second auction at a price equal to the third-highest original valuation. Thus, straightforward bidding is not a dominant strategy in a sequence of second-price auctions. Intelligent bidding requires reasoning about the likely selling price of future auctions, and this in turn makes the auctions' outcomes unpredictable, with the possibility of low social welfare and revenue.

In March 2000, Switzerland auctioned off three blocks of spectrum via a sequence of second-price auctions. The first two auctions were for identical items, 28 MHz blocks, and sold for 121 million and 134 million Swiss francs, respectively. This is already more price variation than one would like for identical items. But the kicker was that in the third auction, where a larger 56 MHz block was being sold, the selling price was only 55 million francs! Some of the bids must have been far from optimal, and both the welfare and revenue achieved by this auction are suspect.[3]

The discussion and history lessons above suggest holding single-item auctions for multiple items *simultaneously* rather than sequentially. What single-item auction format should we choose?

Rookie Mistake #2

Use sealed-bid single-item auctions.

In 1990, the New Zealand government auctioned off essentially identical licenses for television broadcasting using simultaneous (sealed-bid) second-price auctions. It is again difficult for bidders to figure out how to bid in such an auction. Imagine that there are 10 licenses and you want only one of them. How should you bid? One legitimate strategy is to pick one of the licenses—at random, say—and go for it. Another strategy is to bid less aggressively on multiple licenses, hoping that you get one at a bargain price, and that you don't inadvertently win extra licenses that you don't want. The difficulty is trading off the risk of winning too many licenses with the risk of winning too few.

The challenge of bidding intelligently in simultaneous sealed-bid auctions makes the auction format prone to outcomes with low welfare and revenue. For example, suppose there are three bidders and two identical items, and each bidder wants only one. With simultaneous second-price auctions, if each bidder targets only one license, one of the licenses is likely to have only one bidder and will be given away for free (or sold at the reserve price).

The revenue in the 1990 New Zealand auction was only $36 million, a paltry fraction of the projected $250 million. On one license, the high bid was $100,000 while the second-highest bid (and selling price) was $6! On another, the high bid was $7 million and the second-highest was $5,000. To add insult to injury, the winning bids were made available to the public, who could then see just how much money was left on the table!

[3]In addition to the questionable auction format, there were some strategic mergers of potential bidders before the auction, leading to less competition than expected.

8.3.2 The Merits of Simultaneous Ascending Auctions

Simultaneous ascending auctions (SAAs) form the basis of most spectrum auctions run over the last 20 years. Conceptually, SAAs are like a bunch of single-item English auctions being run in parallel in the same room, with one auctioneer per item. More precisely, in each round, each bidder can place a new bid on any subset of items that it wants, subject to an *activity rule*. The activity rule forces all bidders to participate in the auction from the beginning and contribute to the discovery of appropriate prices. For example, such a rule makes it impossible for bidders to "snipe," meaning to enter the auction at last second and place a winning bid. The details of an activity rule can be complex, but the gist is to require that the number of items on which a bidder bids only decreases over time as prices rise. The high bids and bidders are usually visible to all, even though this can encourage signaling and retaliatory bids (Section 8.3.4). The first round with no new bids ends the auction.

The primary reason that SAAs work better than sequential or sealed-bid auctions is *price discovery*. As a bidder acquires better information about the likely selling prices of licenses, she can implement mid-course corrections: abandoning licenses for which competition is fiercer than anticipated, snapping up unexpected bargains, and rethinking which packages of licenses to assemble. The format typically resolves the miscoordination problems that plague simultaneous sealed-bid auctions. For instance, suppose there are two identical items and three bidders. Every round, some bidder will be losing both auctions. When she jumps back in, it makes sense to bid for the currently cheaper item, and this keeps the prices of the two items roughly the same.

Another bonus of the SAA format is that bidders only need to determine their valuations on a need-to-know basis. We've been assuming that valuations are known to bidders at the beginning of the auction, but in practice, determining the valuation for a bundle of items can be costly, involving research and expert advice. In sharp contrast to direct-revelation mechanisms, a bidder can often navigate an SAA with only coarse estimates for most valuations and precise estimates for the bundles that matter.

SAAs are thought to have achieved high social welfare and revenue in numerous spectrum auctions. This belief is not easy to test, since valuations remain unknown after an auction and bids are incomplete and potentially non-truthful. There are a number of sanity checks that can be used to argue good auction performance. First, there should be little or no resale of items after the auction, and any reselling should take place at a price comparable to the auction's selling price. This indicates that speculators did not play a significant role in the auction. Second, similar items should sell for similar prices (cf., the Swiss and New Zealand auctions). Third, revenue should meet or exceed projections. Fourth, there should be evidence of price discovery. For example, prices and provisional winners at the mid-point of the auction should be highly correlated with the final selling prices and winners. Finally, the packages assembled by bidders should be sensible, such as groups of licenses that are adjacent geographically or in frequency range.

8.3.3 Demand Reduction and the Exposure Problem

SAAs have two big vulnerabilities. The first problem is *demand reduction*, and this is relevant even when items are substitutes. Demand reduction occurs when a bidder asks for fewer items than it really wants, to lower competition and therefore the prices paid for the items that it gets.

To illustrate, suppose there are two identical items and two bidders. The first bidder has valuation 10 for one of the items and valuation 20 for both. The second bidder has valuation 8 for one of the items and does not want both (i.e., her valuation remains 8 for both). Giving both items to the first bidder maximizes the welfare, at 20. The VCG mechanism would earn revenue 8 in this example. Now consider how things play out in an SAA. The second bidder would be happy to have either item at any price less than 8. Thus, the second bidder drops out only when both items have price at least 8. If the first bidder stubbornly insists on winning both items, her utility is $20 - 16 = 4$. If, on the other hand, the first bidder targets just one item, then each of the bidders gets one of the items at a near-zero price. The first bidder's utility is then close to 10. In this example, demand reduction leads to a loss of welfare and revenue, relative to the VCG mechanism's outcome. There is ample evidence of demand reduction in many spectrum auctions.

The second big problem with SAAs is relevant when items can be complements, as in many spectrum auctions, and is called the *exposure problem*. As an example, consider two bidders and two non-identical items. The first bidder only wants both items—they are complementary items for the bidder—and her valuation is 100 for them (and 0 otherwise). The second bidder is willing to pay 75 for either item but only wants one item. The VCG mechanism would give both items to bidder 1, for a welfare of 100, and would generate revenue 75. In an SAA, the second bidder will not drop out until the price of both items reaches 75. The first bidder is in a no-win situation: to get both items it would have to pay 150, more than her value. The scenario of winning only one item for a nontrivial price could be even worse. On the other hand, if the second bidder's value for each item is only 40, then the first bidder should just go for it and outlast the second bidder. But how can the first bidder know which scenario is closer to the truth? The exposure problem makes bidding in an SAA difficult for a bidder for whom items are complements, and it leads to economically inefficient allocations for two reasons. First, an overly aggressive bidder might acquire unwanted items. Second, an overly tentative bidder might fail to acquire items for which it has the highest valuation.

8.3.4 Bid Signaling

Iterative auctions like SAAs offer opportunities for strategic behavior that do not exist in direct-revelation mechanisms. In early and relatively uncompetitive spectrum auctions, bidders sometimes used the low-order digits of their bids to effectively send messages to other bidders. In one example, USWest and McLeod were battling it out for license #378 in Rochester, Minnesota, with each repeatedly outbidding the other. Apparently, USWest tired of this bidding war and switched to a retaliatory strategy, bidding on licenses in other geographical areas on which McLeod was the standing high bidder and USWest had shown no interest in previous rounds. McLeod ultimately won back all of these licenses, but had to pay a higher price due to USWest's bids. To make sure its message came through loud and clear, all of USWest's retaliatory bids were a multiple of 1,000 *plus 378*—presumably warning McLeod to get the hell out of the market for Rochester, or else. While this particular type of signaling can be largely eliminated by forcing all bids to be multiples of a suitably large number, it seems impossible to design away all opportunities for undesirable strategic behavior.

8.4 Package Bidding

The exposure problem motivates supplementing the basic SAA format with *package bidding*, meaning bids on sets of items in addition to individual items. Package bidding allows a bidder to bid aggressively on a bundle of items without fear of receiving only a subset of them. There are also scenarios where package bids can remove the incentive for demand reduction.

There has been much discussion about how to implement package bidding, if at all, in wireless spectrum auctions. The conservative viewpoint, which dominated practice until relatively recently, is that package bids add complexity to a quite functional auction format and might lead to unpredictable outcomes. Limited forms of package bidding have been incorporated into spectrum auction designs only over the past 10 years or so, and only outside of the United States.

One design approach is to tack on one extra round after the SAA where bidders can submit package bids on any subsets of items that they want, subject to an activity rule. These package bids compete with each other as well as the winning bids on individual items from the SAA phase of the auction. The final allocation is determined by a welfare maximization computation, treating bids as true valuations. The biggest issue with this approach is that computing the final prices is tricky. The VCG payment rule is not used because of its poor revenue and incentive properties (Section 7.3 and Exercises 7.2–7.4). A more aggressive payment rule, which yields an auction that is not DSIC but does have other good incentive properties, is used instead.

A second approach is to predefine a limited set of allowable package bids rather than allowing bidders to propose their own. Ideally, the predefined package bids should be well aligned with what bidders want, yet structured enough to permit reasonably simple allocation and payment rules. Hierarchical packages have emerged as a sweet spot for this design approach. For example, an auction could allow bids on individual licenses, on regional bundles of licenses, and on nationwide bundles of licenses. The biggest issue with predefined package bids is that they can do more harm than good when they are poorly matched with bidders' goals. For example, imagine a bidder who wants the items $\{A, B, C, D\}$, but the available packages are $\{A, B, E, F\}$ and $\{C, D, H, I\}$. What should her bidding strategy be?

8.5 Case Study: The 2016 FCC Incentive Auction

Wireless spectrum doesn't grow on trees. At this point, in the United States, giving someone a new allocation of spectrum generally requires taking it away from someone else. The U.S. Federal Communications Commission (FCC) is doing precisely this, using a reverse auction (cf., Exercise 2.5) to free up spectrum by buying out television (TV) broadcasters and a forward auction to resell the spectrum to companies that can put it to more valuable use.[4]

The format of the forward auction is similar to past designs (Sections 8.3 and 8.4). The reverse auction is completely new.

After the reverse auction, the FCC will repack the remaining broadcasters so that the newly available spectrum is contiguous. For example, they might buy out a number of TV broadcasters across the nation who were using a UHF channel between 38 and 51, and reassign all of the other broadcasters using the channels to lower channels. This would leave the 84 MHz block of spectrum corresponding to channels 38–51 free to be sold in the forward auction for new uses.

In a very cool development, the reverse auction format can be thought of as a greedy allocation rule, not unlike the knapsack auction allocation rules described in Section 4.2. To describe it, we adopt the following model. Each bidder i (a TV broadcaster) has a private valuation v_i for its broadcasting license. If bidder i loses (that is, is not bought out), then her utility is 0. If bidder i wins (is bought out) at a price of p, then her utility is $p - v_i$. Thus v_i is the "minimum acceptable offer" for buying out i.[5] Letting N denote the set of bidders, a set $W \subseteq N$ of winning bidders—where "winning" means being bought out—is feasible if the remaining bidders $N \setminus W$ can

[4] The auction commenced on March 29, 2016 and is ongoing as of this writing.

[5] This single-parameter model assumes that each TV station is owned by a different strategic agent. This assumption is not entirely true in practice, but it makes the model much easier to reason about.

be repacked in the target range (e.g., the channels below 38).[6] For instance, if $W = N$, then all bidders are bought out and the entire spectrum is freed up, so W is certainly feasible. When $W = \emptyset$, no spectrum is reclaimed, an infeasible outcome. Two TV stations with overlapping geographic areas cannot be assigned the same or adjacent channels, and checking whether or not a given set W is feasible is a medium-size $\mathcal{NP}$-hard problem, closely related to graph coloring (Figure 8.1). State-of-the-art algorithms, building on satisfiability ("SAT") solvers, are used to perform each of these feasibility checks in seconds or less.

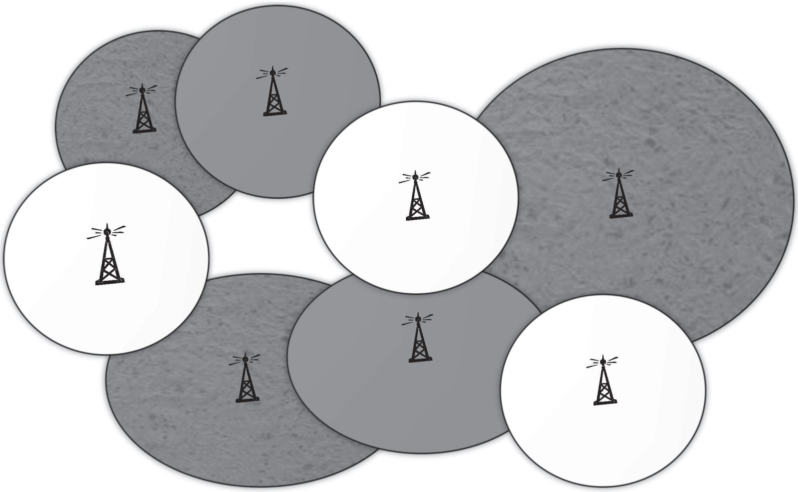

Figure 8.1: Different TV stations with overlapping broadcasting areas must be assigned different channels (indicated by shades of gray). Checking whether or not a given subset of stations can be assigned to a given number of channels without interference is an $\mathcal{NP}$-hard problem.

We next describe the form of the reverse auction allocation rule, which is a *deferred allocation rule.*[7]

[6]One interesting question is how to set this target. The bigger the target, the bigger the expenses per unit of spectrum in the reverse auction and the smaller the revenues per unit of spectrum in the forward auction. The goal is to set the target as large as possible, subject to a lower bound on the net revenue obtained.

[7]This terminology is inspired by the "deferred acceptance" algorithm for computing a stable matching (see Section 10.2).

Deferred Allocation Rule

```
initialize W = N                // initially feasible
while there is an i ∈ W with W \ {i} feasible do
    remove one such i from W // i not bought out
halt with winning bidders W
```

The allocation rule starts with the trivial feasible set (all bidders), and then iteratively removes bidders until a minimal feasible set is reached. This is a "reverse greedy algorithm," since it removes bidders starting from the entire set. In contrast, typical (forward) greedy algorithms iteratively add bidders starting from the empty set (cf., Section 4.2.2).

How should we choose which bidder to remove at each iteration? Natural ideas include removing the bidder with the highest bid (i.e., the least willing to be bought out), or the bidder with the highest ratio of bid to market size. A general way to describe such heuristics is through a *scoring function*, which assigns a score to each remaining bidder at each iteration of the auction. The algorithm can then be implemented by removing the remaining bidder with the highest score, subject to feasibility.[8]

One simple scoring function is the identity. The corresponding allocation rule performs a single pass over the bidders (from the highest to the lowest), removing a bidder whenever doing so preserves feasibility. For example, for the problem of hiring at least one contractor, this allocation rule just chooses the lowest bidder.

If the scoring function is increasing in a bidder's bid and independent of the bids of the other remaining bidders, then the corresponding deferred allocation rule is monotone; in the current context of a reverse auction, this means that bidding lower can only cause a bidder to win (Exercise 8.3). By Myerson's lemma (Theorem 3.7), paying each winner her critical bid—the largest bid at which she would have been bought out—yields a DSIC auction.[9] In the simple case of hiring

[8]The choice of the scoring function in the 2016 FCC Incentive Auction was guided by several factors, including the welfare achieved by different rules on synthetic data, and by constraints on how much price discrimination between bidders was politically feasible.

[9]For ease of participation, the actual FCC auction is iterative, not direct-

at least one contractor and the identity scoring function, this auction is identical to that in Exercise 2.5.

Remarkably, deferred allocation rules lead to mechanisms that have a number of good incentive properties above and beyond DSIC, and which are not shared by their forward-greedy cousins (Problem 8.1).

The Upshot

- ✰ Direct-revelation mechanisms are out of the question for all but the smallest combinatorial auctions.
- ✰ Indirect mechanisms learn information about bidders' preferences only on a need-to-know basis.
- ✰ Selling multiple items separately has the potential to work well when items are substitutes. When items can be complements, selling items separately can produce outcomes with low social welfare.
- ✰ The preferred method in practice of selling items separately is simultaneous ascending auctions (SAAs).
- ✰ SAAs are vulnerable to demand reduction, where a bidder reduces the number of items requested to depress the final selling prices.
- ✰ When items can be complements, SAAs also suffer from the exposure problem, where a bidder that desires a bundle of items runs the risk of acquiring only a useless subset of them.

revelation, and uses descending, bidder-specific prices. In each round, each bidder only has to decide whether to stay in at her current offer, or to drop out and retain her license. The offers at the beginning of the auction are high enough that everyone is happy to participate. For example, for WCBS-TV in New York, the opening offer is $900 million.

- ✰ Package bidding can mitigate the exposure problem but is tricky to implement.
- ✰ The 2016 FCC Incentive Auction is the first to include a reverse auction, where the government buys back licenses from TV broadcasters to reclaim spectrum.
- ✰ Deferred allocation rules are a rich family of reverse auction allocation rules with good incentive properties.

Notes

The history and practice of wireless spectrum auctions are discussed in detail by Cramton (2006) and Milgrom (2004). See also Cramton et al. (2006) and Klemperer (2004) for much more on the theory and implementation of combinatorial auctions, and Rassenti et al. (1982) for an early application of combinatorial auctions to the allocation of airport time slots.

Harstad (2000) demonstrates that bidders are more likely to play their dominant strategies in an English auction than a sealed-bid second-price auction. Cramton and Schwartz (2000) detail collusion and bid signaling in early spectrum auctions. Ausubel and Milgrom (2002) propose using a proxy round to implement package bids, while Goeree and Holt (2010) advocate predefined hierarchical packages. The details of the FCC Incentive Auction design are described in a public notice (Federal Communications Commission, 2015). Milgrom and Segal (2015a,b) discuss the high-level design decisions in the reverse auction, and also define deferred allocation rules. Exercise 8.3 and Problems 8.1 and 8.2 are from Milgrom and Segal (2015a). The algorithms used to implement feasibility checks are described by Fréchette et al. (2016). Problem 8.3 is from Shapley and Shubik (1971).

Exercises

Exercise 8.1 *(H)* The ideal outcome of an SAA (Section 8.3) with item set $M = \{1, 2, \ldots, m\}$ is a *Walrasian equilibrium*, meaning an allocation $S_1, \ldots, S_n \subseteq M$ of bundles to the n bidders and item selling prices $p_1, \ldots, p_m$ that meet the following conditions.

Walrasian Equilibrium

1. Every bidder i gets her preferred bundle, given the prices $\mathbf{p}$:

$$S_i \in \operatorname*{argmax}_{S \subseteq M} \left(v_i(S) - \sum_{j \in S} p_j \right).$$

2. Supply equals demand: every item j appears in at most one bundle S_i, and goes unsold only if $p_j = 0$.

Prove that if an allocation $(S_1, \ldots, S_n)$ and prices $\mathbf{p}$ form a Walrasian equilibrium, then the allocation has the maximum possible social welfare. (This is a form of the "first welfare theorem.")

Exercise 8.2 *(H)* Prove that, even in combinatorial auctions with only two bidders and two items, there need not exist a Walrasian equilibrium.

Exercise 8.3 Consider a deferred allocation rule (Section 8.5) in which bidders are removed according to a scoring function. A scoring function assigns a score to every remaining bidder in the auction, and in each iteration the allocation rule removes, among all bidders whose removal does not destroy feasibility, the bidder with the highest score.

Consider a scoring function that satisfies two properties. First, the score of a bidder i is independent of the bids of the other remaining bidders. (The score can depend on i, on i's bid, on the bids of bidders that have already dropped out, and on the set of remaining bidders.) Second, holding other bids fixed, the score of a bidder is increasing in her bid. Prove that the corresponding deferred allocation rule is monotone: for every bidder i and bids $\mathbf{b}_{-i}$ by the other bidders, if i wins when bidding b_i and $b'_i < b_i$, then i also wins when bidding b'_i.

Problems

Problem 8.1 A direct-revelation mechanism is *weakly group-strategyproof* if for every colluding subset C of bidders, every profile $\mathbf{b}_{-C}$ of bids of the bidders outside C, and every profile $\mathbf{v}_C$ of valuations for C, there is no profile $\mathbf{b}_C$ of bids that results in every bidder of C receiving strictly higher utility than with truthful bids $\mathbf{v}_C$.

(a) In the same setting and with the same assumptions as in Exercise 8.3, prove that the corresponding DSIC mechanism is weakly group-strategyproof.

(b) Prove that the "forward greedy" DSIC mechanism defined in Problem 4.3 is not weakly group-strategyproof in general.

Problem 8.2 In the same setting and with the same assumptions as in Exercise 8.3, give an ascending implementation of the corresponding DSIC mechanism. Your ascending implementation should *not* accept explicit bids. It should proceed in rounds, and at each round, one of the remaining bidders should be given a take-it-or-leave-it offer for dropping out. Prove that straightforward bidding—continuing to the next round if and only if the current offer exceeds her private valuation—is a dominant strategy for every bidder. Prove that the final outcome, assuming straightforward bidding, is the same as the truthful outcome in the direct-revelation DSIC mechanism. For convenience, you can restrict attention to the case where all valuations and scores are positive integers bounded above by a known value $v_{\max}$.

Problem 8.3 *(H)* Prove that in every combinatorial auction in which every bidder has a unit-demand valuation (Exercise 7.5), there exists a Walrasian equilibrium.