Math 447 Assignment 5

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6.12

Let R_t be the number of diners in the resturant at time t.

$$\begin{aligned} P(R_{120} = k) &= \sum_{n=k}^{\infty} P(R_{120} = k \mid N_{120} = n) P(N_{120} = n) \\ &= \sum_{n=k}^{\infty} P(R_{120} = k \mid N_{120} = n) * \frac{e^{-5*120} * (5*120)^n}{n!} \\ &= \sum_{n=k}^{\infty} P(R_{120} = k \mid N_{120} = n) * \frac{e^{-600} * (600)^n}{n!} \end{aligned}$$

Given that n diners enter before 2p.m. with arrival times $S_1, S_2, ..., S_n$. Let $Z_1, Z_2, ...Z_n$ represent the time spent in the diner by each person, where the Z_i's are i.i.d. exponential random variables with parameter 40. Therefore the kth person's exit time can be represented as $S_k + Z_k$. There are k students in the building at time 120 (2p.m.) if and only if k of the exit times $S_1 + Z_1, ..., S_n + Z_n$ exceed t. Therefore

$$= \sum_{n=k}^{\infty} P(\text{ k of the } S_1 + Z_1, ..., S_n + Z_n \text{ exceed } 120 \mid N_{120} = n) * \frac{e^{-600} * (600)^n}{n!}$$

$$= \sum_{n=k}^{\infty} P(\text{ k of the } U_1 + Z_1, ..., U_n + Z_n \text{ exceed } 120) * \frac{e^{-600} * (600)^n}{n!}$$

$$= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{(n-k)} \frac{e^{-600} * (600)^n}{n!}$$

Where,

$$p = P(U_1 + Z_1 > 120) = \frac{1}{120} \int_0^{120} P(Z_1 > 120 - x) dx = \frac{1}{120} \int_0^{120} (1 - (1 - e^{-\frac{x}{40}})) dx = \frac{1}{120} \int_0^{120} e^{-\frac{x}{40}} dx = \frac{1 - e^{-4800}}{3}$$

Therefore,

$$\begin{aligned} P(R_{120} = k) &= \frac{p^k (600)^k}{k!} e^{-600} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} * 600^{n-k}}{(n-k)!} \\ &= \frac{p^k (600)^k}{k!} e^{-600} * e^{600(1-p)} \\ & \frac{e^{-600p} (600p)^k}{k!} \end{aligned}$$

We see that R_{120} has poisson distribution with parameter $600p$, where,

$$p = \frac{1 - e^{-4800}}{3}$$

$$\Rightarrow 600p = 200(1 - e^{-4800}) \approx 200$$

Therefore we conclude that R_{120} has poisson distribution with parameter 200 and,

$$E[R_{120}] = 200$$

$$Var(R_{120}) = 200$$

6.42

```
set.seed(201)

trials = 5000
values = c()

for (a in 0:trials){
  occupancy2pm = 0
  for (i in 0:119){
    enter = rpois(1,5) #number of ppl that enter at time t
    for (j in 1:enter){
      stay = rpois(1,40) #time each new customer will spend in resturant

      if((stay+i) >= 120){#If there arrival + stay time is more than 120 minutes past noon, then they a
        occupancy2pm = occupancy2pm + 1
      }
    }
  }
  values = c(values,occupancy2pm)
}
exp = mean(values)
var = var(values)
exp
```

```
## [1] 200.6587
```

```
var
```

```
## [1] 194.9649
```

Completing 5000 trials of this scenario, the expected number (and variance) of diners in the resturant at 2p.m. is very similar to what we calculated in question 6.12. With these results it confirms our belief that the number of diners in the resturant at 2p.m. is a poisson distribution with parameter 200.

7.19

This is a birth death process with respect to how many taxis are waiting in line. The birth rate (taxi arrivals) is λ and the death rate (passenger arrivals/taxi depatures) is μ. From class we have seen,

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}$$

Since in this case $\lambda_i = \lambda_j \forall i, j$ and $\mu_i = \mu_j \forall i, j$, this simplifies to,

$$\pi_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k}$$

Evaluating the geometric series,

$$\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k = \frac{1}{1-\frac{\lambda}{\mu}} = \frac{\mu}{\mu-\lambda}$$

Therefore,

$$\pi_0 = \frac{1}{\frac{\mu}{\mu-\lambda}} = 1 - \frac{\lambda}{\mu}$$

Here π_0 is the long term probability of no taxis being in line. Therefore the long-term probability that an arriving customer gets a taxi is $1 - \pi_0$.

$$1 - \pi_0 = 1 - (1 - \frac{\lambda}{\mu}) = \frac{\lambda}{\mu}$$

We conclude, the long-term probability that an arriving customer gets a taxi is $\frac{\lambda}{\mu}$

7.26

This a birth death process with respect to how machines are broken. The birth rate (machine breaking) is $\frac{1}{24}$ and the death rate (machine fixed) is $\frac{1}{6}$. From class we have seen,

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}$$

Since in this case $\lambda = \frac{1}{24}$ and $\mu = \frac{1}{6}$ are constant, this simplifies to,

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k}$$

$$= \frac{1}{1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64}}$$

$$= \frac{1}{\frac{85}{64}}$$

$$= \frac{64}{85}$$

Therefore the long-term probability that all machines are working is $\frac{64}{85}$.