McGill University
Department of Mathematics and Statistics
MATH 243 Analysis 2, Winter 2017
Assignment 5

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1 and 2.**

This assignment is due **Tuesday, February 14, at 2:30pm** in class. **Late assignments will not be accepted!**

1. Consider the two functions $f, g : [0,1] \to \mathbb{R}$,

$$f(x) := \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x \leq \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } x = 0 \end{cases}, \qquad g(x) := \begin{cases} \frac{1}{n} & \text{if } \frac{1}{n+1} < x < \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   (a) Prove that f is Riemann integrable on $[0,1]$.
   (b) Prove that g is Riemann integrable on $[0,1]$. Hint: Consider $f - g$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be Thomae's function and let $g : \mathbb{R} \to \mathbb{R}$ be the sign function i.e.

$$g(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

   (a) Prove that g is Riemann integrable on any interval $[a,b]$ with $a < b$.
   (b) However, prove that $g \circ f$ is *not* Riemann integrable on the interval $[0,1]$. (This shows that the composition of two Riemann integrable functions is not necessarily Riemann integrable!)

3. Let f and g be continuous functions on the interval $[a,b]$, $a < b$, such that $\displaystyle\int_a^b f = \int_a^b g$. Prove that there exists a number $c \in [a,b]$ with $f(c) = g(c)$.
   Hint: Use the intermediate value theorem.

4. (a) Let $f : [a,b] \to \mathbb{R}$ be bounded and Riemann integrable on any subinterval $[c,b] \subseteq]a,b]$. Prove that f is Riemann integrable on $[a,b]$ and that $\displaystyle\int_a^b f = \lim_{c \to a^+} \int_c^b f$.
   Hint: Use the squeeze theorem and the additivity of the Riemann integral.
   (b) Show by providing a concrete example that the boundedness condition on f in part (a) cannot be dropped.

5. (a) Let E be a finite subset of the interval $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function which is continuous at all $x \in [a,b] \setminus E$. Prove that f is Riemann integrable on $[a,b]$.
   (b) Show by providing a concrete example that the boundedness condition on f in part (a) cannot be dropped.

6. Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be continuous. Prove that there exists a number $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f$$

   Remark: This result is often called the *mean value theorem for integrals* or the *first mean value theorem for integrals*. $f(c)$ is often called the *average value of f on $[a,b]$*.

   Hint: Use the intermediate value theorem and the extreme value theorem i.e. the existence of absolute extrema of a continuous function on a closed and bounded interval.