Econ 546 Assignment 1

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Problem 1. Game Theory is the analysis of rational agents and their interactions with each other under a well defined set of game rules.

Problem 2a. Yes, the strategy below is a winning strategy for Player 1

- Player 1 removes 1 straw from Pile A (leaving 2 piles of 1 straw each)
- Without loss of generality, Player 2 removes 1 straw from Pile A (leaving only Pile B with 1 straw)
- Player 1 removes 1 straw from Pile B (leaving no piles remaining)
- Player 1 has ended the game. Player 1 wins the game

Problem 2b. Yes, the strategy below is a winning strategy for Player 1

- Player 1 removes 2 straws from Pile A (leaving only Pile B with 1 straw)
- Player 2 removes 1 straw from Pile B (leaving no piles remaining)
- Player 2 has ended the game. Player 1 wins the game

Problem 3. Using backward induction, we examine the equilibrium of the game starting with the scenario where only Pirate 6 remains.

Proposed Distribution {Pirate : Payoff}	Yes Vote(s)
$\{6:20\}$	6
$\{5:20, 6:0\}$	5
$\{4:19, 5:0, 6:1\}$	4,6
$\{3:19, 4:0, 5:1, 6:0\}$	3,5
$\{2:18, 3:0, 4:1, 5:0, 6:1\}$	2,4,6
$\{1:18, 2:0, 3:1, 4:0, 5:1, 6:0\}$	1,3,5

Therefore the Captain would propose the following coin share: {18,0,1,0,1,0}. The Captain receives 18 coins and votes yes, Pirates 3 and 5 both receive 1 coin and vote yes, therefore this plan will be accepted and this payoff scheme is the equilibrium.

Problem 4. We begin by underlying the best responses for Player 1, with Player 2's strategies fixed, and similarly underlying the best responses for Player 2, with Player 1's strategies' fixed

	L	M	R
T	$(\underline{3},\underline{2})$	$(\underline{4},0)$	(0,0)
M	(2,0)	$(3,\underline{3})$	(0,0)
B	(0,0)	(0,0)	$(\underline{3},\underline{3})$

Therefore we conclude that {T,L} and {B,R} are both Nash equilibria in this game.

Problem 5.1. Suppose K=1, and the strategy profile is (1,1,1). Without loss of generality, assume Player 3 chooses to deviate from this strategy and select a new number J, where J $\neq$ 1. Now the average of the 3 numbers is $\frac{2+J}{3}$ and the two-thirds average is $\frac{4+2J}{9}$.

Now we can calculate the absolute distance from the two-thirds average to 1 and J to see which is closer, and therefore who will receive the reward.

$$\left|1 - \frac{4+2J}{9}\right| = \left|\frac{5-2J}{9}\right|$$

$$\left|J - \frac{4+2J}{9}\right| = \left|\frac{7J-4}{9}\right|$$

Further, it is clear that for all values of J,

$$\left|\frac{5-2J}{9}\right| < \left|\frac{7J-4}{9}\right|$$

In other words the two-thirds average is still closer to 1 then it is to J. Therefore Player 3 does not benefit from deviating from the strategy profile and we conclude (1,1,1) is a Nash equilibrium.

Problem 5.2. Summary: Below is a proof that demonstrates that (1,1,1) is the only Nash equilibrium of this game.

Let the number selected by Player 1,2 and 3 be K_1, K_2 and K_3 respectively. Without loss of generality assume $K_1 \leq K_2 \leq K_3$

Suppose $K_1 = K_2 = K_3 \neq 1$, and the strategy profile is (K_3,K_3,K_3). Without loss of generality, assume Player 3 chooses to deviate from this strategy and select a new number $K_3 - 1$. Now the average of the 3 numbers is $\frac{3K_3-1}{3} = K_3 - \frac{1}{3}$ and the two-thirds average is $\frac{2}{3}K_3 - \frac{2}{9}$.

Now we can calculate the absolute distance from the two-thirds average to K_3 and K_3-1 to see which is closer, and therefore who will receive the reward.

$$\left|K_3 - \left(\frac{2}{3}K_3 - \frac{2}{9}\right)\right| = \left|\frac{1}{3}K_3 + \frac{2}{9}\right|$$

$$\left|(K_3 - 1) - \left(\frac{2}{3}K_3 - \frac{2}{9}\right)\right| = \left|\frac{1}{3}K_3 - \frac{7}{9}\right|$$

Further, it is clear that for all values of K_3,

$$\left|\frac{1}{3}K_3 + \frac{2}{9}\right| > \left|\frac{1}{3}K_3 - \frac{7}{9}\right|$$

In other words the two-thirds average is now closer to $K_3 - 1$ then it is to K_3. Therefore Player 3 does benefit from deviating from the strategy profile and we conclude (K,K,K) is not a Nash equilibrium when $K \neq 1$.

Suppose $K_1 = K_2$, and the strategy profile is (K_2,K_2,K_3). The average of the 3 numbers is $\frac{2K_2+K_3}{3}$ and the two-thirds average is $\frac{4K_2+2K_3}{9}$.

Now we can calculate the absolute distance from the two-thirds average to K_2 and K_3 to see which is closer, and therefore who will receive the reward.

$$\left|K_2 - \frac{4K_2 + 2K_3}{9}\right| = \left|\frac{5K_2 - 2K_3}{9}\right|$$

$$\left|K - 3 - \frac{4K_2 + 2K_3}{9}\right| = \left|\frac{7K_3 - 4K_2}{9}\right| = \frac{7K_3 - 4K_2}{9}$$

In the case where $5K_2 \geq 2K_3$ we have:

$$5K_2 - 2K_3 \leq 7K_3 - 4K_2$$

$$\Rightarrow 9K_2 \leq 9K_3$$

This is true since $K_2 \leq K_3$. Next in the case where $5K_2 \leq 2K_3$ we have:

$$2K_3 - 5K_2 \leq 7K_3 - 4K_2$$

$$\Rightarrow -K_2 \leq 5K_3$$

This is true since $-K_2 < 0$ and $K_3 > 0$. Therefore we conclude that the two-thirds average is closer to K_2 and thus Player 3 would benefit from deviating from their current strategy and choosing K_2. Thus this scenario is not a Nash equilibrium.

Finally, suppose the strategy profile is (K_1,K_2,K_3). The average of the 3 numbers is $\frac{K_1+K_2+K_3}{3}$ and the two-thirds average is $\frac{2K_1+2K_2+2K_3}{9}$.

Now we can calculate the absolute distance from the two-thirds average to K_1, K_2 and K_3 to see which is closest, and therefore who will receive the reward.

$$\left|K_1 - \frac{2K_1 + 2K_2 + 2K_3}{9}\right| = \left|\frac{7K_1 - 2K_2 - 2K_3}{9}\right|$$

$$\left|K_2 - \frac{2K_1 + 2K_2 + 2K_3}{9}\right| = \left|\frac{7K_2 - 2K_1 - 2K_3}{9}\right|$$

$$\left|K_3 - \frac{2K_1 + 2K_2 + 2K_3}{9}\right| = \left|\frac{7K_3 - 2K_1 - 2K_2}{9}\right|$$

Examine K_1 and K_3 distances:

$$|7K_1 - 2K_2 - 2K_3| \leq |7K_3 - 2K_2 - 2K_1|$$

$$\Rightarrow |7K_1 - 2K_3| \leq |7K_3 - 2K_1|$$

$$\Rightarrow |7K_1 - 2K_3| \leq 7K_3 - 2K_1$$

In the case where $7K_1 \geq 2K_3$ we have:

$$7K_1 - 2K_3 \leq 7K_3 - 2K_1$$

$$\Rightarrow 9K_1 \leq 9K_3$$

This is true since $K_1 \leq K_3$. Next in the case where $7K_1 \leq 2K_3$ we have:

$$2K_3 - 7K_1 \leq 7K_3 - 2K_1$$

$$\Rightarrow -5K_1 \leq 5K_3$$

This is true since $-K_1 < 0$ and $K_3 > 0$. Therefore we conclude that the two-thirds average is closer to K_1 than K_3. Therefore Player 3 would benefit from deviating from their current strategy and choosing K_1. Thus this scenario is not a Nash equilibrium.

We conclude that the only Nash equilibrium of this game is (1,1,1).

Problem 5.3. Suppose the strategy profile of the class is (0,0,0,....,0,0). Without loss of generality, assume Player 60 chooses to deviate from this strategy and select a new number J, where J $\neq$ 0. Now the average of the 60 numbers is $\frac{J}{60}$ and the two-thirds average is $\frac{J}{90}$.

Now we can calculate the absolute distance from the two-thirds average to 0 and J to see which is closer, and therefore who will receive the reward.

$$\left|0 - \frac{J}{90}\right| = \left|\frac{J}{90}\right|$$

$$\left|J - \frac{J}{90}\right| = \left|\frac{89J}{90}\right|$$

Further, it is clear that for all values of J,

$$\left|\frac{J}{90}\right| < \left|\frac{89J}{90}\right|$$

In other words the two-thirds average is closer to 0 then it is to J. Therefore Player 60 does benefit from deviating from the strategy profile and we conclude (0,0,0,....,0,0) is a Nash equilibrium.

Now consider the strategy profile of the class is (1,1,1,....,1,1). Here the two-thirds average is $\frac{2}{3}$. If Player 60 switches from 1 to 0, the two thirds average becomes $\frac{59}{90} > \frac{1}{2}$. Therefore Player 60 does not benefit from deviating from this strategy profile and we conclude (1,1,1,....,1,1) is a Nash equilibrium.

Now more generally, suppose K $\neq$ 0 and K $\neq$ 1, and the strategy profile of the class is (K,K,K,....,K,K). Without loss of generality, assume Player 60 chooses to deviate from this strategy and select a new number K-1. Now the average of the 60 numbers is $\frac{60K-1}{60}$ and the two-thirds average is $\frac{60K-1}{90}$.

Now we can calculate the absolute distance from the two-thirds average to K and K-1 to see which is closer, and therefore who will receive the reward.

$$\left|K - \frac{60K-1}{90}\right| = \left|\frac{30K+1}{90}\right|$$

$$\left|(K-1)-\frac{60K-1}{90}\right| = \left|\frac{30K-89}{90}\right|$$

Further, it is clear that for all values of K,

$$|30K+1| > |30K-89|$$

In other words the two-thirds average is now closer to K-1 then it is to K. Therefore Player 60 does benefit from deviating from the strategy profile and we conclude (K,K,K,....,K,K) is not a Nash equilibrium when K $\neq$ 1 and K $\neq$ 0.

Proving in general that (0,0,0,....,0,0) and (1,1,1,....,1,1) are the only Nash equilibrium for the game played in class is much more difficult and I will not attempt to do that formally. I will simply state that from 5.2 we see that whenever there is a strategy profile with two Players who pick different numbers either one or both of them will not win the game and therefore changing their strategy while fixing all other player's strategies can improve their outcome. In other words any strategy profile that contains 2 players who pick different numbers is not a Nash equilibrium. Therefore (0,0,0,....,0,0) and (1,1,1,....,1,1) are the only Nash equilibrium for the game played in class.

There are few reasons why the outcome in class was not a Nash equilibrium. The first being that the assumptions made by Nash equilibria, are too strong, in particular assuming that all agents are rational. The class clearly demonstrated non rational behaviour, this included one pair of students choosing the number 69 which is a weakly dominated strategy. Even if the class had acted completely rational and was well aware of what a Nash equilibrium was, the next problem in this game is that there are two Nash equilibria. Therefore without prior coordination the probability of everyone committing to the same equilibrium profile would have been incredibly low.

Problem 6. This problem can be expressed with a table, where each axis represents the agents (animals) actions and the inside entries express the payoffs for each strategy.

	Aggressive	Passive
Aggressive	(0,0)	(3,1)
Passive	(1,3)	(2,2)

We can find the equilibrium by underlying the best responses for Player 1 (Hawk), with Player 2's (Dove) strategies fixed, and similarly underlying the best responses for Player 2, with Player 1's strategies fixed

	Aggressive	Passive
Aggressive	(0,0)	$(\underline{3},\underline{1})$
Passive	$(\underline{1},\underline{3})$	(2,2)

Therefore we conclude that {Aggressive,Passive} and {Passive,Aggressive} are both Nash equilibria in this game.

Problem 7. We begin by finding the best response functions for Player 1 and Player 2 by calculating the derivatives of the players' payoff functions and setting to 0.

$$\frac{du_1}{da_1} = a_2 - 2a_1 = 0$$

$$\Rightarrow b_1(a_2) = \frac{a_2}{2}$$

As well,

$$\frac{du_2}{da_2} = 1 - 2a_2 - a_1 = 0$$

$$\Rightarrow b_2(a_1) = \frac{1 - a_1}{2}$$

Now we solve for (a_1^*, a_2^*) such that $a_1^* = b_1(a_2^*)$ and $a_2^* = b_2(a_1^*)$. We can create a system of equations:

$$\begin{cases} a_1^* = \frac{a_2^*}{2} \\ a_2^* = \frac{1-a_1^*}{2} \end{cases}$$

Solving this system we get $a_1^* = \frac{1}{5}$ and $a_2^* = \frac{2}{5}$, we conclude the Nash equilibrium occurs at $(a_1, a_2) = (\frac{1}{5}, \frac{2}{5})$.

Problem 8. Without loss of generality assume voter V has the following candidate preference, A over B over C. We will show that voting for C is V's only weakly dominated strategy.

Assume that V votes for C, and fix the voting decisions of all other voters. Now if V changes their vote from C to B, then either the results remain the same, or candidate A or B improve their result, either moving from a loss to a tie, a loss to win or a tie to a win. In other words changing their vote from C to B cannot have a positive effect on candidate C, and thus voting for B weakly dominates voting for C. The exact same argument holds for changing their vote from C to A. Therefore we conclude that C is a weakly dominated strategy.

The above argument shows that voting for B is not weakly dominated by voting for C. Now assume V votes for A, and fix the voting decisions of all other voters. Suppose that B and C are tied, V changing their vote from A to B will create a victory for B over C, which V prefers to a tie between B and C. Therefore there exists a scenario where voting for B is preferred to voting for A and thus voting for B is not weakly dominated by voting for A. Therefore voting for B is not a weakly dominated strategy.

It is clear that voting for A is not a weakly dominated strategy. We conclude that voting for C is the only weakly dominated strategy for V.

Assume that V votes for B and all other voters vote for B. Assuming there is more than one voter other than V (i.e. at least 3 people total) then this is a Nash equilibrium, we have V not voting for their favourite candidate and as shown above voting for B is not a weakly dominated strategy.