McGill University
Department of Mathematics and Statistics
MATH 243 Analysis 2, Winter 2016
Assignment 2

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 2 and 5.**

This assignment is due **Tuesday, January 24, at 1:30pm** in class. **Late assignments will not be accepted!**

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x + 2x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Show that f is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and differentiable at 0 with $f'(0) = 1$.

   (b) Prove that, nonetheless, f isn't increasing on any neighborhood of 0 i.e. show that f isn't increasing on $]-\delta, \delta[$ for any $\delta > 0$.
   Hint: Prove that for any $\delta > 0$ there exists an $x \in]-\delta, \delta[$, $x \neq 0$, such that $f'(x) < 0$. Then, using the fact that f' is continuous at x, prove that there exists an $\eta > 0$ such that f is decreasing on $]x - \eta, x + \eta[\subseteq]-\delta, \delta[$.

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) := \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   (a) Show that f is continuously differentiable.

   (b) Let $g(x) := 2x^4 + f(x)$. Show that g has an absolute minimum at 0 but that, nonetheless, there does not exist any $\delta > 0$ such that g is decreasing on $]-\delta, 0[$ and increasing on $]0, \delta[$.

3. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Prove that if $f'(x) \neq 0$ for all $x \in I$, then either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

4. Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Prove that f satisfies a Lipschitz condition on I if and only if f' is bounded on I (recall that a function $f : I \to \mathbb{R}$ is said to satisfy a Lipschitz condition on I if there exists a $K > 0$ such that $|f(x) - f(u)| \leq K|x - u|$ for all $x, u \in I$).

5. Let $f : [0, 2] \to \mathbb{R}$ be differentiable with $f(0) = 0$, $f(1) = 2$ and $f(2) = 1$.

   (a) Prove that there exists a $c_1 \in]0, 2[$ with $f'(c_1) = \frac{1}{2}$.

   (b) Prove that there exists a $c_2 \in]0, 2[$ with $f'(c_2) = -\frac{1}{2}$.