GUÍA 4

CYN

4.1) X V.A $\left\{\frac{k}{8} : k=0,1,\dots,8\right\}$ $P_X(x)=\frac{2}{9}x$

a) $f_Y(y) \,/\, Y=2X-1$

$$f_X(x)=P(X=x)=\begin{cases}\frac{2}{9}x & \text{si } x=\frac{k}{8}\ ,\ k=0,1,\dots,8\\ 0 & \text{en otro caso}\end{cases}$$

$$Y=g(X)=2X-1 \Rightarrow \frac{Y+1}{2}=X=g^{-1}(Y)$$

$$\boxed{g^{-1}(Y)=\frac{Y+1}{2}}$$

$$f_Y(y)=P(Y=y)=P(2X-1=y)=P\left(X=\frac{Y+1}{2}\right)=$$

$$f_X\left(\frac{Y+1}{2}\right)\Rightarrow f_Y(y)=\frac{2}{9}\left(\frac{Y+1}{2}\right)=\frac{Y+1}{9}$$

$$f_Y(y)=\begin{cases}\frac{Y+1}{9} & \text{si } y=\frac{2k}{8}-1 \text{ con } k\in(0,8)\\ 0 & \text{en otro caso}\end{cases}$$

b) $f_Y(y) \,/\, Y=128X^2$

$$Y=g(X)=128X^2 \Rightarrow \sqrt{\frac{Y}{128}}=X=g^{-1}(X)$$

$$f_Y(y)=P(Y=y)=P(128X^2=y)=P\left(X=\sqrt{\frac{Y}{128}}\right)=$$

$$f_X\left(\sqrt{\frac{Y}{128}}\right)=\frac{2}{9}\sqrt{\frac{Y}{128}}=\frac{\sqrt{2Y}}{72}$$

$$f_Y(y)=\begin{cases}\frac{\sqrt{2Y}}{72} & \text{si } y=128\left(\frac{k}{8}\right)^2 \text{ con } k\in(0,8)\\ 0 & \text{en otro caso}\end{cases}$$

c) $f_Y(y) \,/\, Y=-64X^2+64X+2$

$$Y=g(X)=-64X^2+64X+2 \Rightarrow \frac{1}{2}+\frac{\sqrt{18-Y}}{8}=g^{-1}(Y)$$

$$f_Y(y)=P(Y=y)=P\left(X=\frac{1}{2}+\frac{\sqrt{18-Y}}{8}\right)=f_X\left(\frac{1}{2}+\frac{\sqrt{18-Y}}{8}\right)$$

$$=\frac{2}{9}\left(\frac{1}{2}+\frac{\sqrt{18-Y}}{8}\right)=\frac{1}{9}+\frac{\sqrt{18-Y}}{36}$$

$$p_Y(y) = \begin{cases} \frac{1}{9} + \frac{\sqrt{18-y}}{36} & si\ y = 2, 9, 14, 17 \\ 0 & en\ otro\ caso \end{cases}$$

d) $p_Y(y) \ / \ Y = 64X^2 - 96X + 128$

$$64X^2 - 96X + (128 - Y) = 0$$

$$\frac{96 \pm \sqrt{9216 - 256(128-Y)}}{128} = \frac{3}{4} \pm \frac{\sqrt{256Y - 23552}}{128}$$

$$X = \frac{3}{4} + \frac{\sqrt{256Y - 23552}}{128} \qquad Y = 128, \frac{10156}{81}, \ldots$$

$$p_Y(y) = P(Y=y) = P\left(X = \frac{3}{4} + \frac{\sqrt{256y - 23552}}{128}\right) =$$

$$p_X\left(\frac{3}{4} + \frac{\sqrt{256y - 23552}}{128}\right) = \frac{2}{9}\left(\frac{3}{4} + \frac{\sqrt{256y - 23552}}{128}\right) =$$

$$\frac{1}{6} + \frac{\sqrt{256y - 23552}}{576}$$

$$p_Y(y) = \begin{cases} \frac{1}{6} + \frac{\sqrt{256y - 23552}}{576} & si\ y = 128, \frac{10156}{81}, \ldots \\ 0 & en\ otro\ caso \end{cases}$$

4.2) $X \sim P_o(2)$ $\qquad Y = \left|\operatorname{sen}\left(\frac{\pi X}{2}\right)\right|$

$$Y = \left|\operatorname{sen}\left(\frac{\pi X}{2}\right)\right| = \begin{cases} 0 & X = 2n \\ 1 & X = 2n+1 \end{cases}, \ n \in \mathbb{N}_0$$

$$p_Y(0) = P(Y=0) = P(X=2n) = \sum_{n=0}^{\infty} P(X=2n) =$$

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n)!} e^{-2} = e^{-2} \cdot ch(2)$$

$$p_Y(1) = P(Y=1) = P(X=2n+1) = \sum_{n=0}^{\infty} P(X=2n+1)$$

$$= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} e^{-2} = e^{-2} sh(2)$$

$$p_Y(y) = \begin{cases} e^{-2} ch(2) & si\ y = 0 \\ e^{-2} sh(2) & si\ y = 1 \end{cases}$$

4.3) $f_X(x) = \frac{12x}{\pi^2(e^x+1)} \mathbb{1}\{x>0\}$

a) $Y = aX+b$ $(a\neq 0, b\in\mathbb{R})$ $\quad f_Y(y)$

$F_Y(y) = P(Y\le y) = P(aX+b\le y) = P(aX \le y-b) =$
$P\left(X \le \frac{y-b}{a}\right)$

$f_Y(y) = f_X(x(y)) \left|\frac{dx(y)}{dy}\right| \qquad x = \frac{y-b}{a} \Rightarrow \frac{dx(y)}{dy} = \frac{a}{a^2} =$

$f_Y(y) = \frac{12\left(\frac{y-b}{a}\right)}{a\pi^2\left(e^{\left(\frac{y-b}{a}\right)}+1\right)} \mathbb{1}\{y>b\}$

b) $f_Y(y) \,/\, Y = -X^3 \qquad \sqrt[3]{-y} = x$

$f_Y(y) = f_X(x(y))\left|\frac{dx(y)}{dy}\right| \qquad \frac{dx(y)}{dy} = \frac{1}{3}(-y)^{-2/3}(-1)$

$\frac{dx(y)}{dy} = \frac{-1}{3\sqrt[3]{(-y)^2}}$

$\Rightarrow f_Y(y) = \frac{12\sqrt[3]{-y}}{\pi^2\left(e^{\sqrt[3]{-y}}+1\right)3\sqrt[3]{y^2}} \mathbb{1}\{y>0\}$

c) $f_Y(y) \,/\, Y = X + X^{-1}$

$Y = X + \frac{1}{X} = \frac{X^2+1}{X} \Rightarrow 0 = X^2+1-YX \begin{cases} X_1 = \frac{Y+\sqrt{Y^2-4}}{2} \\ X_2 = \frac{Y-\sqrt{Y^2-4}}{2} \end{cases}$

$\frac{dx(y)}{dy} = \frac{1}{2}+\frac{1}{4}(y^2-4)^{-1/2}\,2y = \frac{1}{2} + \frac{y}{2\sqrt{y^2-4}}$

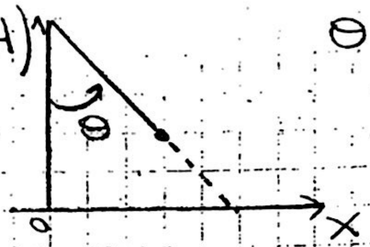
$f_Y(y) = f_X(x(y))\left|\frac{dx(y)}{dy}\right| = \frac{12\left(\frac{y+\sqrt{y^2-4}}{2}\right)}{\pi^2 e^{\left(\frac{y+\sqrt{y^2-4}}{2}\right)}}\left(\frac{1}{2}+\frac{y}{2\sqrt{y^2-4}}\right)$

d) $f_Y(y) \,/\, Y = X^2-3X$

$0 = X^2-3X-Y \Rightarrow X = \frac{3\pm\sqrt{9+4Y}}{2}$

$\frac{dx(y)}{dy} = \frac{d}{dy}\left(\frac{3+\sqrt{9+4y}}{2}\right) = \frac{1}{2}(9+4y)^{-1/2}\cdot 4 = \frac{2}{\sqrt{9+4y}}$

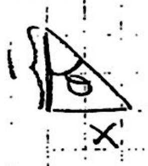
$f_Y(y) = \frac{24\left(\frac{3+\sqrt{9+4y}}{2}\right)}{\pi^2 e^{\frac{3+\sqrt{9+4y}}{2}}\sqrt{9+4y}}$

4.4) 

$\Theta \sim U(-\pi/2 ; \pi/2)$

$f_X(x) = f_\Theta(\theta(x)) \left| \frac{d\theta}{dx} \right|$

$f_\Theta(\theta) = \frac{1}{\pi} \mathbb{1}\{ -\frac{\pi}{2} < \theta < \frac{\pi}{2} \}$



$tg(\theta) = \frac{O}{A} = \frac{X}{1} \Rightarrow tg(\theta) = X \Rightarrow \Theta = arctg(X)$

$\frac{d\theta}{dx} = \frac{1}{1+x^2} \frac{dx}{dx} \Rightarrow \left| \frac{d\theta}{dx} \right| = \frac{1}{1+x^2}$

$\Rightarrow f_X(x) = \frac{1}{\pi(1+x^2)} \mathbb{1}\{ tg(-\pi/2) < x < tg(\pi/2) \}$

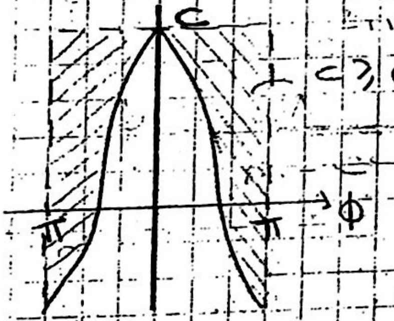
4.5) ϕ = "fase de un generador eléctrico"

$\phi \sim U(-\pi, \pi)$

a) $C = \cos(\phi) \Rightarrow \phi = arc\cos(C)$

$f_\phi(\varphi) = \frac{1}{2\pi} \mathbb{1}\{ -\pi < \varphi < \pi \}$

$F_\phi(\varphi) = \frac{\varphi + \pi}{2\pi} \mathbb{1}\{ -\pi < \varphi < \pi \} + \mathbb{1}\{ \varphi \geq \pi \}$



$F_C(c) = P(C \leq c) = P(\cos(\phi) \leq c) =$

$P(|\phi| \geq arc\cos(c)) = P(arc\cos(c) \leq \phi \leq$

$-arc\cos(c)) = F_\theta(-arc\cos(c)) - F_\theta(arc\cos(c)) =$

$\frac{-arc\cos(c) + \pi}{2\pi} - \frac{arc\cos(c) + \pi}{2\pi} = \frac{-arc\cos(c)}{\pi}$

$\Rightarrow F_C(c) = \frac{-arc\cos(c)}{\pi} \mathbb{1}\{ -1 < c < 1 \} + \mathbb{1}\{ c \geq 1 \}$

$f_C(c) = \frac{1}{\pi \sqrt{1-c^2}} \mathbb{1}\{ -1 < c < 1 \}$

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b) $P(|C| < 0,5) = P(-0,5 < C < 0,5) = F_C(0,5) - F_C(-0{,}5)$

$= -\dfrac{\pi}{3\pi} - \dfrac{-2\pi}{3\pi} = -\dfrac{1}{3} + \dfrac{2}{3} = \dfrac{1}{3}$

$\Rightarrow P(|C| < 0,5) = \dfrac{1}{3}$

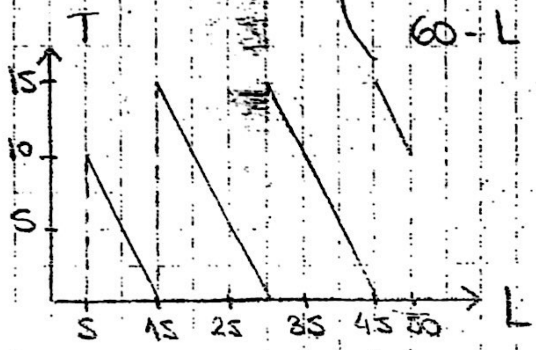
4.6) L = "horario de llegada de Lucas"

$L \sim U(5, 50)$ sube cada 15

T = "tiempo que espera Lucas"

$f_L(l) = \dfrac{1}{45} \mathbb{1}\{5 < l < 50\}$

$$g(t) = T = \begin{cases} 15 - L & \text{si } 7{:}05 \leq l \leq 7{:}15 \\ 30 - L & \text{si } 7{:}15 \leq l \leq 7{:}30 \\ 45 - L & \text{si } 7{:}30 < l \leq 7{:}45 \\ 60 - L & \text{si } 7{:}45 < l \leq 7{:}50 \end{cases}$$



$f_T(t) = f_L(l(t)) \left|\dfrac{dl(t)}{dt}\right|$

$T = 15 - L \Rightarrow L = 15 - T$

$\dfrac{dl(t)}{dt} = -1 \Rightarrow \left|\dfrac{dl(t)}{dt}\right| = 1$

(7:05 → 7:15) (7:15 → 7:30)

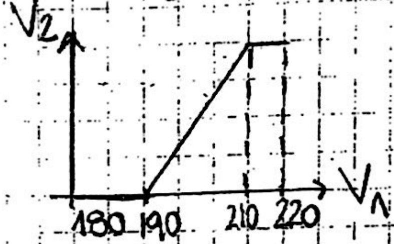
$f_T(t) = \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 10\} + \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 15\} + \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 15\} + \dfrac{1}{45} \mathbb{1}\{10 \leq T \leq 15\} = \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 10\} + \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 10\} \mathbb{1}\{10 \leq T \leq 15\} + \dfrac{1}{45} \mathbb{1}\{0 \leq T \leq 10\} \mathbb{1}\{10 \leq T \leq 15\}$

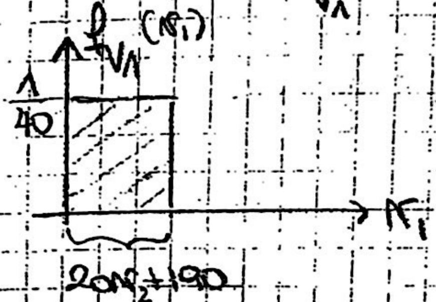
$f_T(t) = \dfrac{1}{15} \mathbb{1}\{0 \leq T \leq 10\} + \dfrac{1}{15} \mathbb{1}\{10 \leq T \leq 15\}$

4.7) $V_1 =$ " voltaje medido " $V_1 \sim \mathcal{U}(180, 220)$

$$g(v_1) = \frac{v_1 - 190}{20} \mathbb{1}\{190 \le v_1 \le 210\} + \mathbb{1}\{210 < v_1\}$$

$V_2 = g(V_1)$ $\quad F_{V_2}(v_2) = ?$ $\quad f_{V_1}(v_1) = \frac{1}{40} \mathbb{1}\{180 < v_1 < 220\}$





$$F_{V_2}(v_2) = P(V_2 \le v_2) = \begin{cases} 0 & \text{si } v_2 < 0 \\ 1 & \text{si } v_2 \geq 1 \\ *^1 & \text{si } 0 \le v_2 < 1 \end{cases}$$

$$P\left(\frac{V_1 - 190}{20} \le v_2\right) = P(V_1 \le 20v_2 + 190) = \int_{180}^{20v_2+190} \frac{1}{40}\, dv_1 =$$

$$\frac{20v_2 + 190 - 180}{40} = \frac{20v_2 + 10}{40}, \quad \text{si } v_2 = 0 \Rightarrow \frac{1}{4}$$

$$\Rightarrow F_{V_2}(v_2) = \begin{cases} 0 & \text{si } v_2 < 0 \\ \frac{1}{4} & \text{si } v_2 = 0 \\ \frac{v_2}{2} + \frac{1}{4} & \text{si } 0 \le v_2 < 1 \\ 1 & \text{si } v_2 \geq 1 \end{cases}$$

4.8) $X =$ " duración de una llamada telefónica "

$X \sim \exp(1/8)$ min

$Y =$ " pulsos por fracción "

$F_Y(y) = ?$

Y	X
1	$(0; 2]$
2	$(2; 4]$
3	$(4; 6]$
$\vdots$	$\vdots$
y	$((y-1)2; y2]$

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$P(Y=y) = P((y-1)2 \le X \le 2y) = F_X(2y) - F_X(2y-2) =$
$1 - e^{-\frac{2y}{8}} - (1 - e^{-\frac{2y+2}{8}}) = -e^{-\frac{y}{4}} + e^{-\frac{y+1}{4}} = (e^{-\frac{1}{4}})^{y+1}(1 - e$

$Y \sim Geo\left(1 - e^{-\frac{1}{4}}\right)$

$\Rightarrow p_Y(y) = (1-(1-e^{-1/4}))^{y-1}(1-e^{-1/4})$

$\Rightarrow p_Y(y) = (e^{-1/4})^{y-1}(1-e^{-1/4})$

4.9) a) $U = X \;,\; V = X+Y$

$(X,Y) = (-2,-2) \Rightarrow (U,V) = (-2,-4)$
$(-2,-1) \Rightarrow (-2,-3)$
$(-2,1) \Rightarrow (-2,-1)$
$(-2,2) \Rightarrow (-2,0)$
$(-1,-2) \Rightarrow (-1,-3)$
$(-1,-1) \Rightarrow (-1,-2)$
$(-1,1) \Rightarrow (-1,0)$
$(-1,2) \Rightarrow (-1,1)$

$(1,-2) \Rightarrow (1,-1)$
$(1,-1) \Rightarrow (1,0)$
$(1,1) \Rightarrow (1,2)$
$(1,2) \Rightarrow (1,3)$
$(2,-2) \Rightarrow (2,0)$
$(2,-1) \Rightarrow (2,1)$
$(2,1) \Rightarrow (2,3)$
$(2,2) \Rightarrow (2,4)$

V \ U	-2	-1	1	2
-4	1/16	0	0	0
-3	1/16	1/16	0	0
-2	0	1/16	0	0
-1	0	0	1/8	0
0	0	0	1/8	1/8
1	0	0	0	1/8
2	0	0	1/16	0
3	0	0	1/16	1/16
4	0	0	0	1/16

b) $U = \min(X,Y) \;,\; V = \max(X,Y)$

$(X,Y) = (-2,-2) \Rightarrow (U,V) = (-2,-2)$
$(-2,-1) = (-2,-1)$
$(-2,1) = (-2,1)$
$(-2,2) = (-2,2)$

$(-1,-2) = (-2,-1)$
$(-1,-1) = (-1,-1)$
$(-1,1) = (-1,1)$
$(-1,2) = (-1,2)$

$(1,-2) = (-2,1)$
$(1,-1) = (-1,1)$
$(1,1) = (1,1)$
$(1,2) = (1,2)$

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$(2;-2)=(-2;2)$
$(2;-1)=(-1;2)$
$(2;1)=(1;2)$
$(2;2)=(2;2)$

V \ U	-2	-1	1	2
-2	1/16	0	0	0
-1	1/16	1/16	0	0
1	1/8	3/16	1/16	0
2	1/8	1/8	2/16	1/16

c) $U = X^2+Y^2 \;;\; V = \frac{Y}{X}$

$(X,Y)=(-2;-2) \Rightarrow (U,V)=(8;1)$
$(-2;-1) \Rightarrow (5;1/2)$
$(-2;1) \Rightarrow (5;-1/2)$
$(-2;2) \Rightarrow (8;-1)$

$(-1;-2) \Rightarrow (5;2)$
$(-1;-1) \Rightarrow (2;1)$
$(-1;1) \Rightarrow (2;-1)$
$(-1;2) \Rightarrow (5;-2)$

$(1;-2) \Rightarrow (5;-2)$
$(1;-1) \Rightarrow (2;-1)$
$(1;1) \Rightarrow (2;1)$
$(1;2) \Rightarrow (5;2)$

$(2;-2) \Rightarrow (8;-1)$
$(2;-1) \Rightarrow (5;-1/2)$
$(2;1) \Rightarrow (5;1/2)$
$(2;2) \Rightarrow (8;1)$

V \ U	2	5	8
-2	0	1/8	0
-1	1/8	0	1/8
-1/2	0	1/8	0
1/2	0	1/8	0
1	1/8	0	1/8
2	0	1/8	0

4.10) $X,Y \,/\, f_{XY}(x,y)$

a) $(U,V) = A(X,Y)^t + B \,,\; A \in \mathbb{R}^{2\times 2}$ y $B \in \mathbb{R}^2$

$\therefore (X,Y) = A^{-1}(U,V) - B$

$\Rightarrow f_{UV}(u,v) = |\det(A)|\, f_{XY}(A\,u\,v)$

$$\Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \underbrace{A^{-1}\left[\begin{pmatrix} U \\ V \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right]}_{g^{-1}(u,v)} \qquad |J_{g(x,y)}| = |\det A|$$

$$f_{UV}(u,v) = f_{XY}\Big|_{g^{-1}(u,v)} \cdot \frac{1}{|\det A|} \Rightarrow f_{UV}(u,v) = \frac{f_{XY}\left(A^{-1}\left[\binom{u}{v} - B\right]\right)}{\det A}$$

1) $Z_1 \sim N(0;1)$ $\quad Z_2 \sim N(0;1)$ $\quad \overline{Z_1, Z_2 \text{ indep}}$

$$f_{Z_1Z_2}(z_1,z_2) \underset{\uparrow}{=} f_{Z_1}(z_1)\, f_{Z_2}(z_2)$$

Por ser indep

$$f_{Z_1Z_2}(z_1,z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} \quad 1\{(z_1,z_2)\in\mathbb{R}^2\}$$

a) $U = Z_1 + Z_2$, $V = Z_1 - Z_2$

$U \sim N(0;2)$ $\quad V \sim N(0;1)$

$U - Z_1 = Z_2$

$V = Z_1 - U + Z_1 \Rightarrow V = 2Z_1 - U$

$\frac{V+U}{2} = Z_1$

$U = \frac{V+U}{2} + Z_2$

$U - \frac{V+U}{2} = Z_2$

$$|J| = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 1-\frac{1}{2} & -\frac{1}{2} \end{vmatrix} =$$

$$g^{-1}(z_1,z_2) = \left(\frac{V+U}{2}, U - \frac{V+U}{2}\right)$$

$$-\frac{1}{4} - \left(\frac{1}{2} - \frac{1}{4}\right) = -\frac{1}{4} - \frac{1}{2} + \frac{1}{4} = -\frac{1}{2}$$

$$\Rightarrow f_{UV}(u,v) = f_{Z_1Z_2}(g^{-1}(z_1,z_2))\ \frac{1}{2}$$

$$\Rightarrow f_{UV}(u,v) = \frac{1}{4\pi} e^{-\frac{1}{2}\left(\left(\frac{v+u}{2}\right)^2 + \left(u - \frac{v+u}{2}\right)^2\right)}$$

$$f_{UV}(u,v) = f_U(u)\, f_V(v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{u^2}{4}} \ \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} =$$

2) $U = \cos(\theta) Z_1 - \operatorname{sen}(\theta) Z_2 = X_1 + X_2$; $X_1 \sim N(0, \cos^2(\theta))$

$V = \operatorname{sen}(\theta) Z_1 + \cos(\theta) Z_2$

$X_2 \sim N(0, \operatorname{sen}^2(\theta))$

$\Rightarrow U \sim N(0;1)$

$\Rightarrow V \sim N(0,1)$ $\quad f_{UV}(u,v) = \frac{1}{2\pi} e^{\frac{1}{2}(u^2+v^2)}\ 1\{(u,v)\in\mathbb{R}\}$

3) $U = Z_1^2 + Z_2^2$, $V = \frac{Z_2}{Z_1}$

$g(x,y) = \left(z_1^2 + z_2^2, \frac{z_2}{z_1}\right) = (u,v)$ $\quad G = \{(u,v) : u > 0\}$, $G_1 = \{(z_1,z_2) : z_1 > 0\}$ $\quad G_2 = \{(z_1,z_2) : z_1 < 0\}$

$P((Z_1,Z_2) \in G_1 \cup G_2) = 1$

$$J_1(u,v) = \left(\begin{vmatrix} 2z_1 & 2z_2 \\ -\frac{z_2}{z_1^2} & \frac{1}{z_1} \end{vmatrix}\right)^{-1} = \left(2 + \frac{2z_2^2}{z_1^2}\right)^{-1} = \frac{1}{2(v^2+1)}$$

$$J_2(u,v) = -\frac{1}{2(v^2+1)}$$

$$\Rightarrow f_{U,V}(u,v) = \left(f(h^{(1)}(u,v)) + f(h^{(2)}(u,v))\right) \frac{1}{2(v^2+1)}\ 1\{(u,v)\in$$

b) $(U,V) = \min(X,Y), \max(X,Y)$

$$(U,V) = \begin{cases} X,Y & \text{si } X \leq Y \\ Y,X & \text{si } Y \leq X \end{cases}$$

$g(x,y) = (\min\{x,y\}, \max\{x,y\})$ $<$ $G_1(X<Y) \Rightarrow g_1(x,y) = (x,y)$; $G_2(Y<X) \Rightarrow g_2(x,y) = (y,x)$

$g_1^{-1}(u,v) = (x,y)$ $\quad g_2^{-1}(u,v) = (y,x)$

$\text{Jac} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ $\quad \text{Jac} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$$\Rightarrow f_{UV}(u,v) = \left(f_{XY}(g_1^{-1}(u,v)) + f_{XY}(g_2^{-1}(u,v))\right) \mathbb{1}\{u \leq v\}$$

c) $(U,V) = (X^2+Y^2, Y/X)$

$g(x,y) = (x^2+y^2, y/x) = (u,v) \Rightarrow u = x^2+y^2 \Rightarrow u = x^2 + v^2x^2$

$v = \frac{y}{x} \Rightarrow vx = y$

$u = x^2(1+v^2) \Rightarrow |x| = \sqrt{\frac{u}{(1+v^2)}}$, $y = \pm v\sqrt{\frac{u}{(1+v^2)}}$

~~$G_1 = \{(x,y) \mid x \geq 0\} \Rightarrow g_1(x,y) = \{x^2+y^2, \frac{y}{x}\}$~~

$\Rightarrow g^{-1}(u,v) = \left\{\sqrt{\frac{u}{1+v^2}}, v\sqrt{\frac{u}{1+v^2}}\right\}$

como $f_{Z_1 Z_2}(z_1,z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} = \frac{1}{2\pi} e^{-\frac{u}{2}}$

$\Rightarrow f_{U,V}(u,v) = \cancel{2}\left(\frac{1}{2\pi} e^{-\frac{u}{2}}\right) \frac{1}{\cancel{2}(v^2+1)} \mathbb{1}\{u>0,\ v \in \mathbb{R}\} =$

$\frac{1}{2} e^{-\frac{u}{2}} \mathbb{1}\{u>0\} \frac{1}{\pi(v^2+1)} \mathbb{1}\{v \in \mathbb{R}\}$

$\Rightarrow U \sim \exp(1/2)$ $\quad V \sim \text{Cauchy}(0;1)$

b) U, V indep para 1), 2) y 3)

c) $P(Z_1^2 + Z_2^2 > 4) = P(U>4) = 1 - e^{-\frac{4}{2}} = 1 - e^{-2} \simeq 0{,}86$

$P(Z_2 > \sqrt{3} Z_1) = P\left(\frac{Z_2}{Z_1} > \sqrt{3}\right) = P(V > \sqrt{3}) =$

4.12) $X_1 \sim U(0;2)$; $X_2 \sim U(0;2)$

$U = \min(X_1, X_2)$ $\quad V = \max(X_1, X_2)$

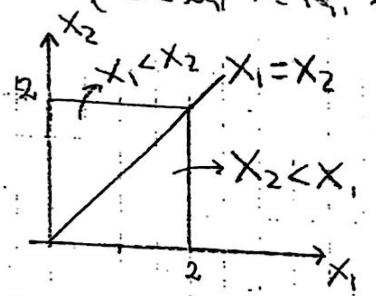
a) $f_{X_1 X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{2}\mathbb{1}\{0<x<2\} \frac{1}{2}\mathbb{1}\{0<x<2\} \Rightarrow$

$f_{X X_2}(x_1,x_2) = \frac{1}{4}\mathbb{1}\{(x,y) \in (0,2)\}$

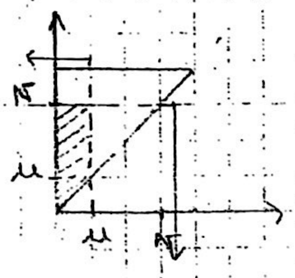
$(U,V) = \begin{cases} (X_1, X_2) & \text{si } X_1 < X_2 \\ (X_2, X_1) & \text{si } X_2 \le X_1 \end{cases}$

$F_{UV}(u,v) = P((U,V) \le (u,v)) = P((\min(X_1,X_2), \max(X_1,X_2)) \le (\min(x_1,x_2), \max \ldots$

$= P(U \le u, V \le v, X_1 < X_2) + P(U \le u, V \le v, X_2 \le X_1) = (1) + (2)$



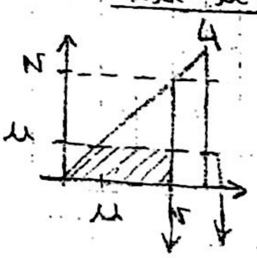
(1)



$u(v-u) = (vu - u^2) +$

$\frac{u^2}{2} = vu - \frac{u^2}{2}$

$\Rightarrow \frac{vu - u^2/2}{4} = \frac{1}{4} vu - \frac{1}{8} u^2$



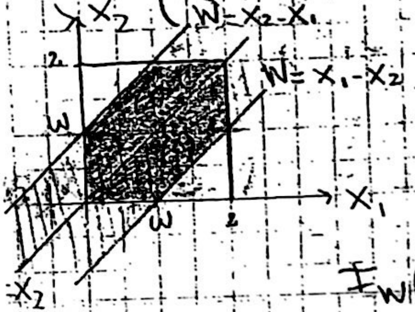
$\frac{u^2}{2} + u(v-u) = \frac{u^2}{2} + uv - u^2 = \frac{uv - u^2/2}{4}$

$\Rightarrow \frac{1}{4} vu - \frac{1}{8} u^2 + \frac{1}{4} vu - \frac{1}{8} u^2 = \frac{1}{2} vu - \frac{1}{4} u^2$

$$F_{UV}(u,v) = \begin{cases} 0 & si\ u<0,\ v<0 \\ \frac{1}{2}vu - \frac{1}{4}u^2 & si\ (u,v)\in(0,2) \\ 1 & si\ u>2,\ v>2 \end{cases}$$

b) $W = \max(X_1,X_2) - \min(X_1,X_2)$

$$W = \begin{cases} X_2 - X_1 & si\ X_2 > X_1 \\ X_1 - X_2 & si\ X_1 > X_2 \end{cases}$$

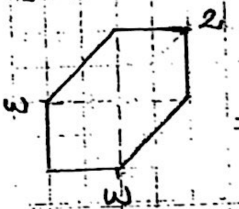


$F_W(w) = P(W \le w) = P(X_1 - X_2 \le w,\ X_1 > X_2) + P(X_2 - X_1 \le w,\ X_2 > X_1)$

$= (1) + (2)$

$F_W(w) = \text{Area} = w^2 + (2-w)^2 + \frac{w(2-w)}{2} + \frac{(2-w)w}{2} =$

$w^2 + 4 - 4w + \cancel{w^2} + 2w - \cancel{w^2} = w^2 - 2w + 4$



$$\Rightarrow F_W(w) = \begin{cases} 0 & si\ w<0 \\ w^2 - 2w + 4 & si\ 0 \le w \le 2 \\ 1 & si\ w > 2 \end{cases}$$

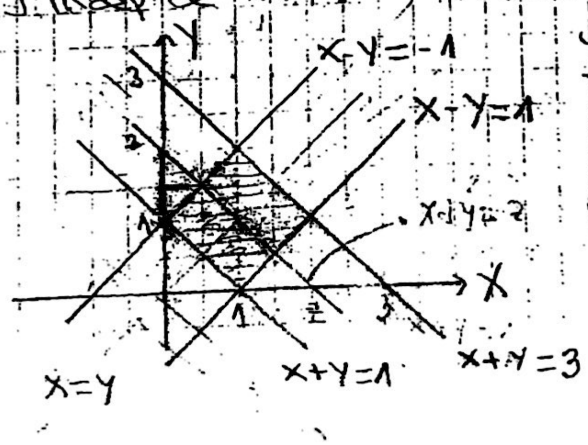
c) $P(U > 1/2,\ V < 3/2)$ y $P(V > 1 + U)$

$P(U > 1/2,\ V < 3/2) =$  $= \frac{1}{2}\cdot\frac{3}{2} - \left(\frac{1}{2}\cdot\frac{1}{2}\right) : 2 = \frac{3}{4} : \frac{1}{8} = \frac{5}{8}$

$P(V - U > 1) = P(X_2 - X_1 > 1,\ X_2 > X_1) + P(X_1 - X_2 > 1,\ X_1 > X_2) = \quad = \frac{1}{2} + \frac{1}{2} = 1$

4.13) $(X,Y) \sim U(\Delta)$ $\quad \Delta = \{(x,y) : 1 \le x+y \le 3,\ -1 \le x-y \le 1\}$, $J = 1\{X<Y\}$

J. indep de $X+Y$?



$f_{XY}(x,y) = \frac{1}{2} 1\{1 \le x+y \le 3,\ -1 \le x-y \le 1\}$

$$F_Z(z) = P(Z \le z) = P(Y + X \le z) = \begin{cases} 0 & si\ z<1 \\ \circledast & si\ 1 \le z \le \\ 1 & si\ 3<z \end{cases}$$

$\circledast$ $h^2 = 1^2 + 1^2 \Rightarrow h = \sqrt{2}$

$h^2 = \left(\frac{z-1}{2}\right)^2 + \left(\frac{z-1}{2}\right)^2 = \sqrt{2\left(\frac{z-1}{2}\right)}$

$\Rightarrow \frac{2\left(\frac{z-1}{2}\right)}{2} = \left(\frac{z-1}{2}\right)$

$F_Z(z) =$

$$F_Z(z) = \begin{cases} 0 & \text{si } z < 0 \\ \frac{z-1}{2} & \text{si } 0 \le z \le 3 \\ 1 & \text{si } z > 3 \end{cases}$$

$Z = X + Y \Rightarrow Z - X = Y$

$W = X - Y$

$W = X - (Z - X)$

$W = 2X - Z$

$W + Z = 2X$

$X = \frac{W+Z}{2}$

$W = \frac{W+Z}{2} - Y \Rightarrow Y = \frac{W}{2} + \frac{Z}{2} - W$

$Y = \frac{Z - W}{2}$

$|Jac| = \begin{vmatrix} \frac{dx}{dw} & \frac{dx}{dz} \\ \frac{dy}{dw} & \frac{dy}{dz} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$

$f_{W,Z}(w,z) = f_{XY}(x,y)\big|_{g^{-1}(w,z)} |Jac|$

$f_{W,Z}(w,z) = \frac{1}{4} \mathbb{1}\{1 \le z \le 3, -1 \le w \le 1\} = \frac{1}{4}\mathbb{1}\{1 \le z \le 3\} \frac{1}{4}\mathbb{1}\{-1 \le w \le 1\}$

Como $W = J$ cuando J vale $1 \Rightarrow J$ y $X+Y$ son independientes

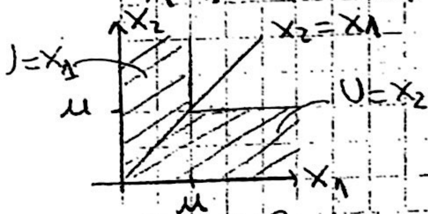
4.14) $X_1 \sim exp(\lambda_1)$; $X_2 \sim exp(\lambda_2)$

$U = min(X_1, X_2)$, $V = max(X_1, X_2)$, $W = V - U$, $J = \mathbb{1}\{U = X_1\} + 2\mathbb{1}\{U = X_2\}$

a) $f_U(u) = ?$

$F_U(u) = P(U \le u) = P(min(X_1, X_2) \le u) = P((X_1 \le u, X_1 < X_2) \cup (X_2 \le u, X_2 < X_1)) =$

$P(X_1 \le u, X_1 < X_2) + P(X_2 \le u, X_2 < X_1) = \begin{cases} 0 & \text{si } u < 0 \\ \iint f_{X_1X_2}(x_1,x_2)\,dx_1dx_2 & \text{si } 0 \le u \end{cases}$



$f_U(u) = (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)u} \mathbb{1}\{u > 0\}$

$\Rightarrow U = min(X_1, X_2) \sim exp(\lambda_1 + \lambda_2)$

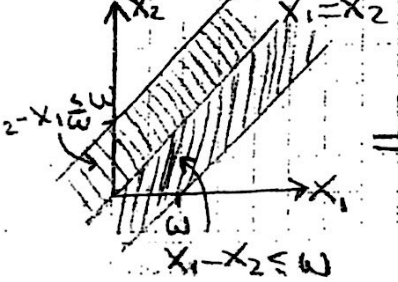
b) $J = \begin{cases} 1 & \text{si } X_1 < X_2 \\ 2 & \text{si } X_2 < X_1 \end{cases}$, J es discreta $\Rightarrow$ competencia

$P(J = 1) = P(X_1 < X_2) = \iint f_{X_1X_2}\,dx_1dx_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2}$

$P(J = 2) = P(X_2 < X_1) = \iint f_{X_1X_2}\,dx_1dx_2 = \frac{\lambda_2}{\lambda_2 + \lambda_1}$

c) $W = V - U = max(X_1, X_2) - min(X_1, X_2) \Rightarrow W = \begin{cases} X_1 - X_2 & \text{si } X_2 < X_1 \\ X_2 - X_1 & \text{si } X_1 < X_2 \end{cases}$

$F_W(w) = P(W \le w) = P(X_1 - X_2 \le w, X_1 > X_2) + P(X_2 - X_1 \le w, X_2 > X_1)$



$f_W(w) = \frac{\lambda_1}{\lambda_2 + \lambda_1} \lambda_2 e^{-\lambda_2 w} + \frac{\lambda_2}{\lambda_2 + \lambda_1} \lambda_1 e^{-\lambda_1 w} \mathbb{1}\{w$

$\Rightarrow f_W(w) = \frac{\lambda_1}{\lambda_2 + \lambda_1} f_{X_2}(w) + \frac{\lambda_2}{\lambda_2 + \lambda_1} f_{X_1}(w)$

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d) U y J indep → "el tiempo del ganador y el que gana"

$P(U \le u, J=1) = P(U \le u)\, p_J(1)$

$P(U \le u, J=2) = P(U \le u)\, p_J(2)$

e) U y W indep → "el tiempo del ganador y cuanto tiempo más después de que el otro gana"

$f_{U,W}(u,w) = f_{U,V}(u, u+w) \Rightarrow$ U y W son independientes

4.15) L_J "llamadas consecutivas de Juan" $L_J \sim \exp(5)$ x hora

L_P "llamadas consecutivas de Pedro" $L_P \sim \exp(10)$ x hora

Sin atender cuando $L_J < 5$ U $L_P < 5$ 4.14

a) $P(\min(L_J, L_P) < \frac{1}{12}h) = P(U < \frac{1}{12}h) = 1 - e^{-\frac{1}{12}(15)} = 1 - e^{-\frac{5}{4}} \approx 0{,}7135$

b) $P(L_J < L_P) = \frac{\lambda_J}{\lambda_P + \lambda_J} = \frac{5}{5+10} = \frac{1}{3} = P(J=1)$

c) $P(U < \frac{1}{12} | J=1) \underset{\substack{\uparrow \\ \text{U y J indep}}}{=} P(U < \frac{1}{12}) = 1 - e^{-\frac{5}{4}} \approx 0{,}7135$

d) $P(J=1 | U \ge \frac{1}{12}) = P(J=1) = \frac{1}{3}$

e) $W = \underbrace{\max\{L_J, L_P\}}_{V} - \underbrace{\min\{L_J, L_P\}}_{U} \Rightarrow P(W > \frac{1}{12}) = P(J=1) P(L_P > 1/12)$

$+ P(J=2) P(L_J < 1/12) = \frac{1}{3}(1 - F_{L_P}(1/12)) + \frac{2}{3}(1 - F_{L_J}(1/12)) =$

$\frac{1}{3}(\cancel{1} - (\cancel{1} - e^{-\frac{5}{6}})) + \frac{2}{3}(1 - (1 - e^{-5/12})) = +\frac{1}{3}e^{-\frac{5}{6}} + \frac{2}{3}e^{-5/12} \approx 0{,}5841$

f) $P(W < 1/12 | U \ge 1/12) \underset{\substack{\uparrow \\ \text{W y U indep}}}{=} P(W < 1/12) = 1 - \left(\frac{1}{3}e^{-\frac{5}{6}} + \frac{2}{3}e^{-\frac{5}{12}}\right) \approx$
0,4156

g) $P(V < 1/12) + P(U \ge 1/12, W < 1/12) = P(L_J < 1/12, L_P < 1/12)$
$+ P(U \ge 1/12) P(W < 1/12) = P(L_J < 1/12) \cdot P(L_P < 1/12) +$
$P(U \ge 1/12) P(W < 1/12) = (1 - e^{-5/12})(1 - e^{-\frac{10}{12}}) + (1 - e^{-\frac{1}{12}(15)}) \cdot$
$(1 - (\frac{1}{3}e^{-\frac{5}{6}} + \frac{2}{3}e^{-\frac{5}{12}})) \approx 0{,}1927 + 0{,}2966 \approx 0{,}4893$

4.16) $X \sim \exp(\lambda)$, $Y \sim \exp(\lambda)$

$U = X + Y$, $V = \frac{X}{X+Y}$ $\lambda > 0$

$g(x,y)$

$g(x,y) = (U,V) \qquad U = X+Y \qquad V = \frac{X}{X+Y}$

$X = U - Y \qquad V = \frac{U-Y}{U-Y+Y} = \frac{U-Y}{U} \rightarrow VU = U - Y$

$Y = U - VU$

$X = U - U + UV$

$\boxed{Y = U(1-V)}$

$\boxed{X = UV}$

$\Rightarrow g^{-1}(u,v) = (uv, u(1-v))$

$\Rightarrow f_{UV}(u,v) = f_{XY}(x,y)\Big|_{g^{-1}(u,v)} |Jac|$

$|Jac| = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -vu - (u(1-v)) = -vu - u + vu = -u$

$f_{XY}(x,y) = \lambda^2 e^{-\lambda(x+y)} \mathbb{1}\{x>0, y>0\}$

$\Rightarrow f_{UV}(u,v) = \lambda^2 e^{-\lambda(uv + u(1-v))} u \mathbb{1}\{uv>0, u(1-v)>0\}$

$f_{UV}(u,v) = u\lambda^2 e^{-\lambda(u)} \mathbb{1}\{uv>0, u(1-v)>0\}$

$uv > 0 \begin{cases} u>0 \wedge v>0 \\ u<0 \wedge v<0 \end{cases} \qquad u - uv > 0 \begin{cases} u>0 \wedge v<1 \\ u<0 \wedge v>1 \end{cases}$

$\Rightarrow f_{UV}(u,v) = u\lambda^2 e^{-\lambda u} \mathbb{1}\{u>0, 0<v<1\}$

$\Rightarrow f_{UV}(u,v) = u\lambda^2 e^{-\lambda u} \mathbb{1}\{u>0\} \mathbb{1}\{0<v<1\}$

$U \sim \Gamma(2,\lambda) \qquad V \sim \mathcal{U}(0,1)$

$f_U(u) = \lambda^2 u e^{-\lambda u} \mathbb{1}\{u>0\}$

$f_V(v) = \mathbb{1}\{0<v<1\}$

b) U y V V.A independientes

4.17) $(X \sim N(0;1)$; $Y \sim N(0;1))$

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} \mathbb{1}\{(x,y) \in \mathbb{R}^2\}$$

$X = \rho \cos(\theta)$ $\quad Y = \rho \,\mathrm{sen}(\theta)$

$$|Jac| = \begin{vmatrix} \frac{dx}{d\rho} & \frac{dx}{d\theta} \\ \frac{dy}{d\rho} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & \rho \\ \mathrm{sen}(\theta) & \rho \end{vmatrix} = \rho$$

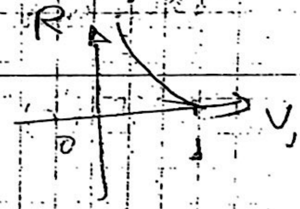
$$\Rightarrow f_{\rho\theta}(\rho,\theta) = f_{XY}(x,y)\Big|_{g^{-1}(\rho,\theta)} |Jac| = \frac{1}{2\pi} e^{-\frac{\rho^2}{2}} \cdot \rho \; \mathbb{1}\{\rho > 0, 0 (0$$

$$\Rightarrow f_{\rho\theta}(\rho,\theta) = \frac{\rho}{2\pi} e^{-\frac{\rho^2}{2}} \mathbb{1}\{\rho > 0, 0 \le \theta \le 2\pi\}$$

$$f_\rho(\rho) = \rho e^{-\frac{\rho^2}{2}} \mathbb{1}\{\rho > 0\} \; ; \quad f_\theta(\theta) = \frac{1}{2\pi} \mathbb{1}\{0 \le \theta \le 2\pi\}$$

4.18) $U_1, U_2 \sim \mathcal{U}(0,1)$ $\quad (Z_1, Z_2) = (R\cos(\theta), R\,\mathrm{sen}(\theta))$

$R = \sqrt{-2\log(U_1)}$, $\theta = 2\pi U_2$



a) $f_\theta(\theta) = \frac{1}{2\pi} \mathbb{1}\{0 < \theta < 2\pi\}$

$R^2 = -2\log(U_1) \Rightarrow u_1 = e^{-\frac{1}{2} r^2}$

$$f_R(r) = \frac{1}{\left| \frac{1}{\sqrt{-2\log(u_1)}} \cdot \frac{-1}{u} \right|}\Bigg|_{u = e^{-r^2/2}}$$

$$f_R(r) = r e^{-\frac{1}{2} r^2} \mathbb{1}\{r > 0\} \quad R \text{ y } \theta \text{ independientes}$$

b) $Z_1 = R\cos(\theta)$, $Z_2 = R\,\mathrm{sen}(\theta)$ $\Rightarrow R^2 = Z_1^2 + Z_2^2$

$$f_{Z_1 Z_2} = \frac{f_{R\theta}(r,\theta)}{|J|}\Bigg|_{g^{-1}}$$

$$f_{R\theta}(r,\theta) = \frac{1}{2\pi} r e^{-\frac{1}{2} r^2} \mathbb{1}\{r > 0, 0 < \theta < 2\pi\}$$

$|Jac| = r$

$$\Leftrightarrow f_{Z_1Z_2}(z_1,z_2) = \frac{1}{2\pi} r e^{-\frac{1}{2}(z_1^2+z_2^2)} \cdot \frac{1}{r} \mathbb{1}\{(z_1,z_2)\in\mathbb{R}$$

$$\Rightarrow f_{Z_1Z_2}(z_1,z_2) = \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} \mathbb{1}\{z_1,z_2\in\mathbb{R}$$

c) Z_1 y Z_2 son indep

$Z_1 \sim N(0,1)$; $Z_2 \sim N(0,1)$

4.19) X es uniforme sobre $\{1,2,3,\ldots,36\}$, Y es unif sobre $\{1,2,3$

X e Y indep

$W = X+Y$ $\quad P(W=w) = P(X+Y=w)$, $2,3,\ldots,40$

$P(X=1)=P(X=2)=\ldots=P(X=36)=\frac{1}{36}$

$P(Y=1)=P(Y=2)=P(Y=3)=P(Y=4)=\frac{1}{4}$

$P(W=w) = \begin{cases} P(W=2) = P(X=1,Y=1) = \frac{1}{144} \\ P(W=3)=P(X=1,Y=2)+P(X=2,Y=1) = \frac{2}{144} = \frac{1}{72} \\ P(W=4) = P(X=1,Y=3)+P(X=2,Y=2)+P(X=3,Y=1) = \\ \frac{3}{144} = \frac{1}{48} \\ P(W=5) = P(X=1,Y=4)+P(X=4,Y=1)+P(X=2,Y=3)+ \\ P(X=3,Y=2) = \frac{1}{36} \\ P(W=6) = P(X=3,Y=3)+P(X=2,Y=4)+P(X=4,Y=2)+ \\ P(X=5,Y=1) = \frac{1}{36} \\ P(W=7) = P(X=3,Y=4)+P(X=4,Y=3)+P(X=5,Y=2)+ \\ P(X=6,Y=1) = \frac{1}{36} \\ P(W=8) = P(X=4,Y=4)+P(X=5,Y=3)+P(X=6,Y=2) \\ +P(X=7,Y=1) = \frac{1}{36} \\ \vdots \\ P(W=n) = \frac{1}{36} \end{cases}$

4.20) L = "cantidad de langostos que arriban a ... de un arado" $L \sim Po(2)$

M = "cantidad de moscas que arriban a la mesa de un arado" $M \sim Po(8)$

L y M son independientes

a) $Z = L + M$ $\quad p_Z(z) = P(Z = z) = P(L+M = z) = P(L=0, M=z) + P(L=1, M=z-1) + \dots + P(L=z-1, M=1) + P(L=z, M=0) \overset{\text{indep}}{=}$

$P(L=0)P(M=z) + P(L=1)P(M=z-1) + \dots + P(L=z)\cdot P(M=0) =$

$$\frac{e^{-(2+8)}}{z!}\sum_{i=0}^{z} 2^i\, 8^{z-i}\binom{z}{i}$$

$$\Rightarrow p_Z(z) = \frac{e^{-10}}{z!}\sum_{i=0}^{z}\binom{z}{i}2^i 8^{z-i} \rightarrow \text{Binomio de Newton}$$

$$(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}a^i b^{n-i}$$

$\Rightarrow Z \sim Po(10)$

$$P(Z=z) = \frac{10^z}{z!}e^{-10}$$

b) $P(M \mid L+M=10) = P(M \mid M = 10-L) = P(M \cap \{M = 10-L\}) =$

$$\frac{P(M=m, M=10-l)}{P(M+L=10)} = \frac{P(M=m)P(M=10-l)}{P(M+L=10)} = \frac{\frac{8^m}{m!}e^{-8}\,\frac{8^{10-l}}{(10-l)!}e^{-8}}{\frac{10^{10}}{10!}e^{-10}}$$

$$= \frac{e^{-16}\,\frac{8^m\, 8^{10}}{8^l\, m!\,(10-l)!}}{e^{-10}\,\frac{10^{10}}{10!}} = \frac{10!\, e^{-6}\, 8^{10}\, 8^{m-l}}{10^{10}\,(m!)(10-l)!}$$

$$\Rightarrow P(M \mid L+M=10) = \left(\frac{4}{5}\right)^{10}\frac{10!\,e^{-6}\,8^{m-l}}{m!\,(10-l)!}$$

c) $P(M>2 \mid L+M=10) = \dfrac{P(M>2 \cap (L+M=10))}{P(M+L=10)} = 1 - [P(M \le 2 \mid L+M=10)]$

$= 1 - [P(M=0 \mid L+M=10) + P(M=1 \mid L+M=10) + P(M=2 \mid L+M=10)]$

$$= 1 - \left[\frac{P(M=0)P(L=10)}{P(M+L=10)} + \frac{P(M=1)P(L=9)}{P(M+L=10)} + \frac{P(M=2)P(L=8)}{P(M+L=10)}\right]$$

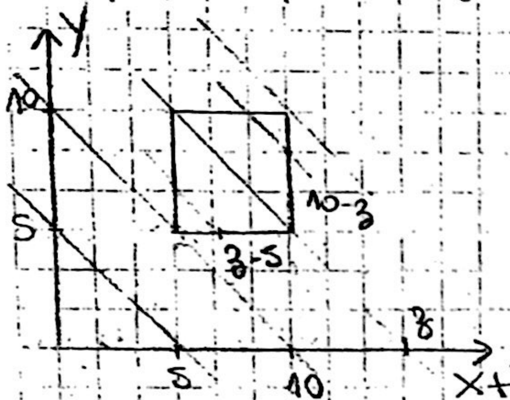
$$= 1 - \left[\frac{\cancel{e^{-8}}\, 2^{10}\,\cancel{e^{-2}}/10!}{\cancel{e^{-10}}\,\frac{10^{10}}{10!}} + \frac{8\cancel{e^{-8}}\cdot\cancel{e^{-2}}\,2^9/9!}{\cancel{e^{-10}}\,\frac{10^{10}}{10!}} + \frac{8^2\cancel{e^{-8}}/2!\;\cdot\cancel{e^{-2}}\,2^8/8!}{\cancel{e^{-10}}\,\frac{10^{10}}{10!}}\right]$$

$= 1 - \left[\left(\frac{1}{5}\right)^{10} + \frac{8 \cdot 2^{9} \, 10!}{9! \, 10^{10}} + \frac{32 \cdot 2^{8} \, 10!}{8! \, 10^{10}} \right] \simeq 0{,}9999$

4.21) X = "tiempo en que se produce el 1er proceso"

Y = "tiempo en que se produce el 2do proceso"

X, Y indep $\sim U(5,10)$



a) $Z = X + Y$

$f_Z(z) = ?$

$f_{XY}(x,y) = \frac{1}{25} \mathbb{1}\{(x,y) \in (5,10)\}$

$$F_Z(z) = P(Z \leq z) = P(X+Y \leq z) = \begin{cases} 0 & \text{si } z < 10 \\ \iint_{\triangle} & \text{si } 10 \leq z < 15 \\ \iint & \text{si } 15 \leq z < 20 \\ 1 & \text{si } z \geq 20 \end{cases}$$

$\square - \triangle$

$$\triangle = \frac{(b.h)/2}{25} = \frac{(z-10)^2}{50} = \frac{z^2}{50} - \frac{2}{5}z + 2$$

$$\triangle = \frac{(b.h)/2}{25} = \frac{(20-z)^2}{50} = \frac{z^2}{50} - \frac{4}{5}z + 8$$

$\Rightarrow \square - \triangle = 1 - \left(\frac{z^2}{50} - \frac{4}{5}z + 8\right)$

$$\Rightarrow F_Z(z) = \begin{cases} 0 & \text{si } z < 10 \\ \frac{z^2}{50} - \frac{2}{5}z + 2 & \text{si } 10 \leq z < 15 \\ -\frac{z^2}{50} + \frac{4}{5}z - 7 & \text{si } 15 \leq z < 20 \\ 1 & \text{si } z \geq 20 \end{cases}$$

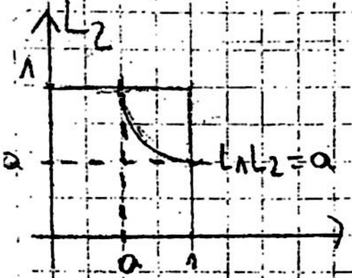
$f_Z(z) = \frac{z}{25} - \frac{2}{5} \mathbb{1}\{10 \leq z < 15\} + \frac{4}{5} - \frac{z}{25} \mathbb{1}\{15 \leq z < 20\} + \mathbb{1}\{z \geq 2$

b) $P(Z < 16) = F_Z(16) = -\frac{(16)^2}{50} + \frac{4 \times 16}{5} - 7 = \frac{17}{25} = 0{,}68$

4.44) L_i = ... $(0,1)$

$L_i \sim U(0,1)$ $\quad f_{L_1L_2}(l_1,l_2) = \mathbb{1}\{l_1 \in (0,1)\}\,\mathbb{1}\{l_2 \in (0,1)\}$

a) $A = L_1 L_2 \Rightarrow f_A(a)$? $\quad F_A(a) = P(A \leq a) = P(L_1L_2 \leq a)$



$$F_A(a) = \begin{cases} 0 & \text{si } a \leq 0 \\ 1 - \iint f_{L_1L_2}\, dl_2\, dl_1 & \text{si } 0<a<1 \\ 1 & \text{si } a \geq 1 \end{cases}$$

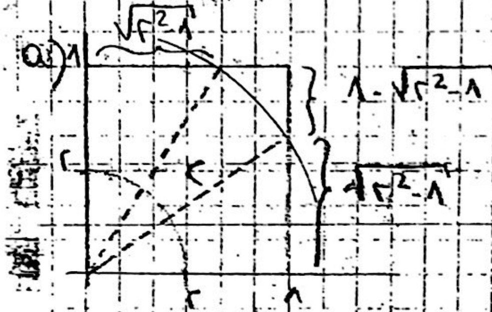
$$1 - \int_a^1 \int_{a/l_1}^1 1\, dl_2\, dl_1 = 1 - \int_a^1 (l_2)\Big|_{a/l_1}^1 dl_1 = 1 - \int_a^1 1 - \frac{a}{l_1}\, dl_1 = 1 - \left(l_1 - a\ln(l_1)\right)\Big|_a^1$$

$$= 1 - (1 - a\ln(1) - a + a\ln(a)) = a - a\ln(a) \Rightarrow F_A(a) = \begin{cases} 0 & \text{si } a \leq 0 \\ a(1-\ln(a)) & \text{si } 0<a<1 \\ 1 & \text{si } a \geq 1 \end{cases}$$

4.23) $X \sim U[0,1]$ $\quad Y \sim U[0,1]$ $\quad X$ e Y indep

$R = \sqrt{X^2+Y^2}$

a)



$F_R(r) = P(R \leq r) = P\left(\sqrt{X^2+Y^2} \leq r\right) =$
$P(X^2+Y^2 \leq r^2) = P\left(Y \leq \sqrt{r^2-X^2}\right)$

$$F_R(r) = \begin{cases} 0 & \text{si } r=0 \\ *^1 & \text{si } 0<r\leq 1 \\ *^2 & \text{si } 1<r\leq\sqrt{2} \\ 1 & \text{si } r\geq\sqrt{2} \end{cases}$$

$$*^1 = \frac{\pi r^2}{4}$$

$$*^2 = 2\sqrt{r^2-1} - r^2 + 1 + \frac{\pi}{4} - \frac{\pi}{4}\sqrt{r^2-1} = 1 + \frac{\pi}{4} + \left(2-\frac{\pi}{4}\right)\sqrt{r^2-1} - r^2$$

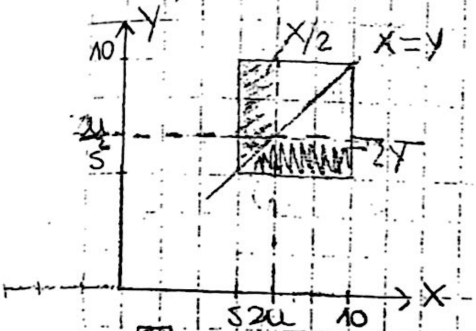
$$\Rightarrow F_R(r) = \begin{cases} 0 & \text{si } r=0 \\ \frac{\pi r^2}{4} & \text{si } 0<r\leq 1 \\ 1+\frac{\pi}{4} + \left(2-\frac{\pi}{4}\right)\sqrt{r^2-1} - r^2 & \text{si } 1<r\leq\sqrt{2} \\ 1 & \text{si } r\geq\sqrt{2} \end{cases}$$

b) $P(R > 1/2) = 1 - P(R \leq 1/2) = 1 - \dfrac{\pi(1/2)^2}{4} = 1 - \dfrac{\pi}{16} \approx 0.803$

$\frac{X}{2} \le u \Rightarrow u \in [2{,}5, 5]$

$2Y \le u \Rightarrow u \in [10, 20]$

4.24) $X, Y \sim U(5, 10)$



a) $U = \frac{X}{2} \mathbb{1}\{X \le Y\} + 2Y \mathbb{1}\{X > Y\}$

$F_U(u) = P(U \le u) = P\left(\frac{X}{2} \le u, X \le Y\right) + P(2Y \le u, X > Y) = \begin{cases} 0 & \text{si } u < 2{,}5 \\ *^1 & \text{si } 2{,}5 \le u < 5 \\ *^2 & \text{si } 5 \le u < 10 \\ *^3 & \text{si } 10 \le u < 20 \\ 1 & \text{si } u \ge 20 \end{cases}$

$*^1 = \frac{1}{2} - \frac{(10-2u)^2}{2} \cdot \frac{1}{25}$

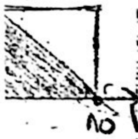
$*^3 = 1 - \frac{(10-u/2)^2}{2} \cdot \frac{1}{25}$

$F_U(u) = \begin{cases} 0 & \text{si } u < 2{,}5 \\ \frac{1}{2} - \frac{(10-2u)^2}{50} & \text{si } 2{,}5 \le u < 5 \\ \frac{1}{2} & \text{si } 5 \le u < 10 \\ 1 - \frac{(10-u/2)^2}{50} & \text{si } 10 \le u < 20 \\ 1 & \text{si } u \ge 20 \end{cases}$

$*^2 = \text{cte} = 1/2$

$(*^1)' = \frac{-2(10-2u)}{50}(-2)$

$(*^3)' = \frac{-2(10-u/2)}{50}\left(-\frac{1}{2}\right)$



$f_U(u) = \frac{20-4u}{25} \mathbb{1}\{2{,}5 \le u < 5\} + \frac{10-u/2}{50} \mathbb{1}\{10 \le u < 20\} + \mathbb{1}\{u \ge 20\}$

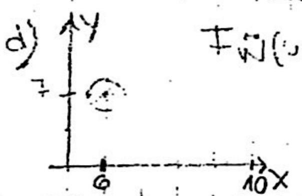
b) $V = \mathbb{1}\{X+Y \le 10\} \Rightarrow V = 0$ si $X+Y > 10$ y $V = 1$ si $X+Y \le 10$

$P(V=v) = \begin{cases} P(V=0) = P(X+Y > 10) = 1/2 \\ P(V=1) = P(X+Y \le 10) = 1/2 \end{cases}$

c)

$\tilde{W} = W \mid W \le 9/10$

d)



$F_{\tilde{W}}(w) = P(\ldots \le w \mid W \le 9/10) = P((X-5)\ldots)$

$= \frac{\pi w^2}{\pi (9/10)^2} \mathbb{1}\{w \le 9/10\}$

4.25) X, Y indep $\quad f_X(x) = 2x \mathbb{1}\{0 \le x \le 1\} \quad f_Y(y) = (2-2y)\mathbb{1}\{0 \le y \le 1\}$

$Z = X + Y \quad$ a) $F_Z(z) = P(Z \le z) = P(X + Y \le z)$

$F_Z(z) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{z-x} f_{XY}(x,y)\, dy \right) dx$

$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\, dx = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} 2x(2-2(z-x))$

$= \int_{-\infty}^{\infty} 4x - 4xz - 4x^2\, dx = \int_0^1 4x(1-z) - 4x^2\, dx = (2x^2(1-z) - 2x^2)$

$2(1-z)-1 = 2-2z-1 = 1-2z$

$\Rightarrow f_Z(z) = 1-2z \;\; \Delta \{ z$

b) $U \sim \mathcal{U}(0,1)$, $X = \sqrt{U}$, $Y = 1-\sqrt{U}$

$$f_X(x) = f_U(u(x)) \left| \frac{du(x)}{dx} \right| \qquad |U| = x^2 \left\langle \begin{array}{l} x^2 = U \\ -x^2 = U \end{array} \right.$$

$$\left| \frac{du(x)}{dx} \right| = 2x$$

$$f_X(x) = 2x \;\; \Delta \{$$

$$\sqrt{U} = 1-y \left\langle \begin{array}{l} U = (1-y)^2 \\ U = -(1-y)^2 \end{array} \right. \qquad \left| \frac{du(y)}{dy} \right| = \left| 2(1-y)\cdot(-1) \right| = 2(1-y)$$

$$f_Y(y) = 2(1-y) \;\; \Delta \{$$