Math 4109

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Preface

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Circles

Chapter 2

The Conic Sections

Throughout this chapter, the following variables will represent special values concerning conic sections.

Table 2.1: Special Values in Conic Sections

a	The length of the semi-major axis.[a]
b	The length of the semi-minor axis.[b]
h	The x-offset of the conic from the origin.
k	The y-offset of the conic from the origin.

[a]($\frac{1}{2}$ of the length of the major axis)
[b]($\frac{1}{2}$ of the length of the minor axis)

2.1 The Circle

Definiton 2.1.1 (The Standard Form for an Equation of the Circle)**.** $(x-h)^2+(y-k)^2=c^2$

In the circle, the length of the semi-major axis (and by extension the semi-minor axis) equivalent to as the radius. Thus, this form is also known as the *Center Radius Form.* In this equation the coordinates (h,k) represent the center of the circle and c represents the radius. In light of this, a circle can be considered a special case of the Ellipse.

Definiton 2.1.2 (The Expanded Form of an Equation for the Circle)**.**

$$ax^2+by^2+cx+dy+e=0$$
$$\text{where } a=b \text{ and } \operatorname{sign} a=\operatorname{sign} b=\operatorname{sign} c$$

Note that this form of the equation can be manipulated back into the center radius form through completing the square.

The steps below demonstrate conversion from Expanded to Center Radius Form:

$$ax^2 + by^2 + cx + dy + e = 0 \tag{2.1a}$$
$$ax^2 + ay^2 + cx + dy + e = 0 \tag{2.1b}$$
$$ax^2 + ay^2 + cx + dy = -e \tag{2.1c}$$
$$ax^2 + ay^2 + cx + dy = -e \tag{2.1d}$$
$$x^2 + y^2 + \frac{c}{a}x + \frac{d}{a}y = -\frac{e}{a} \tag{2.1e}$$
$$\left(x^2 + \frac{c}{a}x + \left(\frac{c}{2a}\right)^2\right) + \left(y^2 + \frac{d}{a}x + \left(\frac{d}{2a}\right)^2\right) = -\frac{e}{a} + \left(\frac{c}{2a}\right)^2 + \left(\frac{d}{2a}\right)^2 \tag{2.1f}$$
$$\left(x + \frac{c}{2a}\right)^2 + \left(y + \frac{d}{2a}\right)^2 = \left(-\frac{e}{a} + \left(\frac{c}{2a}\right)^2 + \left(\frac{d}{2a}\right)^2\right) \tag{2.1g}$$

Here is a concrete example of the above process:

$$4x^2 + 4y^2 - 16x - 24y + 51 = 0 \tag{2.2a}$$
$$4x^2 + 4y^2 - 16x - 24y = -51 \tag{2.2b}$$
$$x^2 + y^2 - 4x - 6y = -\frac{51}{4} \tag{2.2c}$$
$$(x^2 - 4x + 4) + (y^2 - 6y + 9) = -\frac{51}{4} + 4 + 9 \tag{2.2d}$$
$$(x-2)^2 + (y-3)^2 = \frac{1}{4} \tag{2.2e}$$

2.2 The Ellipse

Definiton 2.2.1 (The Standard Form for an Equation of the Ellipse)**.** $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

2.3 The Parabola

Definiton 2.3.1 (The Standard Form for an Equation of the Parabola)**.** $y = 4ax$

2.4 The Hyperbola

Definiton 2.4.1 (The Standard Form for an Equation of the Ellipse)**.** $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

Chapter 3

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Chapter 5

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Chapter 6

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Appendix A

The Laws of the Euclidean Geometries

Acknowledgements Thanks to Cadence Weddle, Leona Liu, and Zoya Yan.

Note Please note that all angles are in degrees and all symbolic values are within the set $\{x \in \mathbb{R} \mid x \geq 0\}$.

Primitive Notions

1. Point
2. Line
3. Plane

Definition of Basic Objects

1. A segment is the set of all points between two distinct points (called the endpoints)
2. A ray is a segment together with the set of all points beyond one of the endpoints
3. Opposite rays are rays which lie on the same line and whose only point of intersection is their common endpoint
4. Collinear points are points which lie on the same line

Definitions

1. Property of Betweenness: the whole equals the sum of its parts
2. Given collinear points C, D, and E, if D is a point between C and E, then $CD + DE = CE$.
3. Midpoint: divides segment into 2 equal parts in half
4. Segment bisector: meets a segment at its midpoint
5. Angle Addition Postulate: the whole equals the sum of its parts
6. Angle bisector: divides an angle into 2 equal parts in half

7. Complementary angle: 2 angles whose sum is 90 degrees
8. Supplementary angle: 2 angles whose sum is 180 degrees
9. Adjacent angles: 2 angles that satisfy the following
   (a) Common vertex
   (b) Common side
   (c) Dont share common interior points
10. Vertical angles: 2 non-adjacent angles formed by the intersection of 2 lines
11. The distance between a point and a line (or subset of a line) is the length of the segment perpendicular to the line with the point as an endpoint

Properties

1. All vertical angles are equal
2. Substitution: $\forall a, b, c, d \in X : (a = b \wedge a = c \wedge b = d) \Rightarrow (c = d)$
3. Transitive: $\forall a, b, c \in X : (aRb \wedge bRc) \Rightarrow aRc$ where R is some equality operator.
4. Reflexive Property
5. Addition Property of Equality
6. Subtraction Property of Equality
7. Division Property of Equality: halves of equals are equal
8. Multiplication Property of Equality

Perpendicularity and Right Angles

1. If 2 lines meet to form 2 equal adjacent angles, then the lines are perpendicular
2. Perpendicular lines meet to form right angles
3. All right angles are equal

Complements and Supplements

1. If the exterior sides of 2 adjacent angles are opposite rays, then the angles are supplementary
2. If the exterior sides of 2 adjacent angles are perpendicular, then the angles are complementary
3. If 2 angles are complementary to the same angle, then they are equal to each other
4. If 2 angles are complementary to equal angles, then they are equal to each other
5. If 2 angles are supplementary to the same angle, then are equal to each other
6. If 2 angles are supplementary to equal angles, then are equal to each other

Triangle Congruence

1. SSS (side-side-side)
2. SAS (side-angle-side)
3. ASA (angle-side-angle)
4. AAS (angle-angle-side)
5. RHL (right-hypotenuse-leg)
6. CPCTE: Corresponding Parts of Congruent Triangles are Equal

Triangles

1. Median: a segment drawn from the vertex of a triangle to the midpoint of the opposite side
2. Altitude: a segment drawn from a vertex of a triangle perpendicular to the opposite side
3. The sum of the interior angles of a triangle is 180 degrees

Isosceles Triangles

1. If a triangle has 2 equal sides, then it is isosceles
2. In a triangle, if 2 sides are equal, then the 2 angles opposite them are equal
3. In a triangle, if 2 angles are equal then the 2 sides opposite them are equal
4. An exterior angle of a triangle equals the sum of its remote interior angles
5. In an isosceles triangle, the median is an altitude

Inequality

1. A whole is greater than either one of its parts
2. In a triangle, the greater angle lies opposite the greater side
3. Transitive property of inequality: $\forall a, b, c \in X : (aRb \wedge bRc) \Rightarrow aRc$ where R is some inequality operator.
4. An exterior angle of a triangle is greater than either of its remote angles

Parallel Lines and Transversals

1. Parallel lines: 2 coplanar lines which never intersect

2. Transversal: a line that intersects 2 other lines in 2 distinct locations

3. Alternate interior angles: non-adjacent angles in the interior of a transversal on opposite sides

4. Same-side interior angles: non-adjacent angles in the interior of a transversal on the same side

5. Corresponding angles definition: one interior and one exterior non-adjacent angle on the same side of the transversal

6. If alternate interior angles are equal, then the lines are parallel

7. If corresponding angles are equal, then the lines are parallel

8. If same side interior angles are supplementary, then the lines are parallel

9. If 2 lines are perpendicular to the same line, then they are parallel

10. Parallel Postulate: Through a point not on a given line, there exists one and only one line parallel to the given line

11. If lines are parallel, then alternate interior angles are equal

12. If lines are parallel, then corresponding angles are equal

13. If lines are parallel, then same side interior angles are supplementary

14. If 2 lines are parallel to the same line, then they are parallel to each other

15. If 3 or more parallel lines cut off equal segments on a transversal, then they do so on any transversal

Midline/Midsegment

1. A midline is a segment which connects the midpoints of 2 sides of a triangle

2. A midline is parallel to the third side

3. A midline is half the length of the third side

4. If a line is drawn from the midpoint of a side of a triangle and is parallel to another side, then it intersects the 3rd side of the triangle at its midpoint

Parallelogram Properties

1. A diagonal is a segment that is formed by connecting 2 non-connecting two nonconsecutive vertical angles
2. A parallelogram is a quadrilateral where both pairs of opposite sides are parallel
3. If a quadrilateral is a parallelogram, then opposite sides are equal
4. If a quadrilateral is a parallelogram, then opposite angles are equal
5. If a quadrilateral is a parallelogram the diagonals bisect each other

Parallelogram Converses

1. If both pairs of opposite sides are parallel, then a quadrilateral is a parallelogram
2. If both pairs of opposite sides are equal, then a quadrilateral is a parallelogram
3. If both pairs of opposite angles are equal, then a quadrilateral is a parallelogram
4. If diagonals bisect each other, then a quadrilateral is a parallelogram
5. If the same pair of opposite sides is both parallel and equal, then a quadrilateral is a parallelogram

Rectangle Properties

1. # Inherit from Parallelogram
2. Rectangle: parallelogram with a right angle
3. A rectangle has all right angles
4. A rectangle is equiangular
5. A rectangle has equal diagonals
6. Rectangle Converses
7. If a parallelogram has a right angle, then it is a rectangle
8. If a quadrilateral is equiangular, then it is a rectangle
9. If a parallelogram has equal diagonals, then it is a rectangle

Rhombus Properties

1. #Inherit from Parallelogram
2. A rhombus is a parallelogram with 2 equal consecutive sides
3. A rhombus is equilateral
4. A rhombus has diagonals that bisect the angles
5. A rhombus has perpendicular diagonals
6. Rhombus Converses
7. A parallelogram with 2 equal consecutive angles is a rhombus
8. If a quadrilateral is equilateral, then it is a rhombus
9. If the diagonals of a parallelogram bisect an angle, then it is a rhombus
10. If the diagonals of a parallelogram are perpendicular, then it is a rhombus

Squares

1. #Inherit from Parallelogram
2. #Inherit from Rhombus
3. #Inherit from Rectangle
4. Square: parallelogram with 2 equal consecutive sides and a right angle
5. A square is a rhombus with a right angle
6. A square is a rectangle with two equal consecutive sides

Trapezoids

1. A trapezoid is quadrilateral with exactly one pair of opposite sides parallel
2. An isosceles trapezoid is a trapezoid with non-parallel sides that are equal
3. An isosceles trapezoid has base angles that are equal
4. An isosceles triangle has diagonals that are equal
5. Median of a trapezoid: segment which connects the midpoints of 2 non-parallel sides
6. A median of a trapezoid is parallel to both bases of the trapezoid
7. The measure of the median of a trapezoid is equal to the arithmetic mean of the measures of the bases

Interior and Exterior Angles of Polygons

1. The sum of the interior angles of a simple convex n-gon is $180 * (n - 2)$ degrees

2. The sum of the exterior angles of a simple convex n-gon is always 360 degrees

Properties of Proportion

1. Cross Multiplication (Means Extremes Rule) $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$

2. Alternation $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$

3. Inversion $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{b}{a} = \frac{d}{c}$

4. Addition Property of Equality $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$

Similarity

1. Similar polygons have corresponding angles that are equal

2. Similar polygons have corresponding sides in proportion

3. AA(A): Angle Angle Similarity

4. CASTE: Corresponding Angles of Similar Triangles are Equal

5. CSSTP: Corresponding Sides of Similar Triangles are Proportional

Right Triangles and Trigonometry

1. In a right triangle, the midpoint of the hypotenuse is equidistant from all of the vertices

Triangle Proportionality

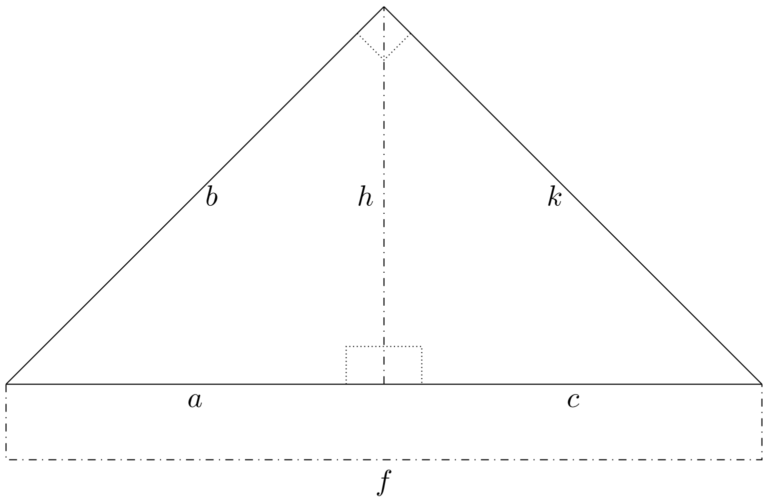
1. Triangle Proportionality Theorem: If a line parallel to one side of a triangle intersects the other two sides, then it divides them proportionally

2. Converse of the Triangle Proportionality Theorem: If a line divides two sides of a triangle proportionally, then it is parallel to the third side

3. The bisector of an angle of a triangle divides the sides opposite the angle in the ratio of the lengths of the other two sides of the triangle

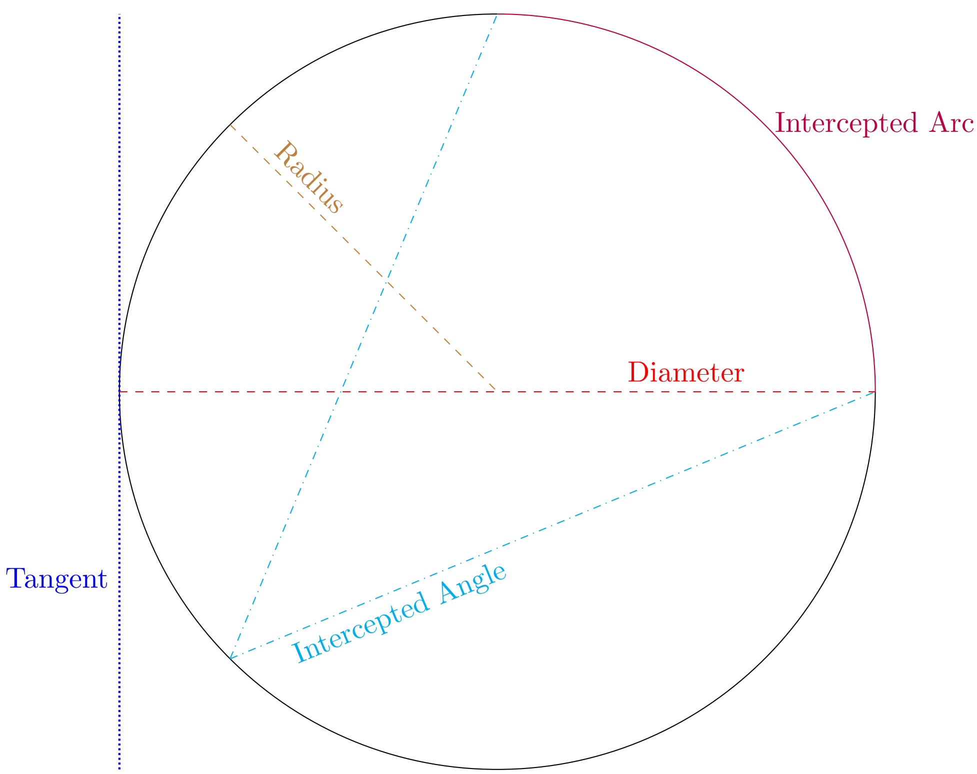
4. Three (or more) parallel lines cut off proportional segments on any two transversals

Proportions in a Right Triangle Divided by an Altitude



1. $\frac{c}{k} = \frac{k}{f}$
2. $\frac{a}{b} = \frac{b}{f}$
3. $\frac{a}{h} = \frac{h}{c}$

Circle Definitions



1. Circle: The set of all points in a plane a given distance from a given point (the center of the circle)

2. Radius: Any segment whose endpoints are the center and a point on the circle
3. The length of any such segment is called the radius of the circle
4. Chord: any segment whose endpoints are two points of a circle
5. Diameter: A chord that contains the center of the circle
6. The length of any such segment is called the diameter of the circle
7. Secant: Any line (or ray or segment) that has as a subset a chord of the circle
8. Tangent: Any line in the plane of the circle that intersects the circle in exactly one point, called the point of tangency
9. Concentric circles: Circles in the same plane with the same center but different radii
10. Congruent circles: Circles with congruent radii
11. Central angle: An angle whose vertex is the center of the circle
12. Inscribed angle: An angle whose vertex lies on the circle and whose sides have as subsets chords of the circle
13. Inscribed polygon: A polygon all of whose vertices lie on the same circle
14. Quadrilateral ABCD is inscribed in circle O Circle O is circumscribed about quadrilateral ABCD
15. Circumscribed polygon: A polygon all of whose sides are tangent to the same circle
16. If R and Q are endpoints of a diameter of a circle, then R, Q, and either half of the circle with endpoints R and Q is a semicircle
17. If P and Q are not endpoints of a diameter of O, then the arc consisting of P, Q, and all points in the interior of $\sphericalangle POQ$ is a minor arc of O, denoted ab
18. If P and Q are not endpoints of a diameter of O, then the arc consisting of P, Q, and all the points on the circle in the exterior of $\sphericalangle POQ$ is a major arc
19. To name a semicircle or major arc, you must use three letters
20. An angle intercepts an arc if each side of the angle contains an endpoint of the arc
21. All other points of the arc lie in the interior of the angle

Properties of Circle Components

1. Radii of the same circle are equal
2. The measure of a minor arc is equal to the measure of the central angle that intercepts it
3. The measure of a semicircle is 180°
4. The measure of an entire circle is 360°

5. The measure of a major arc is the difference between 360° and the measure of the corresponding minor arc
6. Two arcs of the same circle or congruent circles are congruent if they have the same measure
7. An angle inscribed in a semicircle is a right angle
8. The measure of an inscribed angle is half the measure of its intercepted arc
9. If a quadrilateral is inscribed within a circle, then angles opposite each other are supplementary
10. If 2 arcs in a circle are equal, then the inscribed angles which intercept them are equal
11. If 2 angles of the same type intercept the same arc, then the two angles are equal
12. If 2 inscribed angles formed by 2 arcs are equal, then the 2 arcs are equal
13. A tangent to a circle is perpendicular to the radius/diameter drawn to the point of tangency
14. A tangent-chord angle measures half its intercepted arc
15. Angles Formed by Intersections of Chords, Tangents, and Secants
16. An angle formed by two chords measures half the sum of its intercepted arcs
17. An angle formed by two secants measures half the difference of its intercepted arcs
18. An angle formed by a secant and a tangent measures half the difference of its intercepted arcs
19. An angle formed by 2 tangents measures half the difference of its intercepted arcs
20. An angle formed by 2 tangents measures 180° minus the measure of its intercepted minor arc

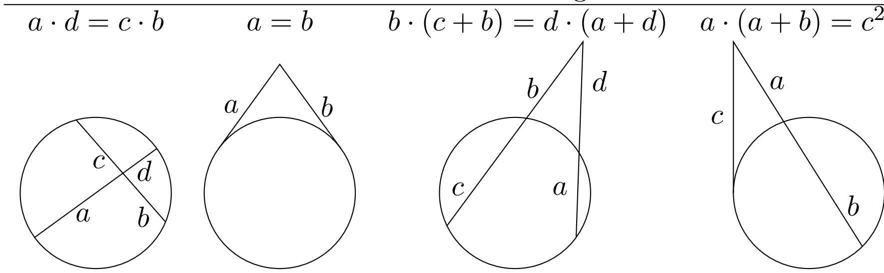
Chord Theorems

1. If 2 chords are parallel, then they cut off equal arcs between them
2. If 2 chords are equal, then the arcs they cut off are equal
3. If a radius is drawn perpendicular to a chord, then it bisects the chord (and arc it intersects)
4. If 2 chords are equal, then they are equidistant from the center of the circle

Segment Length

Table A.1: An Illustration of the Various Segments related to a Circle

$a \cdot d = c \cdot b$	$a = b$	$b \cdot (c+b) = d \cdot (a+d)$	$a \cdot (a+b) = c^2$



1. If two chords intersect, then the product of segments of one chord is equal to the product of segments of the other
2. If 2 tangents are drawn to a circle from the same external point, then the tangents are equal
3. If two secants are drawn to a circle from the same external point, then the product of the secant and its external segment equals the product of the other secant and its external segment
4. If a tangent and secant are drawn to a circle from the same external point, then the tangent is the geometric mean between the external segment of the secant and the secant

Cartesian Geometry

1. Midpoint Formula: The coordinate of the midpoint of two points is equal to the element wise arithmetic means of the coordinates of the initial two points.
2. Slope Formula for lines: The slope of a line can be computed as the difference between the ordinates divided by the difference in the abscissae.
3. The Euclidean distance between two points in two dimensions is the positive root of the sum of squares for the abscissae and the ordinates.
4. If the product of the slopes of two lines is equal to -1 then the two lines are perpendicular
5. If the slope of a line is the negative reciprocal of the slope of another line, then the two lines are perpendicular.
6. If the slope of two lines are equal, then the lines are parallel

Appendix B

Creating Graphics