Lecture Notes by Jonathan Alcaraz (UCR)

Topology

Math 205A
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Based on Lectures by

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Introduction

This course is about the abstraction of continuity.

A subset $O \subseteq \mathbb{R}$ is said to be *open* if for every point $x \in O$, there is a value $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq O$.

This idea of openness in Euclidean space can be used to define continuity of functions on Euclidean space.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous* if for any open $O \subseteq \mathbb{R}$, $f^{-1}(O)$ is open in $\mathbb{R}^n$.

One could prove that this definition is equivalet to the standard ε-δ definition of continuity on Euclidean space.

Point-Set Topology

Definition 1.1 Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a *topology* on X if

(i) $\emptyset$ and X are in $\mathcal{T}$;

(ii) Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$;

(iii) The intersection of finitely many elements of $\mathcal{T}$ is in $\mathcal{T}$.

A set together with a topology $\mathcal{T}$ is called a *topological space* denoted $(X, \mathcal{T})$ (or just X if the topology is understood). Elements of $\mathcal{T}$ are said to be *open sets in X*.

Example

- The *discrete topology* on a set is simply the collection of all subsets.
- The *indisccrete topology* on a set X is the most trivial topolgy, $\{\emptyset, X\}$.

Definition 1.2 If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on the same set with the property $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is said to be *finer* than $\mathcal{T}_1$. If neither $\mathcal{T}_1 \subseteq \mathcal{T}_2$ nor $\mathcal{T}_2 \subseteq \mathcal{T}_1$, these topologies are said to be *incomparable*.

Definition 1.3 A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called *continuous* if $f^{-1}(V) \in \mathcal{T}$ for every $V \in \mathcal{T}'$, that is, the preimage of open sets are open. A *homeomorphism* is a bijective continuous funcction whose inverse is continuous.

The Fundamental Question of Topology Given two topological spaces, determine if they are homeomorphic.

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Definition 2.1 Let X be a set. A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that

(i) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \bigcap B_2$, then there is some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \bigcap B_2$.

Definition 2.2 Given a basis $\mathcal{B}$ on a set X, the *topology generated by* $\mathcal{B}$ is defined by: U is open if and only if for every $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Note It is left as an exercise to confirm that sets with this property do indeed form a topology on X.

Example

(i) The open intervals of $\mathbb{R}$ generate the standard topology on $\mathbb{R}$.

(ii) Open balls in $\mathbb{R}^n$.

Lemma 2.3 The topology generated by a basis $\mathcal{B}$ is precisely the collection of unions of sets in $\mathcal{B}$.

Lemma 2.4 The topology $\mathcal{T}'$ is finer than the topology $\mathcal{T}$ if and only if for every $x \in X$ and $B \in \mathcal{B}$ with $x \in B$, there is a set $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Lemma 2.5 Suppose $\mathcal{C}$ is a collection of open sets of X such that for every $x \in X$ and neighborhood U of x, there is a set $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is a basis of X that generates the same topology of X.

Definition 2.6 A collection $\mathcal{S}$ is a *subbasis* for a topology if $X = \bigcup_{S \in \mathcal{S}} S$.

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Definition 2.7 The *product topology* on the cartesian product of topology spaces $X \times Y$ has a basis $\{U \times V : U \subseteq X, V \subseteq Y \text{ open respectively}\}$.

Note One could check this indeed generates a topology on $X \times Y$. In fact, the finite intersections of elements of this collection are again in this collection.

Theorem 2.8 If $\mathcal{B}$ and $\mathcal{C}$ are bases for X and Y respecively, then

$$\mathcal{D} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$$

is a basis for $X \times Y$ with the product topology.

> Let $W \subseteq X \times Y$ open and $(x, y) \in W$. By the definition of the product topology, there is a product of open sets in X and Y respectively such that $(x, y) \in U \times V \subseteq W$. Since $x \in U$ and U is open, there is a set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Similarly, there is a set $C \in \mathcal{C}$ such that $y \in C \subseteq V$. So
>
> $$(x, y) \in B \times C \subseteq U \times V \subseteq W$$
>
> and $B \times C \in \mathcal{D}$.

Definition 2.9 The maps $\pi_1 : X \times Y \to X : (x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y : (x, y) \mapsto y$ are called *projection maps* and are continuous since $\pi_1^{-1}(U) = U \times Y$.

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Definition 2.10 Let $(X, \mathcal{T})$ be a topological space and Y a subset of X, then the collection

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

is the *subspace topology* on Y.

Lemma 2.11 The subspace topology is indeed a topology on Y.

The axioms we desire follow from basic facts of set theory:

(i) $\emptyset = Y \cap \emptyset$

(ii) $Y = Y \cap X$

(iii) $\bigcap_{n=1}^{N}(U_n \cap Y) = \left(\bigcap_{n=1}^{N} U_n\right) \cap Y$

(iv) $\bigcup_{\alpha \in J}(U_\alpha \cap Y) = (\bigcup U_\alpha) \cap Y$

Lemma 2.12 If $\mathcal{B}$ is a basis for the topology X, then $\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

Let $U \cap Y$ be open in Y with U open in X and $y \in U \cap Y$. Then there is a set $B \in \mathcal{B}$ such that $y \subseteq B \subseteq U$. So $B \cap Y \in \mathcal{B}_Y$ and $y \in B \cap Y \subseteq U \cap Y$.

Note Since we have this notion of the subspace topology, the concept of openness is a relative property.