Lecture Notes by Jonathan Alcaraz (UCR)

Complex Analysis

Math 210A
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Based on Lectures by

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The Topology of the Complex Plane

Definition 1.1 Given $a \in \mathbb{C}$, $r > 0$, define an *open ball* by

$$B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$$

and a *closed ball* by

$$\overline{B}(a,r) = \{z \in \mathbb{C} : |z-a| \leq r\}$$

Definition 1.2 Take sets $A \subseteq G \subseteq \mathbb{C}$. A is said to be *open in* G if for any $a \in A$, there is some $r > 0$ such that $B(a,r) \bigcap G \subseteq A$. A is said to be *closed in* G if $G \backslash A$ is open in G.

Definition 1.3 A subset $G \subseteq \mathbb{C}$ is said to be *connected* if it has either of the following properties:

- If $G = A \bigcup B$ where A, B are open and disjoint, the $A = \emptyset$ or $B = \emptyset$.
- If $A \subseteq G$ is both open in G and closed in G, then $A = \emptyset$ or $A = G$.

Definition 1.4 A *segment* between complex numbers z and w, denoted $[z,w]$ is the set $\{tw + (1-t)z : t \in [0,1]\}$.

Definition 1.5 A *polygon* from a to b is a set $[a, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_n, b]$.

Theorem 1.6 An open set G is connected if and only if, for every $a, b \in G$ there is a polygon from a to b.

Definition 1.7 Given a subset $A \subseteq \mathbb{C}$, we say $z \in \mathbb{C}$ is a *limit point* of A if there exists a sequence $\{a_n\}$ of distinct points in A such that $z = \lim_{n\to\infty} a_n$.

Corollary 1.8 A subset A is closed if and only if A contains all of its limit points.

Definition 1.9 A subset $A \subseteq \mathbb{C}$ is *complete* if every Cauchy sequencce in A converges in A.

Corollary 1.10 A is complet if and only if A is closed.

Definition 1.11 A subset A of $\mathbb{C}$ is *compact* if every open cover of A has a finite subcover. A is *sequentially compact* if every sequencec in A has a subsequence which converges in A.

Definition 1.12 A set $A \subseteq \mathbb{C}$ is *totally bounded* if for every $\varepsilon > 0$ there exists $a_1, \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

Theorem 1.13 The following are equivalent:

(i) A is compact;

(ii) Every infinie set in A has limit point in A;

(iii) A is sequentially compact;

(iv) A is complete and totally bounded.

Corollary 1.14 A is compact if and only if A is closed and bounded.

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Theorem 2.1 Let $A \subseteq \mathbb{C}$ and $f : A \to \mathbb{C}$ be a continuous function. If A is a compact (resp. connected), then $f(A)$ is compact (resp. connected). Moreover, if A is compact, then $f(A)$ is bounded and attains its bounds.

Definition 2.2 A fuunction $f : A \to \mathbb{C}$ is said to be *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon|$ whenever $|z - w| < \delta$.

Note Uniform continuity implies standard pointwise continuity, but the converse need not be true. However, the converse is true when A is compact.

Definition 2.3 Let $f : A \to \mathbb{C}$ and $f_n : A \to \mathbb{C}$ be functions. We say the sequence $\{f_n\}$ *converges uniformly* to f is for every $\varepsilon > 0$, there is some N such that every $n \geq N$ has the property $|f_n(z) - f(z)| < \varepsilon$ for every $z \in A$.

Theorem 2.4 If f_n converges uniformly to f on A, and f_n are continuous, then f is continuous.

Fix $a \in A$. We want to show that f is continuous at a. Let $\varepsilon > 0$. There is n sufficiently large so that $|f(z) - f_n(z)| < \frac{\varepsilon}{3}$ for any $z \in \mathbb{C}$. Moreover, since f_n continuous, there is a $\delta > 0$ such that $|f_n(z) - f_n(a)| < \frac{\varepsilon}{3}$ whenever $|z - a| < \delta$. So if $|z - a| < \delta$, then

$$|f(z) - f(a)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)| < \varepsilon$$

Definition 2.5 Let $u_n : A \to \mathbb{C}$ be a sequence of functions and define $f = \sum_{n=1}^{\infty} u_n$ by

$$f(z) = \lim_{N \to \infty} \sum_{n=1}^{N} u_n(z)$$

We say $\sum_{n=1}^{\infty} u_n$ *converges uniformly* if the partial sums $\sum_{n=1}^{N} u_n$ converge uniformly in the aforementioned sense.

Theorem 2.6 (Weirstrass M-test) Let $A \subseteq \mathbb{C}$ and $u_n : A \to \mathbb{C}$. Suppose there are numbers $\{M_n\}$ such that $|u_n(z)| \leq M_n$ for every $z \in A$ and every n. If $\sum_{n=1}^{\infty} M_n$, then $\sum_{n=1}^{\infty}$ converges uniformly.

Define f_N to be the Nth partial sum of u_n. Since

$$\begin{aligned} |f_n(z) - f_m(z)| &\leq |u_{m+1}(z)| + \cdots + |u_n(z)| \\ &\leq M_{m+1} + \cdots + M_n \end{aligned}$$

Since $\sum M_n$ converges, the RHS can be made arbitrarily small for sufficiently large m, n, f_n is pointwise Cauchy and hencce converges pointwise. Define f to be the pointwise limit of f_n. So,

$$\begin{aligned} |f(z) - f_n(z)| &= |u_{n+1}(z) + u_{n+2} + \cdots| \\ &\leq M_{n+1} + M_{n+2} + \cdots \end{aligned}$$

Again, since M_n converges, this can be made arbitrarilly small.

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Power Series

Definition 3.1 Let $a_n \in \mathbb{C}$ We say the series $\sum_{n=0}^{\infty} a_n$ *converges* to $z \in \mathbb{C}$ if for every $\varepsilon > 0$, there is an N such that for every $m \geq N$, $|\sum_{n=0}^{m} a_n - z| < \varepsilon$. We say the series *converges absolutely* if $\sum_{n=0}^{\infty} |a_n|$ converges. If a series does not converge, we say the series *diverges*.

Lemma 3.2 Absolute converges implies convergence.

> Define z_n to be the n-th partial sum of $\{a_n\}$. If $m > k$, then
>
> $$\begin{aligned} |z_m - z_k| &= |a_{k+1} + \cdots + a_m| \\ &\leq |a_{k+1}| + \cdots + |a_m| \end{aligned}$$
>
> Since the series is absolutely convergent, then this value can be made arbitrarily small. So z_n is Cauchy and hence converges.

Definition 3.3 A *power series* about a is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

Theorem 3.4 For any power series $\sum a_n (z-a)^n$, there is a nonnegative real number R such that:

(1) $|z - a| < R$ implies the series converges absolutely.

(2) $|z - a| > R$ implies the series divverges.

Moreover, if $0 < r < R$, then the series converges uniformly on $\overline{B}(a, r)$. The unique R satisfying these properties is called the *radius of convergence* of the power series.

> Recall the *limit supremum* of a sequence defined by
>
> $$\limsup a_n = \lim_{n\to\infty} (\sup\{a_m : m \geq n\})$$

We need only consider the case where $a = 0$. Define R implicitly by

$$\frac{1}{R} = \limsup |a_n|^{1/n}$$

Suppose $|z| < R$ and choose r so that $|z| < r < R$. Since $\frac{1}{r} > \frac{1}{R}$, there exists N such that for every $n \geq N$, $|a_n|^{\frac{1}{n}} < \frac{1}{r}$. So,

$$\sum_{n=N}^{\infty} |a_n z^n| = \sum_{n=N}^{\infty} \left|(a_n^{\frac{1}{n}})^n\right| |z^n| < \sum_{n=N}^{\infty} \left(\frac{|z|}{r}\right)^n$$

Since $|z| < r$, this is a convergent geometric series. This proves (1).
Suppose $|z| > R$. Choose r so that $R < r < |z|$. Then $\frac{1}{r} < \frac{1}{R}$, so there are infinitely many n such that $\frac{1}{r} < |a_n|^{\frac{1}{n}}$. Therefore the sequence $|a_n z^n| > \left(\frac{|z|}{r}\right)^n$ diverges and hence the corresponding series diverges.
Suppose $0 < r < R$. Choose ρ such that $r < \rho < R$. So there exists N such that for every $n \geq N$, $|z|^{\frac{1}{n}} < \frac{1}{\rho}$. Therefore $n \geq N$, $|z| \leq r$ so $|a_n z^n| \leq \left(\frac{r}{\rho}\right)^n$. So by the Weirstrass M-test, the series converges uniformly.

Theorem 3.5 If $\lim \left|\frac{a_n}{a_{n+1}}\right|$ exists, then $R = \lim \left|\frac{a_n}{a_{n+1}}\right|$

Again, we may assume that $a = 0$. Let $a = \lim \left|\frac{a_n}{a_{n+1}}\right|$. If $|z| < r < a$, then there is some N such that for every $n \geq N$, $r < \left|\frac{a_n}{a_{n+1}}\right|$. Let $B = \left|a_N\, r^N\right|$. Then

$$B > \left|a_{N+1}\, r^{N+1}\right| > \cdots$$

Hence, if $n \geq N$, then

$$|a_n z^n| = |a_n z^n| \left|\frac{z^n}{r^n}\right| \leq B \left|\frac{z}{r}\right|^n$$

Similarly, if $|z| > r > a$, then there exists N such that for every $n \geq N$,

$\left|\frac{a_n}{a_{n+1}}\right| < r$. Let $B = \left|a_N r^N\right|$. Then

$$B = \left|a_N r^N\right| < |a_{N+1}| < \cdots$$

So, if $n \geq N$, then

$$|a_n z^n| = |a_n z^n| \left|\frac{z^n}{r^n}\right| \geq B \left|\frac{z}{r}\right|^n$$

Example Consider

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The radius of convergence is $\lim \left|\frac{n+1}{n}\right| = \infty$.
Since this power series converges for any complex number, it defines a well-defined function called the *exponential function* denoted:

$$\exp(z) := e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$