

Lecture Notes by Jonathan Alcaraz (UCR)

Complex Analysis

Math 210A
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Based on Lectures by

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THE TOPOLOGY OF THE COMPLEX PLANE

Definition 1.1 Given $a \in \mathbb{C}$, $r > 0$, define an *open ball* by

$$B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$$

and a *closed ball* by

$$\overline{B}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}$$

Definition 1.2 Take sets $A \subseteq G \subseteq \mathbb{C}$. A is said to be *open in G* if for any $a \in A$, there is some $r > 0$ such that $B(a, r) \cap G \subseteq A$. A is said to be *closed in G* if $G \setminus A$ is open in G .

Definition 1.3 A subset $G \subseteq \mathbb{C}$ is said to be *connected* if it has either of the following properties:

- If $G = A \cup B$ where A, B are open and disjoint, then $A = \emptyset$ or $B = \emptyset$.
- If $A \subseteq G$ is both open in G and closed in G , then $A = \emptyset$ or $A = G$.

Definition 1.4 A *segment* between complex numbers z and w , denoted $[z, w]$ is the set $\{tw + (1 - t)z : t \in [0, 1]\}$.

Definition 1.5 A *polygon* from a to b is a set $[a, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_n, b]$.

Theorem 1.6 An open set G is connected if and only if, for every $a, b \in G$ there is a polygon from a to b .

Definition 1.7 Given a subset $A \subseteq \mathbb{C}$, we say $z \in \mathbb{C}$ is a *limit point* of A if there exists a sequence $\{a_n\}$ of distinct points in A such that $z = \lim_{n \rightarrow \infty} a_n$.

Corollary 1.8 A subset A is closed if and only if A contains all of its limit points.

Definition 1.9 A subset $A \subseteq \mathbb{C}$ is *complete* if every Cauchy sequence in A converges in A .

Corollary 1.10 A is complete if and only if A is closed.

Definition 1.11 A subset A of \mathbb{C} is *compact* if every open cover of A has a finite subcover. A is *sequentially compact* if every sequence in A has a subsequence which converges in A .

Definition 1.12 A set $A \subseteq \mathbb{C}$ is *totally bounded* if for every $\varepsilon > 0$ there exists $a_1, \dots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^n B(a_i, \varepsilon)$.

Theorem 1.13 The following are equivalent:

- (i) A is compact;
- (ii) Every infinite set in A has limit point in A ;
- (iii) A is sequentially compact;
- (iv) A is complete and totally bounded.

Corollary 1.14 A is compact if and only if A is closed and bounded.

Lecture 2 2 Oct 2017

Theorem 2.1 Let $A \subseteq \mathbb{C}$ and $f : A \rightarrow \mathbb{C}$ be a continuous function. If A is a compact (resp. connected), then $f(A)$ is compact (resp. connected). Moreover, if A is compact, then $f(A)$ is bounded and attains its bounds.

Definition 2.2 A function $f : A \rightarrow \mathbb{C}$ is said to be *uniformly continuous* if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$.

Note Uniform continuity implies standard pointwise continuity, but the converse need not be true. However, the converse is true when A is compact.

Definition 2.3 Let $f : A \rightarrow \mathbb{C}$ and $f_n : A \rightarrow \mathbb{C}$ be functions. We say the sequence $\{f_n\}$ *converges uniformly* to f if for every $\varepsilon > 0$, there is some N such that every $n \geq N$ has the property $|f_n(z) - f(z)| < \varepsilon$ for every $z \in A$.

Theorem 2.4 If f_n converges uniformly to f on A , and f_n are continuous, then f is continuous.

Fix $a \in A$. We want to show that f is continuous at a . Let $\varepsilon > 0$. There is n sufficiently large so that $|f(z) - f_n(z)| < \frac{\varepsilon}{3}$ for any $z \in \mathbb{C}$. Moreover, since f_n continuous, there is a $\delta > 0$ such that $|f_n(z) - f_n(a)| < \frac{\varepsilon}{3}$ whenever $|z - a| < \delta$. So if $|z - a| < \delta$, then

$$|f(z) - f(a)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(a)| + |f_n(a) - f(a)| < \varepsilon$$

Definition 2.5 Let $u_n : A \rightarrow \mathbb{C}$ be a sequence of functions and define $f = \sum_{n=1}^{\infty} u_n$ by

$$f(z) = \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n(z)$$

We say $\sum_{n=1}^{\infty} u_n$ *converges uniformly* if the partial sums $\sum_{n=1}^N u_n$ converge uniformly in the aforementioned sense.

Theorem 2.6 (WEIRSTRASS M-TEST) Let $A \subseteq \mathbb{C}$ and $u_n : A \rightarrow \mathbb{C}$. Suppose there are numbers $\{M_n\}$ such that $|u_n(z)| \leq M_n$ for every $z \in A$ and every n . If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} u_n$ converges uniformly.

Define f_N to be the N th partial sum of u_n . Since

$$\begin{aligned} |f_n(z) - f_m(z)| &\leq |u_{m+1}(z)| + \cdots + |u_n(z)| \\ &\leq M_{m+1} + \cdots + M_n \end{aligned}$$

Since $\sum M_n$ converges, the RHS can be made arbitrarily small for sufficiently large m, n , f_n is pointwise Cauchy and hence converges pointwise. Define f to be the pointwise limit of f_n . So,

$$\begin{aligned} |f(z) - f_n(z)| &= |u_{n+1}(z) + u_{n+2}(z) + \cdots| \\ &\leq M_{n+1} + M_{n+2} + \cdots \end{aligned}$$

Again, since M_n converges, this can be made arbitrarily small.