Lecture Notes by Jonathan Alcaraz (UCR)

Geometry

Math 232A
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Based on Lectures by

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Introduction

Given a smooth surface $S \subseteq \mathbb{R}^3$, we'd like to compare and contrast with $\mathbb{R}^n$. In doing so, we may ask for analogs to lines, angles, volume, distance, length, et cetera.

In fact, we can even consider these analogs in more abstract smooth manifolds. For example:

- $\mathbb{R}P^n$
- the Torus
- the Klein Bottle

While these space can be embedded into $\mathbb{R}^n$, we don't usually consider them this way, but rather thin of them abstractly.

Riemannian Manifolds

Definition 1.1 A *Riemannian Manifold* (M, g) is a C^∞-manifold together with a smoothly varying inner product g on M.

Definition 1.2 An *Inner Product* is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that is bilinear, symmetric, and positive definite.

Note Really there are inifinitely many inner products in a Riemmanian Manifold since each point has its own tangent space. We sometimes say g_p is the inner product at p. We say that g is a *Riemannian metric*, though this shouldn't be confused with standard notions of a metric.

Definition 1.3 Define the *angle* between two points $v, w \in T_pM$ for some $p \in M$ by

$$\sphericalangle(v, w) = \cos^{-1}\left(\frac{g_p(v, w)}{\|v\|\|w\|}\right)$$

where $\|v\|^2 := g_p(v, v)$.

In Euclidean space, given a path $c : [a, b] \to \mathbb{R}^n$, one can define the length of c by

$$len(c) = \int_a^b \|c'\|dt$$

Definition 1.4 In general Riemannian Manifolds, we can similarly define the *length* of a path $\tilde{c} : [a, b] \to M$ by

$$len(c) = \int_a^b \sqrt{g(\tilde{c}', \tilde{c}')} dt$$

In Euclidean space, we have the concept of distance between points p and q defined by $dist(p, q) = len(c)$ where c is the unique line segment conecting p and q. However, in an abstract manifold, there may not be such a shortest path.

Definition 1.5 Given points p and q in a Riemannian Manifold M, the *distance* is defined by

$$dist(p, q) = \inf\{len(c) : c \text{ piece wise smooth between p and q}\}$$

Example

1. $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ with $g((x, v), (x, w)) = vw$
2. Any real inner product space.

Definition 1.6 $f : (M, g) \to (N, h)$ is said to be an *isometry* is f is a diffeomorphism and $f^\star(h) = g$ that is $g(x, y) = h(df(x), df(y))$.

Note Given any Riemannian Manifold (M, g), T_pM is an n-dimensional vector space, hence (T_pM, g_p) is isometric to $(\mathbb{R}^n, can)$

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