Lecture Notes by Jonathan Alcaraz (UCR)

Topology

Math 205A
Fall 2017

Based on Lectures by

Dr. Frederick Willhelm
University of California, Riverside

last updated October 3, 2017

Lecture 1 28 Sep 2017

Introduction

This course is about the abstraction of continuity.

A subset $O \subseteq \mathbb{R}$ is said to be *open* if for every point $x \in O$, there is a value $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq O$.

This idea of openness in Euclidean space can be used to define continuity of functions on Euclidean space.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous* if for any open $O \subseteq \mathbb{R}$, $f^{-1}(O)$ is open in $\mathbb{R}^n$.

One could prove that this definition is equivalet to the standard ε-δ definition of continuity on Euclidean space.

Point-Set Topology

Definition 1.1 Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a *topology* on X if

(i) $\emptyset$ and X are in $\mathcal{T}$;

(ii) Any union of elements of $\mathcal{T}$ is in $\mathcal{T}$;

(iii) The intersection of finitely many elements of $\mathcal{T}$ is in $\mathcal{T}$.

A set together with a topology $\mathcal{T}$ is called a *topological space* denoted $(X, \mathcal{T})$ (or just X if the topology is understood). Elements of $\mathcal{T}$ are said to be *open sets in X*.

Example

- The *discrete topology* on a set is simply the collection of all subsets.
- The *indisccrete topology* on a set X is the most trivial topolgy, $\{\emptyset, X\}$.

Definition 1.2 If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on the same set with the property $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is said to be *finer* than $\mathcal{T}_1$. If neither $\mathcal{T}_1 \subseteq \mathcal{T}_2$ nor $\mathcal{T}_2 \subseteq \mathcal{T}_1$, these topologies are said to be *incomparable*.

Definition 1.3 A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is called *continuous* if $f^{-1}(V) \in \mathcal{T}$ for every $V \in \mathcal{T}'$, that is, the preimage of open sets are open. A *homeomorphism* is a bijective continuous funcction whose inverse is continuous.

The Fundamental Question of Topology Given two topological spaces, determine if they are homeomorphic.