Dependently Typed Languages in Statix

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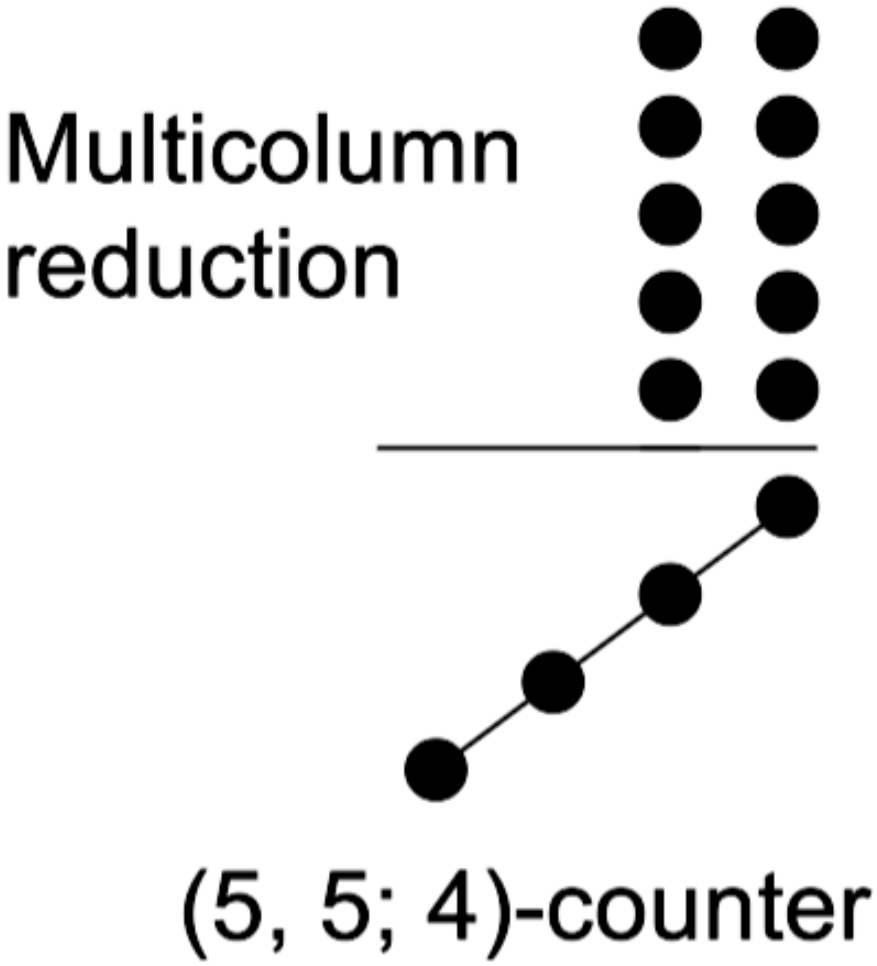

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Dependently Typed Languages in Statix

THESIS

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by

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Dependently Typed Languages in Statix

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Abstract

Static type systems can greatly enhance the quality of programs, but implementing a type checker that is both expressive and user-friendly is challenging and error-prone. The Statix meta-language (part of the Spoofax language workbench) aims to make this task easier by automatically deriving a type checker from a declarative specification of the type system. However, so far Statix has not been used to implement dependent types, an expressive class of type systems which require evaluation of terms during type checking. In this paper, we present an implementation of a simple dependently typed language in Statix, and discuss how to extend it with several common features such as inductive data types, universes, and inference of implicit arguments. While we encountered some challenges in the implementation, our conclusion is that Statix is already usable as a tool for implementing dependent types.

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Chapter 1

Introduction

Spoofax is a textual language workbench: a collection of tools that enable the development of textual languages [1]. When working with the Spoofax workbench, the Statix meta-language can be used for the specification of static semantics. To provide these advantages to as many language developers as possible, Statix aims to cover a broad range of languages and type systems. However, no attempts have been made to express dependently typed languages in Statix.

Dependently typed languages are different from other languages, because they allow types to be parameterized by values. This allows types to express properties of values that cannot be expressed in a simple type system, such as the length of a list or the well-formedness of a binary search tree. This expressiveness also makes dependent type systems more complicated to implement. Especially, deciding equality of types requires evaluation of the terms they are parameterized by.

1.1 Contributions

This goal of this paper is to investigate how well Statix is fit for the task of defining a simple dependently-typed language. We want to investigate whether typical features of dependently typed languages can be encoded concisely in Statix. The goal is not only to show that Statix can implement it, but to investigate whether implementing a dependent type checker is easier in Statix than in a general-purpose programming language.

The technical contributions of this thesis are the following:

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1.2 Outline

TODO navigation

a

Chapter 2

Calculus of Constructions

In this section, we will describe how to implement a dependently typed language in Statix. In section 2.1 we will describe the syntax of the language, then in section 2.2 we will describe how scope graphs are used to type check the language. Section 2.3 describes the dynamic semantics of the language, and finally 2.4 how to type check the language. The implementation of the language is available on GitHub. `https://github.com/JonathanBrouwer/master-thesis/` in the "lody" folder.

2.1 The language

The base language that has been implemented is the Calculus of Constructions [2], the language at the top of the lambda cube [3]. One extra feature was added that is not present in the Calculus of Constructions: let bindings. Let bindings could be desugared by substituting, but this may grow the program size exponentially, so having them in the language is useful. The abstract syntax of the language is available in figure 2.1.

```
Type        : Expr
Let         : ID * Expr * Expr -> Expr
Var         : ID -> Expr
FnType      : ID * Expr * Expr -> Expr
FnConstruct : ID * Expr * Expr -> Expr
FnDestruct  : Expr * Expr -> Expr
```

Figure 2.1: The syntax for the base language. FnConstruct is a lambda function, FnDestruct is application of a lambda function.

An example program is the following, which defines a polymorphic identity function and applies it to a function:

```
let f = \T: Type. \x: T. x;
f (_: Type -> Type) (\x: Type. x)
```

The AST of this program in Aterm format[4] would be:

```
Let(
  "f",
  FnConstruct("T", Type(), FnConstruct("x", Var("T"), Var("x"))),
  FnDestruct(
    FnDestruct(Var("f"), FnConstruct("_", Type(), Type())),
    FnConstruct("x", Type(), Var("x"))
  )
)
```

2.2 Scope Graphs

To type check the base language, we need to store information about the names that are in scope at each point in the program. There are two different cases, names that do not have a known value (only a type), such as function arguments, and names that do have a known value, such as let bindings.[1]

In Statix, all this information can be stored in a *scope graph* [5], which is a feature of Statix. It is a graph consisting of nodes for scopes, labeled edges for visibility relations, scoped declarations for a relation, and queries for references. We only use a single type of edge, called `P` (parent) edges. It also only has a single relation, called `name`. This name stores a `NameEntry`, which can be either a `NType`, which stores the type of a name, or a `NSubst`, which stores a name that has been substituted with a value.

Next, we will introduce some Statix predicates that can be used to interact with these scope graphs:

```
sPutType  : scope * ID * Expr -> scope
sPutSubst : scope * ID * (scope * Expr) -> scope
sGetName  : scope * ID -> NameEntry
sGetNames : scope * ID -> list((path * (ID * NameEntry)))
sEmpty    : -> scope
```

The `sPutType` and `sPutSubsts` predicates generate a new scope given a parent scope and a type or a substitution respectively. To query the scope graph, use `sGetName` or `sGetNames`, which will return a `NameEntry` or a list of NameEntries respectively that the query found. Finally, `sEmpty` returns a fresh empty scope.

We will define a *scoped expression*, as a pair of a scope and an expression. The scope acts as the environment of the expression, containing all of the context needed to evaluate the expression.

2.3 Beta Reductions

A unique requirement for dependently typed languages is beta reduction during type checking, since types may require evaluation to compare.

[1] In non-dependent languages there is no such distinction, but because we may need to value of a binding to compare types, this is needed in dependently typed languages.

We implemented beta reduction using a Krivine abstract machine[6]. The machine can head evaluate lambda expressions with a call-by-name semantics. It works by keeping a stack of all arguments that have not been applied yet. This turned out to be the more natural way of expressing this over substitution-based evaluation relation. We originally tried to implement the latter, which works fine for the base language. However, it runs into trouble when implementing inductive data types; more information about this will be in the full master thesis. An additional benefit is that abstract machines are usually more efficient than substitution-based approaches.

In conventional dependently typed languages, evaluation is often done using De Bruijn indices. However, we chose to use names rather than De Bruijn indices, because scope graphs work based on names rather. Using De Bruijn indices would also prevent us from using editor services that rely on `.ref` annotations, such as renaming.

We need to define multiple predicates that will be used later for type checking. First, the primary predicate is `betaReduceHead`, that takes a scoped expression and a stack of applications, and returns a head-normal expression. The scope acts as the environment from [6], using `NSubst` to store substitutions. All rules for `betaReduceHead` are given in figure 2.2. We use the syntax $\langle s1|e1\rangle t \underset{\beta h}{\Rightarrow} \langle s2|e2\rangle$ to express `betaReduceHead((s1, e1), t) == (s2, e2)`. Figure 2.2 contains the rules necessary for beta head reduction of the language. One predicate that is used for this is the `rebuild` predicate, which takes a scoped expression and a list of arguments and converts it to an expression by adding `FnDestruct`s.

Additionally, we define `betaReduce` which fully beta reduces a term. It works by first calling `betaReduceHead` and then matching on the head, calling `betaReduce` on the sub-expressions of the head recursively.

Finally, we define `expectBetaEq`. This rule first beta reduces the heads of both sides, and then compares them. If the head is not the same, the rule fails. Otherwise, it recurses on the sub-expressions. One special case is when comparing two `FnConstruct`s. Here we need to take into account alpha equality: two expressions which only differ in the names that they use should be considered equal. We implement this by substituting in the body of the functions, replacing their argument names with placeholders.

2.4 Type checking the Calculus of Constructions

We will define a Statix predicate `typeOfExpr` that takes a scope and an expression and type checks the scope in the expression. It returns the type of the expression.

```
typeOfExpr : scope * Expr -> Expr
```

We can then start defining type checking rules for the language. We introduce a number of judgements for typing and equality together with their counterparts in Statix.

1. $\langle s \mid e\rangle : t$ is the same as `typeOfExpr(s, e) == t`
2. $\langle s1 \mid e1\rangle \underset{\beta}{=} \langle s2 \mid e2\rangle$ is the same as `expectBetaEq((s1, e1), (s2, e2))`

$$\frac{}{\langle s \mid \mathsf{Type}() \rangle\ [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{Type}() \rangle} \qquad \frac{\langle \mathsf{sPutSubst}(s, n, (s, v)) \mid b \rangle\ t \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle}{\langle s \mid \mathsf{Let}(n, v, b) \rangle\ t \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle}$$

$$\frac{\mathsf{sGetName}(s, n) = \mathsf{NSubst}(se, e) \qquad \langle se \mid e \rangle\ t \underset{\beta h}{\Rightarrow} \langle se' \mid e' \rangle}{\langle s \mid \mathsf{Var}(n) \rangle\ t \underset{\beta h}{\Rightarrow} \langle se' \mid e' \rangle}$$

$$\frac{\mathsf{sGetName}(s, n) = \mathsf{NType}(t)}{\langle s \mid \mathsf{Var}(n) \rangle\ t \underset{\beta h}{\Rightarrow} \mathsf{rebuild}(s, \mathsf{Var}(n), t)} \qquad \frac{}{\langle s \mid \mathsf{FnType}(n, a, b) \rangle\ [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{FnType}(n, a, b) \rangle}$$

$$\frac{}{\langle s \mid \mathsf{FnConstruct}(n, a, b) \rangle\ [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{FnConstruct}(n, a, b) \rangle}$$

$$\frac{\langle \mathsf{sPutSubst}(s, n, t) \mid b \rangle\ ts \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle}{\langle s \mid \mathsf{FnConstruct}(n, a, b) \rangle\ (t :: ts) \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle} \qquad \frac{\langle s \mid f \rangle\ (a :: ts) \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle}{\langle s \mid \mathsf{FnDestruct}(f, a) \rangle\ ts \underset{\beta h}{\Rightarrow} \langle s' \mid e' \rangle}$$

Figure 2.2: Rules for beta head reducing the Calculus of Constructions

3. $\langle s1 \mid e1 \rangle\ t \underset{\beta h}{\Rightarrow} \langle s2 \mid e2 \rangle$ is the same as `betaReduceHead((s1, e1)) == (s2, e2)` (The same as in section 2.3)
4. $\langle s1 \mid e1 \rangle \underset{\beta}{\Rightarrow} e2$ is the same as `betaReduce((s1, e1)) == e2`
5. $\langle sEmpty \mid e \rangle$ is the same as e (empty scopes can be left out)

The inference rules above can be directly translated to Statix rules. For example, the rule for `Let` bindings is expressed like this in Statix:

```
typeOfExpr(s, Let(n, v, b)) = typeOfExpr(s', b) :-
    typeOfExpr(s, v) == vt, sPutSubst(s, n, (s, v)) == s'.
```

$$\frac{}{\langle s \mid \mathsf{Type}() \rangle : \mathsf{Type}()} \qquad \frac{\langle s \mid v \rangle : vt \qquad \langle \mathsf{sPutSubst}(s, n, (s, v)) \mid b \rangle : t}{\langle s \mid \mathsf{Let}(n, v, b) \rangle : t}$$

$$\frac{\mathsf{sGetName}(s, n) = \mathsf{NType}(t)}{\langle s \mid \mathsf{Var}(n) \rangle : t} \qquad \frac{\mathsf{sGetName}(s, n) = \mathsf{NSubst}(se, e) \qquad \langle se \mid e \rangle : t}{\langle s \mid \mathsf{Var}(n) \rangle : t}$$

$$\frac{\langle s \mid a \rangle : at \qquad at \underset{\beta}{=} \mathsf{Type}() \qquad \langle s \mid a \rangle \underset{\beta}{\Rightarrow} a' \qquad \langle \mathsf{sPutType}(s, n, a') \mid b \rangle : bt \qquad bt \underset{\beta}{=} \mathsf{Type}()}{\langle s \mid \mathsf{FnType}(n, a, b) \rangle : \mathsf{Type}()} \qquad \frac{\langle s \mid a \rangle : at \qquad at \underset{\beta}{=} \mathsf{Type}() \qquad \langle s \mid a \rangle \underset{\beta}{\Rightarrow} a' \qquad \langle \mathsf{sPutType}(s, n, a') \mid b \rangle : bt}{\langle s \mid \mathsf{FnConstruct}(n, a, b) \rangle : \mathsf{FnType}(n, a', bt)}$$

$$\frac{\langle s \mid f \rangle : ft \qquad \langle s \mid ft \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s' \mid \mathsf{FnType}(da, dt, db) \rangle \qquad \langle s \mid a \rangle : at \qquad at \underset{\beta}{=} \langle sf \mid dt \rangle \qquad \langle \mathsf{sPutSubst}(s', n, (s, arg)) \mid db \rangle \underset{\beta}{\Rightarrow} db'}{\langle s \mid \mathsf{FnDestruct}(f, a) \rangle : db'}$$

Figure 2.3: Rules for type checking the Calculus of Constructions

Chapter 3

Avoiding Variable Capturing

TODO Add reference to types and programming languages

The implementation of the base language shown in chapter 2 has one big problem, that is variable capture. This section will explore several ways of solving this. An example is the following: What is the type of this expression (a polymorphic identity function)?

```
\T : Type. \T : T. T
```

The algorithm so far would tell you it is `T : Type -> T : T -> T`. Given the scoping rules of the language, that is equivalent to `T : Type -> x : T -> x`. However, the correct answer would be `T : Type -> x : T -> T`. There is no way of expressing this type without renaming a variable.

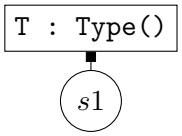
3.1 In depth: Why does this happen?

In this section, we will step through the steps that happen during the type checking of the term above, to explain why the incorrect type signature is returned. To find the type, the following is evaluated:

```
typeOfExpr(_, FnConstruct("T", Type(),
    FnConstruct("T", Var("T"), Var("T")))
)
```

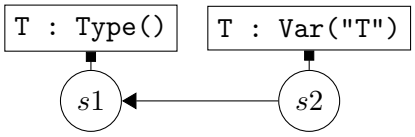
This first handles the outer `FnConstruct`, it creates a new node in the scope graph, and then type checks the body with this scope.

```
typeOfExpr(s1, FnConstruct("T", Var("T"), Var("T")))
```



The same thing happens, the body of the `FnConstruct` is typechecked with a new scope. Note that `Var("T")` in the type of the second `T` is ambiguous, does it refer to the first or second node?

```
typeOfExpr(s2, Var("T"))
```



Finally, we need to find the type of `Var("T")`. This finds the lexically closest definition of `T` (the one in s2), which is correct. But the type of `T` is `T`, which does NOT refer to the lexically closest `T`, but instead to the `T` in s1. This situation, in which a type can contain a reference to a variable that is shadowed, is the problem. We need to find a way to make sure that shadowing like this can never happen.

3.2 Alternative Solutions

De Bruijn Indices

Almost all compilers that typecheck dependently typed languages use de Bruijn representation for variables. Using de Bruijn indices in statix is possible, but sacrifices a lot. It would require a transformation on the AST before typechecking. Modifying the AST like this causes problems, because it changes AST nodes. All editor services that rely on `.ref` annotations, such as renaming, can no longer be used. It also loses a lot of the benefits of using Spoofax, since using scope graphs relies on using names.

Uniquifying names

The first solution that was attempted was having a pre-analysis transformation that gives each variable a unique name. This doesn't work for a variety of reasons. The simplest being, it doesn't actually solve the problem. Names can be duplicated during beta reduction of terms, so we still don't have the guarantee that each variable has a unique name. Furthermore, this is a pre-analysis transformation, so similarly to using de Bruijn indices, it breaks editor services such as renaming.

Renaming terms dynamically

Anytime that we introduce a new name in a type, we could check if the name already exists in the environment, and if it does, choose a different unique name. This approach is possible but tedious to implement in Statix. It requires a new relation to traverse through the type and rename. It also increases the time complexity, as constant traversals of the type are needed.

3.3 Using scopes to distinguish names

The solution we found to work best in the end is to change the definition of `ID`. To be precise, at the grammar level we have two sorts, `RID` is a "Raw ID", being just a string. `ID`

will have two constructors, one being `Syn`, a syntactical `ID`. The second one is `ScopedName`, it is defined in statix, so there is no syntax for it. It will be generated by `typeOfExpr`.

```
context-free sorts ID
lexical sorts RID
context-free syntax
   ID.Syn = RID
signature constructors
  ScopedName : scope * RID -> ID
```

The `ScopedName` constructor has a scope and a raw ID. The scope is used to uniquely identify the name. The main idea is that whenever we encounter a syntactical name, we replace it with a scoped name, so it unambiguous. The scope graph will never have a syntactical name in it. However, when querying the scope graph for a syntactical name, we return the lexically closest name.

The example revisited

In this section, we will step through the steps that happen during the type checking of the term above, with name collisions solved. To find the type, the following is evaluated, note that the names are now wrapped in a `Syn` constructor:

```
typeOfExpr(_, FnConstruct(Syn("T"), Type(),
    FnConstruct(Syn("T"), Var(Syn("T")), Var(Syn("T")))))
```

The name in the `FnConstruct` is replaced with a scoped name. The scope of the name is the scope that the name is first defined in. We then type check the body with this scope.

```
typeOfExpr(s1, FnConstruct(Syn("T"), Var(Syn("T")), Var(Syn("T"))))

(ScopedName(s1, T) : Type)
           [s1]
```

The same thing happens, the body of the `FnConstruct` is typechecked with a new scope. Note that the type of the new `T` now specifies which `T` it means, so it is no longer ambiguous.

```
typeOfExpr(s2, Var(Syn("T")))

(ScopedName(s1, T) : Type)     (ScopedName(s2, T) : Var(ScopedName(s1, T)))
           [s1]<----------------------------------[s2]
```

Finally, we need to find the type of `T`. This finds the lexically closest definition of `T` (the one in s2), as defined earlier. The type of this `T` is `ScopedName(s1, T)`, which explicitly defined which `T` it is. A name can now never shadow another name, since each scope uniquely identifies a name. The final type of the expression is now:

```
FnType(ScopedName(s1, T), Type(),
    FnType(ScopedName(s2, T), Var(ScopedName(s1, T)), Var(ScopedName(s1, T))))
```

3.4 Improving the readability of types

Because the expression above, with `ScopedName`s, is not particularly readable, we add a new post-analysis Stratego pass that converts the `ScopedName`s to ticked named. For example, the above would be transformed to:

```
FnType(T, Type(), FnType(T', Var(T)), Var(T)))
```

Ticks are added to names where necessary. We do this by following these rules:

1. When we encounter a `ScopedName` for the first time, we keep adding ticks to the name until we find a name that has not been used before.
2. We define a dynamic rule `Rename :: string -> string` and we store the new name we generated using this rule.
3. When we encounter a `ScopedName`, use the `Rename` rule to find what the name was transformed to.

Chapter 4

Extending the language

In this chapter, we will add booleans, postulates and type asserts to the language. The goal of this is to show that this language is easy to extend, and to add some features that make testing the language easier.

4.1 Booleans

This section describes how to add Booleans to the language. We will add the type of booleans `Bool`, `true`, `false` and `if`. The `if` expression is not dependent, it expects both branches to have the same type. An example of a program with booleans:

```
let and = \x: Bool. \y: Bool. if x then y else false end
```

The constructors for the language are:

```
BoolType : Expr
BoolFalse : Expr
BoolTrue : Expr
BoolIf : Expr * Expr * Expr -> Expr
```

Then, the rules for beta reducing the language are in figure 4.1. There is one particularly interesting case, that is how to beta-reduce if statements. Converting this rule to Statix is not entirely trivial, since you need to choose which rule to apply based on what `c` evaluates to. A new rule is needed, which has 3 cases. One for if `c` evaluates to true, one for if `c` evaluates to false, and finally a third case for other cases (such as when `c` is a variable that does not have a substitution). These rules are stated below: (Remember that `rebuild` is the rule introduced in chapter 2, which takes a scoped expression and a list of arguments and converts it to an expression by adding `FnDestructs`)

```
betaReduceHead((s, BoolIf(c, b1, b2)), t) =
  betaReduceHeadIf(s, betaReduceHead((s, c), []), c, b1, b2, t).
betaReduceHeadIf(s, (_, BoolTrue()), _, b1, _, t) =
  betaReduceHead((s, b1), t).
betaReduceHeadIf(s, (_, BoolFalse()), _, _, b2, t) =
  betaReduceHead((s, b2), t).
```

```
betaReduceHeadIf(s, _, c, b1, b2, t) =
  rebuild((s, BoolIf(c, b1, b2)), t).
```

$$\frac{}{\langle s \mid \mathsf{BoolTrue}() \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{BoolTrue}() \rangle} \qquad \frac{}{\langle s \mid \mathsf{BoolFalse}() \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{BoolFalse}() \rangle}$$

$$\frac{}{\langle s \mid \mathsf{BoolType}() \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s \mid \mathsf{BoolType}() \rangle}$$

$$\frac{\langle s \mid c \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s' \mid \mathsf{BoolTrue}() \rangle \qquad \langle s \mid b1 \rangle \; t \underset{\beta h}{\Rightarrow} \langle s'' \mid b1' \rangle}{\langle s \mid \mathsf{BoolIf}(c, b1, b2) \rangle \; t \underset{\beta h}{\Rightarrow} \langle s \mid b1' \rangle}$$

$$\frac{\langle s \mid c \rangle \; [] \underset{\beta h}{\Rightarrow} \langle s' \mid \mathsf{BoolFalse}() \rangle \qquad \langle s \mid b2 \rangle \; t \underset{\beta h}{\Rightarrow} \langle s'' \mid b2' \rangle}{\langle s \mid \mathsf{BoolIf}(c, b1, b2) \rangle \; t \underset{\beta h}{\Rightarrow} \langle s \mid b2' \rangle}$$

Figure 4.1: Rules for beta head reducing booleans

Next, the rules for type-checking booleans are in figure 4.2, which are relatively simple. The if expression checks that both branches have the same type, as it is a non-dependent if statement.

$$\frac{}{\langle s \mid \mathsf{BoolType}() \rangle : \mathsf{Type}()} \qquad \frac{}{\langle s \mid \mathsf{BoolTrue}() \rangle : \mathsf{BoolType}()}$$

$$\frac{}{\langle s \mid \mathsf{BoolFalse}() \rangle : \mathsf{BoolType}()} \qquad \frac{\begin{array}{c}\langle s \mid c \rangle : ct \qquad ct \underset{\beta}{=} \mathsf{BoolType}() \\ \langle s \mid b1 \rangle : tb1 \qquad \langle s \mid b2 \rangle : tb2 \qquad tb1 \underset{\beta}{=} tb2\end{array}}{\langle s \mid \mathsf{BoolIf}(c, b1, b2) \rangle : tb1}$$

Figure 4.2: Rules for type checking booleans

4.2 Postulate

Next we will add `Postulate` to the language. A postulate declares that there is a variable with a certain type, without specifying a value. Through the view of the Curry-Howard correspondence, this is equivalent to an axiom. At the current stage, this is useful for testing the language. For example, this is a test with a postulate:

```
postulate T : Type;
```

```
if true then Bool else T end
```

The following rules are used to implement the feature:

$$\frac{\langle s \mid b\rangle\ a \underset{\beta h}{\Rightarrow} \langle s' \mid b'\rangle}{\langle s \mid \mathsf{Postulate}(n,t,b)\rangle\ a \underset{\beta h}{\Rightarrow} \langle s' \mid b'\rangle} \qquad \frac{\langle s \mid t\rangle : tt \qquad tt \underset{\beta}{=} \mathsf{Type}() \qquad \langle s \mid t\rangle \underset{\beta}{\Rightarrow} t' \qquad \langle \mathsf{sPutType}(s,n,t') \mid b\rangle : bt}{\langle s \mid \mathsf{Postulate}(n,t,b)\rangle : bt}$$

Figure 4.3: Rules for postulates

4.3 Type Assert

Finally, we will add `TypeAssert` to the language.

4.4 Extensibility of the approach

Chapter 5

Term Inference

Inference is an important feature of dependent programming languages, that allows redundant parts of programs to be left out. For example, it allows you to infer the arguments of a function, if they can be inferred by the arguments that follow, like here where the type of the argument can be inferred, since we pass it `true` which is a boolean:

```
let id = (\T : Type. \x: T. x);
id _ true
```

We call it *term inference* rather than *type inference*, because we can infer values other than types. For example, it can infer that the `_` in this example must be `true`:

```
postulate f: Bool -> Type;
postulate g: f true -> Type;
\x: f _. g x
```

5.1 Different algorithms for inference

There are a lot of different algorithms for inference[7], some algorithms can solve more inferences than others. However, we would like to avoid implementing an algorithm at all, instead using Statix's built-in first-order unification to do the type inference for us. Implementing an inference algorithm in Statix is theoretically possible but this would be a lot of ugly code, and the goal is to use Statix in a way that is clean and declarative, not to do optimal inference.

The inference that we will build is a very simple unification system, where if at any point during type checking we assert that $e_1 \underset{\beta}{=} e_2$ and either e_1 or e_2 is a free variable, we set the free variable to be equal to the other variable. This process is called *first-order unification*.

There are some situations in which this approach fails, but in most real-world scenarios it works perfectly. For example, it can infer both programs in the introduction of this chapter, but it fails to infer the following program:

```
let f = _;
```

```
\x : Type.
\g: (_: (f x) -> Bool).
g true
```

We know that `f` is a function from `Type -> Type`, but it fails to infer the value of `f`. Because of the way that `g` is used, the type checker asserts that $fx \underset{\beta}{=} Bool$. Since `x` is declared as a function argument and it is completely free, this means that for any `x`, `f x = Bool`. But the rule above is not powerful enough to derive all of this, so it fails.

5.2 Using Statix's first-order unification for inference

First, we introduce a new constructor, `Infer : Expr -> Expr`. This will be used to store metavariables which we want to infer. This is a hack we use, ideally we would like to query statix to see if a metavariable is initialized, but as discussed in section 5.1 this is not possible.

The constructor is introduced when we encounter a `_` variable, a sign that something needs to be inferred. When type-checking this, we create an infer constructor. The constraint at the bottom cannot be solved immediately, since the `q` variable is unbound, but if `q` is bound later the constraint will be evaluated.

```
typeOfExpr_(s, Var(Syn("_"))) = (Infer(q), qt) :-
    (_, qt) == typeOfExpr(sEmpty(), q).
```

When an `Infer` needs to be type-checked, the logic is very similar, except that we don't need to generate a new metavariable:

```
typeOfExpr_(s, Infer(q)) = (Infer(q), t) :-
    typeOfExpr_(s, q) == (_, t).
```

When we encounter a `Infer` in `betaReduceHead`, we keep it intact, because we still want to know that it is a `Infer` in the `expectBetaEq` rule.

The `expectBetaEq` is where it gets interesting. We have a rule for each constructor in the language. For simple cases like `BoolTrue`, we assert that `e2` must be equal.

```
expectBetaEq_((s1, e1@BoolTrue()), (_, Infer(e2))) :- e1 == e2.
```

For more complicated constructors such as `FnType`, we generate new metavariables for the subexpressions, and assert they it must be beta-equal to the provided ones:

```
expectBetaEq_(
    (s1, e1@FnType(arg_name1, arg_type1, body1)),
    (_, Infer(e2))) :- {arg_type2 body2}
  e2 == FnType(arg_name1, Infer(arg_type2), Infer(body2)),
  expectBetaEq_((s1, e1), (sEmpty(), e2)).
```

Finally, there is a case for if we encounter two `Infer`s. Ideally, we would look at whether one infer is instantiated, and apply the rules above. However, this is not possible since we cannot do such queries in Statix. Instead, we say that they must be exactly equal and we

hope for the best. There are situations where this rule fails (where e1 and e2 are beta-equal but not identical), but this works well enough for practical use.

```
expectBetaEq_((_, Infer(e1)), (_, Infer(e2))) :-
    e1 == e2.
```

Chapter 6

Inductive Data Types

Another useful feature is support for inductive data types. An example of a simple inductive datatype is the `Nat` type:

```
data Nat : -> Type where
    zero : Nat,
    suc : Nat -> Nat;
```

We will design data types to have the same features as in Agda:

1. Parameters, which are datatypes that are polymorphic, and are required to be the same for all constructors, such as

   ```
   data Maybe (T : Type) : -> Type where
       None : Maybe T,
       Some : T -> Maybe T;
   ```

2. Indices, which are datatypes that are polymorphic, that may vary from constructor to constructor

   ```
   data Eq : (e1: Bool) (e2 : Bool) -> Type where
       refl : e : Bool -> Eq e e;
   ```

3. We can also combine parameters and indices, for example to create the generic `Eq` type. Note that parameters and indices may also depend on earlier parameters and indices.

   ```
   data Eq (T : Type) : (e1 : T) (e2 : T) -> Type where
       refl : e : T -> Eq e e;
   ```

4. Data types must be strictly positive. This ensures that we cannot make non-terminating programs[1]. For example, the following data type is forbidden, because it refers to itself in a negative position. We will explain exactly how this check works in section 6.2.

[1] After we add support for universes in chapter 7

```
data Bad : -> Type where
    bad : (Bad -> Bool) -> Bad;
```

6.1 In statix

6.2 Positivity Checking

Chapter 7

Universes

Chapter 8

A comparison with conventional implementations

In this chapter, we implement the language defined in chapter 2 in Haskell, and then compare the implementation with the implementation in Statix. We want the design of the implementation in Haskell to be similar to conventional implementations of dependently typed languages, so we can compare the Statix implementation with them.

8.1 Defining the AST

To implement the calculus of constructions in Haskell, we first define the `Expr` datatype. We chose to define the language using De Bruijn indices, since this is the convention when implementing dependently typed languages, and the goal of this implementation in Haskell is to compare the Statix implementation with conventional ones.

```
data Expr =
    Type
    | Let Expr Expr
    | Var Int
    | FnType Expr Expr
    | FnConstruct Expr Expr
    | FnDestruct Expr Expr
```

8.2 Defining environments

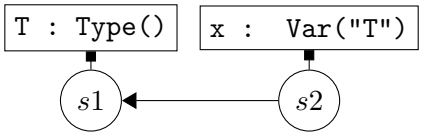
Next, we need to decide how we want to store the environment. We still need to store both function arguments and substitutions, which we called `NType` and `NSubst` in chapter 2. We will use the same names, and store the environment as a list, with the head of the list representing De Bruijn index 0. Finally, we define the type of a scoped expression `SExpr`, as a tuple of an environment and an expression.

```
data EnvEntry = NType Expr | NSubst SExpr
type Env = [EnvEntry]
```

```
type SExpr = (Env, Expr)
```

The way these environments are defined is isomorphic to the way we defined the way we use scope graphs in section 2.2. Nodes in the scope graph have at most one parent, and each node stores one entry, which is exactly the structure of a list. For example, the following scope graph and list have the same meaning:

```
NType(Var(0)) :: NType(Type()) :: Nil
```



The nodes in scope graphs may have multiple children, but we never query the children of a node. We only follow the edges, we don't go in the opposite direction. Similarly, part of a list may be shared, but this is fine in Haskell, since values are immutable.

Finally, we define the two functions `sPutSubst` and `sPutType` to mimic the Statix relations with the same name.

```
sPutSubst :: Env -> SExpr -> Env
sPutSubst env v = NSubst v : env
sPutType :: Env -> Expr -> Env
sPutType env v = NType v : env
```

8.3 Defining beta reduction

Now we want to define beta head reduction. Remember that in Statix, beta head reduction is a relation with the signature

```
betaReduceHead :
    (scope * Expr) * list((scope * Expr)) -> (scope * Expr)
```

In Haskell, a slightly different structure was used. We defined two functions, one returning a `Maybe SExpr` and the other checking if the result `Nothing`, in which case it returns the original expression (unreduced). This structure could also be implemented in Statix, but Haskell makes it a bit easier for us by providing the `Maybe` type and functions on it.

```
brh :: SExpr -> SExpr
brh e = fromMaybe e (brh_ e [])
brh_ :: SExpr -> [SExpr] -> Maybe SExpr
```

Chapter 9

A comparison with implementations in logical frameworks

Chapter 10

Related Work

The implementation in this paper requires performing substitutions in types immediately, as types don't have a scope. Van Antwerpen et al. [8, sect 2.5] present an implementation of System F that does lazy substitutions, by using scopes as types. It would be interesting to see if this approach could also apply to the Calculus of Constructions, where types can contain terms.

Another interesting comparison is to see how implementing a dependently typed language in Statix differs from implementing it in a general purpose language. The pi-forall language[9] is a good example of a language with a similar complexity to the language presented in this paper. In principle, the implementations are very similar. For example, the inference rules presented in [9] are similar to the inference rules presented in figure 2.3 from this paper. The primary difference is that they use a bidirectional type system, whereas this paper does not.

There exist several so-called *logical frameworks*, tools designed specifically for implementing and experimenting with dependent type theories, such as ALF [10], Twelf [11], Dedukti [12], Elf [13] and Andromeda [14]. Since these tools are designed specifically for the task, implementing the type system takes less effort in them compared to Spoofax, but for other tasks such as defining a parser or editor services they are not as well equipped. Some logical frameworks such as Twelf and Dedukti support Miller's *higher-order pattern unification* [15], which can be used as a more powerful way of inferring implicit arguments than the first-order unification built into Statix. Andromeda 2 also supports *extensionality rules* that can match on the type of an equality. We expect that adding extensionality rules to our implementation would be possible to do in Statix, but we leave an actual implementation to future work.

Chapter 11

Conclusion

We have demonstrated that the Calculus of Constructions can be implemented concisely in Statix, by storing substitutions in the scope graph. We have also presented a few extensions to the Calculus of Constructions and discussed how they could be implemented.

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