DGP : Exercise 2

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1 Theory Exercise

1.1

Let $f : \Omega_1 \to \Omega_2$ be a continuous function between two connexe spaces. We write in one dimension $\Omega_1 = [a, b]$. We know the value $f(x = a) = f_a$ and $f(x = b) = f_b$. The first linear interpolant I_1 is just a straight line between the two points f_a and f_b. It can be parametrized as :

$$I_1(t) = (1 - t)f_a + tf_b \quad t \in [0, 1]. \tag{1}$$

Its length L_1 is easily given by the Pythagorean theorem :

$$L_1^2 = (f_b - f_a)^2 + (b_a)^2. \tag{2}$$

We put now a new interpolation point $c \in [a, b]$ and measure $f(c) = f_c$. The interpolation I_3 is piece-wise defined as two segment of I_1-like functions. To compute the length, we use the Pythagorean theorem twice and obtain :

$$L_3^2 = (f_c - f_a)^2 + (c_a)^2 + (f_b - f_c)^2 + (b_c)^2. \tag{3}$$

We observe that if we introduce $0 = c - c$ and $0 = f_c - f_c$ in L_1 definition we obtain :

$$L_1^2 = (f_b - f_c + f_c - f_a)^2 + (b - c + c - a)^2. \tag{4}$$

Now by the triangular inequality :

$$(f_b - f_c + f_c - f_a)^2 + (b - c + c - a)^2 \leq (f_b - f_c)^2 + (f_c - f_a)^2 + (b - c)^2 + (c - a)^2 \tag{5}$$

$$\leq (f_c - f_a)^2 + (c - a)^2 + (f_b - f_c)^2 + (b - c)^2 = L_3^2. \tag{6}$$

Thus, $L_2 \leq L_3$. This result is trivially generalized to any segment in a discrete curve since every refinement of the grid will induce this inequality. This leads to the conclusion that, as the number of points increases, the chord length monotonically increases.

1.2

We can define a discrete curve passing through the set of $N + 2$ points $S_N(x_i)$ with $x_i \in [a, b] \quad i = 1, 2, 3, \ldots, N + 2$ such that :

$$S_N(x_i) = \begin{cases} 0 & \text{if } x_i = 2k, \quad k = 0, 1, 2, 3... \\ \Delta x_i/2 & \Delta x_i = x_{i+1} - x_i \quad \text{otherwise} \end{cases} \tag{7}$$

The resulting curve C_N is a queue of N rectangle triangles with the x-axis as hypotenuse. If we compute the chord length of this curve, we obtain easily the sum of each adjacent side d_i of the triangles :

$$L = \sum_{i=0}^{N} d_i = \sum_{i=0}^{N} \frac{2\Delta x_i}{\sqrt{2}} = \frac{2(b - a)}{\sqrt{2}}. \tag{8}$$

The last equality is found by remarking that the number of points does not affect the chord length of the curve.
Thus this curve has a chord length which stay constant but its points tend to fit a line, here the x-axis.

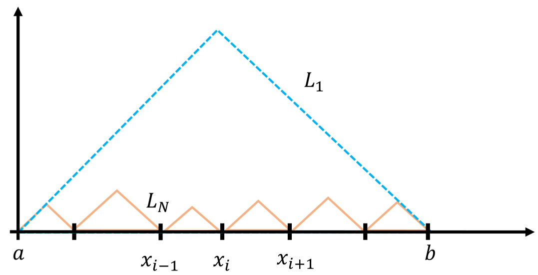

Figure 1: Geometrical representation of two curves C_N

1.3

A sharp corner can be created by stopping the parameter in a intersection of the curve. Here for example we used the curve :

$$\mathbf{K}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t^6 - t^4 - 1 \\ t^4 - t^2 - 1 \end{pmatrix} \quad t \in [-1, 1] \tag{9}$$

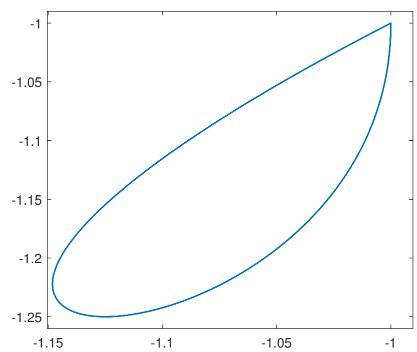

Figure 2: Graph of the curve $\mathbf{K}(t)$