Causality in Hypergraph/Wolfram Model

Rewriting Systems:

$H = (V, E) \sim$ hypergraph,

$V \sim$ vertex (multi)set; $E \sim$ hyperedge (multi)set

Graph: $E \subseteq V \times V$

Hypergraph: $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$.

Loosely (very): WM rule: $\underbrace{H_1 = (V_1, E_1)}_{\text{input hypergraph "pattern"}} \xrightarrow[\substack{e \\ \text{"event"}}]{} \underbrace{H_2 = (V_2, E_2)}_{\text{output hypergraph "pattern"}}$

Causality: $In(e) = E_1 \setminus E_2$; $Out(e) = E_2 \setminus E_1$

$(e_1, e_2) \sim$ causally related $\iff Out(e_1) \cap In(e_2) \neq \emptyset$

$\iff (e_1 \to e_2) \in E_{cau}$.

$G_{cau} = (V_{cau}, E_{cau})$

(Hyper)graph "Production":

$p = (L \xleftarrow{l} K \xrightarrow{r} R) \sim$ span of monomorphisms.

(Hyper)graph "Match": $m : L \to G$

$$\begin{tikzcd} L \arrow[d, "m"'] & K \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, "k"] & R \arrow[d, "n"] \\ G & D \arrow[l, "f"] \arrow[r, "g"'] & H \end{tikzcd}$$

$\sim$ "double-pushout" rewrite

$G \overset{p,m}{\Longrightarrow} H \sim$ (direct) hypergraph derivation.

$f \circ k = m \circ l \qquad g \circ k = n \circ r$

$p_1 = (L_1 \xleftarrow{l_1} K_1 \xrightarrow{r_1} R_1)$,

$p_2 = (L_2 \xleftarrow{l_2} K_2 \xrightarrow{r_2} R_2)$,

$\left(G_0 \overset{*}{\Longrightarrow} G_n\right) \sim$ "multiway system"/ "abstract rewriting system".

$In(p_2) = L_2 \setminus K_2$,

$Out(p_1) = R_1 \setminus K_1$

$(L_2 \setminus K_2) \cap (R_1 \setminus K_1) \neq \emptyset$.

Concurrency Theorem: p_1, p_2 ~ "E-related" hypergraph productions

E-related transformation sequence: $G \overset{p_1}{\Longrightarrow} H \overset{p_2}{\Longrightarrow} G'$

"E-concurrent" production: $G \Longrightarrow G'$

$$\begin{tikzcd}
 & H \arrow[dr, Rightarrow, "p_2"] & \\
G \arrow[ur, Rightarrow, "p_1"] \arrow[rr, Rightarrow, "p_1 *_E p_2"'] & & G'
\end{tikzcd}$$

Parallelism Theorem: p_1, p_2 ~ "sequentially-independent":

$G \overset{p_1}{\Longrightarrow} H_1 \overset{p_2}{\Longrightarrow} G'$

$G \overset{p_2}{\Longrightarrow} H_2 \overset{p_1}{\Longrightarrow} G'$

$p_1 + p_2$ ~ "parallel production":

$G \overset{p_1+p_2}{\Longrightarrow} G'$

$$\begin{tikzcd}
 & G \arrow[dl, Rightarrow, "p_1"'] \arrow[dr, Rightarrow, "p_2"] \arrow[dd, Rightarrow, "p_1+p_2"] & \\
H_1 \arrow[dr, Rightarrow, "p_2"'] & & H_2 \arrow[dl, Rightarrow, "p_1"] \\
 & G' &
\end{tikzcd}$$