Karlsruher Institut für Technologie

Practical SAT Solving

Lecture 2

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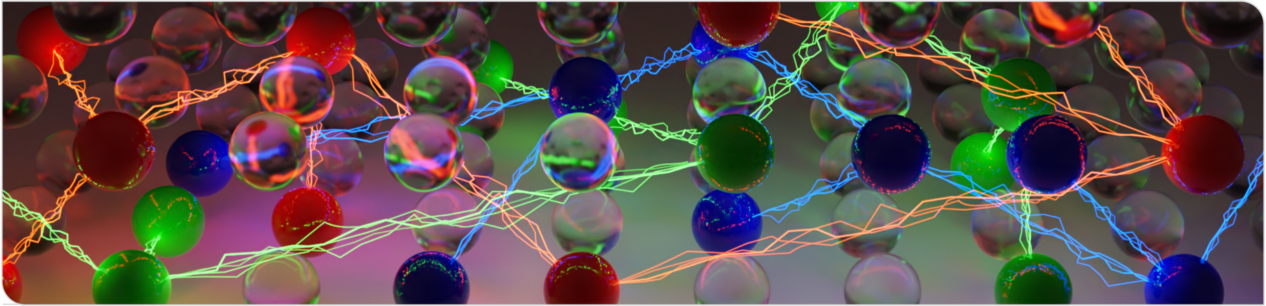


Overview

Recap. Lecture 1

- Satisfiability: Propositional Logic, CNF Formulas, NP-completeness, Applications
- Examples: Pythagorean Triples, Arithmetic Progressions, k-Colorability
- Incremental SAT: IPASIR, Sample Code

Today's Topics

- Tractable Subclasses
- Constraint Encodings
- Encoding Techniques

Tractable Subclasses

Tractable Subclasses (cf. Schaefer, 1978)

- **2-SAT**
  Exactly two literals per clause
- **HORN-SAT**
  At most one positive literal per clause
- **Inverted HORN-SAT**
  At most one negative literal per clause
- **Positive** / **Negative**
  Literals occur only pure (either positive or negative)
- **XOR-SAT**
  No clauses, only XOR constraints

2-SAT

aka. Binary or Quadratic SAT

Each clause has exactly two literals.

Example (2-SAT Formulas)

$F_5 = \{\{x_1, x_2\}, \{\overline{x_1}, x_2\}, \{x_1, \overline{x_2}\}, \{\overline{x_1}, \overline{x_2}\}\}$

$F_7 = \{\{\overline{x_1}, x_2\}, \{\overline{x_2}, x_3\}, \{\overline{x_3}, x_1\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_3\}\}$

Linear Time Algorithm for 2-SAT (cf. Aspvall et al., 1979)

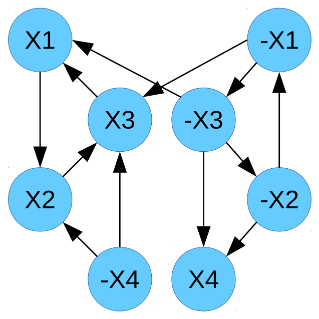
- Construct Implication Graph
- Find Strongly Connected Components (SCC) with Tarjan's Algorithm
  Complexity: $\mathcal{O}(n + m)$, where m is the number of clauses
- Check for Complementary Literals in the same SCC

Implication Graph

An **implication graph** of a 2-SAT formula F is a directed graph with a vertex for each literal of F and 2 edges for each clause $(l_1 \vee l_2)$: $\bar{l}_1 \rightarrow l_2$ and $\bar{l}_2 \rightarrow l_1$.

Example (Implication Graph)

$F_7 = \{\{\overline{x_1}, x_2\}, \{\overline{x_2}, x_3\}, \{\overline{x_3}, x_1\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_3\}\}$



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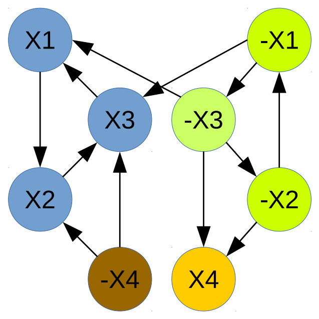
- Find Strongly Connected Components (SCC)
- SCC: There is a path from every vertex to every other vertex
- Tarjan's algorithm finds SCCs in $\mathcal{O}(|V| + |E|)$



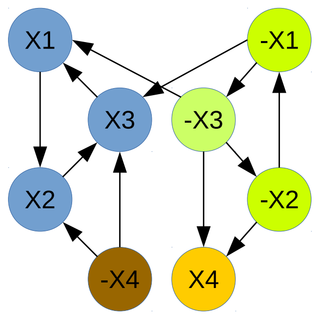
Implication Graph

An **implication graph** of a 2-SAT formula F is a directed graph with a vertex for each literal of F and 2 edges for each clause $(l_1 \vee l_2)$: $\overline{l}_1 \to l_2$ and $\overline{l}_2 \to l_1$.

Example (Implication Graph)

$F_7 = \{\{\overline{x_1}, x_2\}, \{\overline{x_2}, x_3\}, \{\overline{x_3}, x_1\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_3\}\}$

- If an SCC contains both x and $\overline{x}$, the formula is UNSAT
  - x implies its own negation!
  - Literals in an SCC must be either all true or all false



Implication Graph

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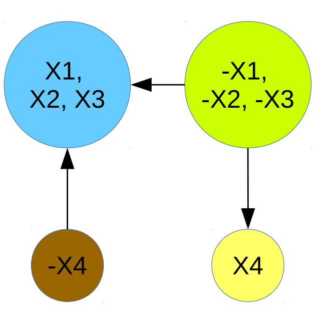
Example (Implication Graph)

$F_7 = \{\{\overline{x_1}, x_2\}, \{\overline{x_2}, x_3\}, \{\overline{x_3}, x_1\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_3\}\}$

- If an SCC contains both x and $\overline{x}$, the formula is UNSAT
  - x implies its own negation!
  - Literals in an SCC must be either all true or all false
- What about SAT? How to get a solution?
  - Contract each SCC into one vertex
  - In reverse topological order, set unassigned literals to `true`.



HornSAT

Each clause contains at most one positive literal.

Example (Horn Formula)

Each clause can be written as an implication with positive literals only and a single consequent:

$$F_6 = \{\{\overline{x_1}, x_2\}, \{\overline{x_1}, \overline{x_2}, x_3\}, \{x_1\}\}$$
$$\equiv (x_1 \rightarrow x_2) \wedge ((x_1 \wedge x_2) \rightarrow x_3) \wedge (\top \rightarrow x_1)$$

HornSAT

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Solving Horn Formulas

- Propagate until fixpoint
- If $\top \rightarrow \bot$ then the formula is UNSAT. Otherwise it is SAT.
- Construct a satisfying assignment by setting the remaining variables to `false`

Hidden Horn

aka. Renamable or Disguised Horn

A CNF formula is Hidden Horn if it can be made Horn by flipping the polarity of some of its variables.

Example (Hidden Horn Formula)

$$
\begin{aligned}
F_8 =& \{\{x_1, x_2, x_4\}, \{x_2, \overline{x_4}\}, \{x_1\}\} \\
\rightsquigarrow& \{\{\overline{x_1}, \overline{x_2}, x_4\}, \{\overline{x_2}, \overline{x_4}\}, \{\overline{x_1}\}\}
\end{aligned}
$$

How to recognize a Hidden Horn formula? And how to hard is it?

Hidden Horn
aka. Renamable or Disguised Horn

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$$\rightsquigarrow \{\{\overline{x_1}, \overline{x_2}, x_4\}, \{\overline{x_2}, \overline{x_4}\}, \{\overline{x_1}\}\}$$

How to recognize a Hidden Horn formula? And how to hard is it?

Recognizing Hidden Horn Formula F

Construct 2-SAT formula R_F that contains the clause $\{l_1, l_2\}$ iff there is a clause $C \in F$ such that $\{l_1, l_2\} \subseteq C$.

- Example: $R_{F_8} = \{\{x_1, x_2\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, \overline{x_4}\}\}$
- If the 2-SAT formula is satisfiable, then F is Hidden Horn
- If $x_i = \texttt{true}$ in ϕ, then x_i needs to be renamed to $\overline{x}_i$

Mixed Horn

A CNF formula is Mixed Horn if it contains only binary and Horn clauses.

Example (Mixed Horn Formula)

$F_9 = \{\{\overline{x_1}, \overline{x_7}, x_3\}, \{\overline{x_2}, \overline{x_4}\}, \{x_1, x_5\}, \{x_3\}\}$

How to solve a Mixed Horn formula? And how to hard is it?

Mixed Horn

A CNF formula is Mixed Horn if it contains only binary and Horn clauses.

Example (Mixed Horn Formula)

$F_9 = \{\{\overline{x_1}, \overline{x_7}, x_3\}, \{\overline{x_2}, \overline{x_4}\}, \{x_1, x_5\}, \{x_3\}\}$

How to solve a Mixed Horn formula? And how to hard is it?

Mixed Horn is NP-complete

Proof: Reduce SAT to Mixed Horn SAT

For each non-Horn non-binary clause $C = \{l_1, l_2, l_3, \dots\}$,

- for each but one positive $l_i \in C$ introduce a new variable l_i' and replace l_i in C by $\overline{l_i'}$
- add clauses $\{l_i', l_i\}, \{\overline{l_i'}, \overline{l_i}\}$ to establish $l_i = \overline{l_i'}$

Next up: CNF Encodings

Elementary Encodings

- Tseitin Transformation
- Cardinality Constraints
- Finite Domain Encodings

Properties of Encodings

- Size: Number of Variables and Clauses
- Propagation consistency: Can the encoding ensure consistency through propagation?

Encoding Circuits

Given a propositional formula F with operations $\wedge$, $\vee$, and $\neg$, how can it be encoded in CNF?

Example (CNF Conversion)

$$F = \neg((\neg x \vee y) \wedge (\neg z \wedge \neg(x \wedge \neg w))) \qquad \text{(Given Formula)}$$

Naive Conversion

Encoding Circuits

Given a propositional formula F with operations $\wedge$, $\vee$, and $\neg$, how can it be encoded in CNF?

Example (CNF Conversion)

$$F = \neg((\neg x \vee y) \wedge (\neg z \wedge \neg(x \wedge \neg w))) \qquad \text{(Given Formula)}$$

$$= (x \wedge \neg y) \vee z \vee (x \wedge \neg w) \qquad \text{(Negation Normal Form)}$$

Naive Conversion

- Convert to Negation Normal Form (NNF)

Encoding Circuits

Given a propositional formula F with operations $\land$, $\lor$, and $\neg$, how can it be encoded in CNF?

Example (CNF Conversion)

$$\begin{aligned} F &= \neg((\neg x \lor y) \land (\neg z \land \neg(x \land \neg w))) && \text{(Given Formula)} \\ &= (x \land \neg y) \lor z \lor (x \land \neg w) && \text{(Negation Normal Form)} \\ &= (x \lor z) \land (x \lor z \lor \neg w) \land (\neg y \lor z \lor x) \land (\neg y \lor z \lor \neg w) && \text{(Conjunctive Normal Form)} \end{aligned}$$

Naive Conversion

- Convert to Negation Normal Form (NNF)
- Apply distributive law to get CNF
- Problem: Applying the distributive law may result in an exponential blow-up.

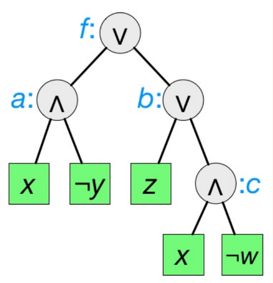
Tseitin Encoding

Idea: Introduce new variables for subformulas.

Example (Tseitin Conversion)

$F = (x \wedge \neg y) \vee z \vee (x \wedge \neg w)$ (Negation Normal Form)

$\overset{\text{SAT}}{=} (c \leftrightarrow x \wedge \neg w) \wedge \cdots \wedge (f \leftrightarrow a \vee b) \wedge f$ (Tseitin Encoding)



- Define new variables: $a \leftrightarrow x \wedge \overline{y}, \quad f \leftrightarrow a \vee b, \quad \ldots$
- Encode definitions in CNF: $(\overline{f} \vee a \vee b) \wedge (f \vee \overline{a}) \wedge (f \vee \overline{b}) \wedge \ldots$
- One additional clause (f) to assert that F must be true

Tseitin Encoding

The Tseitin-Encoding $\mathcal{T}(F)$ of a propositional formula F over connectives $\{\land, \lor, \neg\}$ is specified as follows.

Definition of Tseiting Encoding

$$\mathcal{T}(F) = d_F \land \mathcal{T}^*(F) \qquad \text{(Root Formula)}$$

$$\mathcal{T}^*(F) = \begin{cases} \mathcal{T}_{\text{def}}(F) \land \mathcal{T}^*(G) \land \mathcal{T}^*(H), & \text{if } F = G \circ H \text{ and } \circ \in \{\land, \lor\} \\ \mathcal{T}_{\text{def}}(F) \land \mathcal{T}^*(G), & \text{if } F = \neg G \\ \mathit{True}, & \text{if } F \in \mathcal{V} \end{cases} \qquad \text{(Recursion)}$$

$$\mathcal{T}_{\text{def}}(F) = \begin{cases} (\overline{d_F} \lor d_G) \land (\overline{d_F} \lor d_H) \land (d_F \lor \overline{d_G} \lor \overline{d_H}), & \text{if } F = G \land H \\ (\overline{d_F} \lor d_G \lor d_H) \lor (d_F \lor \overline{d_G}) \land (d_F \lor \overline{d_H}), & \text{if } F = G \lor H \\ (\overline{d_F} \lor \overline{d_G}) \land (d_F \lor d_G), & \text{if } F = \neg G \end{cases} \qquad \text{(Definitions)}$$

$\mathcal{T}(F)$ introduces a new variable d_S for each subformula S of F and is satisfiable iff F is satisfiable.

Tseitin Encoding

Example (Tseitin Encoding)

$$F = \underbrace{\overbrace{(x \wedge \neg y)}^{a,\ S_a} \vee \underbrace{(z \vee \overbrace{(x \wedge \neg w)}^{c,\ S_c})}_{b,\ S_b}}_{f}$$

(Encoding / Auxiliary Variables)

$$\overset{\text{SAT}}{=} \mathcal{T}_{\text{def}}(S_c) \wedge \mathcal{T}_{\text{def}}(S_b) \wedge \mathcal{T}_{\text{def}}(S_a) \wedge \mathcal{T}_{\text{def}}(F) \wedge f$$

$$\overset{\text{SAT}}{=} \cdots \wedge \underbrace{(f \vee \overline{a}) \wedge (f \vee \overline{b}) \wedge (\overline{f} \vee a \vee b)}_{\mathcal{T}_{\text{def}}(F)} \wedge f$$

(Tseitin Encoding)

$$\overset{\text{SAT}}{=} (c \leftrightarrow x \wedge \neg w) \wedge \cdots \wedge (f \leftrightarrow a \vee b) \wedge f$$

Simplification: treat negative literals like variables in $\mathcal{T}(F)$

Tseitin Encoding: Plaisted-Greenbaum Optimization

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Example (Plaisted-Greenbaum Optimization)

$$\begin{aligned}\mathcal{T}(F) &= f \land (f \leftrightarrow a \lor b) \land (a \leftrightarrow x \land \neg y) \land (b \leftrightarrow z \lor c) \land (c \leftrightarrow x \land \neg w) \\ &= f \land (\overline{f} \lor a \lor b) \land (f \lor \overline{a}) \land (f \lor \overline{b}) \\ &\quad \land (\overline{a} \lor x) \land (\overline{a} \lor \overline{y}) \land (a \lor \overline{x} \lor y) \\ &\quad \land (\overline{b} \lor z \lor c) \land (b \lor \overline{z}) \land (b \lor \overline{c}) \\ &\quad \land (\overline{c} \lor x) \land (\overline{c} \lor \overline{w}) \land (c \lor \overline{x} \lor w)\end{aligned}$$

Relaxed Transformation: Exploit *Don't Cares* in monotonic functions

Model Duplication: Unconstrained encoding variables introduce additional models

Semantic Coupling: $\mathcal{T}(F) \models \mathcal{T}^{PG}(F) \models F$

Tseitin Encoding: Plaisted-Greenbaum Optimization

Example (Plaisted-Greenbaum Optimization)

$$\mathcal{T}^{PG}(F) = f \wedge (f \rightarrow a \vee b) \wedge (a \rightarrow x \wedge \neg y) \wedge (b \rightarrow z \vee c) \wedge (c \rightarrow x \wedge \neg w)$$

$$\begin{aligned}
&= \cancel{f} \wedge (\cancel{\overline{f}} \vee a \vee b) \wedge \cancel{(f \vee \overline{a})} \wedge \cancel{(f \vee \overline{b})} \\
&\quad \wedge (\overline{a} \vee x) \wedge (\overline{a} \vee \overline{y}) \wedge \cancel{(a \vee \overline{x} \vee y)} \\
&\quad \wedge (\overline{b} \vee z \vee c) \wedge \cancel{(b \vee \overline{z})} \wedge \cancel{(b \vee \overline{c})} \\
&\quad \wedge (\overline{c} \vee x) \wedge (\overline{c} \vee \overline{w}) \wedge \cancel{(c \vee \overline{x} \vee w)} \\
&\overset{\text{SAT}}{=} (a \vee b) \wedge (\overline{a} \vee x) \wedge (\overline{a} \vee \overline{y}) \wedge (\overline{b} \vee z \vee c) \wedge (\overline{c} \vee x) \wedge (\overline{c} \vee \overline{w})
\end{aligned}$$

Relaxed Transformation: Exploit *Don't Cares* in monotonic functions

Model Duplication: Unconstrained encoding variables introduce additional models

Semantic Coupling: $\mathcal{T}(F) \models \mathcal{T}^{PG}(F) \models F$

Tseitin Encoding: Plaisted-Greenbaum Optimization

Definition of Plaisted Greenbaum Encoding

$$\mathcal{T}(F) = d_F \wedge \mathcal{T}^1(F)$$

$$\mathcal{T}^p(F) = \begin{cases} \mathcal{T}^p_{\text{def}}(F) \wedge \mathcal{T}^p(G) \wedge \mathcal{T}^p(H), & \text{if } F = G \circ H \text{ and } \circ \in \{\wedge, \vee\} \\ \mathcal{T}^p_{\text{def}}(F) \wedge \mathcal{T}^{p \oplus 1}(G), & \text{if } F = \neg G \\ \mathit{True}, & \text{if } F \in \mathcal{V} \end{cases}$$

$$\mathcal{T}^1_{\text{def}}(F) = \begin{cases} (\overline{d_F} \vee d_G) \wedge (\overline{d_F} \vee d_H), & \text{if } F = G \wedge H \\ (\overline{d_F} \vee d_G \vee d_H), & \text{if } F = G \vee H \\ (\overline{d_F} \vee \overline{d_G}), & \text{if } F = \neg G \end{cases}$$

$$\mathcal{T}^0_{\text{def}}(F) = \begin{cases} (d_F \vee \overline{d_G} \vee \overline{d_H}), & \text{if } F = G \wedge H \\ (d_F \vee \overline{d_G}) \wedge (d_F \vee \overline{d_H}), & \text{if } F = G \vee H \\ (d_F \vee d_G), & \text{if } F = \neg G \end{cases}$$

Recap

Elementary Encodings

- Tseitin Transformation
  - Tseitin encoding allows to carry over structure to CNF
  - Formula size linear in the number of subformulas (of bounded arity)
- Cardinality Constraints
- Finite Domain Encodings

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Next Up

Cardinality Constraints

At-Most-One Constraints

Notation: $\text{AtMostOne}(x_1, \ldots, x_n)$ **or** $\leq 1\,(x_1, \ldots, x_n)$ **or** $\sum_i^n x_i \leq 1$

Not more than one literal from $x_1, \ldots, x_n$ is set to True.

Naive / Pairwise Encoding

$\mathcal{E}\left[\leq 1\,(x_1, \ldots, x_n)\right] = \left\{\{\overline{x_i}, \overline{x_j}\} \mid 1 \leq i < j \leq n\right\}$

Size: $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ clauses

Size Complexity

Encoding	Clauses	Enc. Variables	Consistency
Pairwise Encoding	$\mathcal{O}(n^2)$	0	direct
Tree Encoding	$\mathcal{O}(n \log n)$	$\log n$	propagate
Ladder Encoding	$\mathcal{O}(n)$	n	propagate

Cardinality Constraints

Notation: $\leq k\,(x_1, \ldots, x_n)$ **or** $\sum_i^n x_i \leq k$

Not more than k literals from $x_1, \ldots, x_n$ are set to True.

Naive Encoding

$\mathcal{E}\left[\leq k\,(x_1, \ldots, x_n)\right] = \left\{\{\overline{x_{i_1}}, \ldots, \overline{x_{i_{k+1}}}\} \mid 1 \leq i_1 < \cdots < i_{k+1} \leq n\right\}$ Size: $\binom{n}{k+1}$ clauses[a]

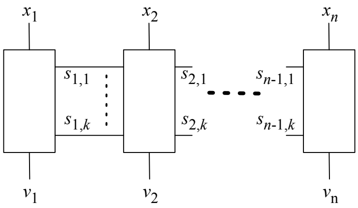
[a] $\approx 2^n/\sqrt{n}$ by Stirling's Approx. for the worst case $k = \lceil n/2 \rceil$

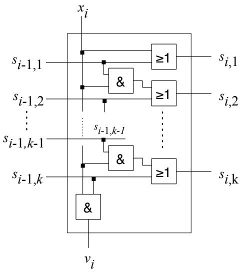
Size Complexity

Encoding	Clauses	Enc. Variables	Consistency
Naive Encoding	$\binom{n}{k+1}$	0	direct
Sequential Counter Encoding	$\mathcal{O}(n \cdot k)$	$\mathcal{O}(n \cdot k)$	propagate
Parallel Counter Encoding	$\mathcal{O}(n)$	$\mathcal{O}(n)$	search

Cardinality Constraints: Sequential Counter Encoding

Idea: encode count-and-compare hardware circuit (cf. Sinz, 2005)





$$(\neg x_1 \vee s_{1,1})$$

$$(\neg s_{1,j}) \qquad \text{for } 1 < j \leq k$$

$$\left.\begin{array}{l} (\neg x_i \vee s_{i,1}) \\ (\neg s_{i-1,1} \vee s_{i,1}) \\ \left.\begin{array}{l} (\neg x_i \vee \neg s_{i-1,j-1} \vee s_{i,j}) \\ (\neg s_{i-1,j} \vee s_{i,j}) \end{array}\right\} \quad \text{for } 1 < j \leq k \\ (\neg x_i \vee \neg s_{i-1,k}) \end{array}\right\} \text{for } 1 < i < n$$

$$(\neg x_n \vee \neg s_{n-1,k})$$

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  - Size of encoding vs. Complexity of consistency
  - Choice of encoding matters
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Next Up

Finite Domain Encodings

Finite-Domain Variables

Common in combinatorial problems: finite domain variables, e.g.: $x \in \{v_1, \ldots, v_n\}$
Relationships between them expressed as equality-formulas, e.g.: $x = v_3 \Rightarrow y \neq v_2$.

Direct encoding / “one-hot-encoding”

- Boolean variables x_v: “x takes value v”
- Must encode that each variable takes exactly one value from its domain (using at-least-one/at-most-one constraints)
- Encoding of variables’ constraints simple

Finite-Domain Variables

Common in combinatorial problems: finite domain variables, e.g.: $x \in \{v_1, \ldots, v_n\}$
Relationships between them expressed as equality-formulas, e.g.: $x = v_3 \Rightarrow y \neq v_2$.

Log-encoding / binary encoding

- Boolean variables b_i^x for $0 \leq i < \lceil \log_2 n \rceil$
- Each value gets assigned a binary number, e.g. $v_1 \rightarrow 00$, $v_2 \rightarrow 01$, $v_3 \rightarrow 10$
- Inadmissible values must be excluded, e.g.:
  $x \in \{v_1, v_2, v_3\}$ requires $(\overline{b_0^x} \vee \overline{b_1^x})$
- Encoding of constraints can become complicated

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- Cardinality Constraints
  - Size of encoding vs. Complexity of consistency
  - Choice of encoding matters
- Finite Domain Encodings
  - One-hot encoding vs. Log encoding
  - One-hot often simpler w.r.t. interaction between encodings