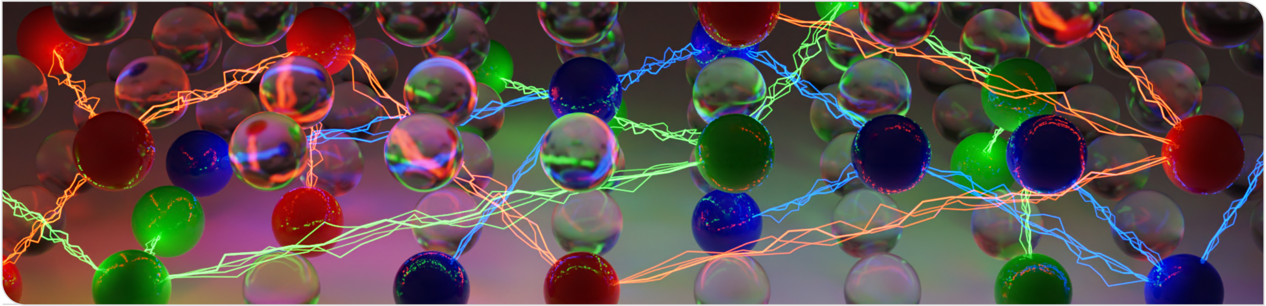
Karlsruher Institut für Technologie

Practical SAT Solving

Lecture 3

Markus Iser, Dominik Schreiber, Tomáš Balyo | April 29, 2024



Overview

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Recap. Lecture 2

- Tractable Subclasses
- Constraint Encodings and their Properties

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- Constraint Encodings and their Properties

Today's Topics: Elementary SAT Algorithms

- Local Search
- Resolution
- DP Algorithm
- DPLL Algorithm

Stochastic Local Search (SLS)

Minimize the Number of Unsatisfied Clauses

Start with a random complete variable assignment α:

0	0	1	0	1	1	0	0	1	1	0	1	0	1	0

Repeatedly flip variables in α to decrease the number of unsatisfied clauses:

0	0	1	0	1	0	0	0	1	1	0	1	0	1	0

Stochastic Local Search (SLS)

Properties of SLS Algorithms

Local search algorithms are incomplete: They cannot show unsatisfiability!

Challenges:

- Which variable should be flipped next?

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  - select variable from an unsatisfied clause
  - select variable that maximizes the number of satisfied clauses
- How to avoid getting stuck in local minima?

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Challenges:

- Which variable should be flipped next?
  - select variable from an unsatisfied clause
  - select variable that maximizes the number of satisfied clauses
- How to avoid getting stuck in local minima?
  - randomization

Classic SLS Algorithms

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GSAT (Selman et al., 1992)

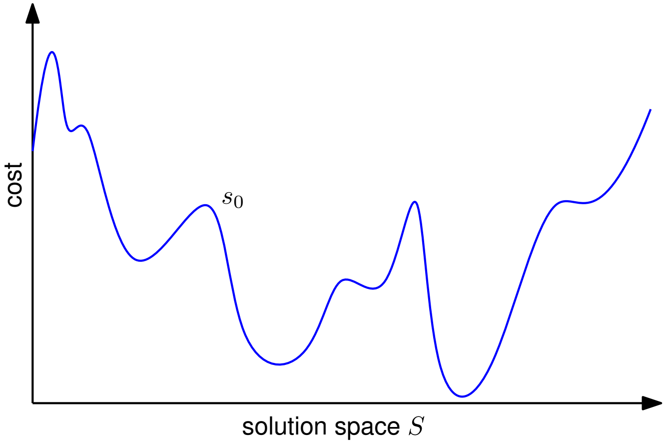
Greedy local search algorithm

Algorithm 1: GSAT

Input: ClauseSet S
Output: Assignment α, or Nothing

```
for i = 1 to MAX_TRIES do
    α = random-assignment to variables in S
    for j = 1 to MAX_FLIPS do
        if α satisfies all clauses in S then return α
        x = variable that produces least number of unsatisfied
            clauses when flipped
        flip x
return Nothing                          // no solution found
```

SLS: Local Minima



[Illustration Adapted from: Alan Mackworth, UBC, Canada]

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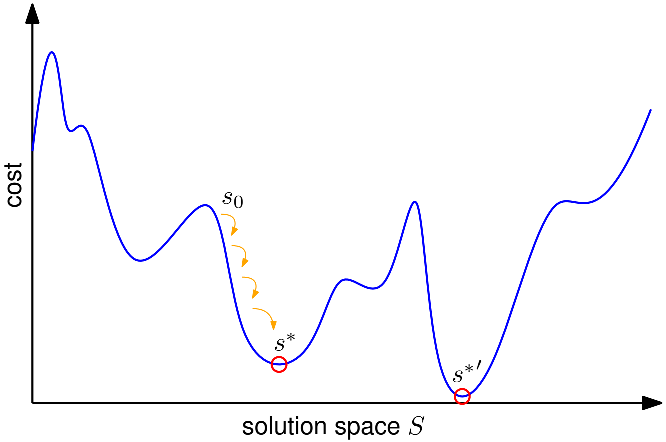

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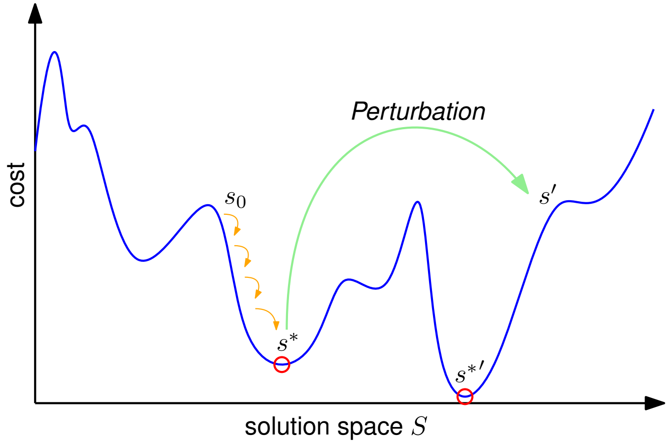

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Classic SLS Algorithms

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WalkSAT (Selman et al., 1993)

Variant of GSAT

Try to avoid local minima by introducing random noise.

Algorithm 2: WalkSAT(S)

```
for i = 1 to MAX_TRIES do
    α = random-assignment to variables in S
    for j = 1 to MAX_FLIPS do
        if α satisfies all clauses in S then return α
        C = random unsatisfied clause in S
        if by flipping an x ∈ C no new unsatisfied clauses
          emerges then flip x
        else with probability p flip an x ∈ C at random
        otherwise, flip a variable that changes the least number
          of clauses from satisfied to unsatisfied
return Nothing                          // no solution found
```

SLS: Important Notions

Consider a flip taking α to α'

breakcount	number of clauses satisfied in α, but not satisfied in α'
makecount	number of clauses not satisfied in α, but satisfied in α'
diffscore	# unsatisfied clauses in α $-$ # unsatisfied clauses in α'

Typically, breakcount, makecount, and/or diffscore are used to select the variable to flip.

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Recap using new nomenclature

GSAT select variable with highest diffscore

WalkSAT select variable with minimal breakcount

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Legacy of SLS

- Extremely successful and popular in early days of SAT
  - SLS outperformed early resolution-based solvers, e.g., based on DP or DPLL
  - for example, state of the art engine for automated planning in the 90s

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- Today, sophisticated resolution-based systematic search solvers dominate in most practical applications
  - Faster, more reliable, and complete!

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  - for example, state of the art engine for automated planning in the 90s
- Today, sophisticated resolution-based systematic search solvers dominate in most practical applications
  - Faster, more reliable, and complete!
- Still useful as a component in more complex solvers
  - Part of (parallel) algorithm portfolios
  - Control branching heuristics in complete search algorithms
  - Detection of autarkies in formula simplification algorithms
  - In combination with complete solvers for optimization problems (e.g., MaxSAT)

Recap

Elementary Algorithms

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  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore

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Next Up

- Resolution

Resolution

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The Resolution Rule

$$\frac{P_1 \cup \{x\}, \quad P_2 \cup \{\neg x\}}{P_1 \cup P_2}$$

Resolution is a logical inference rule to infer a conclusion (resolvent) from given premises (input clauses).

Example (Resolution)

$$\{x_1, x_3, \neg x_7\}, \{\neg x_1, x_2\} \quad \vdash \quad \{x_3, \neg x_7, x_2\}$$

$$\{x_4, x_5\}, \{\neg x_5\} \quad \vdash \quad \{x_4\} \qquad \text{(Fact)}$$

$$\{x_1, x_2\}, \{\neg x_1, \neg x_2\} \quad \vdash \quad \{x_1, \neg x_1\} \qquad \text{(Tautological Resolvent)}$$

$$\{x_1\}, \{\neg x_1\} \quad \vdash \quad \{\} \qquad \text{(Empty Clause)}$$

Resolution

Theorem: Resolution is Sound

Given a CNF formula F with two resolvable clauses $C_1, C_2 \subseteq F$ with resolvent $\mathrm{R}(C_1, C_2)$, the following holds:

$$F \quad \equiv \quad F \wedge \mathrm{R}(C_1, C_2)$$

Proof

Let $C_1 := \{x\} \cup P_1$ and $C_2 := \{\neg x\} \cup P_2$ such that $\mathrm{R}(C_1, C_2) = P_1 \cup P_2 =: D$.

Soundness: $F \vdash F \wedge D \implies F \models F \wedge D$

Any satisfying assignment ϕ of F is also a satisfying assignment of D: Since ϕ satisfies both C_1 and C_2, it necessarily satisfies at least one literal in D. If ϕ satisfies x then it satisfies some literal in P_2. Otherwise, if ϕ satisfies $\neg x$ then it satisfies some literal in P_1.

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Equivalence: $F \vdash F \wedge D \implies F \wedge D \models F$

Since D does not introduce new variables, $F \wedge D$ can not have more satisfying assignments than F.

Resolution

Resolution is Sound and Refutation Complete

- If we manage to infer the empty clause from a CNF formula F, then F is unsatisfiable. (sound)
- If F is unsatisfiable, then there exists a refutation by resolution. (complete)
- Not all possible consequences of F can be derived by resolution. ("only" refutation complete)

Resolution Proof

A resolution proof for F is a sequence of clauses $\langle C_1, C_2, \ldots, C_{k-1}, C_k = \emptyset \rangle$ where each C_i is either an original clause of F or a resolvent of two earlier clauses.

Example (Resolution Proof)

$$F = \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\} \qquad \text{(Formula)}$$

(Refutation)

Resolution

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Example (Resolution Proof)

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$$\equiv \{x_1, x_2\}, \{\neg x_1, x_2\}, \{x_1, \neg x_2\}, \{\neg x_1, \neg x_2\}, \{x_2\}, \{\neg x_2\}, \{\} \qquad \text{(Refutation)}$$

Saturation Algorithm

Properties

- sound and complete – always terminates and answers correctly
- exponential time and space complexity

Algorithm 3: Saturation Algorithm

Input: CNF formula F

Output: $\{$SAT, UNSAT$\}$

while $true$ **do**
  $R :=$ resolveAll(F)
  if $R \cap F \neq R$ **then** $F := F \cup R$
  else **break**
if $\bot \in F$ **then** **return** UNSAT
else **return** SAT

Unit Propagation

Unit Resolution

Resolution where at least one of the resolved clauses is a unit clause, i.e. has size one.

Example (Unit Resolution)

$$R((x_1 \lor x_7 \lor \neg x_2 \lor x_4), (x_2)) = (x_1 \lor x_7 \lor x_4)$$

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Unit Propagation

Apply unit resolution until fixpoint is reached.

Example (Unit Propagation)

Usually, we are only interested in the inferred facts (unit clauses) and conflicts (empty clauses).

$$\{x_1, x_2, x_3\}, \{x_1, \neg x_2\}, \{\neg x_1\} \quad \vdash_1 \quad \{\neg x_2\}, \{x_3\}$$

Recap

Elementary Algorithms

- Local Search
  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore
- Resolution
  - Soundness and Completeness
  - Saturation Algorithm (Exponential Complexity)
  - Unit Propagation

Recap

Elementary Algorithms

- Local Search
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- Resolution
  - Soundness and Completeness
  - Saturation Algorithm (Exponential Complexity)
  - Unit Propagation

Next Up

Davis Putnam (DP) Algorithm (Improving upon saturation-based resolution)

Davis-Putnam Algorithm (Davis & Putnam, 1960)

Presented in 1960 as a SAT procedure for first-order logic.

Deduction Rules of DP Algorithm

- **Unit Resolution:** If there is a unit clause $C = \{l\} \in F$, simplify all other clauses containing l
- **Pure Literal Elimination:** If a literal l never occurs negated in F, add clause $\{l\}$ to F
- **Case Splitting:** Put F in the form $(A \vee l) \wedge (B \vee \bar{l}) \wedge R$, where A, B, and R are clause sets free of l. Replace F by the clausification of $(A \vee B) \wedge R$

Apply above deduction rules (prioritizing rules 1 and 2) until one of the following situations occurs:

- $F = \emptyset \quad \rightarrow$ SAT
- $\emptyset \in F \quad \rightarrow$ UNSAT

Davis-Putnam Algorithm

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Example (DP Algorithm)

$F = \{\{x, y, \neg z, u\}, \{\neg x, y, u\}, \{x, \neg y, \neg z\}, \{z, v\}, \{z, \neg v\}, \{\neg z, \neg u\}, \{\neg x, \neg y, u\}\}$ (Split by x)

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Davis-Putnam Algorithm

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Example (DP Algorithm)

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$F_4 = \{\{v\}, \{\neg v\}\} \vdash_1 \{\emptyset\}$ (Unit Resolution)

DP Variant: Bucket Elimination

Bucket Elimination

- Fix order $\prec$ on variables.
- **Bucket:** set of clauses with same $\prec$-maximal variable
- **Bucket Elimination:** process buckets in decreasing $\prec$-order
  - resolve all clauses in bucket
  - put resolvents in fitting bucket

DP Variant: Bucket Elimination

Example (Bucket Elimination)

$$F = \{(x, y, \overline{z}, u), (\overline{x}, y, u), (x, \overline{y}, \overline{z}), (z, v), (z, \overline{v}), (\overline{z}, \overline{u}), (\overline{x}, \overline{y}, u)\} \qquad (x \succ y \succ z \succ u \succ v)$$

Variable	Bucket
x	$(x, y, \overline{z}, u), (\overline{x}, y, u), (x, \overline{y}, \overline{z}), (\overline{x}, \overline{y}, u)$
y	
z	$(z, v), (z, \overline{v}), (\overline{z}, \overline{u})$
u	
v	

DP Variant: Bucket Elimination

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Example (Bucket Elimination)

$$F = \{(x, y, \overline{z}, u), (\overline{x}, y, u), (x, \overline{y}, \overline{z}), (z, v), (z, \overline{v}), (\overline{z}, \overline{u}), (\overline{x}, \overline{y}, u)\} \qquad (x \succ y \succ z \succ u \succ v)$$

Variable	Bucket
x	processed
y	$(y, \overline{z}, u), (\overline{y}, \overline{z}, u)$
z	$(z, v), (z, \overline{v}), (\overline{z}, \overline{u})$
u	
v	

DP Variant: Bucket Elimination

Example (Bucket Elimination)

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Variable	Bucket
x	processed
y	processed
z	processed
u	$(\overline{u}, v), (u, v), (\overline{u}, \overline{v}), (u, \overline{v})$
v	

DP Variant: Bucket Elimination

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Example (Bucket Elimination)

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Variable	Bucket
x	processed
y	processed
z	processed
u	processed
v	$(v), (\overline{v})$

DP: Discussion

The superiority of the present procedure over those previously available is indicated in part by the fact that a formula on which Gilmore's routine for the IBM 704 causes the machine to compute for 21 minutes without obtaining a result was worked successfully by hand computation using [DP] in 30 minutes.

—from Davis' and Putnam's Paper

- Does DP improve on saturation's average time complexity?

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DP: Discussion

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- Does DP improve on saturation's average time complexity?

  ⇒ yes — if we split over the right variables

- Does DP avoid saturation's exponential space complexity?

  ⇒ no — quadratic blowup in size for eliminating one variable

DPLL Algorithm (Davis et al., 1962)

Davis Putnam Logemann Loveland (DPLL) Algorithm

- DPLL is a backtracking search over partial variable assignments.
- Case splitting over a variable x branches the search over two cases x and $\neg x$: resulting in the simplified formulas $F_{|x=\texttt{true}}$ and $F_{|x=\texttt{false}}$
- Simplification rules:
  - **Unit Propagation:** If $\{l\} \in F$, l must be set to $\texttt{true}$.
  - **Pure Literal Elimination:** If x occurs only positively (or only negatively), it may be fixed to the respective value.

DPLL Algorithm

start with simplifications

recurse on subformulas obtained by case-splitting

stop if satisfying assignment found or all branches are unsatisfiable

Algorithm 4: DPLL(ClauseSet S)

while *S contains a unit clause $\{L\}$* **do**
  delete from S clauses containing L `// unit-subsumption`
  delete $\neg L$ from all clauses in S `// unit-resolution`
if $\emptyset \in S$ **then return** false `// empty clause`
while *S contains a pure literal L* **do**
  delete from S all clauses containing L `// pure literal elimination`
if $S = \emptyset$ **then return** true `// no clauses`
choose a literal L occurring in S `// case-splitting`
if *$DPLL(S \cup \{\{L\}\})$* **then return** true `// first branch`
else if *$DPLL(S \cup \{\{\neg L\}\})$* **then return** true `// second branch`
else return false

DPLL Algorithm with Trail

(S, α) is the clause set S as "seen" under partial assignment α

No pure literal elimination (it is too slow for the benefit it provides)

trailDPLL() leads to efficient iterative DPLL implementation

Algorithm 5: trailDPLL(ClauseSet S, PartialAssignment α)

while (S, α) *contains a unit clause* $\{L\}$ **do**
    add $\{L = 1\}$ to α `// Unit Propagation`
if *a literal is assigned both* 0 *and* 1 *in* α **then**
    return false `// Conflict`
if *all literals assigned* **then**
    return true `// Assignment found`
choose a literal L not assigned in α occurring in S `// Case Splitting`
if *trailDPLL(S,* $\alpha \cup \{\{L = 1\}\}$*)* **then**
    return true `// first branch`
else if *trailDPLL(S,* $\alpha \cup \{\{L = 0\}\}$*)* **then**
    return true `// second branch`
else **return** false

DPLL Algorithm

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Properties

- DPLL always terminates
  - Each recursion eliminates one variable
  - Worst case: binary tree search of depth $|V|$
- DPLL is sound and complete
  - If clause set S is SAT, we eventually find a satisfying α
  - If clause set S is UNSAT, the entire space of (partial) variable assignments is searched (but variable selection still matters!)
- Space complexity: linear!
  — systematic search avoids blowup of “unfocused” DP

Recap

Elementary Algorithms

- Local Search
  - Examples: GSAT, WalkSAT
  - Terminology: breakcount, makecount, diffscore
- Resolution
  - Soundness and Completeness
  - Saturation Algorithm (Exponential Complexity)
- DP Algorithm
  - Systematized Resolution
  - Improved Average Time Complexity
- DPLL Algorithm
  - Case Splitting and Unit Propagation
  - Linear Space Complexity

Next Steps

Coming Lectures

- How can we implement unit propagation efficiently?
- Which literal L to use for case splitting?
- How can we efficiently implement the case splitting step?