Practical SAT Solving

Lecture 7

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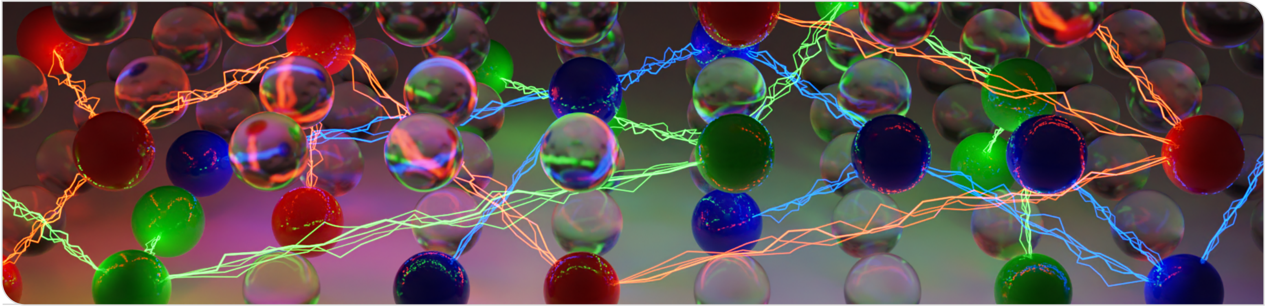

Recap

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Lecture 6: Modern SAT Solving 2

- Efficient Unit Propagation
- Clause Forgetting
- Modern Decision Heuristics: VSIDS & Co.

Today

Preprocessing

Conflict-driven Clause Learning (CDCL) Algorithm

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Last Time

- Efficient Unit Propagation
- Clause Forgetting
- Modern Decision Heuristics

Today

- Preprocessing

Algorithm 1: CDCL(CNF Formula F, &Assignment A $\leftarrow \emptyset$)

```
if not PREPROCESSING then return UNSAT
while A is not complete do
    UNIT PROPAGATION
    if A falsifies a clause in F then
        if decision level is 0 then return UNSAT
        else
            (clause, level) ← CONFLICT-ANALYSIS
            add clause to F and backtrack to level
            continue
    if RESTART then backtrack to level 0
    if CLEANUP then forget some learned clauses
    BRANCHING
return SAT
```

Preprocessing

Preprocessing takes place between problem encoding and its solution.

Preprocessing is ...

- a form of reencoding a problem: to fix bad encodings
- a form of reasoning itself: inprocessing

Conjecture: Smaller problems are easier to solve $\Longrightarrow$ Try to reduce the size of the formula.

Classic Preprocessing Techniques

- Subsumption
- Self-subsuming Resolution
- (Bounded) Variable Elimination (BVE)

Preprocessing: Subsumption

A clause C is subsumed by D iff $D \subseteq C$.

Subsumed clauses can be removed from the formula without changing satisfiability: $\forall D \subseteq C, D \models C$

Example

$\{a, b\}$ subsumes $\{a, b, c\}$ and $\{a, b, d\}$

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Implementation 1: Forward Subsumption

Select clause C and check if it is subsumed by any other clause $D \subseteq C$.

- Temporarily mark all literals in C as unsatisfied, and use one-watched literal data-structure to find subsumed clauses
- **Optimization 1:** Watch literals with the fewest occurrences
- **Optimization 2:** Keep literals sorted and perform merge-sort style subset check

Preprocessing: Subsumption

Implementation 2: Backward Subsumption

Select clause D and check if it subsumes any other clause $C \supseteq D$.

Learned clauses are never subsumed but can subsume other clauses, e.g., recently learned clauses

- **Optimization 1:** Check the clauses of the variable with the fewest occurrences (scales to large formulas)
- **Optimization 2:** Use signatures to skip the majority of subsumption checks (cf. Bloom filters)

Algorithm 2: Signature-based Subsumption Check

`// Initialization:`

for *clause* $\in$ *formula* **do**
    clause.signature = 0
    for *lit* $\in *$*clause* **do**
        clause.signature |= 1ull $<<$ $(id(lit)\%64)$

`// Subsumption Check:`

if $D.signature \,\&\, invert(C.signature) == 0$ **then**
    `// Check if` D `subsumes` C

Preprocessing: Self-Subsuming Resolution

Applicable if the resolvent of C and another clause D subsumes C.

If $C \otimes_x D \subseteq C$ then C can be replaced by $C \otimes_x D$.

Example

Let $\otimes_f$ be the resolution operator on variable f.

$C := \{\neg b, \neg e, f, \neg h\}$ $\quad D := \{\neg b, \neg e, \neg f\}$ $\quad E := C \otimes_f D = \{\neg b, \neg e, \neg h\}$

$\longrightarrow$ Replace C by E (“clause strengthening”)

Implementation

- **Integrate with subsumption:** Allow at most one literal of D to occur negated in C
- **Variant:** On-the-fly subsumption/strengthening of reason clauses during conflict analysis

Preprocessing: Bounded Variable Elimination

Let $S_x, S_{\overline{x}} \subseteq F$ be the sets of all clauses containing x resp. $\overline{x}$, and let $R = \{C \otimes_x D \mid C \in S_x, D \in S_{\overline{x}}\}$ be the set of all resolvents on x. The formulas F and $F' := (F \setminus (S_x \cup S_{\overline{x}})) \cup R$ are equisatisfiable but not equivalent.

Most important preprocessing technique in practical SAT solving

Bounded Variable Elimination (BVE)

Eliminate variable only if the formula size does not increase (too much).

- **Note 1:** Variables of removed clauses have to be rescheduled for further elimination attempts
- **Note 2:** Resolvent can trigger further subsumptions and vice versa
- Particularly effective in presence of functional definitions (cf. Tseitin encoding)
- **Variant:** Incrementally Relaxed BVE: Increase bound each round if formula size did not increase too much
- **Optimizations:** Perform check only for bounded clause size, resolvent size, or variable occurrence count

Preprocessing: Blocked Clause Elimination (BCE)

A clause $\{x\} \cup C$ is blocked in F by x if either x is pure in F or
for every clause $\{\neg x\} \cup D$ in F the resolvent $C \cup D$ is a tautology.

$\rightarrow$ Dead ends in the resolution graph, no proof beyond this point.

Blocked clause elimination (BCE) has a unique fixpoint, and preserves satisfiability.

Example

$F := (a \vee b) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee c)$

First clause is not blocked, second is blocked by both a and $\neg c$, third is blocked by c.

- Effectiveness of BVE can be increased by interleaving it with BCE
- Relationship with circuit-level simplification techniques
- **Generalization: Covered Clauses**
  A clause is covered if it can be turned into a blocked clause by adding a covered literal. A literal x is covered by a clause C, if it contains a literal y such that all non-tautological resolvents of C on y contain x.

Preprocessing: Solution Reconstruction

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Many preprocessing techniques remove clauses or variables from a formula in a mere satisfiability-preserving way, such that the solution to the preprocessed formula might not be a solution to the original formula.

Reconstruction Algorithm

Keep track of eliminated variables (BVE) and clauses (BCE) in a solution reconstruction stack S, and if a model is found, use it to reconstruct a solution to the original formula.

Algorithm 3: Solution Reconstruction

Data: Assignment A, Stack S

while *S is not empty* **do**
- remove the last literal-clause pair (l, C) from S;
- **if** *C is not satisfied by A* **then**
  - $A := (A \setminus \{l = 0\}) \cup \{l = 1\}$

If variables remain unassigned in A, then assign them an arbitrary value.

Recap.

Preprocessing: Classic Techniques

- Subsumption and Self-subsuming Resolution
- Bounded Variable Elimination
- Blocked Clause Elimination
- Solution Reconstruction

Next Up

Relationship between preprocessing techniques and gate encodings

Preprocessing: Relationship with Gate Encodings

Tseitin encoding E of a gate with output o, function g, and input literals $x_1, \dots, x_n$:

$$E \equiv o \leftrightarrow g(x_1, \dots, x_n)$$

Properties of Gate Encodings

Let a Tseitin encoding $E \equiv o \leftrightarrow g(x_1, \dots, x_n)$ be given, and let $A(X) := \{T \cup \{\overline{x} \mid x \in X \setminus T\} \mid T \in 2^X\}$ denote the set of all assignments to variables in X.

For each input assignment $I \in A(\{x_1, \dots, x_n\})$,

a) there exists at least one output assignment $O \in \{o, \overline{o}\}$ such that $I \cup O \models E$ (left-totality)

b) there exists at most one output assignment $O \in \{o, \overline{o}\}$ such that $I \cup O \models E$ (right-uniqueness)

$\rightarrow$ The output is uniquely determined by the input, such that either $I, o \models E$ and $I, \overline{o} \not\models E$ or vice versa.

Preprocessing: Relationship with Gate Encodings

From the left-totality it follows that a Tseitin encoding E is a satisfiable set of blocked clauses.

Left-Totality of Gate Encodings

Let a Tseitin encoding $E \equiv o \leftrightarrow g(x_1, \ldots, x_n)$ be given, it holds that

a) for each clause $C \in E$, either $o \in C$ or $\overline{o} \in C$

Proof: The existence of a clause $C \in E$ such that $o \notin \text{vars}(C)$ would contradict left-totality, because the assignment falsifiying C, falsifies E for any assignment to o.

b) and all resolvents $R \in E_o \otimes_o E_{\overline{o}}$ are tautological.

Proof: The existence of a non-tautological resolvent $R \in E_o \otimes_o E_{\overline{o}}$ would contradict left-totality, because $E \models R$ and $o \notin \text{vars}(R)$, such that the assignment falsifying R, falsifies E for any assignment to o.

Preprocessing: Relationship with Gate Encodings

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From the left-totality it follows that a Tseitin encoding E is a satisfiable set of blocked clauses.

Example (Tseitin encoding $E \equiv o \leftrightarrow x \land y$)

Let a Tseitin encoding $E := \big\{\{\neg o, x\}, \{\neg o, y\}, \{o, \neg x, \neg y\}\big\} \equiv o \leftrightarrow x \land y$ be given, it holds that

a) all resolvents in $E_o \otimes_o E_{\overline{o}} = \big\{\{x, \neg x, \neg y\}, \{y, \neg x, \neg y\}\big\} \equiv \top$ are tautological,

b) and Blocked Clause Elimination (BCE) would remove all clauses from E.

Questions:

- What does BCE do to $F = \big\{\{o\}\big\} \cup E$?
- What does BCE do to $F = \big\{\{\neg o\}\big\} \cup E$?
- What does BCE do to $F = \big\{\{q\}, \{\neg q, o, p\}, \{\neg q, \neg o, \neg p\}\big\} \cup E$?

Preprocessing: Relationship with Gate Encodings

Resolving the clauses of a gate encoding on the output literal o results in a set of tautological clauses.

Idea: Optimized Variable Elimination for Gate Encodings E

Let a formula $F = E \cup R$ with gate clauses E and remainder R be given.
Apply variable elimination as follows:

$$
\begin{aligned}
(E_x \cup R_x) \otimes (E_{\overline{x}} \cup R_{\overline{x}}) &\equiv (E_x \otimes R_{\overline{x}}) \cup (R_x \otimes E_{\overline{x}}) \cup (R_x \otimes R_{\overline{x}}) \cup (E_x \otimes E_{\overline{x}}) \\
&\equiv (E_x \otimes R_{\overline{x}}) \cup (R_x \otimes E_{\overline{x}}) \cup (R_x \otimes R_{\overline{x}}) && (E_x \otimes E_{\overline{x}} \equiv \top) \\
&\equiv (E_x \otimes R_{\overline{x}}) \cup (R_x \otimes E_{\overline{x}}) && ((E_x \otimes R_{\overline{x}}) \cup (R_x \otimes E_{\overline{x}}) \models R_x \otimes R_{\overline{x}})
\end{aligned}
$$

Proof Idea: Each clause $c \in R_x \otimes R_{\overline{x}}$, derived by resolving $c_x \in R_x$ and $c_{\overline{x}} \in R_{\overline{x}}$, can also be derived by resolving clauses in $R_{\overline{x}} \otimes E_x$ and $E_{\overline{x}} \otimes R_x$.

Recap.

Recap.

- Classic Preprocessing Techniques: Subsumption, Self-subsuming Resolution, Bounded Variable Elimination, Blocked Clause Elimination
- Relationship between Preprocessing Techniques and Gate Encodings

Next Time

Propagation-based Techniques and Proof Checking