MATH0010 Methods 2 Summary

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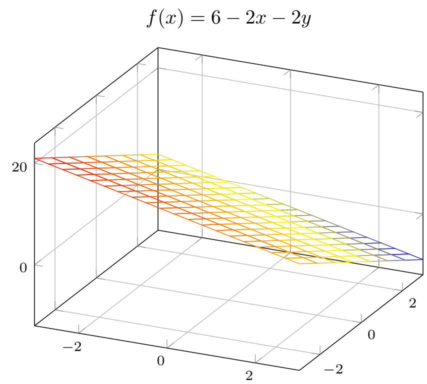
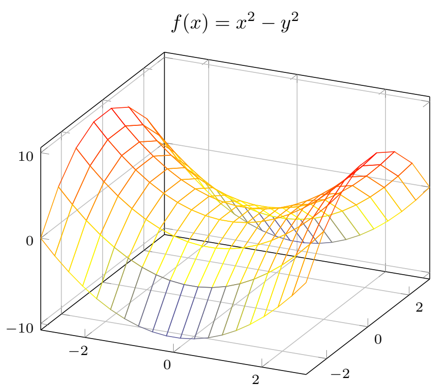
1 Introduction

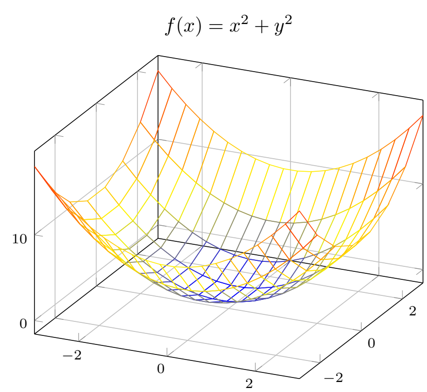
One way to intepret a function of two variables is to think of z=$f(x, y)$ as the height:
Example: $z = 6 - 2x - 2y$



Example: $z = x^2 - y^2$



Example: $z = x^2 + y^2$



Level Sets: for functions of three variables g(x,y,z) one of the most intuitive ways to understand its meaning is to think about its **level surfaces.**

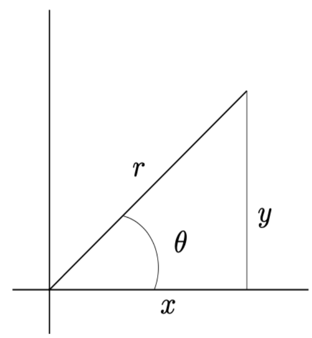
For example: $g(x, y, z) = x^2 + y^2 + z^2$ has level sets which are concentric spheres. $g(x, y, z) = x + y + z$ has level sets which are planes

1.1 Coordinate Systems

1.1.1 2-dimensions

Cartesian coordinates

$$\boxed{\underline{r} = (x, y, z) = x\underline{i} + y\underline{j} + z\underline{k}} \tag{1}$$



Plane polar coordinates In a fixed frame of reference the rectangular coordinates (x,y) are related to the polar coordinates (r,θ) through the relations

$$x = rcos(\theta), y = rsin(\theta), \quad r > 0, \theta \in, 2\pi) \tag{2}$$
$$r = \sqrt{x^2 + y^2}, \theta = arctan(\frac{y}{x}) \tag{3}$$

Example Express in polar coordinates the portion of the unit disc that lies in the first quadrant.

Solution: The region may be expressed in polar coordinates as $0 \leq r \leq 1$ and $0 \leq \theta \leq pi/2$

1.1.2 3-dimensions

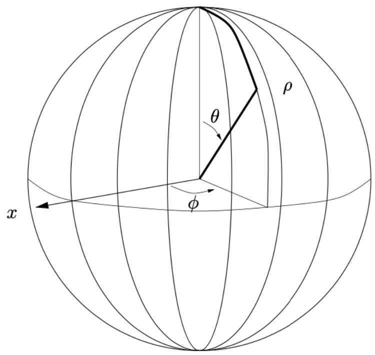
Cylindrical Polar Coordinates

$$\boxed{r = \sqrt{x^2 + y^2}, \theta = arctan(\frac{y}{x}), z = z} \tag{4}$$

Example Express in cylindrical coordinates the solid bounded above by the cone $z = 2 - \sqrt{x^2 + y^2}$ and bounded below by the disc $\Omega : (x - 1)^2 + y^2 \leq 1$.

Solution The region Ω may be expressed as $(rcos(\theta) - 1)^2 + r^2 sin^2(\theta) \leq 1$ which is $-2rcos(\theta) + r^2 \leq 0$ which becomes $r \leq 2cos(\theta)$ and $-\pi/2 \leq \theta \leq \pi/2$. The $z-$limits are $0 \leq z \leq 2 - \sqrt{(x^2 + y^2)}$ which become $0 \leq z \leq 2 - r$

Spherical Coordinates



In a fixed frame of reference the Cartesian coordinates (x,y,z) are related to the spherical coordinates (ρ, θ, ϕ) through the relations:

$$x = \rho sin(\theta)cos(\phi), y = \rho sin(\theta)sin(\phi), z = \phi cos(\theta)$$

$$\rho > 0, \theta \in [0, \pi], \phi \in [0, 2\pi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}, \phi = arctan(\frac{y}{x}), \theta = arccos[\frac{z^2}{\sqrt{x^2 + y^2 + z^2}}]$$

In the above equations,m θ is the latitude or polar angle and ϕ is the longitude

2 Chpt 3: Scalar Functions of several variables

2.1 Functions of two variables

Functions of two variables are where $f(x, y)$ such that $z = f(x, y)$ is the surfce above the xy-plane.

2.1.1 Tangent Plane

- The **tangent plane** at a point A $= (a, b, f(a, b))$ is just the linear approximation to the surface at that point.
- The tangent plane at A is perpendicular to the normal at A

The *linear approximation* to the surface close to A(a,b) is given by the tangent plane so,

$$\boxed{f(x, y) \approx f(a, b) + \begin{pmatrix} x-a \\ y-b \end{pmatrix}^T \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}_A} \tag{5}$$

Example Find the linear approximation to $f(x, y) = x^2 + y^2$ near the point (1,2).

Solution If $f(x, y) = x^2 + y^2$ then $\partial f/\partial x = 2x$ and $\partial f/\partial y = 2y$. At that point (1,2) we have $f = 5, \partial f/\partial x = 2$ and $\partial f/\partial y = 4$. Thus,

$$f(x, y) \approx 5 + (x - 1)2 + (y - 2)4 = 2x + 4y - 5$$

Taylor Series The tangent plane equations is a Taylor Series, truncated at linear order (i.e. the next terms involved all the partial second derivatives of f)

$$f(x, y) \approx f(a, b) + \begin{pmatrix} x-a \\ y-b \end{pmatrix}^T \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}_A + \frac{1}{2}\begin{pmatrix} x-a \\ y-b \end{pmatrix}^T \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\Bigg|_A \begin{pmatrix} x-a \\ y-b \end{pmatrix} \tag{6}$$

where the matrix:

$$H(a, b) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\Bigg|_A$$

is called the Hessian Matrix of the function f at the point (a,b).

2.1.2 Critical points of functions of two variables

In most places, the gradient vector is perpencdicular to contour lines and points "uphill"; but what can we say at places where the gradient is zero?

Points where $\underline{\nabla} f = 0$ are called critical points and to categorise the behaviour of the function near them we use Taylor series. At a *critical point*, the middle term is zero (because $\underline{\nabla} f = 0$) so 6 becomes:

$$f(x, y) \approx f(a, b) + \frac{1}{2}\begin{pmatrix} x-a \\ y-b \end{pmatrix}^T \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\Bigg|_A \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$
$$= f(a, b) + \frac{1}{2}(\underline{x} - \underline{a})^T H(\underline{a})(\underline{x} - \underline{a})$$

Eigenvalues & Eigenvectors
Definition: Suppose we have a square matrix A. Then we say λ is an *eigenvalue* of A with corresponding *eigenvector* $\underline{v}$ if $\underline{v} \neq 0$ and:

$$A\underline{v} = \lambda\underline{v}$$

1. **Finding eigenvalues:**
   Solve $|\mathbf{A} - \lambda\mathbf{I}| = 0$
2. **Finding eigenvectors:**
   Once we know an eigenvalue λ, we can use the relation

   $$(\mathbf{A} - \lambda\mathbf{I})\underline{\mathbf{v}} = \underline{0}$$

   To find out more about finding eigenvalues and eigenvectors: click here.

The hessian matrix (by construction) is a real, symmetric matrix so we know

- H has real eigenvalues: λ_1, λ_2
- The corresponding eigenvectors $\underline{v}_1$ and $\underline{v}_2$ are orthogonal
- The trace of H is the sum $\lambda_1 + \lambda_2$

- The determinant of H is the product $\lambda_1\lambda_2$

Looking at the function approximation, use the unit eigenvectors of H to decompose:

$$\underline{x} - \underline{a} = \alpha \underline{v}_1 + \beta \underline{v}_2$$

which gives (derivation in notes):

$$f(\underline{x}) - f(\underline{a}) = \frac{1}{2}(\alpha^2\lambda_1 + \beta^2\lambda_2)$$

Lambda/'H' Values	Function Values	Outcome
$\lambda_1 > 0$ & $\lambda_2 > 0$	$f(\underline{x}) - f(\underline{a}) > 0$	**Local Min.**
$\|H\| > 0$ & $f_{xx} > 0$		
$\lambda_1 < 0$ & $\lambda_2 < 0$	$f(\underline{x}) - f(\underline{a}) < 0$	**Local Max.**
$\|H\| > 0$ & $f_{xx} < 0$		
λ_1 & λ_2 diff signs	depends	**Saddle Point**
$\|H\| < 0$		

Example Find and classify all the critical points of the function:

$$f(x, y) = x(x^2 - 1)y$$

Solution: We start by finding points where $\underline{\nabla} f = 0$.

$$\frac{\partial f}{\partial x} = 3x^2y - y = y(3x^2 - 1) \qquad \frac{\partial f}{\partial y} = 2x(x^2 - 1)$$

We can satisfy this if either $x = 0$ or $x + \pm 1$. In either cases, $(3x^2 - 1)$ is nonzero so we can only satisfy the first condition by setting y=0. Our three critical points are (0,0), (-1,0) and (1,0).

$$\left.\begin{aligned} &At(0,0), \quad H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ &At(-1,0), \quad H = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \\ &At(1,0), \quad H = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \end{aligned}\right\} \text{All are saddle points}$$

2.2 Chain rule & extended chain rule

Chain Rule

For function of two variables, we obtain chain rule:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}$$

Example If $f(x, y) = x^2 + y^2$ where $x = sin(t), y = t^3$ then,

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt} = 2xcos(t) + 2y3t^2 = 2sin(t)cos(t) + 6t^5$$

Extended chain rule

For $f(x, y)$ suppose that x and y depend on**two** variables s and t. Then changing either s or t changes x and y, so changes f, producing $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ according to the extended chain rule.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Example $f(x, y) = x^2y^3$, where $x = s - t^2, y = s + 2t$ Then,

$$\frac{\partial f}{\partial x} = 2xy^3 \text{ and } \frac{\partial f}{\partial y} = 3x^2y^2$$

$$\begin{aligned} \frac{\partial f}{\partial s} &= 2xy^3 + 3x^2y^2 \\ &= (s - t^2)(s + 2t)^2(5s + 4t - 3t^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial t} &= 2xy^3(-2t) + 3x^2y^2(2) \\ &= 2(s - t^2)(s + 2t)^2(3s - 2st - 7t^2) \end{aligned}$$

2.2.1 The Laplacian

The laplacian $\nabla^2 = \underline{\nabla} \cdot \underline{\nabla}$ which operates on scalar functions $f(x, y, z)$ according to the rule:

$$\boxed{\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}} \tag{7}$$

Example Calculate the Laplacian for $f = x^2 + y^2 + z^2$.

$$\nabla^2 f = 2 + 2 + 2 = 6$$

2.3 Jacobian

The *Jacobian* of the transformation $x = x(s, t), y = y(s, t)$ is the determinant:

$$\boxed{\frac{\partial(x, y)}{\partial(s, t)} = det \begin{vmatrix} \partial x/\partial s & \partial y/\partial s \\ \partial x/\partial t & \partial y/\partial t \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}} \tag{8}$$

2.4 Reference results for polar coordinates:

Jacobians

Following Jacobians can be quoted as standard results:

$$\frac{\partial(x, y)}{\partial(r, \theta)} = r \text{ for plane polar coordinates} \tag{9}$$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r \text{ for cylindrical polar coordinates} \tag{10}$$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \rho^2 sin(\theta) \text{ for spherical polar coordinates.} \tag{11}$$

Grad in polar coordinates

In plane polar coordinates we denote the unit vector in the r-direction by $\underline{e}_r$ and the unit vector in the θ-direction by $\underline{e}_\theta$:

$$\underline{\nabla} = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta$$

For cylindrical polar coordinates:

$$\underline{\nabla} = \frac{\partial f}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{\partial f}{\partial z}\underline{e}_z$$

For **spherical polar coordinates**:

$$\underline{\nabla} f = \frac{\partial f}{\partial r}\underline{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\underline{e}_\theta + \frac{1}{\rho \cdot sin(\theta)}\frac{\partial f}{\partial \phi}\underline{e}_\phi$$

Laplacian in polar coordinates The Laplacian in Cartesian coordinates is given by:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The forms for the Laplacian in different coordinate systems are more complex. They are given by:

Plane polar coordinates:

$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$

Cylindrical polar coordinates:

$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical polar coordinates:

$\nabla^2 f = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2 sin(\theta)}\frac{\partial}{\partial \theta}\left(sin(\theta)\frac{\partial f}{\partial \theta}\right)$
$+ \frac{1}{\rho^2 sin(\theta)}\frac{\partial^2 f}{\partial \phi^2}$

3 Chpt 4: Double and Triple Integrals

3.1 Multi-Integral Notation

Double integrals:

$$\iint_\Omega f(x,y) \cdot dxdy$$

where Ω is some region in the xy-plane.
Triple integral:

$$\iiint_T f(x,y,z) \cdot dxdydz$$

where T is a solid(volume) in the xy-plane.

3.2 Double Integrals

3.2.1 Properties

1. **Area property**

$$\iint_\Omega dxdy = \text{Area of } \Omega$$

   In particular if Ω is the rectangle $\Omega = [a,b] \times [c,d]$ then $\iint_\Omega dxdy = (b-a)(d-c)$

2. **Linearity**

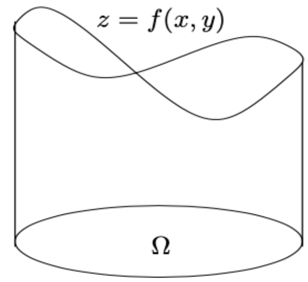
$$\iint_\Omega [\alpha f(x,y) + \beta g(x,y)]dxdy = \alpha \iint_\Omega f(x,y)dxdy + \beta \iint_\Omega g(x,y)dxdy$$

   where α and β are constants.

3. **Additivity** If Ω is broken up into a finite number of nonoverlapping basic regions $\Omega_1, ..., \Omega_n$ then,

$$\iint_\Omega fdxdy = \iint_{\Omega_1} fdxdy + .. + \iint_{\Omega_n} fdxdy$$

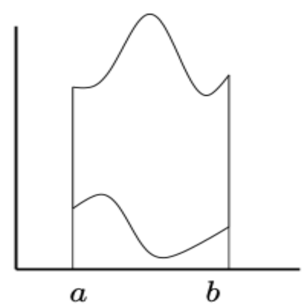
3.2.2 Geometric Interpretation



The double integral over Ω gives the volume of the solid T whose upper boundary is the surface $z = f(x,y)$ and whose lower boundary is the region Ω in the xy-plane.

3.2.3 Evaluating doubled integrals

Horizontally Simple Domain



Here the limits are:

$$a \leq x \leq b, \text{ and } \phi_1(x) \leq y \leq \phi_2(x)$$

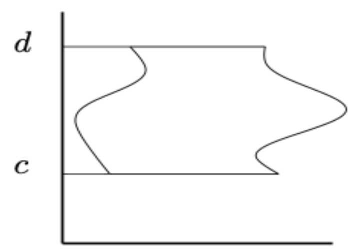
Then,

$$\iint_\Omega f \cdot dxdy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} fdy\right)dx$$

Example
Evaluate $\iint_\Omega (x^4 - 2y)dxdy$, where the domain Ω consists of all points (x,y) with $-1 \leq x \leq 1$ and $-x^2 \leq y^2 \leq x^2$. **Solution:**

$$\begin{aligned}\iint_\Omega (x^4 - 2y)dxdy &= \int_{x=-1}^{x=1}\int_{y=-x^2}^{y=x^2}(x^4 - 2y)dydx \\ &= \int_{x=-1}^{x=1}[x^4y - y^2]_{y=-x^2}^{y=x^2}dx \\ &= \int_{x=-1}^{x=1} 2x^6 dx = \frac{4}{7}\end{aligned}$$

Vertically Simple Domain



Here the limits are:

$$c \leq y \leq d, \text{ and } \psi_1(y) \leq x \leq \psi_2(y)$$

Then,

$$\iint_\Omega f \cdot dxdy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f dx \right) dy$$

Example The triangular region with corners at $(0,0), (1,0)$ and $(1,1)$ can be written either as horizontally simple region:

$$0 \leq x \leq 1 \quad 0 \leq y \leq x$$

or vertically simple region:

$$0 \leq y \leq 1 \quad y \leq x \leq 1$$

3.2.4 Evaluating Double Integrals using Polar Coordinates

Let Ω be a domain formed with all points (x,y) that have polar coordinates (r,θ) in the set where $\beta \leq \alpha + 2pi$

$$\Gamma : \alpha \leq \theta \leq \beta, \quad \rho_1(\theta) \leq r \leq \rho_2(\theta)$$

$$\begin{aligned}
\iint_\Omega f \cdot dxdy &= \iint_\Gamma f(rcos(\theta), rsin(\theta))rdrd\theta \\
&= \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(rcos(\theta), rsin(\theta))rdrd\theta
\end{aligned}$$

Recall that conversion from cartesian to polar coordinates is

$$x = rcos(\theta), \qquad y = rsin(\theta)$$

Example Using the polar coordinates to evaluate $\int\int_\Omega xydxdy$, where Ω is the portion of the unit disc that lies in the first quadrant.

Solution:
The region may be expressed in polar coordinates as $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi/2$. Thus we have

$$\begin{aligned}
\iint_\Omega xy \cdot dxdy &= \int_0^{\pi/2}\int_0^1 rcos(\theta) \cdot rsin(\theta) \cdot rdrd\theta \\
&= \int_0^{\pi/2} cos(\theta)sin(\theta)[r^4/4]_0^1 d\theta \\
&= \frac{1}{4}\int_0^{\pi/2} cos(\theta)sin(\theta)d\theta \\
&= \frac{1}{4}\int_0^{sin(\pi/2)} u \cdot du = \frac{1}{4}[u^2/2]_0^2 = \frac{1}{8}
\end{aligned}$$

One more example in notes...

3.3 Triple Integrals

Check notes to see evaluating triple integrals using cylindrical coordinates, spherical coordinates

3.4 Jacobians and changing variables in multiple integration

3.4.1 Change of variables for double integrals

Consider the change of variables $x = x(u,v)$ and $y = y(u,v)$ which maps the points (u,v) of some domain Γ into the points (x,y) of some other domain Ω. Then

$$\text{The area of } \Omega = \iint_\Gamma \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot dudv$$

Suppose now we want to integrate some function $f(x,y)$ over Ω. If this proves difficult to do directly then we can change variables (x,y) to (u,v) and try to integrate over Γ instead. Then,

$$\iint_\Omega f(x,y) = \iint_\Gamma f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot dudv$$

Example Evaluate the integral $\int\int_\Omega xy \cdot dxdy$ where Ω is the first-quadrant region bounded by the curves $x^2 + y^2 = 4, x^2 + y^2 = 9, x^2 - y^2 = 1 and x^2 - y^2 = 4$.

Solution: The region Ω consists of the intersections of $x > 0$ and $y > 0$ with the four conditions above. We try sub $u = x^2 + y^2, v = x^2 - y^2$. The Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & 2x \\ 2y & -2y \end{vmatrix} = -8xy$$

Hence the inverse Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{8xy}$$

The integral becomes

$$\iint_\Omega xy \cdot dxdy = \frac{1}{8}\int_{u=4}^9 \int_{v=1}^4 dudv = \frac{1}{8}[(9-4)(4-1)] = \frac{15}{8}$$

3.4.2 Change of variables for triple integrals

Consider the change of variables $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$ which maps the points (u,v,w) of some solid S into the points (x,y,z) of some other solid T. Then,

$$\text{The volume of T} = \iiint_S \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \cdot dudvdw$$

Likewise, suppose that we want to integrate some function f over T. If this proves difficult to do directly, we can change the variables (x,y,z) to (u,v,w) and try to integrate over S instead. Then

$$\iiint_T f \cdot dx\,dy\,dz = \iiint_S f \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \cdot du\,dv\,dw$$

Example Calculate the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1$.
Solution: We change the variables to set $u = \frac{x}{a}, v = \frac{y}{b}, w = \frac{z}{c}$.

This has Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$. Then,

$$\iiint_T dx\,dy\,dz = abc \cdot \iiint_{\text{sphere of r=1}} du\,dv\,dw = \frac{4}{3}\pi abc$$

4 Chpt 5: Vector Operators

We've looked at vector functions of single variable, $\underline{r}(t)$ andat scalar functions of several variables $f(\underline{x})$. Now we look at vector functions of several variables $\underline{F}(\underline{x}) = F_1\underline{i} + F_2\underline{j} + F_3\underline{k}$

4.1 Vector differential Operator $\underline{\nabla}(grad)$

Formally defined as:

$$\underline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

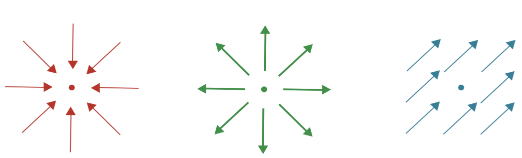
4.1.1 Grad

$$\textbf{Grad} = \underline{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

4.1.2 Divergence

If $\underline{f} = (f_1, f_2, f_3)$ then,

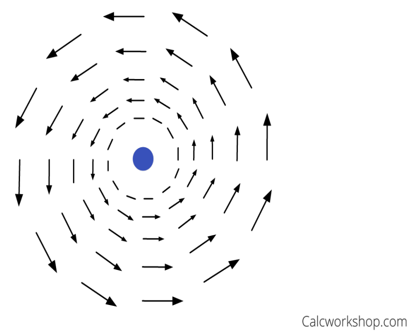
$$\textbf{div}(\underline{\textbf{q}}) = \underline{\nabla} \cdot \underline{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$



4.1.3 Curl

If $\underline{f} = (f_1, f_2, f_3)$ then,

$$\textbf{curl}(\underline{q}) = \underline{\nabla} \times \underline{f} = det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{bmatrix}$$
$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \quad \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \quad \frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial y} \right)$$



4.2 Helmholtz decomposition

Any sufficiently nice vector field $\underline{F}(\underline{x})$ can be written as the sum of a gradient part and a curl part:

$$\underline{F} = \underline{\nabla} f + \underline{\nabla} \times \underline{A}$$

The functions f and A are called the ***scalar potential*** and the ***vector potential*** respectively. Note:

- Irrotational: a vector field which has zero curl
- Solenoidal: a vector field which has zero divergence

5 Chpt 6: Line & Surface Integrals

5.1 Line Integrals

Let $\underline{h} = (h_1, h_2, h_3)$ be a vector function that is continuous over a smooth curve C parameterised by $C : \underline{r}(u) = (x(u), y(u), z(u))$ with $u \in [a,b]$. The $lineintegral$ of $\underline{h}$ over C is the number

$$\int_C \underline{h}(\underline{r}) \cdot \underline{dr} = \int_a^b [\underline{h}(\underline{r}(u)) \cdot \underline{r}'(u)]du$$

Example Calculate $int_C \underline{h}(\underline{r}) \cdot \underline{dr}$ given that $\underline{h}(x,y) = (xy, y^2)$ and the plane curve C is parameterised by $\underline{r}(u) = (u, u^2)$ with u$\in [0,1]$.
Solution:
The derivative of the vector $\underline{r}$ is

$$\underline{r}'(u) = (1, 2u)$$

and on the path of integration, $x = u$ and $y = u^2$ so our integral becomes,

$$\begin{aligned}\int_C \underline{h}(\underline{r})\cdot \underline{dr} &= \int_{u=0}^{1}(xy,y^2)\cdot \underline{r}'(u)du\\ &= \int_{u=0}^{1}(u^3,u^4)\cdot(1,2u)du\\ &= \int_{u=0}^{1}(u^3+2u^5)du\\ &= [u^4/4+2u^6/6]_0^1 = \frac{7}{12}\end{aligned}$$

If the curve C is not smooth but is made up of a finite number of adjoining smooth pieces $C_1, C_2, ...C_n$ it is piecewise smooth. Then we define the integral over C as the sum of the integrals over C_i for $i = 1, ..., n$ that is $\int_C = \int_{C_i} + \cdots + \int_{C_n}$. Example in notes.

5.2 Fundamental Theorem for Line Integrals

In general, if we integrate a vector function $\underline{h}$ from one point to another, the value of the line integral depends on the path chosen.

If the vector function $\underline{h}$ is a *gradient* (i.e. $\exists$ a scalar function f such that $\underline{h} = \underline{\nabla} f$, then the value of the line integral depends only on the endpoints of the path and not on the path itself.

Theorem:

Let C, be parameterised by $\underline{r} = \underline{r}(u)$ with $u \in [a, b]$ be a piecewise smooth curve that begins at $\underline{\alpha} = \underline{r}(a)$ and ends at $\underline{\beta} = \underline{r}(b)$. Then if the vector function $\underline{h}$ is a gradient (i.e. $\underline{h} = \underline{\nabla} f$), we have:

$$\int_C \underline{h}(\underline{r})\cdot \underline{dr} = \int_C \underline{\nabla} f(\underline{r})\cdot \underline{dr} = f(\beta) - f(\alpha)$$

Corollary

If the curve C is closed, i.e. $\underline{\alpha} = \underline{\beta}$, then $f(\alpha) = f(\beta)$ and

$$\int_C \underline{\nabla} f(\underline{r})\cdot \underline{dr} = 0$$

Example Integrate the vector function $\underline{h}(x, y) = (y^2, 2xy - e^{2y}) = \underline{\nabla}(xy^2 - \frac{1}{2}e^{2y})$ over the circular arc $C : \underline{r}(u) = (cos(u), sin(u))$ with $u \in [0, \pi/2]$

Solution:

We note that $\underline{h}$ is a gradient:

$$\underline{h} = (y^2, 2xy - e^{2y}) = \underline{\nabla}(xy^2 - \frac{1}{2}e^{2y})$$

Thus we only need to look at the endpoints:

$$\int_C \underline{h}\cdot \underline{dr} = [xy^2 - \frac{1}{2}e^{2y}]_{(1,0)}^{(0,1)} = \frac{1}{2}(1-e^2)$$

5.2.1 Path Independence

The line integral:

$$\int_C \underline{\nabla} f(\underline{r})\cdot \underline{dr}$$

is a *path-independent integral* meaning that its answer does not depend on the path the curve C takes from its start point to its end point, only on the position of those two points.

In particular, if we take a path-independent integral over a closed curve C, its start and end points are the same so we get:

$$\oint_C \underline{F}\cdot \underline{dr} = 0$$

Note: $\oint$ is the symbol used to denote the line integral over a simple closed curve C taken in the anticlockwise direction.

5.3 Green's Theorem

If $P(x, y)$ and $Q(x, y)$ are scalar functions defined over a domain Ω with piecewise smooth closed boundary C, then

$$\boxed{\iint_\Omega \left[\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right]dx\,dy = \oint P(x,y)dx + Q(x,y)dy}$$

where the integral on the right is a line integral over C taken in the anticlockwise direction.

Example Use Green's theorem to evaluate $\oint_C(3x^2 + y)dx + (2x + y^3)dy$ where C is the circle $x^2 + y^2 = a^2$

Solution In order to fit into the format of Green's Theorem, we need:

$$P(x,y) = 3x^2 + y \quad \text{and} \quad Q(x,y) = 2x + y^3$$

This gives

$$\frac{\partial Q}{\partial x} = 2 \quad \text{and} \quad \frac{\partial P}{\partial y} = 1$$

and so

$$\begin{aligned}\oint_C (3x^2+y)dx + (2x+y^3)dy &= \iint_{x^2+y^2<a^2}[2-1]dxdy\\ &= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r\,d\theta\,dr\\ &= 2\pi[r^2/2]_0^a\\ &= \pi a^2\end{aligned}$$

5.4 The 2D Divergence (Gauss) Theorem

If we introduce a vector field $\underline{q} = (Q, -P)$ we obtain teh divergence theorem in 2D.

Let Ω be a 2D-domain bounded by a piecewise smooth closed curve C. Then for any (continuously differentiable) vector function $\underline{q}(x,y)$ we have that:

$$\boxed{\iint_\Omega (\underline{\nabla} \cdot \underline{q})\, dx\, dy = \oint_C (\underline{q} \cdot \underline{n})\, ds}$$

where $\underline{n}$ is the outer unit normal and the integral on the right is taken wrt arc length.

Example: 2D Divergence Theorem
Verify the 2D Divergence theorem for the vector function $\underline{q} = x\underline{i}$ and the region:

$$\Omega : x^2 + y^2 \leq a^2$$

Solution:
Let's calculate the LHS of Gauss theorem:

$$\underline{\nabla} \cdot \underline{q} = \frac{\partial x}{\partial x} + \frac{\partial 0}{\partial y} = 1$$

so,

$$\iint_\Omega (\underline{\nabla} \cdot \underline{q})\, dx\, dy = \iint_\Omega 1\, dx\, dy = \text{ Area of } \Omega = \pi a^2$$

For the RHS, the curve C is a circle of radius a traversed anti-clockwise:

$$C : \underline{r}(t) = a\, cos(t)\underline{i} + a\, sin(t)\underline{j} \qquad 0 \leq t \leq 2\pi$$

The arc length element is

$$\underline{r}'(t) = -a\, sin(t)\underline{i} + a\, cos(t)\underline{j} \qquad |\underline{r}'(t)| = a \qquad ds = a\, dt$$

We substitute $x = a\, cos(t)$ into the definition of $\underline{\text{q}}$ and note that the outer unit normal to a circle is just the unit radial vector, to obtain.

$$\underline{q} = a\, cos(t)\underline{i} \qquad \underline{n} = cos(t)\underline{i} + sin(t)\underline{j}] \qquad \underline{q} \cdot \underline{n} = acos^2(t)$$

Then the line integral becomes:

$$\begin{aligned}\oint_C (\underline{q} \cdot \underline{n}) &= \int_0^{2\pi} a\, cos^2(t)\, a\, dt \\ &= a^2 \int_0^{2\pi} \frac{1}{2}(1 + cos(t))dt \\ &= \frac{a^2}{2}\left[t + \frac{1}{2}sin(2t)\right]_0^{2\pi} \\ &= \pi a^2\end{aligned}$$

And the two results are the same as required.

5.5 Parameterised Surfaces: Surface Area

- We have seen that a space curve C can be parameterised by a vetor function $\underline{r} = \underline{r}(u)$ where u ranges over some interval I of the u-axis.
- In an analogous manner, we can paramterise a surface S in space by a vector function $\underline{r} = \underline{r}(u,v)$ where (u,v) ranges over some domain Ω of the uv-plane.

Examples

A graph
- Graph of $y = f(x),\ \ x \in [a,b]$ can be parameterised by setting $\underline{r}(u,v) = (u, f(u)),\quad u \in [a,b]$
- Graph of $z = f(x),\ \ (x,y) \in \Omega$ can be parameterised by setting $\underline{r}(u,v) = (u, v, f(u,v)), (u,v) \in \Omega$

A plane If two vectors $\underline{a}$ and $\underline{b}$ are not parallel, then the set of all combinations $u\underline{a} + v\underline{b}$ generates a plane P_0 that passes through the origin. We can parametrise this plane by setting $\underline{\text{r}}$(u,v) $= u\underline{a}+v\underline{b}$.

A sphere The sphere of radius a centered at the origin can be parameterised by setting

$$\underline{r}(u,v) = (a\, sin(u)cos(v), a\, sin(u)sin(v), a\, cos(u))$$

5.5.1 The fundamental vector product

Let S be a surface parameterised by $\underline{r} = \underline{r}(u,v), \quad (u,v) \in \Omega$. The cross product:

$$\underline{N} = \underline{r}'_u \times \underline{r}'_v$$

is called the ***fundamental vector product*** of the surface S. The vector $\underline{N}(u,v)$ is perpendicular to the surface S at the point with position vector $\underline{r}(u,v)$ and if $\neq 0$ it can be taken as the normal to the surface S at that point.

5.5.2 The area of a parameterised surface

The area of a surface S parameterised by $\underline{r} = \underline{r}(u,v), \quad (u,v) \in \Omega$ is given by

$$\text{Area of } S = \iint_S |\underline{N}(u,v)|\, du\, dv$$

Example (Surface area of a sphere)
Using the highlighted equation, we have $\underline{N} = a\, sin(u)\underline{r}$ so $|N| = a^2 sin(u)$ and the area is:

$$\begin{aligned}\iint_S |\underline{N}(u,v)|\, du\, dv = a^2 \int_{u=0}^{\pi} \int_{v=0}^{2\pi} sin(u)\, dv\, du \\ = 2\pi a^2[-cos(u)]_0^{\pi} = 4\pi a^2\end{aligned}$$

Example (The area of a plane domain)
A plane domain may be parameterised as $\underline{r} = (u, v, 0)$ for $(u,v) \in \Omega$. Then $\underline{r}'_u$ = (1,0,0) and $\underline{r}'_v$ = (0,1,0) and so the fundamental vector product is $\underline{N} = (0,0,1)$ which has magnitude 1.

$$\iint_\Omega 1\, du\, dv = \text{Area of } \Omega$$

5.5.3 The Area of a surface $z = f(x, y)$

Let the surface S be the graph of the function $z = f(x, y)$ with $(x, y) \in \Omega$. Then,

$$\text{Area of } S = \iint_\Omega \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1 \right]^{1/2} dx\, dy$$

In this case the parameterisation of S is

$$\underline{r}(u, v) = (u, v, f(u, v)), \quad (u, v) \in \Omega$$

and therefore $\underline{N} = (-f_x, -f_y, 1)$. The unit vector $\underline{n} = \frac{N}{|N|}$ is called the *upper unit normal* because it points upwards.

Example Find the surface area of that part of the parabolic cylinder $z = y^2$ that lies over the triangle with vertices $(0, 0), (0, 1), (1, 1)$ in the xy-plane.

Solution The surface is the graph of the function $f(x, y) = y^2$ and the plane region Ω is $0 \le y \le 1$ and $0 \le x \le y$. Using the equation (6.11) we have

$$\begin{aligned} AreaofS &= \iint_\Omega \sqrt{(2y)^2 + 1} dxdy = \int_{y=0}^1 \sqrt{1 + 4y^2}[x]_{x=0}^y dy \\ &= \int_{y=0}^1 y(1 + 4y^2)dy = \frac{1}{8}\int_{y=0}^1 8y\sqrt{1 + 4y^2}dy \\ &= \frac{1}{8}[\frac{2}{3}(1 + 4y^2)^{3/2}]_{y=0}^1 = \frac{1}{12}[5^{3/2} - 1] \end{aligned}$$

5.6 Surface Integrals

Let $H(x, y, z)$ be a scalar function, continuous over a surface S parameterised by $\underline{r} = \underline{r}(u, v), \quad (u, v) \in \Omega$. The *surface integral* of H over S is the number

$$\iint_S H(x, y, z)d\sigma = \iint_\Omega H(\underline{r}(u, v))|\underline{N}(u, v)|\, du\, dv$$

Example If $\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$, calculate $\iint_S xy\, d\sigma$, where $S : \underline{r}(u, v) = u\underline{a} + v\underline{b}$ with $(u, v) \in [0, 1] \times [0, 1]$.

Solution The fundamental vector product is $\underline{N} = \underline{r}'_u \times \underline{r}'_v = \underline{a} \times \underline{b}$. Thus,

$$\begin{aligned} \iint_S xyd\sigma &= \int_0^1 \int_0^1 (a_1u + b_1v)(a_2u + b_2v)|\underline{a} \times \underline{b}|\, du\, dv \\ &= |\underline{a} \times \underline{b}| \int_0^1 \int_0^1 (a_1a_2u^2 + (a_1b_2 + a_2b_1)uv + b_1b_2v^2)dudv \\ &= |\underline{a} \times \underline{b}| \int_0^1 [\frac{1}{3}a_1a_2u^3 + \frac{1}{2}(a_1b_2 + a_2b_1)u^2v + b_1b_2uv^2]_{u=0}^1 dv \\ &= |\underline{a} \times \underline{b}| \int_0^1 (\frac{1}{3}a_1a_2v + \frac{1}{2}(a_1b_2 + a_2b_1)v) + b_1b_2v^2)dv \\ &= |\underline{a} \times \underline{b}|[\frac{1}{3}a_1a_2v + \frac{1}{4}(a_1b_2 + a_2b_1)v^2 + \frac{1}{3}b_1b_2v^3]_{v=0}^1 \\ &= |\underline{a} \times \underline{b}|(\frac{1}{3}(a_1a_2 + b_1b_2) + \frac{1}{4}(a_1b_2 + a_2b_1)) \end{aligned}$$

5.6.1 Flux of a vector function

Let $\underline{q}(x, y, z)$ be a vector function that is continuous over a smooth surface S parameterised by $\underline{r} = \underline{r}(u, v), \quad (u, v) \in \Omega$. The *flux* of $\underline{q}$ across S in the direction of the unit normal $\underline{n}$ to the surface $\bar{S}$ is the number

$$\iint_S \underline{q} \cdot \underline{n} d\sigma$$

which can be calculated as

$$\iint_S \underline{q} \cdot \underline{n} d\sigma = \iint_\Omega \underline{q}(\underline{r}(u, v)) \cdot \underline{N}\, du\, dv$$

Proposition

If S is the graph of a function $z = f(x, y)$ with $(x, y) \in \Omega$ and $\underline{n}$ is the upper unit normal, the flux of the vector function $\underline{q} = (q_1, q_2, q_3)$ across S in the direction of $\underline{n}$ is

$$\iint_S \underline{q} \cdot \underline{n} d\sigma = \iint_\Omega (-q_1f_x - q_2f_y + q_3))dx\, dy$$

Example Let S be the portion of the paraboloid $z = 1 - x^2 - y^2$ tat lies above the unit disc Ω. Calculate the flux of $\underline{q}(x, y, z) = (x, y, z)$ across this surface in the direction of the upper unit normal.

Solution

Using the formulation above $f(x, y) = 1 - x^2 - y^2$ adn $f_x = -2x$ and $f_y = -2y$. Then,

$$\begin{aligned} Flux &= \iint_\Omega (-x(-2x) - y(-2y) + (1 - x^2 - y^2))dxdy \\ &= \iint_\Omega (x^2 + y^2 + 1)dxdy \\ &= \int_{r=0}^1 \int_{\theta=0}^{2\pi} (r^2 + 1)rd\theta dr = \frac{3}{2}\pi \end{aligned}$$