MATH0016 Methods 3 Summary

Jonathan Lim

1 Chpt 1: Surface Integrals and the Divergence

This chapter focuses on the **Divergence Theorem**. We begin with normal vectors and surface integrals.

1.1 The Unit Normal Vector

Consider a surface $z = f(x, y)$ for a function $f : \mathbb{R}^2 \to \mathbb{R}$. Define

$$g(x, y, z) = z - f(x, y)$$

The normal vector is in the direction of grad g. It follows that the unit normal vector is

$$\boxed{\boldsymbol{n} = \frac{-\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}}{\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1}}}$$

1.2 Surface Integrals

Formula:

$$\boxed{\iint\limits_S g(x, y, z)dS = \iint\limits_R g(x, y, f(x, y))\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1dxdy}}$$

Example: Let S be an octant of sphere of radius 2 centered at the origin. We take the surface $z = f(x, y) := \sqrt{4 - x^2 - y^2}$ and restrict to $x \geq 0$ and $y \geq 0$.

Solution
We want to compute $\iint_S z^2 dS$. To use the double integral formula for the surface integral, we need to find R and the partial derivatives of f.

For R: The projection of S onto the xy-plane is just the set of (x,y) such that $f(x, y) = 0$ with $x \geq 0$ and $y \geq 0$. This can be described as $0 \leq x \leq 2$ and $0 \leq y \leq \sqrt{4 - x^2}$. Now,

$$\sqrt{(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + 1} = \frac{2}{\sqrt{4 - x^2 - y^2}}$$

so,

$$\begin{aligned}
\iint_S z^2 dS &= \iint_R (4 - x^2 - y^2)\frac{2}{\sqrt{4 - x^2 - y^2}}dxdy \\
&= 2\iint_R \sqrt{4 - x^2 - y^2}dxdy \\
&= 2\int_0^{\pi/2}\int_0^2 \sqrt{4 - r^2}rdrd\theta \\
&= \int_0^{\pi/2}(-\frac{2}{3}(4 - 2^2)^{3/2} + \frac{2}{3}(4 - 0^2)^{3/2})d\theta \\
&= \frac{16}{3}\int_0^{\pi/2} d\theta = \frac{8\pi}{3}
\end{aligned}$$

1.3 Flux integrals

General idea: Flux represents the amount of 'flow' of substance across a given surface.
Formula:

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n}dS = \iint_R (-F_1\frac{\partial f}{\partial x} - F_2\frac{\partial f}{\partial y} + F_3)dxdy}$$

1.4 The Divergence Theorem

For a closed surface S and a vector field $\mathbf{F}$ defined in a region of $\mathbb{R}^3$ containing S and its interior V, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}dS = \iiint_V \vec{\nabla} \cdot \mathbf{F}(x, y, z)dV$$

In words: this says that the flux integral of a vector function over a closed surface is equal to the triple integral of the divergence of that vector function over the region enclosed by that surface.

Example Lets evaluate $\iint_S \mathbf{r} \cdot \mathbf{n}dS$, for any closed surface S. Since $\mathbf{r}$ is defined everywhere, we can apply the Divergence Theorem. We also know that the divergence of $\mathbf{r}$ is 3. So,

$$\begin{aligned}
\iint_S \mathbf{r} \cdot \mathbf{n}dS &= \iiint_V \vec{\nabla} \cdot \mathbf{r}dV \\
&= 3\iiint_V dV \\
&= 3\Delta V
\end{aligned}$$

Example Let's calculate the flux of the vector field $\mathbf{F}(x, y, z) := (x^2 + y^2)\mathbf{i} + (y^2 + z^2)\mathbf{j} + (x^2 + z^2)\mathbf{k}$ over the surface of the unit cube centered at (0.5,0.5,0.5) with edges parallel to the coordinate axes.

Again, $\mathbf{F}$ is defined everywhere, so we can apply the Divergence Theorem.

Computing the divergence of $\mathbf{F}$, we get

$$\vec{\nabla} \cdot \mathbf{F} = 2x + 2y + 2z$$

Therefore, the triple integral becomes:

$$\iiint_V \vec{\nabla} \cdot \mathbf{F} = 2\iiint_V (x + y + z)dV$$

This is not a volume integral since the integrand is not 1. To evaluate this triple integral, we must use iterated integration.Because V is a cube, it is described by the inequalities

$0 \leq x \leq 1, 0 \leq y \leq 1$ and $0 \leq z \leq 1$. Therefore,

$$\begin{aligned} 2\iiint_V (x+y+z)dV &= 2\int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz \\ &= 2\int_0^1\int_0^1 \left[\frac{1}{2}x^2+xy+xz\right]_0^1 dydz \\ &= \int_0^1\int_0^1 (1+2y+2z)dydz \\ &= \int_0^1 [y+y^2+2yz]_0^1 dz \\ &= \int_0^1 (2+2z)dz = 3 \end{aligned}$$

2 Chpt 2: Line integrals, the Curl and Stokes' Theorem

2.1 Line Integrals

These describe i.e. the trajectories of particles in a flowing fluid. We call C a smooth curve if we can represent it by a parameterization such as

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad \text{where } a \leq t \leq b$$

The point in $\mathbf{R}^3$ given by $\mathbf{r}(a)$ is called the *initial point* of C and the point $\mathbf{r}(b)$ is called the *terminal point* of C. If $\mathbf{r}(a) = \mathbf{r}(b)$ then we call C a *closed curve.*
Examples

- 2D Example: Let $\mathbf{r} : [0, \pi] \to \mathbb{R}^2$ be given by $\mathbf{r}(t) := cos(t)\mathbf{i} + sin(t)\mathbf{j}$. This curve is the upper semicircle of radius 1 centered at the origin. It is oriented in the anti-clockwise direction.
- 3D Example: Let $\mathbf{r} : [0, 4\pi] \to \mathbb{R}^3$ be given by $\mathbf{r}(t) := t^2\mathbf{i} + cos(t)\mathbf{j} + sin(t)\mathbf{k}$. This curve is a spiral which goes around the x-axis twice.

Line integral Formula

$$\int_C \mathbf{F} \cdot d\mathbf{r} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$$

Several repeated topics from MATH0011.

1. Path Independence
2. Circulation & Curl

2.2 Stokes' Theorem

Given a curve C and a capping surface S, if $\mathbf{F}$ is a smooth vector field defined on C and S, then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\vec{\nabla} \times \mathbf{F} \cdot \mathbf{n})dS$$

Note: Derivation in notes.

2.2.1 Connection with Green's theorem

Succinctly explained here.

2.2.2 Stokes' Theorem, path independence and gradient

Definition: ***Contractible*** A closed curve C in a domain D is called *contractible* if we can shrink the curve to a point in D without leaving the domain.

Definition: ***Simply-connected*** A domain D is called simply-connected if every closed curve lying in D is contractible.

Gradient vector field
If a vector field $\mathbf{F}$ is of the form $\mathbf{F} = \vec{\nabla} f$ for some scalar function f, then f is called a *potential function* for the vector field $\mathbf{F}$. In this cases, $\mathbf{F}$ is called a *gradient vector field.*

Example: Consider the field $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. We find that $f(x, y) = xy$ is a function for which F is the gradient. i.e.:

$$\vec{\nabla} f(x,y) = \frac{\partial(xy)}{\partial x}\mathbf{i} + \frac{\partial(xy)}{\partial y}\mathbf{j} = y\mathbf{i} + x\mathbf{j} = \mathbf{F}$$

Some other stuff to write about...write later (21st dec)

3 Chpt 3: Fourier Series

3.1 Formula

Any sufficiently nice function $F : [-L, L] \to \mathbf{R}$ can be written as a Fourier Series:

$$F(x) = c + \sum_{n=1}^{\infty}\left(a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L})\right)$$

where,

$$\begin{aligned} c &= \frac{1}{2L}\int_{-L}^{L} F(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} F(x) \cdot cos\Big(\frac{n\pi x}{L}\Big)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} F(x) \cdot sin(\frac{n\pi x}{L}\Big)dx \end{aligned}$$

The a_n, b_n, c are called the *Fourier coefficients* of F.

3.1.1 For derivation but sidenote:

If $n \geq 0$ is an integer then,

- $\int_{-L}^{L} sin\left(\frac{n\pi x}{L}\right)dx = 0$
- $\int_{-L}^{L} cos\left(\frac{n\pi x}{L}\right) = 0$

If $m, n > 0$ are integers then,

- $\int_{-L}^{L} sin\Big(\frac{n\pi x}{L}\Big) sin\Big(\frac{m\pi x}{L}\Big) dx = L\delta_{mn}$
- $\int_{-L}^{L} cos\Big(\frac{n\pi x}{L}\Big) cos\Big(\frac{m\pi x}{L}\Big) dx = L\delta_{mn}$
- $\int_{-L}^{L} cos\Big(\frac{n\pi x}{L}\Big) sin\Big(\frac{m\pi x}{L}\Big) dx = 0$

where, δ_{mn} is the Kronecker Delta: $\delta_{mn} = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}$

Example Consider the function $F(x) = x$ on the interval $[-\pi, \pi]$ (i.e. $L = \pi$). Its Fourier coefficients are given by

$$c = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0 \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(nx)dx = \frac{1}{\pi}\left(\left[x\frac{sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{sin(nx)}{n}\right) = 0 + \frac{1}{n^2\pi}[cos(nx)]_{-\pi}^{\pi} = 0$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(x)dx \\ &= \frac{1}{\pi}\left(\left[-x\frac{cos(nx)}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{cos(nx)}{n}\right) \\ &= (-cos(n\pi)/n - cos(n\pi)/n) + \frac{1}{n^2\pi}[sin(nx)]_{-\pi}^{\pi} \\ &= -2cos(n\pi)/n \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

where we have used that $cos(n\pi) = (-1)^n$.
Therefore the Fourier series of $F(x) = x$ on $[-\pi, \pi]$ is

$$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} sin(nx)$$

3.1.2 Even/ Odd Fourier Series

If function is even (Fourier Cosine Series):

$$F(x) = c + \sum_{n=1}^{\infty} a_n cos\Big(\frac{n\pi x}{L}\Big)dx$$

$$\begin{aligned} &where, \\ &b_n = 0 \quad, \forall n \\ &a_n = \frac{2}{L}\int_0^L F(x)cos(\frac{n\pi x}{L})dx \\ &c = \frac{1}{L}\int_0^L F(x)dx \end{aligned}$$

If function is odd (Fourier Sine Series):

$$F(x) = \sum_{n=1}^{\infty} b_n sin\Big(\frac{n\pi x}{L}\Big)dx$$

$$\begin{aligned} &where, \\ &b_n = \frac{2}{L}\int_0^L F(x)sin(\frac{n\pi x}{L})dx \end{aligned}$$

Example Consider L=1, where function is defined by:

$$F(x) = \begin{cases} 0, & if\, x \in [-1, 0), \\ \frac{1}{2}, & if\, x = 0, \\ 1, & if\, x \in (0, 1] \end{cases}$$

The function is neither odd nor even, but if we subtract 1/2 then we get an odd function

$$G(x) = \begin{cases} -\frac{1}{2}, & if\, x \in [-1, 0), \\ 0, & if\, x = 0, \\ \frac{1}{2}, & if\, x \in (0, 1] \end{cases}$$

This has Fourier coefficients

$$\begin{aligned} b_n &= 2\int_0^1 G(x)sin(n\pi x)dx \\ &= \int_0^1 sin(n\pi x)dx \\ &= \frac{1}{n\pi}[-cos(n\pi x)]_0^1 \\ &= \frac{1 - (-1)^n}{n\pi} \end{aligned}$$

This means that $b_n = 2/n\pi$ if n is odd and zero if n is even. Therefore,

$$G(x) = \frac{2}{\pi}sin(\pi x) + \frac{2}{3\pi}sin(3\pi x) + \cdots$$

and

$$F(x) = \frac{1}{2} + \frac{2}{\pi}sin(\pi x) + \frac{2}{3\pi}sin(3\pi x) + \cdots$$

3.2 Half-range Fourier Series

Definition Suppose that $F(x)$ is a function $[0, L] \to \mathbf{R}. Define its odd extension to be the function:$

$$F_{odd}(x) = \begin{cases} F(x) & \text{if} x > 0 \\ -F(-x) & \text{if} x < 0 \end{cases}$$

The half-range sine series of F is then defined to be the Fourier series of F_{even} in other words,

$$F(x) = \sum_{n=1}^{\infty} b_n sin\Big(\frac{n\pi x}{L}\Big)$$

on [0,L] where $b_n = \frac{2}{L}\int_0^L F(x)sin(\frac{n\pi x}{L})dx$.
Analogously one can define the half-range cosine series by taking the Fourier series of the even extension

$$F_{even}(x) = \begin{cases} F(x) & \text{if} x > 0 \\ F(-x) & \text{if} x < 0 \end{cases}$$

3.3 Parseval's Theorem

If $F(x) = c + \sum_{n=1}^{\infty}(a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L}))$ then,

$$\boxed{\frac{1}{L}\int_{-L}^{L} F(x)^2 dx = 2c^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

Example Prove that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Solution

Take $F(x) = x$ on $[-\pi, \pi]$. We computed the Fourier series of $F(x)$ on $[-\pi, \pi]$ and found previously that

$$F(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n} sin(nx)$$

Applying Parseval's Theorem implies

$$\frac{1}{pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

and the LHS equals $\frac{2\pi^2}{3}$ and rearranging it gives the desired relationship.

Hilbert space in notees but NFE

4 Chpt 4: Straight lines are shortest paths

Definition

Let $A : V \to \mathbf{R}$ be a functional on a possibly infinite-dimensional vector space. For each $\gamma \in V$ and each vector ϵ we define the *functional* or *Gâteaux derivative* of A in the ϵ-direction at γ

$$dA(\gamma; \epsilon) = \frac{d}{d\tau}\bigg|_{\tau=0} A(\gamma + \tau\epsilon)$$

Definition

We say that γ is a critical point of A if $dA(\gamma; \epsilon) = 0 \quad \forall \epsilon \in V$.

Theorem 4.6 (Fundamental Theorem)

Suppose that $y : [0, 1] \to \mathbf{R}^n$ is a vector-valued function. If $\int_0^1 y(t)\epsilon(t)dt = 0$ for all smooth functions $\epsilon : [0, 1] \to \mathbf{R}^n$ then $y(t) = 0$ for all $t \in [0, 1]$.

5 Chpt 5: The Euler-Lagrange Equation I

5.1 Definition of Lagrangian

A function $L(x, y(x), y'(x))$ of three variables is called a *Lagrangian*. It defines a function $A : V \to \mathbf{R}$ by:

$$A(y) = \int_a^b L(x, y(x), y'(x))dx$$

Now, we will derive an equation satisfied by the critical points of functionals A defined by a Lagrangian. This equation is a second-order differential equation called the $Euler - Lagrangeequation$.

5.2 Computing the Gâteaux derivative

Theorem 5.2 If A is a functional of the form $\int_a^b L(x, y, y')dx$ defined on a space of functionals y satisfying $y(a) = y_a, y(b) = y_b$ then the Gâteaux derivative $dA(y; \epsilon)$ is

$$dA(y; \epsilon) = \int_a^b \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right)\epsilon(x)dx$$

The function y is a critical point of A iff the Euler-Langrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0$$

Example 5.4 Let $L(x, y, y') = \sqrt{1 + (y')^2}$. Then,

$$A(y) = \int_a^b \sqrt{1 + (y')^2}dx$$

This functional measures the arc-length of the graph of y between (a, y_a) and (b, y_b), so it should be minimised by a straight line graph. We have,

$$\frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}$$

The Euler-Lagrange equation is therefore:

$$\begin{aligned} 0 - \frac{d}{dx}\frac{y'}{\sqrt{1 + (y')^2}} = 0 &\iff \frac{y'}{\sqrt{1 + (y')^2}} = C \\ &\iff y' = \frac{C}{\sqrt{1 - C^2}} \end{aligned}$$

Solution is therefore

$$y = \frac{C}{\sqrt{1 - C^2}} \cdot x + D$$

Using the boundary conditions, $y(a) = y_a$ and $y(b) = y_b$ to get

$$y = \frac{y_a - y_b}{b - a}(x - a) + y_a$$

Example 5.5 Suppose $L(x, y, y') = \frac{1}{2}(m(y')^2 - ky^2)$ for some constants m and w. The functional is

$$A(y) = \frac{1}{2}\int_a^b (m(y')^2 - ky^2)dx$$

We have,

$$\frac{\partial L}{\partial y} = -ky \qquad \frac{\partial L}{\partial y'} = my'$$

The Euler-Lagrange equation is therefore

$$-ky - \frac{d}{dx}(my') = 0 \iff y" = -ky/m$$

This is the simple harmonic oscillator with frequency $\omega = \sqrt{k/m}$. Its solutions are

$$y(x) = Asin(\omega x) + Bcos(\omega x)$$

Using boundary conditions, $y(a) = y_a$ and $y(b) = y_b$ to get

$$y_a = Asin(\omega a) + Bcos(\omega a)$$

$$y_b = Asin(\omega b) + Bcos(\omega b)$$

5.3 Beltrami's Identity

For certain simple Lagrangians, the Euler-Lagrange equation reduces to a *first-order* differential equation.

If $L(x, y, y')$ is independent of x and y is a solution of the Euler-Lagrange equation then,

$$L - y'\frac{\partial L}{\partial y'} = C$$

for some constant C.

Example 5.8.1 In our previous examples: $L(x, y, y') = \sqrt{1 + (y')^2}$ is independent of p, so Beltrami's identity holds:

$$c = L - y'\frac{\partial L}{\partial y'} = \sqrt{1 + (y')^2} - y'\frac{y'}{\sqrt{1 + (y')^2}}$$

so,

$$\frac{1 + (y')^2 - (y')^2}{\sqrt{1 + (y')^2}} = c \Leftrightarrow y' = \sqrt{c^2 - 1}$$

so again y is a straight line.

Example 5.8.2 $L(x, y, y') = \frac{1}{2}(m(y')^2 - ky^2)$ is independent of p, so Beltrami's identity holds

$$c = L - y'\frac{\partial L}{\partial y'} = \frac{1}{2}(m(y')^2 - ky^2) - m(y')^2 = \frac{1}{2}(m(y')^2 + ky^2)$$

This implies that

$$\frac{y'}{\sqrt{2c - y^2}} = \sqrt{k/m}$$

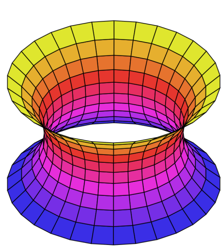
Substituting $y = \sqrt{\frac{2c}{k}}sin(\theta)$ allows us to integrate:

$$\theta = x\sqrt{k/m} + D$$

so $y = Csin(\omega x + D)$ which is another way of writing the previous solutions $Asin(\omega x) + Bcos(\omega x)$

Example 5.9 Catenoid

Let y be a function on $[a, b]$ with $y(a) = y_a, y(b) = y_b$ and suppose that $y(x) > 0$ for all $x \in [a, b]$. Consider the surface of revolution



Its surface area is given by the integral

$$A(y) = 2\pi\int_a^b y\sqrt{1 + (y')^2}dx$$

Which function y minimises this surface area for given y_a, y_b?. A minimiser y will be a critical point of the functional A so it will solve the Euler-Lagrange equation for $L = y\sqrt{1 + (y')^2}$. This has no explicit dependence on x so we can use Beltrami's Identity $L - y'\frac{\partial L}{\partial y'} = c$ for some constant c. This means,

$$c = y\sqrt{1 + (y')^2} - y'\frac{yy'}{\sqrt{1 + (y')^2}}$$

or

$$y' = \sqrt{\frac{y^2}{c^2} - 1}$$

Sub $y = cosh(\theta)$ we get

$$\int \frac{c\,sinh(\theta)d\theta}{\sqrt{cosh^2(\theta) - 1}} = x + D$$

or

$$\theta = \frac{x + D}{c}$$

Therefore, $y(x) = c\,cosh(\frac{x+D}{c})$. Then determine the constants c and D from y_a and y_b (nontrivial task).

Example 5.10 (Brachistochrone)
Brachistochrone curve is the fastest curve from point a to b, given height $a > b$.
Let $L = \sqrt{\frac{1+r^2}{-2gq}}$ and suppose that $a = y_a = 0$. The physical significance of this Lagrangian is the following:

1. Consider a wire suspended in midair underneath the x-axis so that its height at x is $y(x)$ with in particular $y(0) = 0$.
2. A bead sitting on the wire at $(x, y(x))$ and moving along the wire with speed $v(x)$ takes time

$$A(y) = \int_a^b \frac{1}{v(x)}ds$$

to get from a to b where $ds = \sqrt{1 + (y')^2}dx$ is the length of an infinitesimal arc.

3. The Langragian comes from taking $v = \sqrt{-2gy}$ (derived from N2Law: $\frac{1}{2}mv^2 + mgy = 0$)

We seek the configuration of wire y which minimises the time taken for the bead to go from $(0, 0)$ to (b, y_b). This y will solve the Euler-Lagrange equation.
The Lagrangian L is independent of p so Beltrami's equation holds:

$$c = L - y'\frac{\partial L}{\partial y'} = \frac{\sqrt{1 + (y')^2}}{\sqrt{-2gy}} - \frac{(y')^2}{\sqrt{-2gy(1 + (y')^2)}}$$

$$\Leftrightarrow y' = \sqrt{\frac{1}{-2gyc^2} - 1}$$

which gives

$$\int \frac{\sqrt{-y}dy}{\sqrt{\frac{1}{2gc^2} + y}} = x + D$$

which we can integrate by substituting $y = \frac{-sin^2(\theta)}{2gc^2}$:

$$x + D = \frac{1}{2gc^2} sin^{-1}(\sqrt{-2gc^2 y}) - \sqrt{-y}\sqrt{2gc^2 + y}$$

From here, we see that $y(0) = 0$ then $D = 0$. It is not so simple to find y in terms of x or to determine the constant c.

6 Chpt 6: The Euler-Lagrange Equation II - Constraints

We consider constraints of the form

$$G(y) = \int_a^b M(x, y(x), y'(x))dx = 0$$

for some Lagrangian $M(x, y, y')$

Example 6.1 To minimise the arc-length of the graph of $y : [a, b] \to \mathbf{R}$ given that the area underneath the graph is equal to K. Now the arc-length is

$$A(y) = \int_a^b \sqrt{1 + (y')^2}dx$$

and

$$G(y) = \int_a^b \left(y - \frac{K}{b-a} \right) dx$$

measures how far the area underneath the graph is from K. We introduce a Lagrange multiplier λ and minimise the functional

$$F(y, \lambda) = \int_a^b A(y) - \lambda \cdot G(y) = \int_a^b \left(\sqrt{1 + (y')^2} - \lambda \left(y - \frac{K}{b-a} \right) \right) dx$$

Varying wrt λ gives us the constraint $G(y) = 0$ and wrt y gives the Euler-Lagrange equation.

$$-\lambda = \frac{d}{dx} \frac{y'}{\sqrt{1 + (y')^2}} \Leftrightarrow y' = \frac{D - \lambda x}{\sqrt{1 - (D - \lambda x)^2}}$$

Using $D - \lambda x = sin(\theta)$ we get $\lambda y + C = cos(\theta)$ for some constant C, so $(\lambda y + C)^2 + (D - \lambda x)^2 = 1$ and the graph $(x, y(x))$ lies on a circle of radius $\frac{1}{\lambda}$ (it is a segment of circle btw (a, y_a) and (b, y_b). We could also use Beltrami's Identity.

Example 6.2 (Catenary) Consider a chain hanging above the x-axis with its endpoints fixed at $(a, y_a), (b, y_b)$. It will hang so as to minimise its total potential energy. If the chain is uniform with density ρkgm^{-1} then the segment lying over an infinitesimal segment dx has mass $\rho\sqrt{1 + (y')^2}dx$. The potential energy of this segment is $\rho g y\sqrt{1 + (y')^2}dx$ so the functional to be minimised is

$$A(y) = \rho g \int_a^b y\sqrt{1 + (y')^2}dx$$

However the chain is inelastic so its length so its length is fixed at K metres.

$$G(y) = \int_a^b \left(\sqrt{1 + (y')^2} - \frac{K}{b-a} \right) dx = 0$$

thus the modified functional for the constrained problem is

$$\int_a^b \left(\rho g y\sqrt{1 + (y')^2} - \lambda \left(\sqrt{1 + (y')^2} - \frac{K}{b-a} \right) \right) dx$$

which has no explicit x-dependence, so we will solve this constrained problem using Beltrami's identity.

Beltrami's Identity implies

$$\rho g y\sqrt{1 + (y')^2} - \lambda \left(\sqrt{1 + (y')^2} - \frac{K}{b-a} \right) - y'(\rho g y - \lambda)y'/\sqrt{1 + (y')^2} = c$$

for some constant c. This gives

$$\rho g y - \lambda = \frac{c - \lambda K}{b-a}\sqrt{1 + (y')^2}$$

. We define $C := \frac{c - \lambda K}{b-a}$ and we arrange to get

$$y' = \sqrt{\frac{(\rho g y - \lambda)^2}{C^2 - 1}}$$

Substituting $cosh(z) = \frac{\rho g y - \lambda}{C}$ we integrate and get

$$x = \frac{C}{\rho g} cosh^{-1}(\frac{\rho g y - \lambda}{C}) - D \Longleftrightarrow y = \frac{C}{\rho g} cosh(\frac{\rho g}{c}(x + D)) + \frac{\lambda}{\rho g}$$

This curve is called ***catenary curve***

7 Chpt 7: The Euler-Lagrange Equation III - More variables

7.1 Vector-valued functions

Theorem 7.1 Let V be the space of functions $\mathbf{y} : [a,b] \to \mathbf{R}^n$ satisfying the boundary conditions $\mathbf{y}(a) = \mathbf{y}_a$ and $\mathbf{y}(b) = \mathbf{y}_b$. We will write $y(x)$ in coordinates $(y_1(x), ..., y_x(n))$. Let A be a functional defined by a Lagrangian $L(x, y_1, ..., y_n, y'_1, .., y'_n)$ by

$$A(\mathbf{y}) = \int_a^b L(x, y_1(x), ..., y_n(x), y'_1(x), ..., y'_n(x))dx$$

Let $\boldsymbol{\epsilon}(x)$ be a function such that $\boldsymbol{\epsilon}(a) = \boldsymbol{\epsilon}(b) = 0$. Then the Gâteaux derivative of A at $\mathbf{y}$ in the -direction is

$$dA(\mathbf{y}; \boldsymbol{\epsilon}) = \sum_{i=1}^n \int_a^b \left(\frac{\partial L}{\partial y_i} - \frac{d}{dx}\frac{\partial L}{\partial y'_i}\right)\epsilon_i(x)dx$$

which vanishes for all $\boldsymbol{\epsilon}$ iff the n Euler-Lagrange equations hold

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\frac{\partial L}{\partial y'_i} = 0, \qquad i = 1, ..., n$$

Example 7.2 (Isoperimetric Problem) Let $\gamma : \mathbf{R} \to \mathbf{R}^2$ be a curve with coordinates $\gamma(t) = (x(t), y(t))$. We assume that γ is a closed curve $(\gamma(t + 2\pi)) = \gamma(t)$ which is just as good for integrating by parts as assuming $\gamma(0)$ and $\gamma(1)$ are fixed. Supposed that we know it has action $\int_0^{2\pi}(\dot{x} + \dot{y}^2)dt = K$ and we want the maximise the area of the region it bounds.

By Green's theorem, the area of this region U is

$$\int_U dxdy = \int_{\partial U} xdy \quad \text{or} \quad \int_0^{2\pi} x(t)\dot{(y)}(t)dt$$

Therefore we must find the critical points of the constrained problem:

$$\int_0^{2\pi}(x\dot{y} - \lambda(\dot{x}^2 + \dot{y}^2 - \frac{K}{2\pi}))dt$$

The two Euler-Lagrange equations are:

$$\dot{y} = \frac{d}{dt}(-2\lambda\dot{x})$$
$$0 = \frac{d}{dt}(x - 2\lambda\dot{y})$$

This gives

$$\ddot{x} = \frac{\dot{y}}{2\pi}, \quad \ddot{y} = \frac{\dot{x}}{2\pi}$$

Differentiating again allows us to rearrange and get:

$$\dddot{y} = -\frac{\dot{y}}{4\lambda^2}, \qquad \dddot{x} = -\frac{\dot{x}}{4\lambda^2}$$

and so $\dot{x}$ and $\dot{y}$ obey simple harmonic motion. This means that $t \mapsto (x(t), y(t))$ is a circle.

7.2 Functions of several variables

Now, let $U \subset \mathbf{R}^m$ be an open subset whose boundary ∂U is smooth. We will consider functions $\phi : U \to \mathbf{R}$ with fixed boundary values; in other words we will fix a function $\phi_0 : \partial U \to \mathbf{R}$ and consider functions ϕ such that

$$\phi(\mathbf{x}) = \phi_0(\mathbf{x}), \quad \mathbf{x} \in \partial U$$

Perturbations $\epsilon(\mathbf{x})$ satisfying $\epsilon(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial U$. Our Lagrangian L will now depend on $\mathbf{x} = (x_1, ..., x_m)$ on $\phi(\mathbf{x})$ and on the partial derivatives $\partial_i\phi = \frac{\partial\phi}{\partial x_i}$, $i = 1, ..., m$, that is:

$$A(\mathbf{y}) = \int_U L(\mathbf{x}, \phi(\mathbf{x}), \nabla\phi(\mathbf{x})dx_1 \cdots dx_m$$

Theorem 7.3 The Gâteaux derivative of A at ϕ in the ϵ-direction is

$$dA(\phi; \epsilon) = \int_U \left(\frac{\partial L}{\partial\phi} - \sum_{i=1}^m \frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial_i\phi)}\right)\epsilon(\mathbf{x})dx$$

which vanishes for all ϵ iff the Euler-Lagrange equation holds

$$\frac{\partial L}{\partial\phi} - \sum_{i=1}^m \frac{\partial}{\partial x_i}\frac{\partial L}{\partial(\partial_i\phi)} = 0$$

7.2.1 Examples

Example 7.5 Consider $U = [0,1]^2$, the square and functions ϕ with fixed boundary values

$$\phi(x,0) = \phi_0(x,0), \qquad \phi(0,y) = \phi_0(0,y),$$
$$\phi(x,1) = \phi_0(x,1), \qquad \phi(1,y) = \phi_0(1,y).$$

We try to minimise the functional

$$A(\phi) = \int_0^1\int_0^1 \left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right)dxdy$$

We can think of this functional as the total gradient $\int|\nabla\phi|^2 dxdy$ of a temperature distribution ϕ on U. Since heat flows to minimise a gradient, a minimiser for this functional will be a steady-state temperature distribution on the square. We have

$$\frac{\partial L}{\partial\phi} = 0 \qquad \frac{\partial L}{\partial(\partial_x\phi)} = 2\partial_x\phi \qquad \frac{\partial L}{\partial(\partial_y\phi)} = 2\partial_y\phi$$

so the Euler-Lagrange equation is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Example 7.6 Let's use the functional

$$A(\phi) = \int_U \sqrt{1 + (\partial_x\phi)^2 + (\partial_y\phi)^2}dxdy$$

This measures the area of the graph of ϕ. The Euler-Lagrange equation is

$$\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} + \frac{\partial}{\partial y}\frac{\partial L}{\partial \phi_y}$$

and in this case

$$L = \sqrt{1+\phi_x^2+\phi_y^2}$$

so

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\frac{\phi_x}{\sqrt{1+|\nabla\phi|^2}} + \frac{\partial}{\partial y}\frac{\phi_y}{\sqrt{1+|\nabla\phi|^2}} \\ &= ... \\ &\text{and we get,} \end{aligned}$$

$$\frac{\partial^2\phi}{\partial x^2}\left(1+\left(\frac{\partial\phi}{\partial y}\right)^2\right)+\frac{\partial^2\phi}{\partial y^2}\left(1+\left(\frac{\partial\phi}{\partial x}\right)^2\right) = 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}$$

8 Chpt 8: Methods of Characteristics, I: Linear Case

8.1 Linear Change of coordinates

For very simple PDEs, we can change coordinates and turn them into PDEs we already know how to solve.

Example 8.2 Consider the PDE for $\phi(x, y)$:

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 0$$

The expression on the left-hand side looks like the expression:

$$\frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial\phi}{\partial y}$$

coming from the chain rule, provided we pick

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = -1$$

So let's change to a new (linear) system of coordinates (u,v) satisfying the above equations. I.e. we could take

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \text{ or } \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

The only conditions for this to define a suitable coordinate change are that the first column is given by 1 and -1 and that the matrix is invertible.
We use

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

whose inverse is:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

WRT to the new basis, the chain rule tells us that

$$\begin{aligned} \frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial\phi}{\partial y} &= \frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} \\ &= 0(\text{by our equation}) \end{aligned}$$

so the general solution to the equation is $\phi(u, v) = C(v) \iff \phi(x, y) = C(x + y)$ so any function of $v = x + y$ is a solution. For example $sin(x + y), e^{x+y}$

Example 8.3 Consider the PDE for $\phi(x, y)$:

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = x$$

making the same change of coordinates as before

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \iff \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

this equation becomes

$$\begin{aligned} \frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial\phi}{\partial y} &= \frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} \\ &= x(\text{by our equation}) \\ &= u(\text{by our coordinate change}) \end{aligned}$$

We integrate this and get

$$\phi(x, y) = \frac{1}{2}u^2 + C(v)$$

where $C(v)$ is an arbitrary function of $v = x + y$. Translating back into our original coordinates we get

$$\phi(x, y) = \frac{x^2}{2} + C(x + y)$$

Example 8.4 Consider the equation:

$$\frac{\partial\phi}{\partial x} + 2\frac{\partial\phi}{\partial y} = sin(y)$$

Use the coordinate change

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \iff \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x & y \end{pmatrix}$$

This gives:

$$\frac{\partial\phi}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x} + 2\frac{\partial\phi}{\partial y} = sin(y) = sin(2u + v)$$

Fixing v and integrating wrt u, we get

$$\phi(u, v) = -\frac{1}{2}cos(2u + v) + C(v)$$

or

$$\phi(x, y) = -\frac{1}{2}cos(y) + C(y - 2x)$$

8.1.1 Boundary Conditions

Example 8.5 In addition to example 8.4 which asked to solve $\frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial y} = sin(y)$ we subject to the boundary condition $\phi(s, 0) = s^2$.

We already saw that the general solution to this equation is $\phi(x, y) = -\frac{1}{2}cos(y) + C(y - 2x)$. If we substitute this into the BC we get,

$$s^2 = -\frac{1}{2}cos(0) + C(0 - 2s)$$

which means $C(-2s) = s^2 + \frac{1}{2}$. Substituting $w = -2s$ gives $C(w) = \frac{w^2}{4} + \frac{1}{2}$

8.2 Nonlinear change of coordinates

. So far we have only allowed change of coordinates by a linear transformation. What kind of equations do we get if we make more interesting coordinates changes?

Example 8.2 Use plane polar coordinates:

$$x = rcos(\theta) \qquad\qquad y = rcos(\theta)$$

By the chain rule we have:

$$\frac{\partial \phi}{\partial r} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} = \frac{x}{r}\frac{\partial \phi}{\partial x} + \frac{y}{r}\frac{\partial y}{\partial r}$$

In particular, the equation

$$\frac{\partial \phi}{\partial r} = 0$$

becomes (after multiplying out by r)

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = 0$$

In particular, the solutions to this are just functions of $\theta = tan^{-1}(\frac{y}{x})$.

8.2.1 Characteristic Vector Field

Lemma 8.7 Given an expression of the form

$$\sum_{i=1}^{n} A_i(x_1, ..., x_n)\frac{\partial \phi}{\partial x_i},$$

suppose that we can find coordinates $(u_1, ..., u_n)$ such that

$$\frac{\partial x_i}{\partial u_1} = A_i(x_1, ..., x_n)$$

Then,

$$\frac{\partial \phi}{\partial u} = \sum_{i=1}^{n} A_i(\mathbf{x})\frac{\partial \phi}{\partial x_i}$$

Example 8.8 Consider the equation $x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = 0$. We want to solve

$$\begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

For simplicity, we write this as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The solution is $x = Ae^u, y = Be^u$. We want to think of these two equations as giving a coordinate transformation, but we have three new coordinates (u, A, B) so this doesn't quite make sense yet. Let us make an arbitrary choice: set A=1 and $v = B$ and take our new coordinates to be u. That is

$$x = e^u, \quad y = ve^u \iff u = ln(x), \quad v = \frac{y}{x}$$

This arbitrary choice is completely analogous to the way we could choose our matrix entries freely in section 8.1. With these new coordinates, we have:

$$x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}$$

Therefore, the equation is $\frac{\partial \phi}{\partial u} = 0$ and the solution is $\phi(u, v) = C(v)$ (where C is an arbitrary function). Substituting our expression $v = y/x$ we get:

$$\phi(x, y) = C(y/x)$$

Definition 8.9 Consider an equation of the form $A(x, y)\frac{\partial \phi}{\partial x} + B(x, y)\frac{\partial \phi}{\partial y} + C(x, y)\phi + D(x, y) = 0$ ("inhomogeneous linear"). The vector field

$$\begin{pmatrix} A(x, y) \\ B(x, y) \end{pmatrix}$$

is called the *characteristic vector field.* The differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A(x, y) \\ B(x, y) \end{pmatrix}$$

This method for solving first-order PDEs is called *method of characteristics.*

Example 8.11 Consider the equation

$$-y\frac{\partial \phi}{\partial x} + x\frac{\partial \phi}{\partial y} = 0$$

The characteristic vector field is $(-y, x)$ and the characteristic equations are

$$\dot{x} = -y, \quad \dot{y} = x$$

Differentiating again we get $\ddot{x} = -x$, so $x = Acos(u) + Bsin(u)$ and $y = -\dot{x} = Asin(u) - Bcos(u)$. Let us pick $B = 0$ and $v = A$. Now we have

$$(x, y) = (v \cdot cos(u), v \cdot sin(u))$$

. The inverse coordinate transform is

$$v = \sqrt{x^2 + y^2}, \quad u = tan^{-1}(y/x)$$

.

The equation becomes $\frac{\partial \phi}{\partial u} = 0$ which has solution $C(v) = C(\sqrt{x^2 + y^2})$

More examples in notes but lazy to write.. (01/01/23)

9 Chpt 9: Methods of Characteristics, II: Quasilinear Case

We consider first-order quasilinear equations

$$A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0 \tag{1}$$

which are more complicated because all the coefficients are allowed to depend on ϕ. For notational simplicity, we will only consider the case where $\phi(x,y)$ is a function of two variables.

9.1 Characteristic vector field

Definition 9.2 The *characteristic vector field* of

$$A(x,y,\phi)\frac{\partial\phi}{\partial x} + B(x,y,\phi)\frac{\partial\phi}{\partial y} + C(x,y,\phi) = 0$$

is

$$(A(x,y,z), B(x,y,z), -C(x,y,z))$$

This is now a vector field in $\mathbf{R}^3$ (coordinates x,y,z). A *characteristiccurve* is a solution $(x(t),y(t),z(t))$ to

$$\frac{\partial x}{\partial t} = A(x,y,z)$$
$$\frac{\partial y}{\partial t} = B(x,y,z)$$
$$\frac{\partial z}{\partial t} = -C(x,y,z)$$

10 Chpt 11: D'Alembert's Method

In this chapter we have one more situation in which a linear change of coordinates enable us to solve a PDE. $\rightarrow$ We are interested in linear second-order hyperbolic equations with constant coefficients.

10.0.1 Wave equation

The wave equation:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$$

which describes the motion of waves with speed c. This equation simpifies drastically if we change to so-called *light-cone coordinates*:

$$x_+ = x + ct, x_- = x - ct$$

- Axis $x_- = 0$ is a line where $x = ct$ i.e. the trajectory of a particle moving with speed c in the positive x-direction.
- Axis $x_+ = 0$ is the trajectory of a particle moving with speed c in the negative x-direction.
- We have $x = \frac{1}{2}(x_+ + x_-)$ and $t = \frac{1}{2c}(x_+ - x_-)$

Using chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial x_\pm} &= \frac{\partial x}{\partial x_\pm}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x_\pm}\frac{\partial}{\partial t}\\ &= \frac{1}{2}\left(\frac{\partial}{\partial x} \pm \frac{1}{c}\frac{\partial}{\partial t}\right)\end{aligned}$$

so,

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = 4\frac{\partial^2}{\partial x_+\partial x_-}$$

thus,

$$\frac{\partial^2\phi}{\partial x_+\partial x_-} = 0$$

Integrating this directly we see that any wave equation can be written as

$$\phi(x,t) = C_-(x-ct) + C_+(x+ct) \tag{2}$$

Example 11.1 Solve the wave equation for $\phi(x,t)$ with initial conditions:

$$\phi(x,0) = e^{-x^2}, \quad \frac{\partial\phi}{\partial t}(x,0) = 0$$

Using equation 2 the initial condition becomes:

$$\phi(x,0) = C_-(x) + C_+(x) = e^{-x^2}$$

$$\frac{\partial\phi}{\partial t}(x,0) = -cC'_-(x) + \ cC'_+(x) = 0$$

This give us two simultaneous equations for $C_\pm$. Integrating the second equation implies $C_+(x) = C_-(x) + k$ for some constant k.The first then gives,

$$C_-(x) + C_+(x) = 2C_-(x) + k = e^{-x^2}$$

so

$$C_-(x) = \frac{1}{2}(e^{-x^2} - k), \quad C_+(x) = \frac{1}{2}(e^{-x^2} + k)$$

Solution is therefore:

$$\frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$

10.1 Hyperbolic equations

The wave equation belongs to the class of ***hyperbolic second-order linear equations***. We consider the most general of these in two variables x, y:

$$A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = D(x,y)$$

Our objective is to find the coordinates (s,t) so that the equation becomes

$$A\frac{\partial^2\phi}{\partial s\partial t} = A\frac{\partial^2\phi}{\partial x^2} + B\frac{\partial^2\phi}{\partial x\partial y} + C\frac{\partial^2\phi}{\partial y^2} = D$$

How (lemma 11.3)
If we have (or set):

$$x = s + t \quad y = -\beta s - \alpha t$$

then

$$\frac{\partial^2}{\partial s \partial t} = \frac{\partial^2}{\partial x^2} - (\alpha + \beta)\frac{\partial^2}{\partial x \partial y} + \alpha\beta\frac{\partial^2}{\partial y^2}$$

Comparing to the hyperbolci equation, we set

$$\frac{B}{A} = -(\alpha + \beta), \quad \frac{C}{A} = \alpha\beta$$

Notice/recall: if α and β are the roots of the quadratic equation

$$AT^2 + BT + C = 0$$

then $\frac{B}{A} = -(\alpha + \beta), \quad \frac{C}{A} = \alpha\beta$

Definition 11.5 A PDE:

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} = D$$

is said to be **hyperbolic, parabolic or elliptic** if the quantity $B^2 - 4AC$ is respectively positive, zero or negative.

Example 11.8 Solve

$$\frac{\partial^2 \phi}{\partial x^2} + 5\frac{\partial^2 \phi}{\partial x \partial y} + 4\frac{\partial^2 \phi}{\partial y^2} = xy$$

The quadratic equation we need to solve is $T^2 + 5T + 4 = 0$ which has roots $(-5 \pm \sqrt{25 - 16/2}$ that is $\alpha = -4, \beta = -1$. Under the change of coordinates $x = s + t, y = s + 4t$ the PDE becomes

$$\frac{\partial^2 \phi}{\partial s \partial t} = xy = s^2 + 5st + 4t^2$$

Integrating up directly we get

$$\phi(s,t) = \frac{1}{3}s^3 t + \frac{5}{4}s^2 t^2 + \frac{4}{3}st^3 + C_1(s) + C_2(t)$$

where C_1 and C_2 are arbitrary functions. Changing the coordinates back to x, y we have, (using $s = \frac{1}{3}(4x - y), t = \frac{1}{3}(y - x)$

$$\phi(x,y) = \frac{1}{81}\left(\frac{1}{3}(4x-y)^3(y-x) + \frac{5}{4}(4x-y)^2(y-x)^2 + \frac{4}{3}(4x-y)(y-x)^3\right) + C_1((4x-y)/3) + C_2((y-x)/3)$$