MATH0010 Methods 1 Summary

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1 Chpt 3 Differential Calculus

1.1 Defn of Derivatives & Special rules

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Basic Formula

- $(fg)' = f'g + fg'$ - **Product Rule**
- $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$ - **Quotient Rule**
- $\frac{d}{dx}(x^n) = nx^{n-1}$ - **Power Rule**
- $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$ - **Chain Rule**

Common Dervatives:

$\frac{d}{dx}(x) = 1$	$\frac{d}{dx}(cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(sin(x)) = cos(x)$	$\frac{d}{dx}(tan^{-1}(x)) = \frac{1}{1+x^2}$
$\frac{d}{dx}(cos(x)) = -sin(x)$	$\frac{d}{dx}(a^x) = a^x ln(a)$
$\frac{d}{dx}(tan(x)) = sec^2(x)$	$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$
$\frac{d}{dx}(sec(x)) = sec(x)tan(x)$	$\frac{d}{dx}(ln(x)) = \frac{1}{x}$
$\frac{d}{dx}(csc(x)) = -csc(x)cot(x)$	$\frac{d}{dx}(ln[f(x)]) = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(cot(x)) = -csc^2(x)$	$\frac{d}{dx}(log_a(x)) = \frac{1}{xln(a)}$
$\frac{d}{dx}(sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(log_a[f(x)]) = \frac{f'(x)}{f(x)ln(a)}$

1.2 Derivative of inverse functions

Example 3.1 *Find* $\frac{d(arcsin(x))}{dx}$.
Let $y = arcsin(x)$ Then $x = sin(y)$ Differentiating with respect to y gives:

$$\frac{dx}{dy} = cos(y)$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{1}{cos(y)}$$

We want dy/dx as a function of x not y. Recall $x = sin(y)$ so $(cos(y))^2 = 1 - x^2. Therefore,$

$$\frac{dy}{dx} = \frac{1}{cos(y)} = \frac{\pm 1}{\sqrt{1-x^2}}$$

Now $sin^{-1}x$ is an increasing function throughout the domain, so dy/dx ≥ 0 so,

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

1.3 Implicit Differentiation

Example 3.2 *Find the slope of the tangent to the curve:*

$$y^7 + x^4y^3 - 10xy^2 + 3 = 0$$

at the point (2,1). Differentiating the equation wrt x gives:

$$7y^6\frac{dy}{dx} + \Big\{4x^3y^3 + 3x^4y^2\frac{dy}{dx}\Big\} - 10\Big\{y^2 + 2xy\frac{dy}{dx}\Big\} = 0$$

Evaluating at x=2, y=1 we have:

$$(7+48-40)\frac{dy}{dx}\Big|_{(x=2,y=1} + (32-10) = 0$$

Therefore the slope of the tangent at (2,1) is

$$\frac{dy}{dx}\Big|_{(x=2,y=1} = -\frac{22}{15}$$

1.4 Hyperbolic Functions

$cosx = \frac{1}{2}(e^{ix} + e^{-ix})$	$\boxed{coshx = \frac{1}{2}(e^x + e^{-x})}$
$sinx = \frac{1}{2i}(e^{ix} - e^{-ix})$	$\boxed{sinhx = \frac{1}{2}(e^x - e^{-x})}$

Parity of functions

- Even functions: $f(-x) = f(x) \quad \forall x \in domain$
- Odd functions: $f(-x) = -f(x) \quad \forall x \in domain$

Some other identities & derivatives:

$sinh(x) = -i \cdot sin(ix)$	$cosh^2(x) = 1 + sinh^2(x)$
$cosh(x) = cos(ix)$	$\frac{d}{dx}(sinh(x)) = cosh(x)$
$e^{ix} = cos(x) + isin(x)$	$\frac{d}{dx}(cosh(x)) = sinh(x)$
$e^x = cosh(x) + sinh(x)$	$\frac{d}{dx}(tanh(x)) = sech^2(x)$

1.5 Taylor and Maclaurin Series

Deriving geometric progression sum formula:
Now, $\forall n \in \mathbb{Z}^+$ we define:

$$s_n(x) = 1 + x + ... + x^n$$

Multiply by x gets:

$$xs_n(x) = x + x^2 ... + x^{n+1}$$

Taking the gives us $(1-x)s_n(x) = 1 - x^{n+1}$ and we have

$$s_n(x) = \frac{1-x^n}{1-x}$$

For $|x| < 1$ then $x^n \to \infty$ and:

$$\lim_{n\to\infty} s_n(x) = \frac{1}{1-x}$$

We now find similar series expansions for more general functions. Suppose there is a function f defined near some point a s.t. f can be expanded as a series: Power Series:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n + ..$$

Now suppose that f is differentiable at x=a. Here we differentiated a sum and said it is the same as the sum of the derivatives (which is nontrivial and proven in Analysis)

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + ...nc_n(x-a)^{n-1}..$$

Motivated by this, we <u>define</u> the Taylor Series of f at $x = a$ to be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Taylor series at $x = 0$ is called the Maclaurin series of f:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Example 3.4 Find the Maclaurin series for e^x Setting $f(x) = e^x$, we have $f^{(n)}(x) = e^x$ for $n = 0, 1, ...$ and so $f^{(n)}(0) = 1$ for $n = 0, 1, ...$ Therefore the Maclaurin series for e^x is:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{x!}x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example 3.5 (The Binomial Theorem). Find the Maclaurin series for $(1+x)^\alpha$, for any real α Let $f(x) = (1+x)^\alpha$. Then $f'(x) = \alpha(1+x)^{\alpha-1}, f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}, ...$, and subsequently $f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$. Therefore $f(0) = 1$ and for $n > 0$, we have

$$f^{(n)} = \alpha(\alpha-1)\cdots(\alpha-n+1)$$

The maclaurin series is therefore:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{x!}x^n = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$

It can be shown that $\forall x s.t. |x| < 1$ we have:

$$(1+x)^\alpha = \sum_{n=1}^{\infty} \binom{\alpha}{r} x^r$$

where,

$$\binom{\alpha}{0} = 1 and$$

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$$

Useful Maclaurin Series

$$\boxed{\begin{aligned} cos(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} \qquad (1) \\ sin(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1} \qquad (2) \end{aligned}}$$

2 Chpt 4 Functions of several variables

2.1 Partial Derivatives

Formal Definition:

$$\frac{\partial f}{\partial x} = \lim_{h\to0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h\to0} \frac{f(x,y+h) - f(x,y)}{h}$$

Higher Derivatives:

$$\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x} \quad or \quad (f_x)_y = f_{xy}$$

Notation: $f_x = \frac{\partial f}{\partial x}$

Example 4.1 Find all the first and second order partial derivatives of $f(x,y) = sin(x^2y) + xy$. The first order derivatives are

$$\frac{\partial f}{\partial x} = 2xycos(x^2y) \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2cos(x^2y) + x$$

note also: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$

2.2 Tangent plane and linear approximation

For (x,y) near some point (x_0, y_0) we have the approximation:

$$\boxed{\text{f(x,y)} \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

2.3 Directional derivative and gradient

The *directional derivative* of f in the direction $\hat{\mathbf{u}}$ is defined to be

$$D_{\hat{u}}f(x,y) = \lim_{h\to0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$$

Plugging in the points $(x+hu_1, y+hu_2)$ into the $f(x,y)$ approximation and then plugging that into the equation for $D_{\hat{u}}$ gives

$$D_{\hat{u}}f(x,y) = u_1f_x(x_0,y_0) + u_2f_y(x_0,y_0)$$

which can be succinctly written as

$$D_{\hat{u}}f(x,y) = \hat{\mathbf{u}} \cdot \nabla f(x,y), \quad \nabla f(x,y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Note:

- The direction of $\nabla f(x_0, y_0)$ is the direction which $f(x, y)$ increases most rapidly at (x_0, y_0).
- The magnitude (i.e. $|\nabla f(x_0, y_0)|$) is the rate of change of f in this direction (largest rate).

2.4 Examples from Sheet 3

1. Find the tangent plane to the surface $y = xz^2 + z + 1$ at the point $(1, 3, 2)$.

   Solution:

   Let $g(x, y, z) = xz^2 - y + z$. Then the surface we are considering is the level $g(x, y, z) = -1$. Now,

   $$\nabla g(x, y, z) = z^2\mathbf{i} - \mathbf{j} + (2xz + 1)\mathbf{k}$$

   so a normal to the surface at (1,3,-2) is

   $$\mathbf{n} = \nabla g(1, 3, -2) = 4\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

   Since $P_0 := (1, 3, -2)$ is a point on the plane, then any point $P = (x, y, z)$ on the plane satisfies:

   $$0 = \mathbf{n} \cdot \vec{P_0P} = 4(x - 1) - (y - 3) - 3(z + 2)$$

   Hence the Cartesian form of the plane is $4x - y - 3z = 7$.

2. Let
   $$f(x, y, z) = a(x + 1)^2 + b(y - 2)^2 + cz^2$$
   where a,b,c are constants. The greatest rate of change in f at the point (0,3,1) is in the direction

   $$\hat{\mathbf{u}}_1 = \frac{1}{3}(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

   Also the rate of change of f at $(0, 0, 1)$ in the direction

   $$\hat{\mathbf{u}}_2 = \frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$$

   is 8. Find the values of a, b and c. Solution: $\nabla f(x, y, z) = 2a(x + 1)\mathbf{i} + 2b(y - 2)\mathbf{j} + 2cz\mathbf{k}$ The greatest rate of change in f at the point $(0, 3, 1)$ is in the same direction as $\nabla f(0, 3, 1) = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k}$ So there is a positive number α such that $2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k} = \alpha\hat{\mathbf{u}}_1 = \frac{\alpha}{3}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$

   We have:

   $$8 = D_{\hat{\mathbf{u}}_1}(0, 0, 1) = \hat{\mathbf{u}}_2 \cdot \nabla f(0, 0, 1) = (\frac{3}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}) \cdot \frac{\alpha}{3}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = \frac{4}{3}$$

   Therefore $\alpha = 6$, giving $a = 1, b = -2, c = 2$.

3 Chpt 5 Integral Calculus

3.1 Integral Calculus

3.1.1 Integration Rules & Special Cases

$$\left.\begin{array}{l} \int e^x dx = e^x + C \\ \int a^x dx = \frac{a^x}{ln(a)} + C \end{array}\right\} \text{Exponent}$$

$$\left.\begin{array}{l} \int x^n dx = \frac{x^{n+1}}{n+1} + C \\ \int x^{-1} dx = ln(x) + C \end{array}\right\} \text{Power}$$

$\int |x| dx = \frac{x|x|}{2} + C$

$\int sec^2(x) dx = tan(x) + C$

$\int_b^a f(x) dx = -\int_a^b f(x) dx$

$\int sec(x)tan(x) dx = sec(x)$

$\int_b^a f(x) dx = \int_b^c f(x) dx + \int_c^a f(x) dx$

$\int csc(x)cot(x) dx = -csc(x) + C$

$\int \frac{1}{ax+b} dx = \frac{1}{a} ln|ax + b| + C$

$\int csc^2(x) dx = -cot(x) + C$

$\int sin(x) dx = -cos(x) + C$

$\int tan(x) dx = ln|sec(x)| + C$

$\int cos(x) dx = sin(x) + C$

3.1.2 Integration by substitution

If we see an integral of the form

$$F(f(x))f'(x)dx,$$

we can always make the substitution $u = f(x)$ so $du/dx = f'(x)$ and the integral becomes:

$$\int F(f(x))f'(x)dx = \int F(u)\frac{du}{dx}dx = \int F(u)du$$

	Suggested substitution
$\sqrt{a^2 - x^2}$	$x = a \cdot sin(u)$
$\sqrt{a^2 + x^2}$	$x = a \cdot sinh(u)$
$\sqrt{x^2 - a^2}$	$x = a \cdot cosh(u)$
$a^2 + x^2$	$x = a \cdot tan(u)$
$a^2 - x^2$	$x = a \cdot tanh(u)$

Example 5.1 Find $\int \frac{1}{(1+x^2)^{\frac{3}{2}}} dx$

Using substitution $x = sinh(u)$, we have $dx = cosh(u)du$ and $1 + x^2 = 1 + (sinh(u))^2 = cosh^2(u)$ Therefore,

$$\int \frac{1}{(1 + x^2)^{\frac{3}{2}}} dx = \int \frac{cosh(u)}{(cosh(u))^3} du = \int (sech(u))^2 du = tanh(u) + C$$

and we express $tanh(u)$ in terms of $x = sinh(u)$:

$$tanh(u) = \frac{sinh(u)}{cosh(u)} = \frac{sinh(u)}{\sqrt{1 + (sinh(u))^2}} = \frac{x}{\sqrt{1 + x^2}}$$

so finally,

$$\int \frac{1}{(1 + x^2)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{1 + x^2}} + C$$

3.1.3 Integration by parts

For differentiable functions u and v, integration of the product rule$(uv)' = u'v + uv'$ results in the integration by parts formula:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Priority of choosing u:

L	Logarithms
I	Inverse Functions
A	Algebraic
T	Trigonometric
E	Exponential

Example 5.2 *Find* $\int arcsin(x)dx$ We have

$$\int arcsin(x)dx = \int (arcsin(x))\frac{d(x)}{dx}dx = xarcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

Recursion formulas
Example 5.4 Let $I_n = \int_0^{\pi/3} sec^n(x)dx$ Using integration by parts with $u = sec^{n-2}(x)$ and $v = tan(x)$ we have

$$I_n = \int_0^{\pi/3} sec^n(x)dx = \int_0^{\pi/3} sec^{n-2}(x)\frac{d(tan(x))}{dx}dx$$
$$= tan(x)sec^{n-2}(x)\Big|_0^{\pi/3} -$$
$$(n-2)\int_0^{\pi/3} sec^{n-3}(tan(x)sec(x))tan(x)dx$$
$$= \sqrt{3}(2)^{n-2} - (n-2)\int_0^{\pi/3} sec^{n-2}(x)(sec^2(x)-1)dx$$
$$= \sqrt{3}(2)^{n-2} - (n-2)\Big(I_n - I_{n-2}\Big)$$

Partial Fractions

Rational function	Partial fraction form
$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a} + \frac{B}{x-b}$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
$\frac{px+q}{x(x^2+1)}$	$\frac{A}{x} + \frac{Bx+C}{x^2+1}$

3.2 The $tan(\theta/2)$ substitution

$$\boxed{\begin{aligned} sin(2u) &= \frac{2tan(u)}{1+tan^2(u)} \quad (3) \\ cos(2u) &= \frac{1-tan^2(u)}{1+tan^2(u)} \quad (4) \end{aligned}}$$

If we let $u = \theta/2$ and $t = tan(\theta/2)$ then we have

$$sin(\theta) = \frac{2t}{1+t^2} \quad \text{and} \quad cos(\theta) = \frac{1-t^2}{1+t^2}$$

We also note from $\theta = 2arctan(t)$ that

$$d\theta = \frac{2}{1+t^2}dt$$

and so the integral transforms to an integral of a rational function of t.
Example 5.6 *Evaluate* $\int_0^{\pi/2} \frac{1}{2+sin(\theta)}d\theta$.

Let $t = tan(\theta/2)$. Then endpoints $\theta = 0$ and $\theta = \pi/2$ transform to $t = 0$ and $t = 1$. Then we have:

$$\int_0^{\pi/2} \frac{d\theta}{2+sin(\theta)} = \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}}\frac{2dt}{1+t^2}$$
$$= \int_0^1 \frac{dt}{1+t+t^2}$$
$$= \int_0^1 \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$$

Now we make a substitution of $t + \frac{1}{2} = \frac{\sqrt{3}}{2}tan(u)$. Then $dt = \frac{\sqrt{3}}{2}sec^2(u)du$ and the endpoints $t = 0$ and $t = 1$ become $u = \pi/6$ and $u = \pi/3$ respectively.

$$\int_0^{\pi/2} \frac{d\theta}{2+sin(\theta)} = \int_{\pi/6}^{\pi/3} \frac{\frac{\sqrt{3}}{2}sec^2(u)du}{\frac{3}{4}(tan^2(u)+1)}$$
$$= \frac{2}{\sqrt{3}}\int_{\pi/6}^{\pi/3} du = \frac{\pi}{3\sqrt{3}}$$

3.3 Improper integral

will write notes later - 16dec

4 Chpt 6 Differential Equations

4.1 First-order ordinary differential equation

4.1.1 Separable Equations

An ordinary equation of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

is called *separable* if $g(y) \neq 0 then we have,$

$$\int \frac{1}{g(y)}dy = \int f(x)$$

Example 6.1 Solve the initial value problem $xy\frac{dy}{dx} = 1, y(-1) = -6$ Separating variables we have $yy' = x^{-1}$. Integrating gives

$$\int ydy = \int x^{-1}dx \Rightarrow \frac{1}{2}y^2 = ln|x| + c$$

Initial conditions give $\frac{(-6)^2}{2} = ln|-1| + c \implies c = 18$ Therefore, $y^2 = 2ln|x| + 36$ and initial condition $y(-1) = -6$ shows that we must take the minus sign: $y = -\sqrt{ln(x^2)+36}$

4.1.2 First-order linear ordinary differential equations

Such equations are of the form:

$$\frac{dy}{dx} + p(x)y = f(x)$$

for some function p and f. If $f(x)$ is zero then the equation is said to be *homogeneous*.

$$\frac{dy}{dx}+p(x)y=0 \Leftrightarrow \frac{1}{y}\frac{dy}{dx}=-p(x) \Leftrightarrow y(x)=exp\left(-\int p(x)dx\right)$$

Derivation of methods of solution: Integrating Factor

Example 6.2 Solve the equation $x^2y'+2xy=1$.
The key observation here is that the LHS is an exact derivative which means it can be written as:

$$\frac{d(x^2y)}{dx}=1 \Longrightarrow y(x)=x^{-1}+cx^{-2}$$

For more general equations, the idea here is to use the integrating factor $I(x)$ to multiply with the given equation to get the LHS as an exact derivative.
Doing this we get

$$I(x)y'(x)+p(x)I(x)y(x)=f(x)I(x)$$

and the LHS looks abit like

$$\frac{dI(x)}{dx}=p(x)I(x)$$

where we choose

$$I(x)=exp\left(\int p(x)\right)dx$$

Combining equations we get

$$\frac{d}{dx}(I(x)y(x))=f(x)I(x)$$

so,

$$\boxed{\begin{aligned} y(x)&=\frac{1}{I(x)}\int f(x)I(x)dx \qquad (5)\\ \text{with, } I(x)&=exp\left(\int p(x)\right)dx \qquad (6)\end{aligned}}$$

Example 6.3 Find the general solution of $y'+(1+6x)y=xe^{-x}$
Here $p(x)=1+6x$. An integrating factor is given by

$$I(x)=exp\left(\int p(x)dx\right)=exp\left(\int(1+6x)dx\right)=exp(x+3x^2)$$

Hence,

$$\begin{aligned} y(x)&=\frac{1}{e^{x+3x^2}}\int(xe^{-x}\cdot e^{x+3x^2})\\ &=e^{-x}\left(\frac{1}{6}+Ce^{-3x^2}\right)\end{aligned}$$

Example 6.4 Find the general solution of $xy'+(6x^3+1)y=4x^2$
First write the equation in standard form:

$$y'+(6x^{-2}+x^{-1})y=4x$$

We see that $p(x)=6x^2+x^{-1}$ hence:

$$\begin{aligned} I(x)=exp\left(\int p(x)dx\right)&=exp\left(\int(6x^2+x^{-1})dx\right)\\ &=exp(2x^3+ln|x|)=e^{2x^3}e^{ln|x|}=|x|e^{2x^3}\end{aligned}$$

Now for any constant multiple of an integrating factor is still an integrating factor, so for $x<0$ we can multiply by -1 giving the integrating factor xe^{2x^3}. We then get,

$$\left(xe^{2x^3}y\right)'=4x^2e^{2x^3}$$

Which integrates to give $xe^{2x^3}y=(2/3)e^{2x^3}+C$. So the general solution is

$$y(x)=\frac{1}{x}\left(\frac{2}{3}+C\cdot exp(-2x^3)\right)$$

4.1.3 Second-order linear ordinary differential equations (ODE)

Any second-order linear ordinaryb differential equation can be written in the form:

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=f(x).$$

If f = 0 then the equation is *homogenous*.
Note:The set of all solutions forms a 2D-vector space (meaning we only need to find two solutions that are not multiples of each other) $\Longrightarrow$ Let y_1 and y_2 be solutions to the homogeneous second-order ode, then for any constant c_1 and c_2 the function:

$$y(x):=c_1y_1(x)+c_2y_2(x)$$

is also a solution. **Constant coefficient homogeneous second-order linear ordinary differential equations** Consider the case where p and q are constants:

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0$$

We look for a solution of the form (bcus it's convenient):

$$y(x)=e^{\lambda x}$$

Substituting that into our equation gets:

$$(\lambda^2+p\lambda+q)e^{\lambda x}=0$$

where $(\lambda^2+p\lambda+q)$ is the characteristic equation and the solutions of the characteristic equation is given by:

$$\lambda=\frac{-p\pm\sqrt{\Delta}}{2}, \quad \Delta=p^2-4q^2$$

Cases	Num. of Roots	Gen. Soln (y(x)=..)
$\Delta>0$	No distinct roots	$c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$
$\Delta=0$	Repeated real root λ	$c_1+c_2x)e^{\lambda x}$
$\Delta<0$	Complex Roots	$e^{\mu x}(c_1cos(\nu x)+c_2sin(\nu x))$

Inhomogenous equations To solve non-homogeneous second-order ODE, step 1 is to find the ***general solution*** $y_h(x)$ and step 2 is to find the ***particular solution*** $y_p(x)$. Then the general solution will be given by

$$y(x) = y_p(x) + y_h(x)$$

To get the ***particular solution*** we need to guess for a given solutions, called 'ansatz'. This method of guessing is called method of undetermined coefficients

$f(x)$	Choice for particular solution $y_p(x)$
Any constant	A
$3x^2$	$Ax^2 + Bx + C$
$3e^{4x}$	Ae^{4x}
$4cos(x)$	$Acos(x) + Bsin(x)$
$sinh(x)$	$\alpha xe^x + \beta e^{-x}$
$xe^{3x}cos(x)$	$(Ax + B)e^{3x}cos(x) + (Cx + D)e^{3x}sin(x)$

Example 6.9 Find the general solution of $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = sinh(x) + sin(2x)$.

General Solution

The characteristic equation for the homogeneous ODE:

$$y_h" + 5y_h' - 6 = 0 \quad \text{is} \quad \lambda^2 + 5\lambda - 6 = 0$$

so $\lambda = 1$ or $\lambda = -6$ and the general solution of the homogeneous equation is:

$$y_h = c_1e^x + c_2e^{-6x}$$

where c_1 and c_2 are arbitrary constants

Particular Solution

We need to find a particular solution of:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = \frac{1}{2}e^x - \frac{1}{2}e^{-x} + sin(2x)$$

$$y_p = e^x + \beta e^{-x} + \gamma sin(2x) + \delta sin(2x)$$

Hence,

$$y_p' = \alpha(1 + x)e^x - \beta e^{-x} + 2\gamma cos(2x) - 2\delta sin(2x)$$

and,

$$y_p" = \alpha(2 + x)e^x + \beta e^{-x} - 4\gamma sin(2x) - 4\delta cos(2x)$$

Substituting these values for y_p, y_p' & $y_p"$ into the equation we get:

$$\Big\{\alpha([2 + 5] + [1 + 5 - 6]x)e^x + \beta(1 - 5 - 6)e^{-x} +$$

$$(-4\gamma - 10\delta - 6\gamma)sin(x) + (-4\delta + 10\gamma - 6\delta)cos(x)\Big\}$$

$$= \frac{1}{2}e^x - \frac{1}{2}e^{-x} + sin(2x)$$

Equating coeeffients we get $7\alpha = 1/2, 10\beta = 1/2, -10(\gamma+\delta) = 1$ and $10(\gamma - \delta) = 0$. So $\alpha = 1/14, \beta = 1/20, \gamma = \delta = -1/20$. Hence the general solution is:

$$y = y_h+y_p = c_1e^xc_2e^{-6x}+\frac{1}{14}xe^{-x}+\frac{1}{20}e^{-x}-\frac{1}{20}(sin(2x)+cos(2x))$$