The Algebra of Open and Interconnected Systems

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Part I

Mathematical Foundations

Chapter 4

Decorated corelations

When enough structure is available to us, we may decorate corelations too. Furthermore, and key to the idea of 'black-boxing', we get a hypergraph functor from decorated cospans to decorated corelations.

4.1 Introduction

Consider the case

'flow networks' labelled graphs.

Semantics live on boundaries only.

We then introduce a new framework for working with hypergraph categories: decorated corelations.

Decorated corelations adds compositional operations to network-diagram representations, and handles composition of semantics too.

Two main theorems:

Theorem 4.1. *Given a category $\mathcal{C}$ with finite colimits, factorisation system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ is stable under pushouts, and a lax symmetric monoidal functor*

$$\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \longrightarrow \mathrm{Set},$$

we may define a hypergraph category of with morphisms decorated corelations.

Theorem 4.2. *Every hypergraph category can be constructed in this way.*

They apply to functors too.

4.2 Decorated corelations

The key difference is to decorate cospans we need to know how to push decorations up. To decorate corelations we all need to know how to pull decorations back down. This is related to the existence of an extraspecial commutative Frobenius monoid in our main applications.

4.2.1 Adjoining right adjoints

Suppose we have a cospan $X + Y \to N$ with a decoration on N. Reducing this to a corelation requires us to factor this to $X + Y \stackrel{e}{\to} \overline{N} \stackrel{m}{\to} N$. To define a category of decorated corelations, then, we must specify how to take decoration on N and 'pull it back' along m to a decoration on $\overline{N}$.

For decorated cospans, it is enough to have a functor F from a category $\mathcal{C}$ with finite colimits; the image Ff of morphisms f in $\mathcal{C}$ describes how to move decorations forward along f. In this subsection we explain how to expand $\mathcal{C}$ to include morphisms m^{opp} for each $m\mathcal{C}$, so that the image of m^{opp} describes how to move morphisms backwards along m.

Proposition 4.3. *Let $\mathcal{C}$ be a category with finite colimits, and let $\mathcal{M}$ be a subcategory of $\mathcal{C}$ stable under pushouts. Then we define the category $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$ as follows*

The symmetric monoidal category $(\mathcal{C};\mathcal{M}^{\mathrm{opp}}, +)$	
objects	*the objects of $\mathcal{C}$*
morphisms	*isomorphism classes of cospans of the form $\stackrel{c}{\to}\stackrel{m}{\leftarrow}$, where c lies in $\mathcal{C}$ and m in $\mathcal{M}$*
composition	*given by pushout*
monoidal product	*the coproduct in $\mathcal{C}$*
coherence maps	*the coherence maps in $\mathcal{C}$*

Proof. This is a symmetric monoidal subcategory of $\mathrm{Cospan}(\mathcal{C})$. Our data is well-defined: composition because $\mathcal{M}$ is stable under pushouts, and monoidal composition by Lemma 3.9. □

This category can be viewed as a bicategory, with 2-morphisms given by maps of cospans. In this bicategory every morphism of $\mathcal{M}$ has a right adjoint.

Examples 4.4. Our familiar examples:

- $\mathcal{C};\mathcal{C}^{\mathrm{opp}}$ is by definition equal to $\mathrm{Cospan}(\mathcal{C})$.

- $\mathcal{C};\mathcal{I}_{\mathcal{C}}^{\mathrm{opp}}$ is naturally isomorphic to $\mathcal{C}$.
- $\mathrm{Set};\mathrm{Inj}^{\mathrm{opp}}$ is the category of partial functions.

Lemma 4.5. *Let $\mathcal{C}$, $\mathcal{C}'$ be categories with finite colimits, and let $\mathcal{M}$, $\mathcal{M}'$ be subcategories each stable under pushouts. Let $A\colon \mathcal{C} \to \mathcal{C}'$ be functor that preserves colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$. Then A extends to a symmetric strong monoidal functor*

$$A\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \longrightarrow \mathcal{C}';\mathcal{M}'^{\mathrm{opp}}.$$

mapping X to AX and $\stackrel{c}{\to}\stackrel{m}{\leftarrow}$ to $\stackrel{Ac}{\to}\stackrel{Am}{\leftarrow}$.

Proof. Note $A(\mathcal{M}) \subseteq \mathcal{M}'$, so $\stackrel{Ac}{\to}\stackrel{Am}{\leftarrow}$ is indeed a morphism in $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$. This is then a restriction and corestriction of the usual functor $\mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{C}')$ to the above domain and codomain. □

That this 'subcospan category' construction could be defined more generally using any two isomorphism-containing wide subcategories stable under pushout, but the above suffices for decorated corelations.

4.2.2 Decorated corelations

Decorated corelations are constructed from a lax monoidal functor from $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$ to Set.

Definition 4.6. Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose we also have a lax monoidal functor

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times).$$

We define an F-**decorated corelation** to a pair

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}, \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd}\right)$$

where the cospan is jointly $\mathcal{E}$-like. A morphism of decorated corelations is a morphism of decorated cospans between two decorated corelations.

Suppose we have decorated corelations

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}, \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd}\right) \quad \text{and} \quad \left(\begin{tikzcd} & M & \\ Y \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd}, \quad \begin{tikzcd} FM \\ 1 \arrow[u, "s"'] \end{tikzcd}\right).$$

Then their composite is given by the composite corelation

$$\begin{tikzcd} & \overline{N +_Y M} & \\ X \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd}$$

paired with the decoration

$$1 \longrightarrow F(N+M) \longrightarrow F(N +_Y M) \stackrel{F(m^{\mathrm{opp}})}{\longrightarrow} F(\overline{N +_Y M})$$

As composition of corelations and decorated cospans are both well-defined up to isomorphism, this too is well-defined up to isomorphism. Again, we will be lazy about the distinction between a decorated corelation and its isomorphism class.

4.2.3 Categories of decorated corelations

In this subsection we give a definition of the hypergraph category $F\mathrm{Corel}$ of decorated corelations. In analogy with how we defined the hypergraph category of corelations, we leverage the fact that decorated cospans form a hypergraph category, this time using a structure preserving map

$$\square\colon F\mathrm{Cospan} \longrightarrow F\mathrm{Corel}.$$

Here $F\mathrm{Cospan}$ denotes the decorated cospan category constructed from the restriction of the functor $F\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \to \mathrm{Set}$ to the domain $\mathcal{C}$.

Given a cospan $X \to N \leftarrow Y$, write $m\colon \overline{N} \to N$ for the $\mathcal{M}$ factor of the copairing $X+Y \to N$. The functor $\square$ takes a decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN)$$

to the decorated corelation

$$(X \xrightarrow{\bar{i}} \overline{N} \xleftarrow{\bar{o}} Y,\ 1 \xrightarrow{Fm^{\mathrm{opp}}\circ s} F\overline{N}),$$

where the corelation is given by the jointly $\mathcal{E}$-part of the cospan, and the decoration is given by composing s with the F-image $Fm^{\mathrm{opp}}\colon FN \to F\overline{N}$ of the map $N \stackrel{1_N}{\to} N \stackrel{m}{\leftarrow} \overline{N}$ in $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$. We call $Fm^{\mathrm{opp}} \circ s$ the **restricted decoration** of the decorated cospan $(X \to N \leftarrow Y,\ 1 \stackrel{s}{\to} FN)$.

The monoidal product of two decorated corelations is their monoidal product as decorated cospans.

Theorem 4.7. *Let $\mathcal{C}$ be a category with finite colimits and factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M}$ stable under pushout, and let*

$$F : (\mathcal{C}; \mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times)$$

be a lax symmetric monoidal functor. Then we may define

The hypergraph category $(F\mathrm{Corel}, +)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of* F*-decorated corelations in* $\mathcal{C}$
composition	*given by* $\mathcal{E}$*-part of pushout with restricted decoration*
monoidal product	*the coproduct in* $\mathcal{C}$
coherence maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*
hypergraph maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*

Similar to the corelations theorem (Theorem 3.10), we have specified well-defined data and now need to check a collection of coherence axioms. As before, we prove this alongside the theorem about decorated corelation functors in the next section.

Example 4.8. Note that decorated cospans are a special case of decorated corelations: simply use an morphism–isomorphism factorisation system.

Example 4.9. Note that 'undecorated' corelations are a special case of decorated corelations: they are corelations decorated by the functor $1\colon \mathcal{C}; \mathcal{M}^{\mathrm{opp}} \to \mathrm{Set}$ that maps each object to the one element set 1, and each morphism to the identity function on 1. This is a symmetric monoidal functor with the coherence maps all also the identity function on 1.

4.3 Functors between decorated corelation categories

Functors between decorated corelation categories hold no surprises: their requirements combine the requirements of corelations and decorated cospans. Recall that Lemma 4.5 says that we can extend a colimit-preserving functor $\mathcal{C} \to \mathcal{C}'$ to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \to \mathcal{C}'; \mathcal{M}'^{\mathrm{opp}}$.

Proposition 4.10. *Let $\mathcal{C}$, $\mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M})$, $(\mathcal{E}', \mathcal{M}')$, such that $\mathcal{M}$ and $\mathcal{M}'$ are stable under pushout, and suppose that we have lax symmetric monoidal functors*

$$F : (\mathcal{C}; \mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times)$$

and

$$G : (\mathcal{C}'; \mathcal{M}'^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times).$$

Further let $A\colon \mathcal{C} \to \mathcal{C}'$ *be a functor that preserves finite colimits and such that the image of* $\mathcal{M}$ *lies in* $\mathcal{M}'$. *This functor* A *extends to a symmetric monoidal functor* $\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \to \mathcal{C}'; \mathcal{M}'^{\mathrm{opp}}$.

Suppose we have a monoidal natural transformation θ*:*

$$\begin{tikzcd} \mathcal{C}; \mathcal{M}^{\mathrm{opp}} \arrow[d, "A"'] \arrow[drr, "F"] \arrow[drr, phantom, shift right=4ex, "\Downarrow\theta"] & & \\ & & \mathrm{Set} \\ \mathcal{C}'; \mathcal{M}'^{\mathrm{opp}} \arrow[urr, "G"'] & & \end{tikzcd}$$

Then we may define a hypergraph functor $T\colon F\mathrm{Corel} \to G\mathrm{Corel}$ *sending each object* $X \in F\mathrm{Corel}$ *to* $AX \in G\mathrm{Corel}$ *and each decorated corelation*

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN)$$

to

$$(AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY, \quad 1 \xrightarrow{s} FN \xrightarrow{\theta_N} GAN \xrightarrow{Gm^{\mathrm{opp}}_{AN}} G\overline{AN}).$$

The coherence maps $\overline{\kappa_{X,Y}}$) *are given by the coherence maps of* A *with the restricted empty decoration.*

Proof of Theorem 4.7 and Proposition 4.10. In the proof of Theorem 3.10 and Proposition 3.11 we proved that the map

$$\square\colon\ \mathrm{Corel}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C}')$$

preserved composition and had natural coherence maps. Specialising to the case when $\mathrm{Corel}(\mathcal{C}) = \mathrm{Cospan}(\mathcal{C}')$, we saw that this bijective-on-objects, surjective-on-morphisms, composition and monoidal product preserving map proved $\mathrm{Corel}(\mathcal{C}')$ is a hypergraph category, and it immediately followed that $\square$ is a hypergraph functor.

The analogous argument holds here: we simply need to prove

$$\square\colon F\,\mathrm{Corel} \longrightarrow G\,\mathrm{Corel}$$

preserves composition and has natural coherence maps. Theorem 4.7 then follows from examining the map $F\,\mathrm{Cospan} \to F\,\mathrm{Corel}$ obtained by choosing $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}', \mathcal{I}_{\mathcal{C}'})$, F the restriction of G to $\mathcal{C}'$, A the identity functor on $\mathcal{C}'$, and θ the identity natural transformation. Subsequently Proposition 4.10 follows from noting that all the axioms hold for the corresponding maps in $G\mathrm{Cospan}$.

$\square$ preserves composition. Suppose we have decorated corelations

$$f = (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN) \qquad \text{and} \qquad g = (Y \xrightarrow{i_Y} M \xleftarrow{o_Y} Z,\ 1 \xrightarrow{t} FM)$$

We know the functor $\square$ preserves composition on the cospan part; this is precisely the content of Proposition 3.11. It remains to check that $\square(g \circ f)$ and $\square g \circ \square f$ have isomorphic decorations. This is expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
GA\overline{(\overline{N +_Y M})} \arrow[rrrr, "Gn"] & & & & G\overline{(\overline{AN} +_{AY} \overline{AM})} \\
GA(\overline{N +_Y M}) \arrow[u, "Gm^{\mathrm{opp}}_{A(\overline{N+_Y M})}"] & & (**) & & G(\overline{AN} +_{AY} \overline{AM}) \arrow[u, "Gm^{\mathrm{opp}}_{\overline{AN}+_{AY}\overline{AM}}"'] \\
F(\overline{N +_Y M}) \arrow[u, "\theta_{\overline{N+_Y M}}"] & GA(N +_Y M) \arrow[ul, "GAm^{\mathrm{opp}}_{N+_Y M}"'] & & G(AN +_{AY} AM) \arrow[ll, "G\sim"'] \arrow[ur, "G(m^{\mathrm{opp}}_{AN}+_{AY}m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "\text{(C)}"] & G(\overline{AN} + \overline{AM}) \arrow[u, "G{[j_{\overline{AN}},j_{\overline{AM}}]}"'] \\
F(N +_Y M) \arrow[u, "Fm^{\mathrm{opp}}_{N+_Y M}"] \arrow[r, phantom, "\text{(TN)}"] & GA(N + M) \arrow[u, "GA{[j_N,j_M]}"'] & \text{(A)} & G(AN + AM) \arrow[ll, "G\alpha_{N,M}"] \arrow[u, "G{[j_{AN},j_{AM}]}"] \arrow[ur, "G(m^{\mathrm{opp}}_{AN}+m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "\text{(GM)}"] & G\overline{AN} \times G\overline{AM} \arrow[u, "\gamma_{\overline{AN},\overline{AM}}"'] \\
F(N + M) \arrow[u, "F{[j_N,j_M]}"] \arrow[ur, "\theta_{N+M}"'] & & \text{(TM)} & & GAN \times GAM \arrow[ul, "\gamma_{AN,AM}"] \arrow[u, "Gm^{\mathrm{opp}}_{AN}\times Gm^{\mathrm{opp}}_{AM}"'] \\
& & FN \times FM \arrow[ull, "\varphi_{N,M}"] \arrow[urr, "\theta_N\times\theta_M"'] & & \\
& & 1 \arrow[u, "\rho_1\circ(s\times t)"'] & &
\end{tikzcd}$$

This diagram does indeed commute. To check this, first observe that (TM) commutes by the monoidality of θ, (GM) commutes by the monoidality of G, and (TN) commutes by the naturality of θ. The remaining three diagrams commute as they are G-images of diagrams that commute in $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$. Indeed, (A) commutes since A preserves colimits and G is functorial, (C) commutes as it is the G-image of a pushout square in $\mathcal{C}'$, so

$$\xleftarrow{m^{\mathrm{opp}}_{AN}+m^{\mathrm{opp}}_{AM}}\xrightarrow{[j_{\overline{AN}},j_{\overline{AM}}]} \qquad \text{and} \qquad \xrightarrow{[j_{AN},j_{AM}]}\xleftarrow{m^{\mathrm{opp}}_{AN}+_{AY}m^{\mathrm{opp}}_{AM}}$$

are equal as morphisms of $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$, and $(**)$ commutes as it is the G-image of the right-hand subdiagram of $(*)$ used to define n in the proof of Lemma 3.12.

Coherence maps are natural. Let $f = (X \longrightarrow N \longleftarrow Y,\ 1 \to FN)$, $g = (Z \longrightarrow M \longleftarrow W,\ 1 \to FM)$ be F-decorated corelations in $\mathcal{C}$. We wish to show that

$$\begin{tikzcd} AX + AY \arrow[r, "\square f + \square g"] \arrow[d, "\overline{\kappa_{X,Y}}"'] & AZ + AW \arrow[d, "\overline{\kappa_{Z,W}}"] \\ A(X+Y) \arrow[r, "\square(f+g)"] & A(Z+W) \end{tikzcd}$$

commutes in $G\mathrm{Corel}$, where the coherence maps are given by

$$\overline{\kappa_{X,Y}} = \left(\begin{tikzcd} & \overline{A(X+Y)} & \\ AX + AY \arrow[ur] & & A(X+Y) \arrow[ul] \end{tikzcd} , \begin{tikzcd} G(\overline{A(X+Y)}) \\ GA(X+Y) \arrow[u, "Gm^{\mathrm{opp}}_{AX+AY}"'] \\ G\varnothing \arrow[u, "G!"'] \\ 1 \arrow[u, "\gamma_1"'] \end{tikzcd} \right).$$

Lemma 3.14 shows that the composites of corelations agree. It remains to check that the decorations also agree.

Here Lemma 2.5 is helpful. Since $\square$ is composition preserving, we can replace the $\overline{\kappa}$ with the empty decorated coherence maps κ of $G\mathrm{Cospan}$, and compute these composites in $G\mathrm{Cospan}$, before restricting to the $\mathcal{E}'$-parts. Lemma 2.5 then implies that the restricted empty decorations on the isomorphisms $\overline{\kappa}$ play no role in determining the composite decorations. It is thus enough to prove that the decorations of $\square f + \square g$ and $\square(f+g)$ are the same up to the isomorphism $p\colon G(\overline{AN} + \overline{AM}) \to G\overline{A(N+M)}$ between their apices, as defined in the diagram (#) in the proof of Lemma 3.14.

This comes down to proving the following diagram commutes:

$$\begin{tikzcd} & & GAN \times GAM \arrow[r, "Gm \times Gm"] \arrow[dr, "\gamma"] & G\overline{AN} \times G\overline{AM} \arrow[r, "\gamma"] \arrow[d, phantom, "\text{(G)}"] & G(\overline{AN} + \overline{AM}) \arrow[dd, "Gp"', "\sim"] \\ 1 \arrow[r, "\langle s,t \rangle"] & FN \times FM \arrow[ur, "\theta"] \arrow[dr, "\varphi"'] & \text{(T)} & G(AN + AM) \arrow[ur, "G(m+m)"'] \arrow[d, "G\kappa"] & \text{(\#\#)} \\ & & F(N+M) \arrow[r, "\theta"'] & GA(N+M) \arrow[r, "Gm"'] & G\overline{A(N+M)} \end{tikzcd}$$

It is straightforward to check this commutes: (T) by the monoidality of θ, (G) by the monoidality of G, and (##) as it is the G-image of the rightmost square in (#). □

Corollary 4.11. *Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose that we also have a lax monoidal functor*

$$F : (\mathcal{C}; \mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times).$$

Then we may define a category $F\mathrm{Corel}$ *with objects the objects of* $\mathcal{C}$ *and morphisms isomorpism classes of* F*-decorated corelations.*

Write also F *for the restriction of* F *to the wide subcategory* $\mathcal{C}$ *of* $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$*. We can thus also obtain the category* $F\mathrm{Cospan}$ *of* F*-decorated cospans. We moreover have a functor*

$$F\mathrm{Cospan} \to F\mathrm{Corel}$$

which takes each object of $F\mathrm{Cospan}$ *to itself as an object of* $F\mathrm{Corel}$*, and each decorated cospan*

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}, \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd} \right)$$

to its jointly-$\mathcal{E}$*-part*

$$\begin{tikzcd} & \overline{N} & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

decorated by the composite

$$1 \xrightarrow{\ s\ } FN \xrightarrow{Fm_N^{\mathrm{opp}}} F\overline{N}.$$

4.4 All hypergraph categories are decorated corelation categories

Not all hypergraph categories are decorated *cospan* categories. To see this, we can count morphisms. In a decorated cospan category the possible apices and decorations are the same for all morphisms. So unless the category $\mathcal{C}$ and thus $\mathrm{Cospan}(\mathcal{C})$ is trivial, there must be a number of morphisms $0 \to 0$. On the other hand, the hypergraph category of finite sets and equivalence relations, discussed in Subsection 3.4.1, has a unique morphism $0 \to 0$.

Decorated corelation categories, however, are more powerful. In this section we show that all hypergraph categories are decorated corelation categories, and provide some examples for intuition.

4.4.1 The global sections construction

Suppose we have a hypergraph category and wish to construct a hypergraph equivalent decorated corelation category. The main idea to is to take advantage of the compact closed structure: recall that morphisms $X \to Y$ in a hypergraph category are in one-to-one correspondence with morphisms $I \to X \otimes Y$. Using an isomorphism–morphism factorisation system on the category comprising just the objects and the coherence and hypergraph maps, we can use the monoidal global sections functor to decorated each corelation $X \to X \otimes Y \leftarrow Y$ with the monoidal elements of $X \otimes Y$. This recovers the original hypergraph category.

Theorem 4.12. *Every hypergraph category is hypergraph equivalent to a decorated corelation category.*

Proof. Let $(\mathcal{H}, \otimes)$ be a hypergraph category. By Theorem 1.5, we have a hypergraph equivalent strict hypergraph category $\mathcal{H}_{\mathrm{str}}$, with objects finite lists of objects in $\mathcal{H}$. Write $\mathcal{O}$ for the collection of objects in $\mathcal{H}$, and $\mathrm{FinSet}_{\mathcal{O}}$ for $\mathcal{O}$-labelled finite sets (i.e. an object is a finite set S together with each element in S labelled by some object in $\mathcal{O}$, and a morphism is a function that preserves labels). The category $\mathrm{FinSet}_{\mathcal{O}}$ has finite colimits: the colimit is just the colimit set in FinSet with elements labelled by the label of any element that maps to it in the colimit diagram. This labelling exists and is unique by the universal property of the colimit. Write $+$ for the coproduct; we consider $\mathrm{FinSet}_{\mathcal{O}}$ a symmetric monoidal category. Replacing FinSet with an equivalent skeleton, we think of the objects of $\mathrm{FinSet}_{\mathcal{O}}$ as finite lists of objects of $\mathcal{H}$, and hence as the same as the objects of $\mathcal{H}_{\mathrm{str}}$.

We define a wide hypergraph embedding

$$H\colon (\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), +) \longrightarrow (\mathcal{H}_{\mathrm{str}}, \otimes)$$

whose 'image' is the hypergraph structure of $\mathcal{H}$. More precisely, H sends each object of $\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}})$ to the same as an object of $\mathcal{H}_{\mathrm{str}}$. As all functions can be generated under composition and coproduct from the unique functions $\mu\colon 2 \to 1$ and $\eta\colon 0 \to 1$, the morphisms of $\mathrm{FinSet}_{\mathcal{O}}$ are generated by the unique morphisms $\mu_X\colon [X, X] = X + X \to X$ and $\eta_X\colon \varnothing \to X$, and so the morphisms of $\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}})$ are generated by the cospans μ_X, η_X, $\delta_X = \mu_X^{\mathrm{opp}}$, and $\epsilon_X = \eta_X^{\mathrm{opp}}$. The functor H maps these generators to the corresponding Frobenius maps on X in $\mathcal{H}_{\mathrm{str}}$. Since Cospan(FinSet) is the 'generic special commutative monoid' (see Proposition 1.6), it is straightforward to check this defines a hypergraph functor.

Next, recall that the monoidal global sections functor $\mathcal{H}(I, -)\colon \mathcal{H} \to \mathrm{Set}$ is a lax symmetric monoidal functor taking each object X of $\mathcal{H}$ to the homset $\mathcal{H}(I, X)$, and recall we have a hypergraph equivalence $\mathcal{H}_{\mathrm{str}} \to \mathcal{H}$. Write F for the composite of these three functors:

$$F\colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) \xrightarrow{H} \mathcal{H}_{\mathrm{str}} \xrightarrow{\sim} \mathcal{H} \xrightarrow{\mathcal{H}(I,-)} \mathrm{Set}$$

This is lax symmetric monoidal functor mapping X to the homset $\mathcal{H}(I, X)$, and a cospan between lists of objects in $\mathcal{H}$ to the corresponding Frobenius map.

Now consider $\mathrm{FinSet}_{\mathcal{O}}$ with an (isomorphism,morphism)-factorisation system. This implies F defines a decorated corelation category $F\mathrm{Corel}$ with objects those of $\mathcal{H}_{\mathrm{str}}$, and morphisms $X \to Y$ trivial corelations $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$ decorated by some morphism $s \in \mathcal{H}(I, X \otimes Y)$. We will show $(F\mathrm{Corel}, +)$ and $(\mathcal{H}_{\mathrm{str}}, \otimes)$ are isomorphic as hypergraph categories.

First, let us examine composition in $F\mathrm{Corel}$. As morphisms $X \to Y$ are completely specified by their decoration $s \in \mathcal{H}(I, X \otimes Y)$, we will abuse terminology and refer to the decorations themselves as the morphisms. Given morphisms $s \in \mathcal{H}(I, X \otimes Y)$ and $t \in \mathcal{H}(I, Y \otimes Z)$ in $F\mathrm{Corel}$, composition is given by the map

$$H(I, X \otimes Y \otimes Y \otimes Z) \to H(I, X \otimes Z)$$

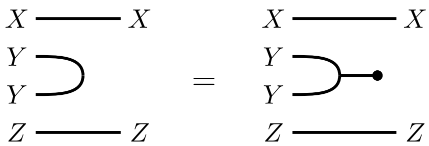
arising as the F-image of the cospan $X+Y+Y+Z \xrightarrow{[j_{X+Y}, j_{Y+Z}]} X+Y+Z \xleftarrow{[\iota_X, \iota_Z]} X+Z$, where these maps come from the pushout square

$$\begin{tikzcd}
 & & X+Y+Z & & \\
 & X+Y \arrow[ur, "j_{X+Y}"] & & Y+Z \arrow[ul, "j_{Y+Z}"'] & \\
X \arrow[ur, "\iota_X"] & & Y \arrow[ul, "\iota_Y"'] \arrow[ur, "\iota_Y"] & & Z \arrow[ul, "\iota_Z"']
\end{tikzcd}$$

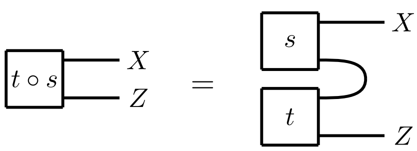
and the trivial isomorphism-morphism factorisation $X+Z = X+Z \xrightarrow{[\iota_X, \iota_Z]} X+Y+Z$ in $\mathrm{FinSet}_{\mathcal{O}}$.

In terms of string diagrams in $\mathcal{H}$, this means composing the maps

with the Frobenius map

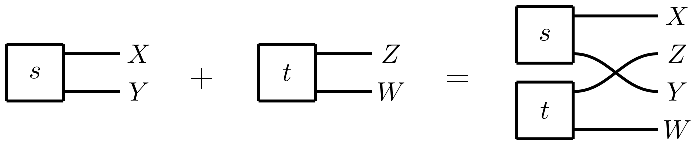


to get



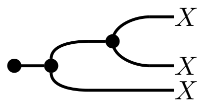
in $\mathcal{H}(I, X \otimes Z)$.

The monoidal product is given by



Taking a hint from the compact closed structure, the isomorphism between FCorel and $\mathcal{H}_{\mathrm{str}}$ is the clear: the functors act as the identity on objects, and on morphisms take $f : A \to B$ in $\mathcal{H}$ to its name $\hat{f} : I \to A + B$ as a morphism of FCorel, and vice versa.

It is straightforward to check that these are strict hypergraph functors. To demonstrate the most subtle aspect, the Frobenius structure, we consider the multiplication on an object X of FCorel. To obtain this, we start with the multiplication in FCospan, the cospan $X + X \xrightarrow{[1,1]} X \xleftarrow{1} X$ decorated with the empty decoration, which in this case is $\eta_X \in \mathcal{H}(I, X)$, and restrict this empty decoration along the map $X \xleftarrow{[1,1,1]} X + X + X$. Thus the multiplication on X in FCorel is



$\in \mathcal{H}(I, X \otimes X \otimes X)$,

which corresponds under our isomorphism to the map $\mu_X \colon X \otimes X \to X \in \mathcal{H}_{\mathrm{str}}$, as required. $\square$

We can also recover hypergraph functors using decorated corelations.

Theorem 4.13. *All hypergraph functors are decorated corelation functors.*

Proof. Let $(T, \tau) \colon \mathcal{H} \to \mathcal{H}'$ be a hypergraph functor between hypergraph categories. The previous theorem implies there exist lax symmetric monoidal functors

$$F \colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) \to \mathrm{Set} \quad \text{and} \quad F' \colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}'}) \to \mathrm{Set}$$

such that we have isomorphisms $\mathcal{H}_{\mathrm{str}} \cong F\mathrm{Corel}$ and $\mathcal{H}'_{\mathrm{str}} \cong F'\mathrm{Corel}$. We will realise T as the functor induced by some colimit-preserving functor $A \colon \mathrm{FinSet}_{\mathcal{O}} \to \mathrm{FinSet}_{\mathcal{O}'}$ and monoidal natural transformation:

$$\begin{tikzcd}
\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) \arrow[dr, "F"] \arrow[dd, "A"'] & \\
\arrow[r, phantom, "\Downarrow\theta"] & \mathrm{Set} \\
\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}'}) \arrow[ur, "F'"'] &
\end{tikzcd}$$

Define the functor A to take each $\mathcal{O}$-labelled set $[X_1, \dots, X_n]$ to the same set now $\mathcal{O}'$-labelled $[TX_1, \dots, TX_n]$, and to take a function between $\mathcal{O}$-labelled sets to the same function, noting that the function now preserves the $\mathcal{O}'$-labels. This functor A clearly preserves colimits: again, colimits in $\mathrm{FinSet}_{\mathcal{O}}$ are just colimits in FinSet with the labels inherited as described above.

Let $S = [X_1, \dots, X_n]$ be an $\mathcal{O}$-labelled set. Then we define

$$\begin{aligned}\theta_S \colon \mathcal{H}(I, X_1 \otimes \dots \otimes X_n) &\longrightarrow \mathcal{H}'(I, TX_1 \otimes \dots \otimes TX_n);\\ \Big(I \xrightarrow{s} X_1 \otimes \dots \otimes X_n\Big) &\longmapsto \Big(I \xrightarrow{\tau} TI \xrightarrow{Ts} T(X_1 \otimes \dots \otimes X_n) \xrightarrow{\tau} TX_1 \otimes \dots \otimes TX_n\Big),\end{aligned}$$

where the τ are the appropriate coherence isomorphisms of T. This collection of maps θ is natural as functions in $\mathrm{FinSet}_{\mathcal{O}}$ act as the Frobenius maps on the homsets $\mathcal{H}(I, X)$ and $\mathcal{H}'(I, TX)$, and the functor T is hypergraph and so maps, loosely speaking, Frobenius maps to Frobenius maps. Moreover θ is monoidal as T is monoidal. Thus θ defines a monoidal natural transformation.

The hypergraph functor $F\mathrm{Corel} \to F'\mathrm{Corel}$ induced by θ is by definition the map taking $s \in \mathcal{H}(I, X \otimes Y)$ to $\tau^{-1} \circ Ts \circ \tau_1 \in \mathcal{H}'(I, TX \otimes TY)$. Composing this functor with the appropriate equivalences we recover $T \colon \mathcal{H} \to \mathcal{H}'$. □

4.4.2 A categorical equivalence

Choose Grothendieck universes so that we may talk about categories Set now of all 'small' sets, and Cat of all 'small' categories. If we restrict our attention to small hypergraph categories, we map summarise these results as a categorical equivalence.

Indeed, given some fixed object set $\mathcal{O}$, we have a category of lax symmetric monoidal functors and monoidal natural transformations

$$\mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set}).$$

This is equivalent to some subcategory of the category HypCat of hypergraph categories. To get a category equivalent to all hypergraph categories, we must patch together these functor categories by varying the object set and specifying morphisms from objects in one category to objects in another. For this we use the Grothendieck construction.

Given a (contravariant) functor $S \colon \mathcal{B} \to \mathrm{Cat}$ from some category $\mathcal{B}$ to Cat, we define the **(contravariant) Grothendieck construction** $\mathcal{B} \int S$ to be the category with objects pairs $(\mathcal{O}, F)$ where $\mathcal{O}$ is an object of $\mathcal{B}$ and F is an object of $S\mathcal{O}$, and morphisms pairs

$$(f, \theta) \colon (\mathcal{O}, F) \longrightarrow (\mathcal{O}', F')$$

where $f\colon \mathcal{O} \to \mathcal{O}'$ is a morphism in $\mathcal{B}$ and $\theta\colon F \to Sr(F')$ is a morphism in $S\mathcal{O}$.

Now, define the functor

$$\mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{-}), \mathrm{Set})\colon \mathrm{Set} \longrightarrow \mathrm{Cat}$$

as follows. On objects let it map a set $\mathcal{O}$ to the lax symmetric monoidal functor category $\mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set})$. For morphisms, recall from the above proof that a function $r\colon \mathcal{O} \to \mathcal{O}'$ induces a functor $R\colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) \to \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}'})$ taking each $\mathcal{O}$-labelled set $N \xrightarrow{l} \mathcal{O}$ to the $\mathcal{O}'$-labelled set $N \xrightarrow{r\circ l} \mathcal{O}'$. This in turn defines a functor

$$\begin{aligned}\rho\colon \mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}'}), \mathrm{Set}) &\longrightarrow \mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set});\\ \Big(F'\colon\ \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}'}) \to \mathrm{Set}\Big) &\longmapsto \Big(R \circ F'\colon\ \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) \to \mathrm{Set}\Big).\end{aligned}$$

The Grothendieck construction

$$\mathrm{Set} \int \mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{-}), \mathrm{Set})$$

thus gives a category where are objects are some label set $\mathcal{O}$ together with an object in $\mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set})$—that is, with a lax symmetric monoidal functor $F\colon (\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), +) \to (\mathrm{Set}, \times)$. The morphisms $(\mathcal{O}, F) \to (\mathcal{O}, F')$ are functions of label sets $r : \mathcal{O} \to \mathcal{O}'$ together with a natural transformation $\theta\colon R \circ F' \Rightarrow F$.

In the case of small categories, it is not difficult to show Theorems 4.12 and 4.13 imply:

Theorem 4.14. *There is an equivalence of categories*

$$\mathrm{HypCat} \cong \mathrm{Set} \int \mathrm{LaxSymMon}(\mathrm{Cospan}(\mathrm{FinSet}_{-}), \mathrm{Set}).$$

The 2-categorical version of the above equivalence is the also the subject of a forthcoming paper by Vagner, Spivak, and Schultz, from their operadic perspective [?].

4.4.3 Factorisations as decorations

We have seen that every hypergraph category can be constructed as a decorated corelations category. More precisely, we have seen that every hypergraph category can be constructed as a decorated corelation category with the factorisation system the trivial isomorphism-morphism factorisation system. But decorated corelation categories may be constructed using other factorisation systems, and these are also hypergraph categories. How do the two constructions relate to each other?

For each hypergraph category there is a poset of lax symmetric monoidal functors that give decorated corelation constructions of an equivalent category. Although the constructed categories are all equivalent, decorated corelations only allows functors decrease the number of corelations available; corelations are ultimately a ‘quotienting’ construction. The utility of decorated corelations is that we can maintain equivalence by putting this extra, quotiented information into the decoration.

For illustration, we first example this interaction for the simplest hypergraph category: Cospan(FinSet), the free hypergraph category on a single object.

Example 4.15. As per Example 4.9, Cospan(FinSet) is the hypergraph category of undecorated (morphism-isomorphism)-corelations in FinSet. Theorem 4.12 shows it is also the partition-decorated (isomorphism-morphism)-corelations in FinSet.

First, the global sections functor $G\colon$ Cospan(FinSet) $\to$ Set takes each finite set X to the set of (equivalence classes of) cospans $0 \to D \leftarrow X$; that is, to the set GX of functions $s\colon X \to D$, where a unique codomain D is chosen for each finite cardinality. Given a cospan $X \xrightarrow{f} N \xleftarrow{g} Y$, its image under the global sections functor is the functor $GX \to GY$ mapping $s\colon X \to D$ in GX to the function $Y \to N +_Y D$ in

$$\begin{tikzcd} & X \arrow[r, "s"] \arrow[d, "f"'] & D \arrow[d] \\ Y \arrow[r, "g"] & N \arrow[r] & N +_Y D \end{tikzcd}$$

where the square is a pushout square. The coherence maps $\gamma_1\colon 1 \to G\varnothing$ map the unique element of 1 to the unique function $!\colon \varnothing \to \varnothing$, and $\gamma_{X,Y}$ maps a pair of functions $a\colon X \to D$, $b\colon Y \to E$ to $a + b\colon X + Y \to D + E$. This defines a lax symmetric monoidal functor (G, γ).

A decorated *cospan* in FinSet with respect to (G, γ) is a cospan of finite sets $X \to N \leftarrow Y$ together with a function of finite sets $N \to D$. Using the isomorphism–morphism factorisation, a decorated *corelation* is thus a cospan $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$ together with a function $X + Y \to D$. A decorated corelation is thus specified by the decoration $X + Y \to D$ alone. Note that these are in one-to-one correspondence with cospans $X \to D \leftarrow Y$ in FinSet via the coproduct maps.

As observed in the proof of Theorem 4.12, the hypergraph structure on GCorel agrees with that on Cospan(FinSet) via this correspondence; the multiplication $\mu_X\colon X + X \to X$ in GCorel is simply given by the decoration $(X + X) + X \to X$, and so on.

The intuition is that in the global sections construction on a cospan category, the shift from cospans to corelations takes the 'factored part' and puts it into the decoration.

This suggests an isomorphism; indeed, the isomorphism given by Theorem 4.12 does precisely this. We can construct one direction, the one from the smaller to larger $\mathcal{M}$—that is, from $\mathcal{M} = \mathcal{I}_{\mathrm{FinSet}}$ to $\mathcal{M} = \mathrm{FinSet}$—as a decorated corelations functor. Note that the identity on FinSet maps the subcategory $\mathcal{I}_{\mathrm{FinSet}}$ into FinSet, and so extends to a morphism FinSet $\to$ Cospan(FinSet). Also recall that the 'undecorated' cospan category Cospan(FinSet) is equal to the decorated cospan category given by the functor $1\colon \mathrm{FinSet} \to \mathrm{Set}$ mapping each finite set to some chosen one element set 1, and each morphism to the identity morphism on 1. Define monoidal natural transformation

$$\begin{tikzcd}
\mathrm{FinSet} = \mathrm{FinSet};\mathcal{I}_{\mathrm{FinSet}}^{\mathrm{opp}} \arrow[d, "\iota"'] \arrow[dr, "1"] \arrow[d, phantom, shift left=8ex, "\Downarrow\theta"] & \\
& \mathrm{Set} \\
\mathrm{Cospan(FinSet)} = \mathrm{FinSet};\mathrm{FinSet}^{\mathrm{opp}} \arrow[ur, "G"'] &
\end{tikzcd}$$

with each $\theta_X\colon 1 \to GX$ mapping the unique element to the identity function $1_X\colon X \to X$. This gives the hypergraph functor we expect, mapping the undecorated cospan $X \to N \leftarrow Y$ to the trivial cospan $X \to X+Y \leftarrow Y$ decorated by $X+Y \to N$. It is now routine to verify this is an isomorphism.

The previous example extends to any finitely cocomplete category $\mathcal{C}$: the hypergraph category Cospan($\mathcal{C}$) can always be constructed as (i) trivially decorated $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$-corelations, or (ii) $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms with domain the apex of the corelation, and moreover the isomorphism of these hypergraph categories is given by the analogous monoidal natural transformation between the decorating functors.

More general still, a category of trivially decorated $(\mathcal{E}, \mathcal{M})$-corelations in $\mathcal{C}$ can always be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms in $\mathcal{E}$ with domain the apex of the corelation, and the isomorphism of these hypergraph categories a decorated corelations functor.

Most generally, we can still perform this construction on decorated corelation categories: Theorem 4.12 implies any category of $(\mathcal{E}, \mathcal{M})$-decorated corelations can be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by codomain decorated morphisms in $\mathcal{E}$. Given some lax symmetric monoidal functor

$$F\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \longrightarrow \mathrm{Set},$$

the decorated $(\mathcal{I}_\mathcal{C}, \mathcal{C})$-corelations are specified by the (global sections) functor

$$F' \colon \mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Set}$$

taking any object $Z \in \mathrm{Cospan}(\mathcal{C})$ to the pair $(Z \stackrel{e}{\to} N, 1 \stackrel{s}{\to} FN)$, where e is a morphism in $\mathcal{E}$.

What is the utility of this variety of constructions? The first is that some categories are very naturally constructed as corelation (or, dually, relation) categories, like the category of equivalence relations, relations, or linear relations. Here the factorisation system has an intuitive interpretation, like epimorphisms retaining only the structure in the apex that is 'accessible' or 'mapped onto' by the feet/boundary. Decorating this corelations retains the same sort of intuition.

On the other hand, the 'fully decorated' latter form, where all interesting structure is carried by the apex of the decorated corelation, is useful for constructing functors into the given decorated corelation category. The ability to construct functors from one hypergraph category to another is essential for the understanding of hypergraph categories as network-type diagrammatic languages, with functors giving rise to notions of semantics, equivalence of diagrams, and reasoning.

We will illustrate these principles in greater depth in the next path, which delves into applications of this philosophy and framework. We conclude this section and Part I with a collection of examples of a more abstract nature.

4.5 Examples

4.5.1 Matrices

Let R be a commutative ring. We will construct matrices over R as decorated corelations over $\mathrm{FinSet}^{\mathrm{opp}}$. Here the coproduct is the Cartesian product $\times$ of sets, the initial object is the one element set 1, and cospans are spans in FinSet. The notation will thus be less confusing if we talk of decorated spans on $(\mathrm{FinSet}, \times)$ given by the contravariant lax monoidal functor

$$\begin{aligned} R^{(-)} : (\mathrm{FinSet}, \times) &\longrightarrow (\mathrm{Set}, \times); \\ N &\longmapsto R^N \\ \Big(f \colon N \to M\Big) &\longmapsto \Big(R^f \colon R^M \to R^N; v \mapsto v \circ f\Big). \end{aligned}$$

The coherence maps $\varphi_{X,Y} \colon R^X \times R^Y \to R^{X \times Y}$ take a pair (s, t) of maps $s \colon X \to R$, $t \colon Y \to R$ to the pointwise product $s \cdot t \colon X \times Y \to R; (x, y) \mapsto s(x) \cdot t(y)$. As in the introduction, we show $R^{(-)}$Cospan is the category of 'multivalued matrices' over $\mathbb{R}$, and $R^{(-)}$Corel is the category of matrices over R.

This becomes a decorated relations example if we extend $R^{(-)}$ to the functor

$$\begin{aligned} R^{(-)}: (\mathrm{Span}(\mathrm{FinSet}), \times) &\longrightarrow (\mathrm{Set}, \times); \\ N &\longmapsto R^N \\ \Big(f\colon N \to M\Big) &\longmapsto \Big(R^f\colon R^M \to R^N; v \mapsto v \circ f\Big) \\ \Big(g^{\mathrm{opp}}\colon N \to M\Big) &\longmapsto \Big(R^f\colon R^M \to R^N; v \mapsto v \circ f\Big). \end{aligned}$$

Many aspects of this example are 'atypical' for the intuition we have been working towards. Note that the monoidal product here is the tensor product of matrices, not the biproduct. Indeed, there is no special commutative Frobenius algebra in Vect if we use the biproduct, but if we use the tensor product then these correspond to orthonormal bases (Vicary). The comultiplication is the diagonal map, multiplication is codiagonal. unit produces basis.

We note that you could take decorations here in the category RMod of R-modules. While Proposition 2.9 shows that the resulting decorated cospans category would be isomorphic, this may hint at an enriched version of the theory.

4.5.2 Two constructions for linear relations

We saw earlier that linear relations are epi-mono corelations in Vect. The hypergraph structure is given by addition. We show how to recover this in another construction. We also get a hypergraph functor between them. This is very useful for compositional linear relations semantics of diagrams.

We can also construct linear relations in Vect$^{\mathrm{opp}}$.

$$\begin{aligned} : \mathrm{Cospan}(\mathrm{FinSet}) &\longrightarrow \mathrm{Set} \\ X &\longmapsto \{\mathrm{subspaces of} k^X\} \\ f: X \to Y &\longmapsto L \mapsto \{v \mid v \circ f \in L\} \\ f^{\mathrm{opp}}: X \to Y &\longmapsto L \mapsto \{v = u \circ f \mid u \in L\} \end{aligned}$$

Then Cospan is cospans decorated by subspaces, and Corel is linear relations. This is important for circuits work [**?**, **?**].

4.5.3 Automata

This construction comes immediately from Walters et al. Automata are alphabet labelled graphs. There is a decorated cospan functor to categories enriched over languages, and this factors nicely to get a decorated corelation category with morphisms languages recognised between points in domain and codomain.

Part II

Applications

Bibliography

[Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page 20.)