The Algebra of Open and Interconnected Systems

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Contents

Part I

Mathematical Foundations

Chapter 1

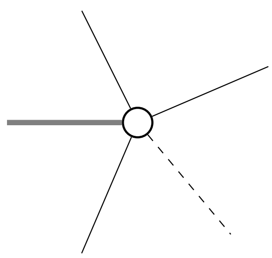
Hypergraph categories: the algebra of interconnection

In this chapter we introduce hypergraph categories, giving a definition, coherence theorem, and graphical language. We then explore a fundamental example of hypergraph categories: categories of cospans.

We assume basic familiarity with category theory and symmetric monoidal categories; although we give a sparse overview of the latter for reference. A proper introduction to both can be found in Mac Lane [?].

1.1 The algebra of interconnection

Our aim is to algebraicise network diagrams. A network diagram is built from pieces like so:



These represent open systems, concrete or abstract; for example a resistor, a chemical reaction, or a linear transformation. The essential feature, for openness and for networking, is that the system may have terminals, perhaps of different 'types', each one depicted by a line radiating from the central body. In the case of a resistor each terminal might represent a wire, for chemical reactions a chemical species, for linear transformations a variable in the domain or codomain. Network diagrams are formed by connecting terminals of systems to build larger systems.

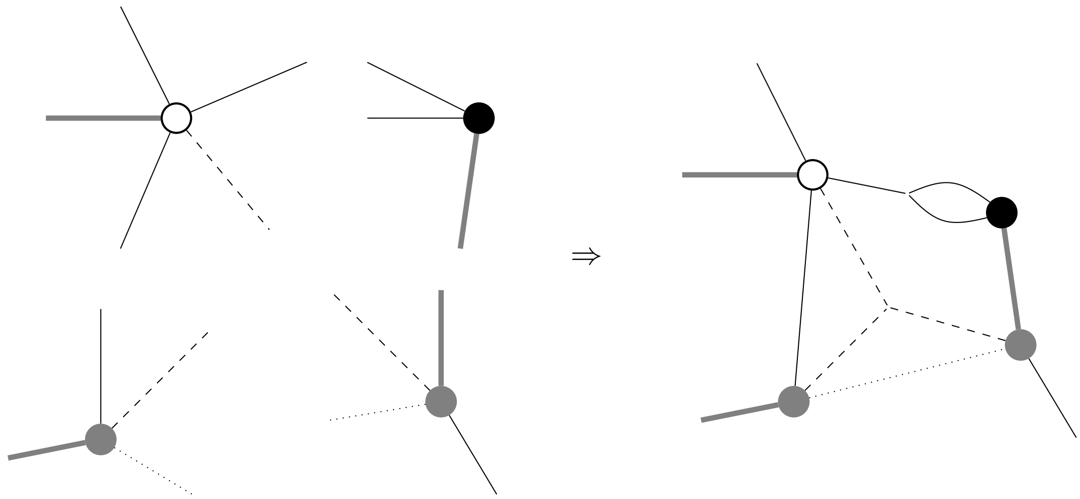


Figure 1.1: Interconnection of network diagrams. Note that we only connect terminals of the same type, but we can connect as many as we like.

A network-style diagrammatic language is a collection of network diagrams together with the stipulation that if we take some of these network diagrams, and connect terminals of the same type in any way we like, then we form another diagram in the collection. The point of this chapter is that hypergraph categories provide a precise formalisation of network-style diagrammatic languages.

In jargon, a hypergraph category is a symmetric monoidal category in which every object is equipped with a special commutative Frobenius monoid in a way compatible with the monoidal product. We will walk through these terms in detail, illustrating them with examples and a few theorems.

The key data comprising a hypergraph category are its objects, morphisms, composition rule, monoidal product, and Frobenius maps. Each of these model a feature of network diagrams and their interconnection. The objects model the terminal types, while the morphisms model the network diagrams themselves. The composition, monoidal product, and Frobenius maps model different aspects of interconnection: composition models the interconnection of two terminals of the same type, the monoidal product models the network formed by taking two networks without interconnecting any terminals, while the Frobenius maps model multi-terminal interconnection.

These Frobenius maps are the distinguishing feature of hypergraph categories as compared to other structured monoidal categories, and are crucial for formalising the intuitive concept of network languages detailed above. In the case of electric circuits

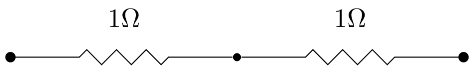
Networks	Hypergraph categories
list of terminal types	object
network diagram	morphism
series connection	composition
juxtaposition	monoidal product
branching	Frobenius maps

Figure 1.2: Corresponding features of networks and hypergraph categories.

the Frobenius maps model the 'branching' of wires; in the case when diagrams simply model an abstract system of equations and terminals variables in these equations, the Frobenius maps allow variables to be shared between many systems of equations.

Examining these correspondences, it is worthwhile to ask whether hypergraph categories permit too much structure to be specified, given that the interconnection rule is now divided into three different aspects, and features such as domains and codomains of network diagrams, rather than just a collection of terminals, exist. The answer is given by examining the additional coherence laws that these data must obey. For example, in the case of the domain and codomain, we shall see that hypergraph categories are all compact closed categories, and so there is ultimately only a formal distinction between domain and codomain objects. One way to think of these data is as scaffolding. We could compare it to the use of matrices and bases to provide language for talking about linear transformations and vector spaces. They are not part of the target structure, but nonetheless useful paraphenalia for constructing it.

Network languages are not only syntactic entities: as befitting the descriptor 'language', they typically have some associated semantics. Circuits diagrams, for instance, not only depict wire circuits that may be constructed, they also represent the electrical behaviour of that circuit. Such semantics considers the circuits

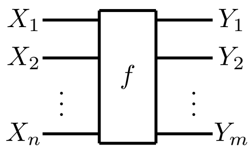


and


the same, even though as 'syntactic' diagrams they are distinct. A cornerstone of the utility of the hypergraph formalism is the ability to also realise the semantics of these diagrams as morphisms of another hypergraph category. This 'semantic' hypergraph category, as a hypergraph category, still permits the rich 'networking' interconnection structure, and a so-called hypergraph functor implies that the syntactic category provides a sound framework for depicting these morphisms. Network languages syntactically are often 'free' hypergraph categories, and much of the interesting structure lies in their functors to their semantic hypergraph categories.

1.2 Symmetric monoidal categories

Suppose we have some tiles with inputs and outputs of various types like so:



These tiles may vary in height and width. We can place these tiles above and below each other, and to the left and right, so long as the inputs on the right tile match the outputs on the left. Suppose also that some arrangements of tiles are equal to other arrangements of tiles. How do we formalise this structure algebraically? The theory of monoidal categories provides an answer.

Hypergraph categories are first monoidal categories, indeed symmetric monoidal categories. A monoidal category is a category with two notions of composition: ordinary categorical composition and monoidal composition, with the monoidal composition only associative and unital up to natural isomorphism. They are the algebra of processes that may occur simultaneously as well as sequentially. First defined by Bénabou and Mac Lane in the 1960s [**?**, **?**], their theory and their links with graphical representation are well explored. We bootstrap on this, using monoidal categories to define hypergraph categories, and so immediately arriving at an understanding of how hypergraph categories formalise our network languages.

Moreover, symmetric monoidal functors play a key role in our framework for defining and working with hypergraph categories: decorated cospans and corelations constructions. For this reason we provide, for quick reference, a definition of symmetric monoidal categories.

1.2.1 Monoidal categories

A **monoidal category** $(\mathcal{C}, \otimes)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object I, and natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\rho_A : A \otimes I \to A$, and $\lambda_A : I \otimes A \to A$ such that for all A, B, C, D in $\mathcal{C}$ the following two diagrams commute:

$$\begin{tikzcd}
\big((A \otimes B) \otimes C\big) \otimes D \arrow[rr, "\alpha_{(A\otimes B),C,D}"] \arrow[d, "\alpha_{A,B,C}\otimes \mathrm{id}_D"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,(C\otimes D)}"] \\
\big(A \otimes (B \otimes C)\big) \otimes D \arrow[r, "\alpha_{A,(B\otimes C),D}"'] & A \otimes \big((B \otimes C) \otimes D\big) \arrow[r, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] & A \otimes \big(B \otimes (C \otimes D)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

We call $\otimes$ the **monoidal product**, I the **monoidal unit**, α the **associator**, ρ and λ the **right** and **left unitor** respectively. The associator and unitors are known collectively as the **coherence maps**.

By Mac Lane's coherence theorem, these two axioms are equivalent to requiring that 'all formal diagrams'—that is, all diagrams in which the morphism are built from identity morphisms and the coherence maps using composition and the monoidal product—commute. Consequently, between any two products of the same ordered list of objects up to instances of the monoidal unit, such as $((A \otimes I) \otimes B) \otimes C$ and $A \otimes ((B \otimes C) \otimes (I \otimes I))$, there is a unique so-called **canonical** map. See Mac Lane [**?**, Corollary of Theorem VII.2.1] for a precise statement and proof.

A **lax monoidal functor** $(F, \varphi) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \boxtimes)$ between monoidal categories consists of a functor $F : \mathcal{C} \to \mathcal{C}'$, and natural transformations $\varphi_{A,B} : FA \boxtimes FB \to F(A \otimes B)$ and $\varphi_1 : 1_{\mathcal{C}'} \to F1_{\mathcal{C}}$, such that for all $A, B, C \in \mathcal{C}$ the three diagrams

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[r, "\varphi_{A,B}\otimes \mathrm{id}_{FC}"] \arrow[d, "\alpha_{FA,FB,FC}"'] & F(A \otimes B) \otimes FC \arrow[r, "\varphi_{A\otimes B,C}"] & F((A \otimes B) \otimes C) \arrow[d, "F\alpha_{A,B,C}"] \\
FA \otimes (FB \otimes FC) \arrow[r, "\mathrm{id}_{FA}\otimes \varphi_{B,C}"'] & FA \otimes F(B \otimes C) \arrow[r, "\varphi_{A,B\otimes C}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes I' \arrow[r, "\rho"] \arrow[d, "\mathrm{id}\otimes\varphi_1"'] & F(A) \\
F(A) \otimes F(I) \arrow[r, "\varphi_{A,I}"'] & F(A \otimes I) \arrow[u, "F\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes F(A) \arrow[r, "\lambda"] \arrow[d, "\varphi_1\otimes\mathrm{id}"'] & F(A) \\
F(I) \otimes F(A) \arrow[r, "\varphi_{I,A}"'] & F(I \otimes A) \arrow[u, "F\lambda"']
\end{tikzcd}$$

commute. We further say a monoidal functor is a **strong monoidal functor** if the φ are isomorphisms, and a **strict monoidal functor** if the φ are identities.

A **monoidal natural transformation** $\theta : (F, \varphi) \Rightarrow (G, \gamma)$ between two monoidal functors F and G is a natural transformation $\theta : F \Rightarrow G$ such that

$$\begin{tikzcd} F1_{\mathcal{C}} \arrow[rr, "\theta_I"] & & G1_{\mathcal{C}} \\ & 1_{\mathcal{C}'} \arrow[ul, "\varphi_1"] \arrow[ur, "\gamma_1"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} FA \boxtimes FB \arrow[r, "\theta_A \otimes \theta_B"] \arrow[d, "\varphi_{A,B}"'] & GA \boxtimes GB \arrow[d, "\gamma_{A,B}"] \\ F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B) \end{tikzcd}$$

commute for all objects A, B.

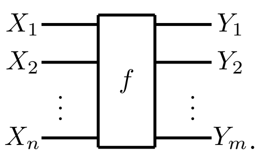
Two monoidal categories $\mathcal{C}, \mathcal{D}$ are **monoidally equivalent** if there exist strong monoidal functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that the composites FG and GF are monoidally naturally isomorphic to the identity functors. (Note that identity functors are immediately strict monoidal functors.)

1.2.2 String diagrams

A **strict monoidal category** category is a monoidal category in which the associators and unitors are all identity maps. In this case then any two objects that can be related by associators and unitors are equal, and so we may write objects without parentheses and units without ambiguity. An equivalent statement of Mac Lane's coherence theorem is that every monoidal category is monoidally equivalent to strict monoidal category.

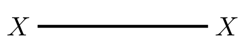
Yet another equivalent statement of the coherence theorem is the existence of a graphical calculus for monoidal categories. As discussed above, monoidal categories figure strongly in our current investigations precisely because of this. We leave the details to discussions elsewhere. The main point is that we shall be free to assume our monoidal categories are strict, writing $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ for objects in $(\mathcal{C}, \otimes)$ without a care for parentheses. We then depict a morphism $f \colon X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$ with the diagram:

$$f \quad =$$



Identity morphisms are depicted by 'wires':

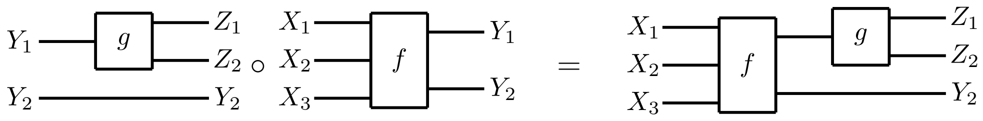
$$\mathrm{id}_X \quad =$$



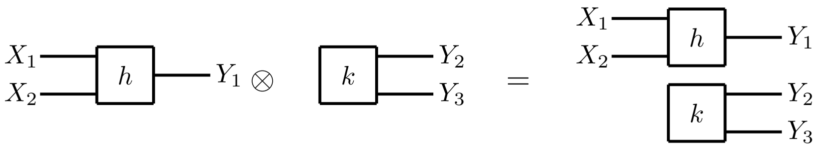
and the monoidal unit is not depicted at all:

$$\mathrm{id}_I \quad =$$

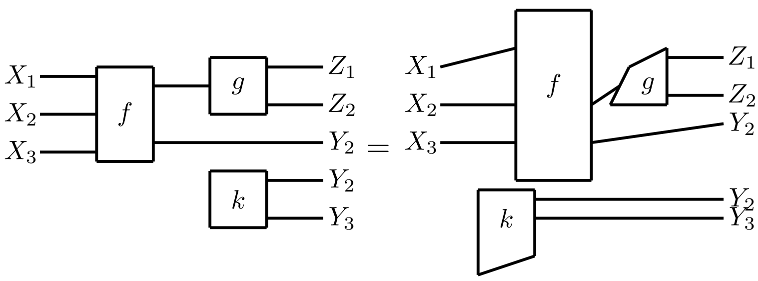
Composition of morphisms is depicted by connecting the relevant ‘wires’:



while monoidal composition is just juxtaposition:



Only the ‘topology’ of the diagrams matters: if two diagrams with the same domain and codomain are homotopy equivalent, they represent the same morphism. On the other hand, two algebraic expressions might have the same diagrammatic representation. For example, the equivalent diagrams



read as all of the equivalent algebraic expressions

$$((g \otimes \mathrm{id}_{Y_2}) \otimes k) \circ f = (g \otimes ((\mathrm{id}_{Y_2} \otimes k)) \circ \rho \circ (f \otimes \mathrm{id}_I) = (g \otimes \mathrm{id}_{Y_2}) \circ f \circ (\mathrm{id}_{X_1 \otimes (X_2 \otimes X_3)} \otimes k)$$

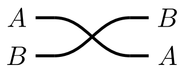
and so on. The coherence theorem says that this does not matter: if two algebraic expressions have the same diagrammatic representation, then the algebraic expressions are equal. In more formal language, the graphical calculus is sound and complete for the axioms of monoidal categories.

The coherence theorem thus implies that the graphical calculi goes beyond visualisations of morphisms: it can provide provide bona-fide proofs of equalities of morphisms. As a general principle, string diagrams are more intuitive than the conventional algebraic language for understanding monoidal categories.

1.2.3 Symmetry

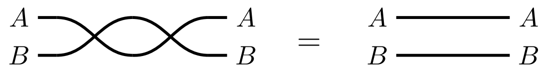
A symmetric braiding in a monoidal category provides the ability to permute objects or, equivalently, cross wires. We define symmetric monoidal categories making use of the graphical notation outlined above, but introducing a new, special symbol .

A **symmetric monoidal category** is a monoidal category $(\mathcal{C}, \otimes)$ together with natural isomorphisms

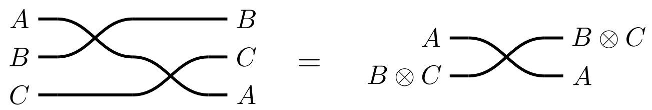


$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

such that



and



for all A, B, C in $\mathcal{C}$. We call σ the **braiding**. We will also talk, somewhat incidentally, of braided monoidal categories in the next chapter; a **braided monoidal category** is a monoidal category with a braiding that only obeys the latter axiom.

A **(lax/strong) symmetric monoidal functor** is a (lax/strong) monoidal functor that further obeys

$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\varphi_{A,B}"] \arrow[d, "\sigma'_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\sigma_{A,B}"] \\
FB \otimes FA \arrow[r, "\varphi_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

Morphisms between symmetric monoidal functors are simply monoidal natural transformations. Thus two symmetric monoidal categories are **symmetric monoidally equivalent** if they are monoidally equivalent by strong *symmetric* monoidal functors. If our categories are merely braided, we refer to these functors as **braided monoidal functors**.

The coherence theorem for symmetric monoidal categories, with respect to string diagrams, states that two morphisms in a symmetric monoidal category are equal according to the axioms of symmetric monoidal categories if and only if their diagrams are equal up to homotopy equivalence and applications of the defining graphical identities above. See Joyal–Street [**?**, Theorem 2.3] for more precision and details.

1.3 Hypergraph categories

Just as symmetric monoidal categories equip monoidal categories with precisely enough extra structure to model crossing of strings in the graphical calculus, hypergraph categories equip symmetric monoidal categories with precisely enough extra structure to model multi-input multi-output interconnections of strings of the same type. For this, we require each object to be equipped with a so-called special commutative Frobenius monoid, which provides chosen maps to model this interaction. These have a coherence result, known as the 'spider theorem', that says exactly how we use the maps to describe the connection of strings does not matter: all that matters is that the strings are connected.

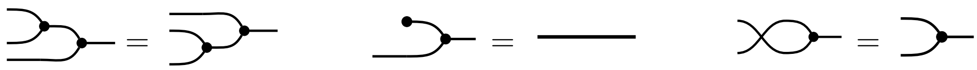
1.3.1 Frobenius monoids

A Frobenius monoid comprises a monoid and comonoid on the same object that interact according to the so-called Frobenius law.

Definition 1.1. A **special commutative Frobenius monoid** $(X, \mu, \eta, \delta, \epsilon)$ in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is an object X of $\mathcal{C}$ together with maps

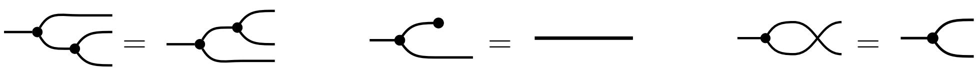
$\mu\colon X \otimes X \to X$ $\quad$ $\eta\colon I \to X$ $\quad$ $\delta\colon X \to X \otimes X$ $\quad$ $\epsilon\colon X \to I$

obeying the commutative monoid axioms



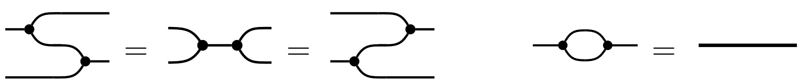
(associativity) (unitality) (commutativity)

the cocommutative comonoid axioms



(coassociativity) (counitality) (cocommutativity)

and the Frobenius and special axioms



(Frobenius) (special)

We call μ the **multiplication**, η the **unit**, δ the **comultiplication**, and ϵ the **counit**.

Special commutative Frobenius monoids were first formulated by Carboni and Walters, under the name commutative separable algebras. The Frobenius law and the special law were termed the S=X law and the diamond=1 law respectively [?, ?].

Alternate axiomatisations are possible. In addition to the 'upper' unitality law above, the mirror image 'lower' unitality law also holds, due to commutativity and the naturality of the braiding. While we write two equations for the Frobenius law, this is redundant: given the other axioms, the equality of any two of the diagrams implies the equality of all three. Further, note that a monoid and comonoid obeying the Frobenius law is commutative if and only if it is cocommutative. Thus while a commutative and cocommutative Frobenius monoid might more properly be called a bicommutative Frobenius monoid, there is no ambiguity if we only say commutative.

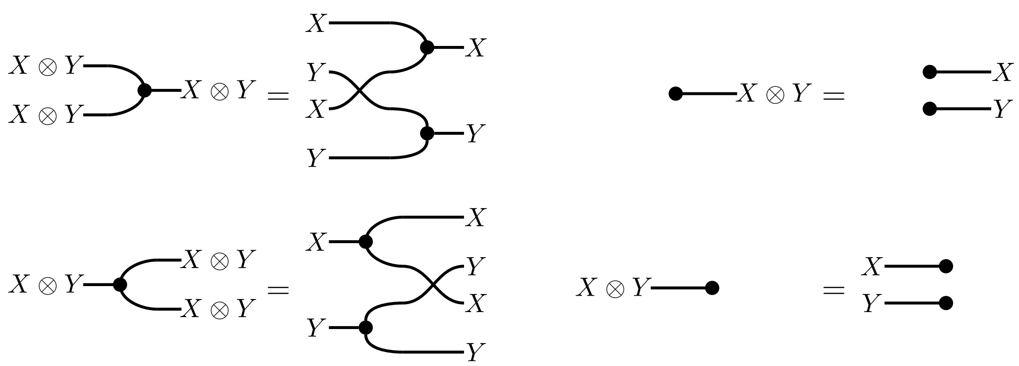
The common feature to these equations is that each side describes a different way of using the generators to connect some chosen set of inputs to some chosen set of outputs. This observation provides a 'coherence' type result for special commutative Frobenius monoids, known as the 'spider theorem'.

Theorem 1.2. *Let* $(X, \mu, \eta, \delta, \epsilon)$ *be a special commutative Frobenius monoid, and let* $f, g\colon X^{\otimes n} \to X^{\otimes m}$ *be map constructed, using composition and the monoidal product, from* μ, η, δ, ϵ, *the coherence maps and braiding, and the identity map on* X. *Then* f *and* g *are equal if and only if given their string diagrams in the above notation, there exists a bijection between the connected components of the two diagrams such that corresponding connected components connect the exact same sets of inputs and outputs.*

See [?, ?] [?, ?] for further details.

1.3.2 Hypergraph categories

Definition 1.3. A **hypergraph category** is a symmetric monoidal category in which each object X is equipped with a special commutative Frobenius structure $(X, \mu_X, \delta_X, \eta_X, \epsilon_X)$ such that



Note that we do *not* require these Frobenius morphisms to be natural in X. While morphisms in a hypergraph category need not interact with the Frobenius structure in any particular way, we do require functors between hypergraph categories to preserve it.

Definition 1.4. A functor (F, φ) of hypergraph categories, or **hypergraph functor**, is a strong symmetric monoidal functor (F, φ) such that for each object X the following diagrams commute:

$$\begin{tikzcd} FX \boxtimes FX \arrow[rr, "\mu_{FX}"] \arrow[dr, "\varphi"'] & & FX \\ & F(X \otimes X) \arrow[ur, "F\mu_X"'] & \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{D}} \arrow[rr, "\eta_{FX}"] \arrow[dr, "\varphi_1"'] & & FX \\ & F1_{\mathcal{C}} \arrow[ur, "F\eta_X"'] & \end{tikzcd}$$

$$\begin{tikzcd} FX \arrow[rr, "\delta_{FX}"] \arrow[dr, "F\delta_X"'] & & FX \boxtimes FX \\ & F(X \otimes X) \arrow[ur, "\varphi^{-1}"'] & \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[rr, "\epsilon_{FX}"] \arrow[dr, "F\epsilon_X"'] & & 1_{\mathcal{D}} \\ & F1_{\mathcal{C}} \arrow[ur, "\varphi^{-1}"'] & \end{tikzcd}$$

Equivalently, a strong symmetric monoidal functor F is a hypergraph functor if for every X the special commutative Frobenius structure on FX is

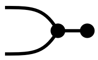
$$(FX,\ \ F\mu_X \circ \varphi_{X,X},\ \ \varphi_{X,X}^{-1} \circ F\delta_X,\ \ F\eta_X \circ \varphi_1,\ \ \varphi_1^{-1} \circ F\epsilon_X).$$

Just as monoidal natural transformations themselves are enough as morphisms between symmetric monoidal functors, so too they suffice as morphisms between hypergraph functors. Two hypergraph categories are **hypergraph equivalent** if there exist hypergraph functors with monoidal natural transformations to the identity functors.

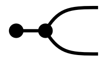
The term hypergraph category was introduced recently [**?**, **?**], in reference to the fact that these special commutative Frobenius monoids provide precisely the structure required to draw graphs with 'hyperedges': edges connecting any number of inputs to any number of outputs. Again first defined by Walters and Carboni [**?**], under the name well-supported compact closed categories, in recent years hypergraph categories have been rediscovered a number of times, also appearing under the names dungeon categories [**?**] and dgs-monoidal categories [**?**].

1.3.3 Hypergraph categories are self-dual compact closed

Note that if an object X is equipped with a Frobenius monoid structure then the maps

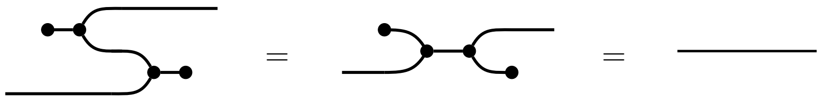


and



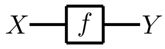
$\epsilon \circ \mu \colon X \otimes X \to 1$ $\qquad\qquad$ $\delta \circ \eta \colon 1 \to X \otimes X$

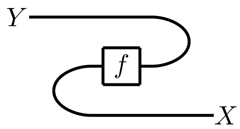
obey both



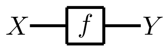
and the reflected equations. Thus if an object carries a Frobenius monoid it is also self-dual, and any hypergraph category is a fortiori self-dual compact closed.

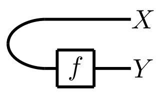
We introduce the notation

As in any self-dual compact closed category, mapping each morphism  to its dual morphism

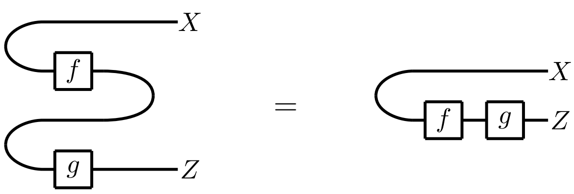


further equips each hypergraph category with a so-called dagger functor—an involutive contravariant endofunctor that is the identity on objects—such that the category is a dagger compact category. Dagger compact categories were first introduced in the context of categorical quantum mechanics [**?**], under the name strongly compact closed category, and have been demonstrated to be a key structure in diagrammatic reasoning and the logic of quantum mechanics.

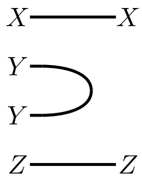
Compactness allows us to blur the distinction between composition and the monoidal product of morphisms. Firstly, there is a one-to-one correspondence between morphisms $X \to Y$ and morphisms $1 \to X \otimes Y$ given by taking  to its so-called **name**



By compactness, we have the equation



Here the right hand side is the name of the composite $f \circ g$, while the left hand side is the monoidal product post-composed with the map



Thus this morphism, the product of a cap and two identity maps, enacts the categorical composition on monoidal products of names. We will make liberal use of this fact.

1.3.4 Coherence

The lack of naturality of the Frobenius maps in hypergraph categories affects some common properties of structured categories. For example, it is not always possible to construct a skeletal hypergraph category hypergraph equivalent to a given hypergraph category: isomorphic objects may be equipped with 'different' Frobenius monoids. Similarly, a fully faithful, essentially surjective hypergraph functor does not necessarily define a hypergraph equivalence of categories.

Nonetheless, in this section we prove that every hypergraph category is hypergraph equivalent to a strict hypergraph category. This coherence result will be important in proving that every hypergraph category can be constructed using decorated corelations.

Theorem 1.5. *Every hypergraph category is hypergraph equivalent to a strict hypergraph category. Moreover, the objects of this strict hypergraph category form a free monoid.*

Proof. Let $(\mathcal{H}, \otimes)$ be a hypergraph category. As $\mathcal{H}$ is a fortiori a symmetric monoidal category, a standard construction (see Mac Lane [**?**, Theorem]) gives an equivalent strict symmetric monoidal category $(\mathcal{H}_{\mathrm{str}}, \cdot)$ with objects finite lists $[x_1, \dots, x_n]$ of objects of $\mathcal{H}$ and morphisms $[x_1, \dots, x_n] \to [y_1, \dots, y_m]$ those morphisms from $((x_1 \otimes x_2) \otimes \dots) \otimes x_n \to ((y_1 \otimes y_2) \otimes \dots) \otimes y_m$ in $\mathcal{H}$. Composition is given by composition in $\mathcal{H}$.

The monoidal structure is given as follows. Given a list X of objects in $\mathcal{H}$, write PX for the corresponding monoidal product in $\mathcal{H}$ with all open parathesis at the front. The monoidal product of two objects is given by concatenation $\cdot$ of lists; the monoidal unit is the empty list. The monoidal product of two morphisms is given by their monoidal product in $\mathcal{H}$ pre- and post-composed with the necessary canonical maps: given $f\colon X \to Y$ and $g\colon Z \to W$, their product $f \cdot g\colon X \cdot Y \to Z \cdot W$ is

$$P(X \cdot Y) \longrightarrow PX \otimes PY \xrightarrow{f \otimes g} PZ \otimes PW \longrightarrow P(Z \cdot W).$$

By design, the associators and unitors are simply identity maps. The braiding $X \cdot Y \to Y \cdot X$ is given by the braiding $PX \otimes PY \to PY \otimes PX$ in $\mathcal{H}$, similarly pre- and post-composed with the necessary canonical maps. This defines a strict symmetric monoidal category [**?**].

To make $\mathcal{H}_{\mathrm{str}}$ into a hypergraph category, we equip each object $[x_1, \dots, x_n]$ with a special commutative Frobenius monoid in a similar way. For example, the multiplication on $[x_1, \dots, x_n]$ is given by

$$\begin{array}{c}
P([x_1, \dots, x_n] \cdot [x_1, \dots, x_n]) = ((((((x_1 \otimes x_2) \otimes \dots) \otimes x_n) \otimes x_1) \otimes x_2) \otimes \dots) \otimes x_n \\
\downarrow \\
(((x_1 \otimes x_1) \otimes (x_2 \otimes x_2)) \otimes \dots) \otimes (x_n \otimes x_n) \\
\downarrow{\scriptstyle ((\mu_{x_1} \otimes \mu_{x_2}) \otimes \dots) \otimes \mu_{x_n}} \\
P([x_1, \dots, x_n]) = ((x_1 \otimes x_2) \otimes \dots) \otimes x_n
\end{array}$$

where the first map is the canonical map such that each pair of x_i's remains in the same order. It is straightforward to check that this defines a hypergraph category.

The strict hypergraph category $(\mathcal{H}_{\mathrm{str}}, \cdot)$	
objects	finite lists $[x_1, \dots, x_n]$ of objects of $\mathcal{H}$
morphisms	$\hom_{\mathcal{H}_{\mathrm{str}}}\big([x_1, \dots, x_n], [y_1, \dots, y_m]\big)$ $= \hom_{\mathcal{H}}\big(((x_1 \otimes x_2) \otimes \dots) \otimes x_n, ((y_1 \otimes y_2) \otimes \dots) \otimes y_m\big)$
composition	composition of corresponding maps in $\mathcal{H}$
monoidal product	concatenation of lists
coherence maps	associators and unitors are strict; braiding is inherited from $\mathcal{H}$
hypergraph maps	lists of hypergraph maps in $\mathcal{H}$

Our standard construction further gives strong symmetric monoidal functors $P\colon \mathcal{H}_{\mathrm{str}} \to \mathcal{H}$ extending the map P above, and $S\colon \mathcal{H} \to \mathcal{H}_{\mathrm{str}}$ sending $x \in \mathcal{H}$ to the string $[x]$ of length 1 in $\mathcal{H}_{\mathrm{str}}$. These extend to hypergraph functors.

In detail, the functor P is given on morphisms by taking a map in $\hom_{\mathcal{H}_{\mathrm{str}}}(X, Y)$ to the same map considered now as a map in $\hom_{\mathcal{H}}(PX, PY)$; its coherence maps are given by the canonical maps $PX \otimes PY \to P(X \cdot Y)$. The functor S is even easier to define: a morphism $x \to y$ in $\mathcal{H}$ is by definition a morphism $[x] \to [y]$ in $\mathcal{H}_{\mathrm{str}}$, so S is a monoidal embedding of $\mathcal{H}$ into $\mathcal{H}_{\mathrm{str}}$.

By Mac Lane's proof of the coherence theorem for monoidal categories these are both strong monoidal functors; by inspection they also preserve hypergraph structure, and so are hypergraph functors. As they already witness an equivalence of symmetric monoidal categories, thus $\mathcal{H}$ and $\mathcal{H}_{\mathrm{str}}$ are equivalent as hypergraph categories. □

1.4 Example: cospan categories

A central example of a hypergraph category is the category $\mathrm{Cospan}(\mathcal{C})$ of cospans in any category $\mathcal{C}$ with finite colimits. We will later see that decorated cospan categories are a generalisation of such categories, and each inherits a hypergraph structure from such.

We first recall the basic definitions. Let $\mathcal{C}$ be a category with finite colimits, writing the coproduct $+$. A **cospan**

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

from X to Y in $\mathcal{C}$ is a pair of morphisms with common codomain. We refer to X and Y as the **feet**, and N as the **apex**. Given two cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ and $X \xrightarrow{i'} N' \xleftarrow{o'} Y$ with the same feet, a **map of cospans** is a morphism $n \colon N \to N'$ in $\mathcal{C}$ between the apices such that

$$\begin{tikzcd} & N \arrow[dd, "n"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd}$$

commutes.

Cospans may be composed using the pushout from the common foot: given cospans $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite cospan is $X \xrightarrow{j_N \circ i_X}$

$N +_Y M \xleftarrow{j_M \circ i_Z} Z$, where the top part of

$$\begin{tikzcd}[column sep=small]
 & & & N +_Y M & & & \\
 & N \arrow[urr, "j_N"] & & & & M \arrow[ull, "j_M"'] & \\
X \arrow[ur, "i_X"] & & & Y \arrow[ull, "o_Y"'] \arrow[urr, "i_Y"] & & & Z \arrow[ul, "o_Z"']
\end{tikzcd}$$

is a pushout square. This composition rule is associative up to isomorphism, and so we may define a category, in fact a symmetric monoidal bicategory, $\mathrm{Cospan}(\mathcal{C})$ with objects the objects of $\mathcal{C}$ and morphisms isomorphism classes of cospans [**?**].

The symmetric monoidal structure is 'inherited' from $\mathcal{C}$. Indeed, we shall consider any category $\mathcal{C}$ with finite colimits a symmetric monoidal category as follows. Given maps $f\colon A \to C$, $g\colon B \to C$ with common codomain, the universal property of the coproduct gives a unique map $A + B \to C$. We call this the **copairing** of f and g, and write it $[f, g]$. The monoidal product on $\mathcal{C}$ is then given by the coproduct $+$, with monoidal unit the initial object $\varnothing$ and coherence maps given by copairing the appropriate identity, inclusion, and initial object maps. For example, the braiding is given by $[\iota_X, \iota_Y]\colon X + Y \to Y + X$ where $\iota_X\colon X \to Y + X$ and $\iota_Y\colon Y \to Y + X$ are the inclusion maps into the coproduct $Y + X$.

The category $\mathrm{Cospan}(\mathcal{C})$ inherits this symmetric monoidal structure from $\mathcal{C}$ as follows. Call a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ **wide** if $\mathcal{C}$ contains all objects of $\mathcal{D}$, and call a functor that is faithful and bijective-on-objects a **wide embedding**. Note then that we have a wide embedding

$$\mathcal{C} \longhookrightarrow \mathrm{Cospan}(\mathcal{C})$$

that takes each object of $\mathcal{C}$ to itself as an object of $\mathrm{Cospan}(\mathcal{C})$, and each morphism $f\colon X \to Y$ in $\mathcal{C}$ to the cospan

$$\begin{tikzcd}
 & Y & \\
X \arrow[ur, "f"] & & Y, \arrow[ul, equal]
\end{tikzcd}$$

where the extended 'equals' sign denotes an identity morphism. Now since the monoidal product $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is left adjoint to the diagram functor, it preserves colimits, and so extends to a functor $+\colon \mathrm{Cospan}(\mathcal{C}) \times \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{C})$. The coherence maps are just the images of the coherence maps in $\mathcal{C}$ under this wide embedding; checking naturality is routine, and clearly they still obey the required axioms.

Write FinSet for the category of finite sets and functions. Replacing FinSet with its equivalent strict skeleton, it is well-known, due to Lack [?], that special commutative Frobenius monoids in a monoidal category $\mathcal{C}$ are in one-to-one correspondence with strict monoidal functors $\mathrm{Cospan}(\mathrm{FinSet}) \to \mathcal{C}$. Over the next few chapters we will further explore this deep link between cospans and special commutative Frobenius monoids and hypergraph categories. To begin, we detail a natural hypergraph structure on $\mathrm{Cospan}(\mathcal{C})$.

This hypergraph structure also comes from copairings of identity morphisms. Call cospans

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & N & \\ Y \arrow[ur, "o"] & & X \arrow[ul, "i"'] \end{tikzcd}$$

that are reflections of each other **opposite** cospans. Given any object X in $\mathcal{C}$, the copairing $[1_X, 1_X]\colon X + X \to X$ of two identity maps on X, together with the unique map $!\colon \varnothing \to X$ from the initial object to X, define a monoid structure on X. Considering these maps as morphisms in $\mathrm{Cospan}(\mathcal{C})$, we may take them together with their opposites to give a special commutative Frobenius structure on X. It is easily verified that this gives a hypergraph category.

Given $f\colon X \to Y$ in $\mathcal{C}$, abuse notation by writing $f \in \mathrm{Cospan}(\mathcal{C})$ for the cospan $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$, and f^{opp} for the cospan $Y \xrightarrow{1_Y} Y \xleftarrow{f} X$. To summarise:

The hypergraph category $(\mathrm{Cospan}(\mathcal{C}), +)$	
objects	the objects of $\mathcal{C}$
morphisms	isomorphism classes of cospans in $\mathcal{C}$
composition	given by pushout
monoidal product	the coproduct in $\mathcal{C}$.
coherence maps	inherited from $(\mathcal{C}, +)$
hypergraph maps	$\mu = [1,1]$, $\eta = !$, $\delta = [1,1]^{\mathrm{opp}}$, $\epsilon = !^{\mathrm{opp}}$.

We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category $\mathrm{Cospan}(\mathcal{C})$; we of course refer instead to the isomorphism class of the said cospan.

It is not difficult to show that we in fact have a functor from the category with objects categories with finite colimits and morphisms colimit preserving functors to the category of hypergraph categories and hypergraph functors. Our next task will be to show that this extends to a functor from the category of symmetric lax monoidal presheaves over categories with finite colimits. This is known as the decorated cospans construction.

The decorated cospans theorem then leads to the decorated corelations construction, which gives a way to build every hypergraph category using cospans. Write Mon_C for the category with objects functions $[n] = \{1, 2, \ldots, n\} \to C$, where C is some set, and morphisms functions $[n] \to [m]$ that commute over C. We will prove the following.

Theorem 1.6. *The category of hypergraph categories is equivalent to the category of lax symmetric monoidal functors*

$$(\mathrm{Cospan}(\mathrm{Mon}_C), +) \to (\mathrm{Set}, \times).$$

varying over sets C.

This theorem, couched in the language of algebras for operads, is also the subject of a forthcoming paper by Vagner, Spivak, and Schultz [**?**].

Chapter 3

Corelations: a tool for black boxing

Black boxing!

USE MATERIAL FROM INTRODUCTION OF CORELATIONS PAPER AND CORELATIONS SECTION IN BLACK BOX PAPER

3.1 The idea of black boxing

Consider a circuit diagram.

We often view such diagrams atomically, representing the complete physical system built as specified. Yet the very process of building such a system involves assembling it from its parts, each of which we might diagram in the same way. The goal of this paper is to develop formal category-theoretic tools for describing and interpreting this process of assembly.

Circuits are constructed to get at some behaviour, with two circuits equivalent if they specify the same behaviour. Here we focus on this 'black boxing' aspect.

Corelations [**?**].

Semantics live on boundaries only.

We then introduce a new framework for working with hypergraph categories: decorated corelations.

Decorated corelations adds compositional operations to network-diagram representations, and handles composition of semantics too.

Two main theorems:

Theorem 3.1. *Given a category $\mathcal{C}$ with finite colimits, factorisation system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ is stable under pushouts, and a symmetric lax monoidal functor*

$$\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \longrightarrow \mathrm{Set},$$

we may define a hypergraph category of with morphisms decorated corelations.

Theorem 3.2. *Every hypergraph category can be constructed in this way.*

They apply to functors too.

3.2 Corelations

Given sets X, Y, a relation $X \rightsquigarrow Y$ is a subset of the product $X \times Y$. More abstractly, we might say a relation is a jointly-monic span in the category of sets (or an isomorphism class thereof). We generalise the dual concept.

3.2.1 Factorisation systems.

The relevant properties of jointly-monic spans come from the fact that monomorphisms form one half of a factorisation system. A factorisation system allows any morphism in a category to be factored into the composite of two morphisms in a coherent way.

Definition 3.3. A **factorisation system** $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ comprises subcategories $\mathcal{E}$, $\mathcal{M}$ of $\mathcal{C}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$.

(ii) every morphism $f \in \mathcal{C}$ admits a factorisation $f = m \circ e$, $e \in \mathcal{E}$, $m \in \mathcal{M}$.

(iii) given morphisms f, f', with factorisations $f = m \circ e$, $f' = m' \circ e'$ of the above sort, for every u, v such that the square

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}$$

commutes, there exists a unique morphism s such that

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[r, "m"] \arrow[d, dotted, "\exists! s"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

commutes.

Examples 3.4. We introduce some factorisation systems of central importance in what follows.

- Write $\mathcal{I}_\mathcal{C}$ for the wide subcategory of $\mathcal{C}$ containing exactly the isomorphisms of $\mathcal{C}$. Then $(\mathcal{I}_\mathcal{C}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{I}_\mathcal{C})$ are both factorisation systems in $\mathcal{C}$.
- The prototypical example of a factorisation system is the epi-mono factorisation system in Set. This follows from a more general fact, true in any category, that if every arrow can be factorised as an epi followed by a split mono, then this results in a factorisation system. The only non-trivial part to check is the uniqueness condition: given epis e_1, e_2, split monos m_1, m_2, and commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "e_1"] \arrow[d, "u"'] & \cdot \arrow[r, "m_1"] \arrow[d, dotted, "\exists! t"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e_2"'] & \cdot \arrow[r, "m_2"'] & \cdot \end{tikzcd}$$

we must show that there is a unique t that makes the diagram commute. Indeed let $t = m_2' v m_1$ where m_2' satisfies $m_2' m_2 = id$. To see that the right square commutes, observe

$$m_2 t e_1 = m_2 m_2' v m_1 e_1 = m_2 m_2' m_2 e_2 u = m_2 e_2 u = v m_1 e_1$$

and since e_1 is epi we have $m_2 t = v m_1$. For the left square,

$$t e_1 = m_2' v m_1 e_1 = m_2' m_2 e_2 u = e_2 u.$$

Uniqueness is immediate, since, e_1 is epi and m_2 is mono.

See [**?**, §14] for more details.

Definition 3.5. Call a factorisation system $(\mathcal{E}, \mathcal{M})$ in a monoidal category $(\mathcal{C}, \otimes)$ a **monoidal factorisation system** if $(\mathcal{E}, \otimes)$ is a monoidal category.

One might wonder why $\mathcal{M}$ does not appear in the above definition. To give a touch more intuition for this definition, we quote a theorem of Ambler. Recall a symmetric monoidal closed category is one in which each functor $-\otimes X$ has a specified right adjoint $[X,-]$. See Ambler for proof and further details [**?**, Lemma 5.2.2].

Proposition 3.6. *Let $(\mathcal{E},\mathcal{M})$ be a factorisation system in a symmetric monoidal closed category $(\mathcal{C},\otimes)$. Then the following are equivalent:*

(i) $(\mathcal{E},\mathcal{M})$ is a monoidal factorisation system.

(ii) $\mathcal{E}$ is closed under $-\otimes X$ for all $X \in \mathcal{C}$.

(iii) $\mathcal{M}$ is closed under $[X,-]$ for all $X \in \mathcal{C}$.

In a category with finite coproducts every factorisation system is a monoidal factorisation system for the coproduct.

Lemma 3.7. *Let $\mathcal{C}$ be a category with finite coproducts, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Then $(\mathcal{E},+)$ is a symmetric monoidal category.*

Proof. The only thing to check is that $\mathcal{E}$ is closed under $+$. That is, given $f\colon A \to B$ and $g\colon C \to D$ in $\mathcal{E}$, we wish to show that $f+g\colon A+C \to B+D$, defined in $\mathcal{C}$, is also a morphism in $\mathcal{E}$.

Let $f+g$ have factorisation $A+C \xrightarrow{e} \overline{B+D} \xrightarrow{m} B+D$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. We will prove that m is an isomorphism. To construct an inverse, recall that by definition, as f and g lie in $\mathcal{E}$, there exist morphisms $x\colon B \to \overline{B+D}$ and $y\colon D \to \overline{B+D}$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, equal] \arrow[d, dotted, "x"] & B \arrow[d] \\ A+C \arrow[r, "e"'] & \overline{B+D} \arrow[r, "m"'] & B+D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C \arrow[r, "g"] \arrow[d] & D \arrow[r, equal] \arrow[d, dotted, "y"] & D \arrow[d] \\ A+C \arrow[r, "e"'] & \overline{B+D} \arrow[r, "m"'] & B+D \end{tikzcd} \qquad (1)$$

The copairing $[x,y]$ is an inverse to m.

Indeed, taking the coproduct of the top rows of the two diagrams above and the copairings of the vertical maps gives the commutative diagram

$$\begin{tikzcd} A+C \arrow[r, "f+g"] \arrow[d, equal] & B+D \arrow[r, equal] \arrow[d, "{[x,y]}"'] & B+D \arrow[d, equal] \\ A+C \arrow[r, "e"] & \overline{B+D} \arrow[r, "m"] & B+D \end{tikzcd}$$

Reading the right-hand square immediately gives $m \circ [x,y] = 1$.

Conversely, to see that $[x,y]\circ m = 1$, remember that by definition $f+g = m\circ e$. So the left-hand square above implies that

$$\begin{tikzcd} A+C \arrow[r, "e"] \arrow[d, equal] & \overline{B+D} \arrow[d, "{[x,y]\circ m}"] \\ A+C \arrow[r, "e"'] & \overline{B+D} \end{tikzcd}$$

commutes. But by the universal property of factorisation systems, there is a unique map $\overline{B+D} \to \overline{B+D}$ such that this diagram commutes, and clearly the identity map also suffices. Thus $[x,y]\circ m = 1$. $\square$

3.2.2 Corelations

Relations, then, may be generalised as spans such that the span maps 'jointly' belong to some class $\mathcal{M}$ of an $(\mathcal{E},\mathcal{M})$-factorisation system. We define corelations in the dual manner.

Definition 3.8. Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. An **$(\mathcal{E},\mathcal{M})$-corelation** $X \rightsquigarrow Y$ is a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ such that the copairing $[i,o]\colon X+Y \to N$ lies in $\mathcal{E}$.

When the factorisation system is clear from context, we simply call $(\mathcal{E},\mathcal{M})$-corelations: 'corelations'.

We also say that a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ with the property that the copairing $[i,o]\colon X+Y \to N$ lies in $\mathcal{E}$ is **jointly $\mathcal{E}$-like**. Note that if a cospan is jointly $\mathcal{E}$-like then so are all isomorphic cospans. Thus the property of being a corelation is closed under isomorphism of cospans, and we again are often lazy with out language, referring to both jointly $\mathcal{E}$-like cospans and their isomorphism classes as corelations.

If $f\colon A \to N$ is a morphism with factorisation $f = m\circ e$, write $\overline{N}$ for the object such that $e\colon A \to \overline{N}$ and $m\colon \overline{N} \to N$. Now, given a cospan $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$, we may use the factorisation system to write the copairing $[i_X, o_Y]\colon X+Y \to N$ as

$$X+Y \xrightarrow{e} \overline{N} \xrightarrow{m} N.$$

From the universal property of the coproduct, we also have maps $\iota_X\colon X \to X+Y$ and $\iota_Y\colon Y \to X+Y$. We then call the corelation

$$X \xrightarrow{e\circ\iota_X} \overline{N} \xleftarrow{e\circ\iota_Y} Y$$

the **$\mathcal{E}$-part** of the above cospan. On occasion we will also write $e\colon X+Y \to \overline{N}$ for the same corelation.

Examples 3.9. Many examples of corelations are already familiar.

- For the morphism-isomorphism factorisation system $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, corelations are just cospans.

- For the isomorphism-morphism factorisation $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$, jointly $\mathcal{I}_{\mathcal{C}}$-like cospans $X \to Y$ are simply isomorphisms $X + Y \xrightarrow{\sim} N$. Thus there is a unique isomorphism class of corelation between any two objects.

- Note that the category Set has finite colimits and an epi-mono factorisation system. Epi-mono corelations from $X \to Y$ in Set surjective functions $X + Y \to N$; thus their isomorphism classes are partitions, or equivalence relations on $X + Y$.

3.2.3 Categories of corelations

We compose corelations by taking the $\mathcal{E}$-part of their composite cospan. That is, given corelations $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite is given by the cospan $X \to \overline{N +_Y M} \leftarrow Z$ in

$$\begin{tikzcd}
 & & & N +_Y M & & & \\
 & & & \overline{N +_Y M} \arrow[u, "m"] & & & \\
 & & N \arrow[ur, "e \circ \iota_N"] & & M \arrow[ul, "e \circ \iota_M"'] & & \\
 & X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z, \arrow[ul, "o_Z"'] &
\end{tikzcd}$$

where $m \circ e$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of $[j_N, j_M] \colon N + M \to N +_Y M$.

It is well-known that this composite is unique up to isomorphism, and that when $\mathcal{M}$ is well-behaved it defines a category with morphisms isomorphism classes of corelations. For example, bicategorical version of the dual theorem, for spans and relations, can be found in [?]. Nonetheless, for the sake of completeness we sketch our own argument here.

Proposition 3.10. *Let $\mathcal{C}$ be a category with finite colimits and with a factorisation system $(\mathcal{E}, \mathcal{M})$. Then the above is a well-defined composition rule on isomorphism classes of corelations.*

Proof. Let $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y)$ and $(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ Y \xrightarrow{i'_Y} M' \xleftarrow{o'_Z} Z)$ be pairs of isomorphic jointly $\mathcal{E}$-like cospans. Their composites *as cospans* are isomorphic, so the factorisation system gives an isomorphism s such that the diagram

$$\begin{tikzcd} X+Z \arrow[r, "e"] \arrow[d, equal] & \overline{N +_Y M} \arrow[r, "m"] \arrow[d, dotted, "\sim"', "s"] & N +_Y M \arrow[d, "\sim"] \\ X+Z \arrow[r, "e'"'] & \overline{N' +_Y M'} \arrow[r, "m'"'] & N' +_Y M' \end{tikzcd}$$

commutes. This s is an isomorphism of the composite corelations. □

Composition of corelations is *not* associative in general. It is, however, associative when $\mathcal{M}$ is **stable under pushout**: that is, whenever

$$\begin{tikzcd} \cdot \arrow[r, "j"] & \cdot \\ \cdot \arrow[u] \arrow[r, "m"] & \cdot \arrow[u] \end{tikzcd}$$

is a pushout square such that $m \in \mathcal{M}$, we also have that $j \in \mathcal{M}$.

Stability under pushout is a powerful property. A key corollary, both for associativity and in general, is that it implies $\mathcal{M}$ is also closed under $+$.

Lemma 3.11. *Let $\mathcal{C}$ be a category with finite colimits, and let $\mathcal{M}$ be a subcategory of $\mathcal{C}$ stable under pushouts and containing all isomorphisms. Then $(\mathcal{M}, +)$ is a symmetric monoidal category.*

Proof. It is enough to show that for all morphisms $m, m' \in \mathcal{M}$ we have $m + m'$ in $\mathcal{M}$. Since $\mathcal{M}$ contains all isomorphisms, the coherence maps are inherited from $\mathcal{C}$. The required axioms—the functoriality of the tensor product, the naturality of the coherence maps, and the coherence laws—are also inherited as they hold in $\mathcal{C}$.

To see $m + m'$ is in $\mathcal{M}$, simply observe that we have the pushout square

$$\begin{tikzcd} A+C \arrow[r, "m+1"] & B+C \\ A \arrow[u, "\iota"] \arrow[r, "m"] & B \arrow[u, "\iota"'] \end{tikzcd}$$

in $\mathcal{C}$. As $\mathcal{M}$ is stable under pushout, $m + 1 \in \mathcal{M}$. Similarly, $1 + m' \in \mathcal{M}$. Thus their composite $m + m'$ lies in $\mathcal{M}$, as required. □

An analogous argument shows that pushouts of maps $m+_Y m'$ also lie in $\mathcal{M}$. Using this lemma it is not difficult to show associativity—the key point is that factorisation 'commutes' with pushouts, and that we have a category $\mathrm{Corel}_{(\mathcal{E},\mathcal{M})}(\mathcal{C})$. Again, this is all well-known, and can be found in [?]. We will incidentally reprove these facts in the following, while pursuing richer structure.

Indeed, for modelling networks, we require not just a category, but a hypergraph category. Corelation categories come equipped with this extra structure. Recall that we gave decorated cospan categories a hypergraph structure by defining a wide embedding $\mathrm{Cospan}(\mathcal{C}) \hookrightarrow F\mathrm{Cospan}$, via which $F\mathrm{Cospan}$ inherited the coherence and Frobenius maps (Theorem 2.4). We will argue similarly here, after showing that the 'quotient' map

$$\mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

taking each cospan to its jointly $\mathcal{E}$-like part is functorial. Indeed, we define the coherence and Frobenius maps of $\mathrm{Corel}(\mathcal{C})$ to be their image under this map. For the monoidal product we again use the coproduct in $\mathcal{C}$.

Theorem 3.12. *Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be factorisation system on $\mathcal{C}$ such that $\mathcal{M}$ is stable under pushout. Then there exists a hypergraph category* $\mathrm{Corel}_{(\mathcal{E},\mathcal{M})}(\mathcal{C})$ *with*

The hypergraph category $(\mathrm{Corel}_{(\mathcal{E},\mathcal{M})}(\mathcal{C}),+)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of* $(\mathcal{E},\mathcal{M})$*-corelations in* $\mathcal{C}$
composition	*given by the* $\mathcal{E}$*-part of pushout*
monoidal product	*the coproduct in* $\mathcal{C}$
coherence maps	*inherited from* $\mathrm{Cospan}(\mathcal{C})$
hypergraph maps	*inherited from* $\mathrm{Cospan}(\mathcal{C})$

Again, we will drop explicit reference to the factorisation system when context allows, simply writing $\mathrm{Corel}(\mathcal{C})$.

Proposition 3.10 shows that our composition rule is a well-defined function, Lemma 3.7 shows likewise for the monoidal product $+\colon \mathrm{Corel}(\mathcal{C})\times\mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$. Thus we have the required data for a hypergraph category. It just remains to check a number of axioms: associativity and unitality of the categorical composition, functoriality of the monoidal product, naturality of the coherence maps, the coherence axioms for symmetric monoidal categories, the Frobenius laws.

Our strategy for this will simply be to show that the (surjective) function taking a cospan to its jointly $\mathcal{E}$-part preserves composition and has natural strict monoidal coherence maps. This implies that to evaluate an expression in the monoidal category of corelations, we may simply evaluate it in the monoidal category of cospans, and then take the $\mathcal{E}$-part. Thus if an equation is true for cospans, it is true for corelations.

Instead of proving just this, however, we will prove a generalisation regarding the analogous 'reduction' map between any two corelation 'categories'. This will reduce to the desired special case by taking the domain to be the 'trivial' $(\mathcal{C}, \mathcal{I}_\mathcal{C})$-corelations, which we know is equal to the hypergraph category of cospans. But the generality is not spurious: it has the advantage of proving the existence of a class of hypergraph functors between corelation categories in the same fell swoop.

Although a touch convoluted, this strategy is worth the pause for thought. We will use it once again for *decorated corelations*, to great economy.

3.2.4 Functors between corelation categories

We have seen that to construct a functor between cospan categories one may start with a colimit-preserving functor between the underlying categories. Corelation categories are similar, but we remove the part of each cospan that lies in $\mathcal{M}$. Hence for functors between corelation categories, we also require that the target category removes at least as much information as the source category.

Proposition 3.13. *Let $\mathcal{C}$, $\mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M})$, $(\mathcal{E}', \mathcal{M}')$, such that $\mathcal{M}$ and $\mathcal{M}'$ are stable under pushout. Further let $A\colon \mathcal{C} \to \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$.*

Then we may define a hypergraph functor $\square\colon \mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C}')$ sending each object X in $\mathrm{Corel}(\mathcal{C})$ to AX in $\mathrm{Corel}(\mathcal{C}')$ and each corelation

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$$

to the jointly-$\mathcal{E}'$-part

$$AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY.$$

of the image cospan. The coherence maps are the isomorphisms $\kappa_{X,Y}\colon AX + AY \to A(X+Y)$ given as A preserves colimits.

As discussed, we are still yet to prove that $\mathrm{Corel}(\mathcal{C})$ is a hypergraph category. We address this first with two lemmas regarding these proposed functors.

Lemma 3.14. *The above function $\square\colon \mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C}')$ preserves composition*

Proof. Let $f = (X \longrightarrow N \longleftarrow Y)$ and $g = (Y \longrightarrow M \longleftarrow Z)$ be corelations in $\mathcal{C}$. By definition, the corelations $\square(g) \circ \square(f)$ and $\square(g \circ f)$ are given by the first arrows in the top and bottom row respectively of the diagram:

$$\begin{tikzcd}
AX + AZ \arrow[r, "\mathcal{E}'"] \arrow[d, equal] & \overline{\overline{AN} +_{AY} \overline{AM}} \arrow[r, "\mathcal{M}'"] \arrow[d, dotted, leftrightarrow, "n"] & \overline{AN} +_{AY} \overline{AM} \arrow[r, "m'_{AN} +_{AY} m'_{AM}"] & AN +_{AY} AM \arrow[d, leftrightarrow, "\sim"] \\
AX + AZ \arrow[r, "\mathcal{E}'"] & \overline{A(\overline{N +_Y M})} \arrow[r, "\mathcal{M}'"] & A(\overline{N +_Y M}) \arrow[r, "Am_{N+_Y M}"] & A(N +_Y M)
\end{tikzcd}$$

The morphisms labelled $\mathcal{E}'$ lie in $\mathcal{E}'$, and similarly for $\mathcal{M}'$; these are given by the factorisation system on $\mathcal{C}'$. The maps $Am_{N+_Y M}$ and $m'_{AN} +_{AY} m'_{AM}$ lie in $\mathcal{M}'$ too: $Am_{N+_Y M}$ as it is in the image of $\mathcal{M}$, and $m'_{AN} +_{AY} m'_{AM}$ as $\mathcal{M}'$ is stable under pushout.

Moreover, the diagram commutes as both maps $AX+AZ \to AN+_{AY}AM$ compose to that given by the pushout of the images of f and g over AY. Thus the diagram represents two $(\mathcal{E}', \mathcal{M}')$ factorisations of the same morphism, and there exists an isomorphism n between the corelations $\square(g) \circ \square(f)$ and $\square(g \circ f)$. This proves that $\square$ preserves composition. $\square$

Lemma 3.15. *The maps κ above are natural.*

Proof. Let $f = (X \longrightarrow N \longleftarrow Y)$, $g = (Z \longrightarrow M \longleftarrow W)$ be corelations in $\mathcal{C}$. We wish to show that

$$\begin{tikzcd}
A(X) + A(Y) \arrow[r, "\square(f)+\square(g)"] \arrow[d, "\kappa_{X,Y}"'] & A(Z) + A(W) \arrow[d, "\kappa_{Z,W}"] \\
A(X+Y) \arrow[r, "\square(f+g)"] & A(Z+W)
\end{tikzcd}$$

commutes. Consider the diagram

$$\begin{tikzcd}
(AX + AY) + (AZ + AW) \arrow[r, "\mathcal{E}'+\mathcal{E}'"] \arrow[d, "\kappa_{X,Y}+\kappa_{Z,W}"'] & \overline{AN} + \overline{AM} \arrow[r, "\mathcal{M}'+\mathcal{M}'"] \arrow[d, dotted, "n"] & AN + AM \arrow[d, "\kappa_{N,M}"] \\
A(X+Y) + A(Z+W) \arrow[r, "\mathcal{E}'"] & \overline{A(N+M)} \arrow[r, "\mathcal{M}'"] & A(N+M)
\end{tikzcd}$$

This diagram represents two different factorisations of the coproduct of the images of f and g, and hence commutes. The top edge is the coproduct of the respective factorisations of f and g, whereas the bottom edge is the factorisation of the coproduct of f and g.

Note that by Lemma 3.7 the coproduct of two maps in $\mathcal{E}'$ is again in $\mathcal{E}'$, while Lemma 3.11 implies the same for $\mathcal{M}'$. Thus the top edge is an $(\mathcal{E}', \mathcal{M}')$-factorisation, and the uniqueness of factorisations gives the isomorphism n. This proves the naturality of the maps κ. $\square$

These lemmas now imply that $\mathrm{Corel}(\mathcal{C})$ is a well-defined hypergraph category.

Proof of Theorem 3.12. To complete the proof then, consider the case of Proposition 3.13 with $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, and $A = 1_{\mathcal{C}}$. Then the domain of $\square$ is $\mathrm{Cospan}(\mathcal{C})$ by definition. In this case, the function $\square\colon \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ is bijective-on-objects and surjective-on-morphisms. Moreover, note that by definition this function maps the coherence and hypergraph maps of $\mathrm{Cospan}(\mathcal{C})$ onto the corresponding maps of $\mathrm{Corel}(\mathcal{C})$. As $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph, and $\square$ preserves composition and respects the monoidal and hypergraph structure, $\mathrm{Corel}(\mathcal{C})$ is also a hypergraph category.

For instance, suppose we want to check the functoriality of the monoidal product $+$. We then wish to show $(g \circ f) + (k \circ h) = (g + k) \circ (f + h)$ for corelations of the appropriate types. But $\square$ preserves composition, and the naturality of κ, here the identity map, implies that for any two cospans the $\mathcal{E}$-part of their coproduct is equal to the coproduct of their $\mathcal{E}$-parts. Thus we may compute these two expressions by viewing f, g, h, and k as cospans, evaluating them in the category of cospans, and then taking their $\mathcal{E}$-parts. Since the equality holds in the category of cospans, it holds in the category of corelations. $\square$

Corollary 3.16. *There is a strict hypergraph functor*

$$\square\colon \mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

that takes each object of $\mathrm{Cospan}(\mathcal{C})$ *to itself as an object of* $\mathrm{Corel}(\mathcal{C})$ *and each cospan to its* $\mathcal{E}$*-part.*

Finally, we complete the proof that we have hypergraph functors.

Proof of Proposition 3.13. As A preserves colimits, it follows immediately that $\square$ respects the monoidal and hypergraph structure. The coherence maps for $\square$ are natural as A is colimit-preserving and the monoidal products are just the coproducts in the underlying categories, and obey the symmetric monoidal functor coherence laws as these are each just diagrams of isomorphisms of coproducts in $\mathcal{C}'$. As for the hypergraph structure, for example the Frobenius multiplication $[1_X, 1_X]$ (THIS IS NOT THE MULT) on an object X of $\mathcal{C}$ obeys $A[1_X, 1_X] \circ \kappa_{X,X} = [1_{AX}, 1_{AX}]$, as required.

$\square$

Example 3.17. NEEDS ATTENTION¿ Note that if both $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ are epi-mono factorisations, then we always have that $F(\mathcal{E}) \subseteq \mathcal{E}'$ and $F(\mathcal{M}) \subseteq \mathcal{M}'$. Indeed, if an (one-sided) inverse exists in the domain category, it exists in the codomain category. Thus colimit-preserving functors between categories with finite colimits and epi-mono factorisation systems also induce a functor between the epi-mono corelation categories.

3.2.5 Example: equivalence relations as corelations in Set

In each factorisation system of Examples 3.9 the right factor $\mathcal{M}$ is stable under pushout. This gives the hypergraph categories of cospans in $\mathcal{C}$, the indiscrete category on the objects of $\mathcal{C}$, and equivalence relations between finite sets.

The last of these examples is perhaps the most instructive for black-boxing open systems.

An **extraspecial commutative Frobenius monoid** $(X, \mu, \eta, \delta, \epsilon)$ in a monoidal category $(\mathcal{C}, \otimes)$ is a special commutative Frobenius monoid that further obeys the extra law

The extra law is a recent discovery, appearing first under this name in the work of Baez and Erbele [**?**], as the 'bone law' in [**?**, **?**], and as the 'irredundancy law' in [**?**].

Observe that each of these equations equate string diagrams that connect precisely the same elements of the domain and codomain. To wit, the associativity, coassociativity, and Frobenius laws show that the order in which we build a connected component through pairwise clustering is irrelevant, the special law shows that having multiple connections between points is irrelevant, and the extra law shows that 'extra' components not connected to the domain or codomain are irrelevant.

In fact the converse holds: two morphisms built from the generators of an extraspecial commutative Frobenius monoid are equal and if and only if their diagrams impose the same connectivity relations on the disjoint union of the domain and codomain. This is an extension of the spider theorem for special commutative Frobenius monoids.

3.2.6 Example: linear relations as corelations in Vect

Recall that a linear relation $L\colon U \rightsquigarrow V$ is a subspace $L \subseteq U \oplus V$. We compose linear relations as we do relations, and vector spaces and linear relations form a category LinRel. This category can be constructed as the category of relations in the category Vect of vector spaces and linear maps with respect to epi-mono factorisations. We show that they may also be constructed as corelations in Vect with respect to epi-mono factorisations.

If we restrict to the full subcategory FinVect of finite dimensional vector spaces this is easy to see: after picking a basis for each vector space the transpose yields an equivalence of FinVect with its opposite category, so the category of $(\mathcal{E},\mathcal{M})$-corelations (jointly-epic cospans) is isomorphic to the category of $(\mathcal{E},\mathcal{M})$-relations (jointly-monic spans) in FinVect. This fact has been fundamental in work on finite dimensional linear systems and signal flow diagrams [**?**, **?**, **?**].

We prove the general case in detail. To begin, note Vect has an epi-mono factorisation system with monos stable under pushouts. This factorisation system is inherited from Set: the epimorphisms in Vect are precisely the surjective linear maps, the monomorphisms are the injective linear maps, and the image of a linear map is always a subspace of the codomain, and so itself a vector space. Monos are stable under pushout as the pushout of a diagram $V \stackrel{f}{\leftarrow} U \stackrel{m}{\rightarrow} W$ is $V \oplus W/\mathrm{Im}\,[f \ \ -g]$. The map $m' \colon V \to V \oplus W/\mathrm{Im}\,[f \ \ -g]$ into the pushout has kernel $f(\ker m)$. Thus when m is a monomorphism, m' is too.

Thus we have a category of corelations Corel(Vect). We show that the map Corel(Vect) $\to$ LinRel sending each vector space to itself and each corelation

$$U \stackrel{f}{\rightarrow} A \stackrel{g}{\leftarrow} V$$

to the linear subspace $\ker[f \ \ -g]$ is a full, faithful, and bijective-on-objects functor.

Indeed, corelations $U \stackrel{f}{\rightarrow} A \stackrel{g}{\leftarrow} V$ are in one-to-one correspondence with surjective linear maps $U \oplus V \to A$, which are in turn, by the isomorphism theorem, in one-to-one correspondence with subspaces of $U \oplus V$. These correspondences are described by the kernel construction above. Thus our map is evidently full, faithful, and bijective-on-objects. It also maps identities to identities. It remains to check that it preserves composition.

Suppose we have corelations $U \stackrel{f}{\rightarrow} A \stackrel{g}{\leftarrow} V$ and $V \stackrel{h}{\rightarrow} B \stackrel{k}{\leftarrow} W$. Then their pushout is given by $P = A \oplus B/\mathrm{Im}\,[g \ \ -h]$, and we may draw the pushout diagram

$$\begin{tikzcd}
U \arrow[dr, "f"'] & & V \arrow[dl, "g"] \arrow[dr, "h"'] & & W \arrow[dl, "k"] \\
& A \arrow[dr, "\iota_A"'] & & B \arrow[dl, "\iota_B"] & \\
& & P \arrow[u, phantom, "\wedge"] & &
\end{tikzcd}$$

We wish to show the equality of relations

$$\ker[f \ \ -g]; \ker[h \ \ -k] = \ker[\iota_A f \ \ -\iota_B g].$$

Now $(\mathbf{u},\mathbf{w}) \in U \oplus W$ lies in the composite relation $\ker[f \ \ -g];\ker[h \ \ -k]$ iff there exists $\mathbf{v} \in V$ such that $f\mathbf{u} = g\mathbf{v}$ and $h\mathbf{v} = k\mathbf{w}$. But as P is the pushout, this is true iff

$$\iota_A f\mathbf{u} = \iota_A g\mathbf{v} = \iota_B h\mathbf{v} = \iota_B k\mathbf{w}.$$

This in turn is true iff $(\mathbf{u},\mathbf{w}) \in \ker[\iota_A f \ \ -\iota_B k]$, as required.

3.3 Decorated corelations

When enough structure is available to us, we may decorate corelations too. Furthermore, and key to the idea of 'black-boxing', we get a hypergraph functor from decorated cospans to decorated corelations.

3.3.1 Adjoining right adjoints

Suppose we have a cospan $X+Y \to N$ with a decoration on N. Reducing this to a corelation requires us to factor this to $X+Y \xrightarrow{e} \overline{N} \xrightarrow{m} N$. Decorated corelations require a method from taking a decoration on N and 'pulling it back' along m to a decoration on $\overline{N}$.

A subcategory stable under pushouts is a useful thing.

Proposition 3.18. *Let $\mathcal{C}$ be a category with finite colimits, and let $\mathcal{M}$ be a subcategory of $\mathcal{C}$ stable under pushouts. Then we define the category $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$ as follows*

The symmetric monoidal category $(\mathcal{C};\mathcal{M}^{\mathrm{opp}},+)$	
objects	*the objects of $\mathcal{C}$*
morphisms	*isomorphism classes of cospans of the form $\xrightarrow{c}\xleftarrow{m}$, where c is a morphism in $\mathcal{C}$ and m a morphism in $\mathcal{M}$*
composition	*given by pushout*
monoidal product	*the coproduct in $\mathcal{C}$*
coherence maps	*the coherence maps in $\mathcal{C}$*

Proof. Composition is well-defined as $\mathcal{M}$ is stable under pushouts. Monoidal composition is well-defined by Lemma 3.11. Necessary laws hold as they are inherited from $\mathcal{C}$. □

This category can be viewed as a bicategory, with 2-morphisms given by maps of cospans. In this bicategory every morphism of $\mathcal{M}$ has a right adjoint.

Examples 3.19.

- Note that $\mathcal{C};\mathcal{C}^{\mathrm{opp}}$ is by definition equal to $\mathrm{Cospan}(\mathcal{C})$.

- Writing $\mathcal{I}_\mathcal{C}$ for the wide subcategory of isomorphisms in $\mathcal{C}$, note that $\mathcal{C};\mathcal{I}_\mathcal{C}^{\mathrm{opp}}$ is naturally isomorphic to $\mathcal{C}$.

Lemma 3.20. *Let $\mathcal{C}$, $\mathcal{C}'$ be categories with finite colimits, and let $\mathcal{M}$, $\mathcal{M}'$ be subcategories each stable under pushouts. Let $A\colon \mathcal{C} \to \mathcal{C}'$ be functor that preserves colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$. Then A extends to a symmetric (strong) monoidal functor*

$$A\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \longrightarrow \mathcal{C}';\mathcal{M}'^{\mathrm{opp}}.$$

mapping X to AX and $\stackrel{c}{\to}\stackrel{m}{\leftarrow}$ to $\stackrel{Ac}{\to}\stackrel{Am}{\leftarrow}$.

Proof. Note $A(\mathcal{M}) \subseteq \mathcal{M}'$, so $\stackrel{Ac}{\to}\stackrel{Am}{\leftarrow}$ is indeed a morphism in $\mathcal{C}';\mathcal{M}'$. The functor A preserves colimits, so composition is preserved. □

Note this could be done more generally with any two isomorphism-containing wide subcategories stable under pushout.

3.3.2 Decorated corelations

Definition 3.21. Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose that we also have a lax monoidal functor

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times).$$

Then we define an F-**decorated corelation** to be the isomorphism class of a pair

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd} \right)$$

where the cospan is jointly-$\mathcal{E}$-like.

Again, we will be lazy about the distinction between a decorated corelation and its isomorphism class.

Suppose we have decorated corelations

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd} \right) \quad \text{and} \quad \left(\begin{tikzcd} & M & \\ Y \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FM \\ 1 \arrow[u, "s"'] \end{tikzcd} \right).$$

Then their composite is given by the composite corelation

$$\begin{tikzcd} & \overline{N +_Y M} & \\ X \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd}$$

paired with the decoration

$$1 \longrightarrow F(N+M) \longrightarrow F(N+_Y M) \stackrel{F(m^{\text{opp}})}{\longrightarrow} F(\overline{N+_Y M})$$

This is well-defined.

3.3.3 Categories of decorated corelations.

Theorem 3.22. *Let $\mathcal{C}$ be a category with finite colimits and factorisation system $(\mathcal{E},\mathcal{M})$ with $\mathcal{M}$ stable under pushout, and let*

$$F : (\mathcal{C};\mathcal{M}^{\text{opp}},+) \longrightarrow (\mathrm{Set},\times)$$

be a symmetric lax monoidal functor. Then we may define

The hypergraph category $(F\mathrm{Corel},+)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of* F*-decorated corelations in* $\mathcal{C}$
composition	*given by* $\mathcal{E}$*-part of pushout with restricted decoration*
monoidal product	*the coproduct in* $\mathcal{C}$ *on objects, coproduct of cospans and pair of decorations on morphisms.*
coherence maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*
hypergraph maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*

Similar to the corelations theorem (Theorem 3.12), we prove this alongside the theorem in the next subsection.

Example 3.23. Note that decorated cospans are a special case of decorated corelations: we use an morphism–isomorphism factorisation system.

Example 3.24. Note that 'undecorated' corelations are a special case of decorated corelations: they are corelations decorated by the functor $1\colon \mathcal{C};\mathcal{M}^{\text{opp}} \to \mathrm{Set}$ that maps each object to the one element set 1, and each morphism to the identity function on 1. This is a symmetric monoidal functor with the coherence maps all also the identity function on 1.

The key difference is to decorate cospans we need to know how to push decorations up. To decorate corelations we all need to know how to pull decorations back down. This is related to the existence of an extraspecial commutative Frobenius monoid in our main applications.

Associativity: To take a decoration on $A+B$ to one on $A+_C \overline{B}$ we may either reduce to the $\mathcal{E}$-part of B and then pushout over C, or pushout over C and then reduce to the $\mathcal{E}$ part of B. This lemma implies that both processes result in the same decoration.

3.3.4 Functors between decorated corelation categories

Proposition 3.25. *Let $\mathcal{C}$, $\mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M})$, $(\mathcal{E}', \mathcal{M}')$, such that $\mathcal{M}$ and $\mathcal{M}'$ are stable under pushout, and suppose that we have symmetric lax monoidal functors*

$$F : (\mathcal{C}; \mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times)$$

and

$$G : (\mathcal{C}'; \mathcal{M}'^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times).$$

Further let $A \colon \mathcal{C} \to \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$. This functor A extends to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \to \mathcal{C}'; \mathcal{M}'^{\mathrm{opp}}$.

Suppose we have a monoidal natural transformation θ:

$$\begin{tikzcd} \mathcal{C}; \mathcal{M}^{\mathrm{opp}} \arrow[dr, "F"] \arrow[d, "A"'] & \\ \mathcal{C}'; \mathcal{M}'^{\mathrm{opp}} \arrow[r, "G"'] \arrow[ur, phantom, "\Downarrow\theta"] & \mathrm{Set} \end{tikzcd}$$

Then we may define a hypergraph functor $T \colon F\mathrm{Corel} \to G\mathrm{Corel}$ sending each object $X \in F\mathrm{Corel}$ to $AX \in G\mathrm{Corel}$ and each decorated corelation

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN$$

to

$$AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY, \quad 1 \xrightarrow{s} FN \xrightarrow{\theta_N} GAN \xrightarrow{Gm^{\mathrm{opp}}_{AN}} G\overline{AN}.$$

The coherence maps are given by

$$\kappa_{X,Y} = \left(\begin{tikzcd} & \overline{A(X+Y)} & \\ AX + AY \arrow[ur] & & A(X+Y) \arrow[ul] \end{tikzcd} \quad , \quad \begin{array}{c} G(\overline{A(X+Y)}) \\ \uparrow Gm^{\mathrm{opp}}_{AX+AY} \\ GA(X+Y) \\ \uparrow G! \\ G\varnothing \\ \uparrow \gamma_1 \\ 1 \end{array} \right).$$

Proof. In the proof of Proposition 3.13 we used the existence of a bijective-on-objects, surjective-on-morphisms, composition preserving map $\mathrm{Cospan}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ to prove the associativity and other properties of $\mathrm{Corel}(\mathcal{C})$. Our proof strategy here is entirely analogous.

Similar to before, we still have not proved that decorated corelations form well-defined hypergraph categories. So we begin by merely showing that the map $\square$ is composition-preserving, and then that $\square$ respects the monoidal and hypergraph structure. We then specialise to the case where the domain forms a decorated cospan category that maps surjectively onto a generic decorated corelations codomain. Since we know decorated cospan categories are well-defined hypergraph categories, we can conclude the same for decorated corelations categories, and hence prove that $\square$ is a hypergraph functor.

We prove that $\square$ preserves composition. Suppose we have decorated corelations

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN \qquad \text{and} \qquad Y \xrightarrow{i_Y} M \xleftarrow{o_Y} Z, \quad 1 \xrightarrow{t} FM$$

We know the functor $\square$ preserves composition on the cospan part; this is precisely the content of Proposition 3.13. It remains to check that $\square(g \circ f)$ and $\square g \circ \square f$ have isomorphic decorations. This is expressed by the commutativity of the following diagram:

$$\begin{tikzcd}
G\overline{A(\overline{N +_Y M})} \arrow[rrrr, "Gn"] & & & & G(\overline{\overline{AN} +_{AY} \overline{AM}}) \\
GA(\overline{N +_Y M}) \arrow[u, "Gm^{\mathrm{opp}}_{A(\overline{N+_Y M})}"] \arrow[rrrr, phantom, "(**)"] & & & & G(\overline{AN} +_{AY} \overline{AM}) \arrow[u, "Gm^{\mathrm{opp}}_{\overline{AN}+_{AY}\overline{AM}}"'] \\
F(\overline{N +_Y M}) \arrow[u, "\theta_{\overline{N+_Y M}}"] \arrow[dr, phantom, "(\mathrm{TN})"] & GA(N +_Y M) \arrow[ul, "GAm^{\mathrm{opp}}_{N+_Y M}"'] \arrow[drr, phantom, "(\mathrm{A})"] & & G(AN +_{AY} AM) \arrow[ll, "G\sim"'] \arrow[ur, "G(m^{\mathrm{opp}}_{AN} +_{AY} m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "(\mathrm{C})"] & G(\overline{AN} + \overline{AM}) \arrow[u, "G[j_{\overline{AN}}, j_{\overline{AM}}]"'] \\
F(N +_Y M) \arrow[u, "Fm^{\mathrm{opp}}_{N+_Y M}"] & GA(N+M) \arrow[u, "GA[j_N, j_M]"'] & & G(AN+AM) \arrow[ll, "G\alpha_{N,M}"] \arrow[u, "G[j_{AN}, j_{AM}]"] \arrow[ur, "G(m^{\mathrm{opp}}_{AN} + m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "(\mathrm{GM})"] & G\overline{AN} \times G\overline{AM} \arrow[u, "\gamma_{\overline{AN},\overline{AM}}"'] \\
F(N+M) \arrow[u, "F{[j_N, j_M]}"] \arrow[ur, "\theta_{N+M}"'] \arrow[rrrr, phantom, "(\mathrm{TM})"] & & & & GAN \times GAM \arrow[ul, "\gamma_{AN,AM}"] \arrow[u, "Gm^{\mathrm{opp}}_{AN} \times Gm^{\mathrm{opp}}_{AM}"'] \\
& & FN \times FM \arrow[ull, "\varphi_{N,M}"] \arrow[urr, "\theta_N \times \theta_M"'] & & \\
& & 1 \arrow[u, "\rho_1 \circ (s \times t)"'] & &
\end{tikzcd}$$

This diagram does indeed commute. To check this, first observe that (TM) commutes by the monoidality of θ, (GM) commutes by the monoidality of G, and (TN) commutes by the naturality of θ. The remaining three diagrams commute as they are G-images of diagrams that commute in $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$. Indeed, (A) commutes since A preserves colimits and G is functorial, (C) commutes as it is the G-image of a pushout square in $\mathcal{C}'$, so

$$\xleftarrow{m^{\mathrm{opp}}_{AN}+m^{\mathrm{opp}}_{AM}} \xrightarrow{[j_{\overline{AN}},j_{\overline{AM}}]} \quad \text{and} \quad \xrightarrow{[j_{AN},j_{AM}]} \xleftarrow{m^{\mathrm{opp}}_{AN}+_{AY}m^{\mathrm{opp}}_{AM}}$$

are equal as morphisms of $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$, and $(**)$ commutes as it is the G-image of the right-hand subdiagram of $(*)$ used to define n.

It is evident that $\square$ is bijective-on-objects and surjective-on-morphisms. This proves the theorem. $\square$

Corollary 3.26. *Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose that we also have a lax monoidal functor*

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times).$$

Then we may define a category $F\mathrm{Corel}$ with objects the objects of $\mathcal{C}$ and morphisms isomorpism classes of F-decorated corelations.

Write also F for the restriction of F to the wide subcategory $\mathcal{C}$ of $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$. We can thus also obtain the category $F\mathrm{Cospan}$ of F-decorated cospans. We moreover have a functor

$$F\mathrm{Cospan} \to F\mathrm{Corel}$$

which takes each object of $F\mathrm{Cospan}$ to itself as an object of $F\mathrm{Corel}$, and each decorated cospan

$$\left(\begin{array}{ccc} & N & \\ & {\scriptstyle i}\nearrow \quad \nwarrow{\scriptstyle o} & \\ X & & Y \end{array} , \quad \begin{array}{c} FN \\ \uparrow{\scriptstyle s} \\ 1 \end{array} \right)$$

to its jointly-$\mathcal{E}$-part

$$\begin{array}{ccc} & \overline{N} & \\ & {\scriptstyle i}\nearrow \quad \nwarrow{\scriptstyle o} & \\ X & & Y \end{array}$$

decorated by the composite

$$1 \xrightarrow{s} FN \xrightarrow{Fm^{\mathrm{opp}}_N} F\overline{N}.$$

3.4 All hypergraph categories are decorated corelation categories

structured categories are algebras over their graphical calculus operad [?]. These equivalences are 2-categorical.

Write $\int$ for the Grothendieck construction. The Grothendieck construction is

Theorem 3.27. *hypergraph categories are symmetric lax monoidal functors cospan to Set.*

$$\mathrm{HypCat} \cong \int^{\mathcal{O}\in\mathrm{Set}} \mathrm{SymLaxMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set})$$

[?]

Remark 3.28. Not all hypergraph categories are decorated *cospan* categories. To see this, we can count morphisms. The possible apices and decorations are the same for all morphisms. So for a decorated cospan category over the prop of finite sets, the number of morphisms $0 \to 1$ cannot be more than countably many times those $0 \to 0$ (we just get to choose an element of the apex). But the skeletal category of vector spaces over $\mathbb{R}$ with monoidal product the tensor product has $\mathbb{R}$ morphisms $0 \to 0$, and $\mathbb{R}^2$ morphisms $0 \to 1$.

Decorated corelation categories, however, are more powerful. We can recover all hypergraph categories by forcing the decorations to be on the coproduct of the domain and codomain itself. For this we use the isomorphism–morphism factorisation. Let $\mathcal{H}$ be a hypergraph category, and let $\mathcal{C}$ be the wide subcategory of all Frobenius morphisms. Then $\mathcal{H}$ can be recovered as the $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by global sections in $\mathcal{H}$.

3.4.1 The global sections construction.

Theorem 3.29. *All hypergraph categories are decorated corelation categories.*

Proof. Let $\mathcal{H}$ be a hypergraph category. Without loss of generality we can assume $\mathcal{H}$ has objects a free monoid under the tensor product; write $\mathcal{O}$ for a collection of generators for this free monoid, and $\mathrm{FinSet}_{\mathcal{O}}$ for $\mathcal{O}$ labelled finite sets (ie an object is a finite set X together with a function $X \to \mathcal{O}$). This is a finitely cocomplete category. Equivalent to finite lists of objects in $\mathcal{H}$.

Define the global sections functor

$$\begin{aligned} G\colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) &\longrightarrow \mathrm{Set} \\ A &\longmapsto \mathcal{H}(I, A) \\ \overset{f}{\to}\overset{g}{\leftarrow} &\longmapsto \text{action of Frobenius maps} \end{aligned}$$

This is symmetric lax monoidal functor. Note that $\mathcal{H}(I, A)$ depends on the order we choose to convert a multiset into an object A of $\mathcal{H}$. Nonetheless, from any two choices A, A' we get a canonical map $A \to A'$. This is really that clique in Set.

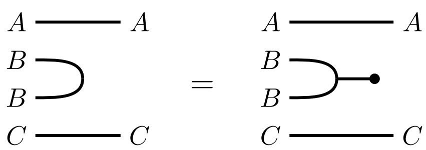
Consider the category $\mathrm{FinSet}_{\mathcal{O}}$ with an (isomorphism,morphism)-factorisation system. We get a decorated corelations category with objects multisets of generating objects of $\mathcal{H}$, and morphisms $A \to B$ trivial corelations $A \to A + B \leftarrow B$ decorated by some morphism $s \in \mathcal{H}(I, A + B)$. Recall that this decorated corelation is only specified up to isomorphism; in the following we always choose representatives such that the apex of the jointly-isomorphic cospan is always of the form $A + B$ for morphisms $A \to B$.

Given morphisms $s \in \mathcal{H}(I, A+B)$ and $t \in \mathcal{H}(I, B+C)$ of types $A \to B$ and $B \to C$ in GCorel, composition is given by the map $H(I, A + B + B + C) \to H(I, A + C)$ arising as the G-image of the cospan $A + B + B + C \overset{[j,j]}{\to} A + B + C \overset{m}{\leftarrow} A + C$ where maps come from the pushout square

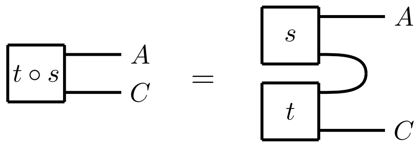
$$\begin{tikzcd}
& & A+C \arrow[d] & & \\
& & A+B+C & & \\
& A+B \arrow[ur] & & B+C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C \arrow[ul]
\end{tikzcd}$$

In terms of string diagrams in $\mathcal{H}$, this means composing the maps

with the Frobenius map

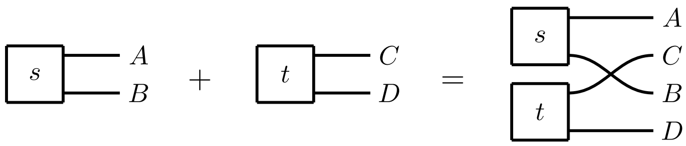


to get



in $\mathcal{H}(I, A+C)$.

The monoidal product is given by



recalling that we have chosen to represent the equivalence class of corelations $A+C \to B+D$ with the apex $A+C+B+D$.

Taking a hint from the compact closed structure, it is straightforward to construct a pair of inverse hypergraph functors between GCorel to $\mathcal{H}$. Indeed, GCorel is constructed to have the same collection of objects as $\mathcal{H}$; simply have the functors be the 'identity' on objects. On morphisms, we take $f : A \to B$ in $\mathcal{H}$ to its 'name' $\hat{f} : I \to A+B$ as a morphism of GCorel. This is a bijection.

To check it is composition and monoidal product preserving, we can easily use diagrammatic reasoning. For example

Therefore $\mathcal{H}$ is isomorphic as a hypergraph category to GCorel. □

Theorem 3.30. *All hypergraph functors are decorated corelation functors.*

Proof. Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraph categories, and $T \colon \mathcal{H} \to \mathcal{H}'$ be a hypergraph functor. By the above theorem, there exist symmetric lax monoidal functors

$$G \colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}_\mathcal{H}}) \to \mathrm{Set}$$

and

$$G' \colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}'}}) \to \mathrm{Set}$$

such that $\mathcal{H} = G$Corel and $\mathcal{H}' = G'$Corel. Furthermore, define a functor $A \colon \mathrm{FinSet}_{\mathcal{O}_\mathcal{H}} \to \mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}'}}$ taking $N \to \mathcal{O}_\mathcal{H}$ to $N \to \mathcal{O}_\mathcal{H} \to \mathcal{O}_{\mathcal{H}'}$, where the second map is that by the functor T on objects of $\mathcal{H}$. We claim this is a well-defined colimit-preserving functor and show that T can be constructed from a monoidal natural transformation between G and $G' \circ A$. □

Compare with Spivak Vagner construction.

3.4.2 Examples.

We give some examples reproducing hypergraph categories as decorated corelations categories.

Example 3.31. Example: empty decorations and equivalence relations.

Consider the hypergraph category Cospan(FinSet). This is the simplest hypergraph category: it is free hypergraph category on the one object discrete category. We show how to recover it as a decorated corelation category.

As per Example 3.24, Cospan(FinSet) is the hypergraph category of undecorated (morphism-isomorphism)-corelations in FinSet. It is also the partition-decorated (isomorphism-morphism)-corelations in FinSet.

First, the global sections functor $G\colon \mathrm{Cospan(FinSet)} \to \mathrm{Set}$ takes each finite set X to the set of (equivalence classes of) cospans $0 \to D \leftarrow X$; that is, to the set of functions $X \to D$ where a unique D is chosen for each finite cardinality. Given a cospan $X \xrightarrow{f} N \xleftarrow{g} Y$, its image under the global sections functor maps a function $a\colon X \to D$, to the function $Y \to N +_Y D$ given by

$$\begin{tikzcd} & X \arrow[r, "d"] \arrow[d, "f"'] & D \arrow[d] \\ Y \arrow[r, "g"] & N \arrow[r] & N +_Y D \end{tikzcd}$$

where the square is a pushout square.

The coherence maps $\gamma_1\colon 1 \to G\varnothing$ map the unique element of 1 to the unique function $!\colon \varnothing \to \varnothing$, and $\gamma_{X,Y}$ maps a pair of functions $a\colon X \to D$, $b\colon Y \to E$ to $a + b\colon X + Y \to D + E$. This is a symmetric lax monoidal functor. We use this functor to decorate cospans in FinSet.

A decorated cospan in FinSet with respect to this functor is a cospan $X \to N \leftarrow Y$ in FinSet together with a function $N \to D$ for some finite set. Using the isomorphism–morphism factorisation, a decorated corelation (a morphism in GCorel) is a cospan $X \to X + Y \leftarrow Y$, together with a function $X + Y \to D$. This is the same as a cospan.

The hypergraph structure is given by the decoration $X + X \to X$ etc, as the shift from cospans to corelations takes the 'factored out part' and puts it into the decoration. At this point the morphisms are specified entirely by their decoration.

It is straightforward to show the two categories are isomorphic. Note that the identity on FinSet maps $\mathcal{I}_{\mathrm{FinSet}}$ into FinSet, and so extends to morphism FinSet $\to$

Cospan(FinSet). We can define a monoidal natural transformation $1(X) = 1 \xrightarrow{\theta_X} GX = \{X \to D\}$ mapping the unique element to the identity function $1_X \colon X \to X$.

$$\begin{tikzcd}
\mathrm{FinSet} = \mathrm{FinSet};\mathcal{I}_{\mathrm{FinSet}}^{\mathrm{opp}} \arrow[d, "\iota"'] \arrow[dr, "1"] & \\
& \mathrm{Set} \\
\mathrm{Cospan}(\mathrm{FinSet}) = \mathrm{FinSet};\mathrm{FinSet}^{\mathrm{opp}} \arrow[ur, "G"'] &
\end{tikzcd}$$

It is easy to verify that this is a monoidal natural transformation. This gives the hypergraph functor we expect, mapping the undecorated cospan $X \to N \leftarrow Y$ to $X \to X + Y \leftarrow Y$ decorated by $X + Y \to N$.

Like so many examples before, it is easy to verify this is a full, faithful, bijective-on-objects hypergraph functor.

Example 3.32. The previous example extends to any finitely cocomplete category $\mathcal{C}$: the hypergraph category Cospan($\mathcal{C}$) can always be constructed as (i) trivially decorated $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$-corelations, or (ii) $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms with domain the apex of the corelation, and moreover the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

More general still, a category of trivially decorated $(\mathcal{E}, \mathcal{M})$-corelations in $\mathcal{C}$ can always be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms in $\mathcal{E}$ with domain the apex of the corelation, and the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

Most general, the theorem implies that any category of decorated corelations can be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by codomain decorated morphisms in $\mathcal{E}$. The latter are specified by a functor Cospan($\mathcal{C}$) $\to$ Set, as in the theorem.

The latter form is good for constructing functors that have image the corelation category.

Note that with decorated corelations we have a method of constructing hypergraph categories from other hypergraph categories. This is black boxing.

3.5 Examples

3.5.1 Path integrals and matrices

Let R be a commutative ring. Take functor

$$\begin{aligned} R^{(-)} : (\mathrm{FinSet}^{\mathrm{opp}}, +) &\longrightarrow (\mathrm{Set}, \times) \\ X &\longmapsto R^X \end{aligned}$$

Then $R^{(-)}$Cospan is path integrals, $R^{(-)}$Corel is matrices over R.

Many aspects of this example are 'atypical', regarding the intuition we have been working towards. Note that the monoidal product here is the tensor product of matrices, not the biproduct. Indeed, there is no special commutative Frobenius algebra in Vect if we use the biproduct, but if we use the tensor product then these correspond to orthonormal bases (Vicary). The comultiplication is the diagonal map, multiplication is codiagonal. unit produces basis.

We note that you could take decorations here in the category RMod of R-modules. While Proposition 2.8 shows that the resulting decorated cospans category would be isomorphic, this hints at an enriched version of the theory.

3.5.2 Two constructions for linear relations

We saw earlier that linear relations are epi-mono corelations in Vect. The hypergraph structure is given by addition. We show how to recover this in another construction. We also get a hypergraph functor between them. This is very useful for compositional linear relations semantics of diagrams.

We can also construct linear relations in $\mathrm{Vect}^{\mathrm{opp}}$.

$$\begin{aligned} : \mathrm{Cospan}(\mathrm{FinSet}) &\longrightarrow \mathrm{Set} \\ X &\longmapsto \{\mathrm{subspaces of} k^X\} \\ f : X \to Y &\longmapsto L \mapsto \{v \mid v \circ f \in L\} \\ f^{\mathrm{opp}} : X \to Y &\longmapsto L \mapsto \{v = u \circ f \mid u \in L\} \end{aligned}$$

Then Cospan is cospans decorated by subspaces, and Corel is linear relations. This is important for circuits work [**?**, **?**].

3.5.3 Automata

This construction comes immediately from Walters et al. Automata are alphabet labelled graphs. There is a decorated cospan functor to categories enriched over languages, and this factors nicely to get a decorated corelation category with morphisms languages recognised between points in domain and codomain.

Part II

Applications

Bibliography

[Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page 20.)