The Algebra of Open and Interconnected Systems

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Contents

Preface

This is a thesis in the mathematical sciences, with emphasis on the mathematics. But before we get to the category theory, I want to say a few words about the scientific tradition in which this thesis is situated.

Mathematics is the language of science. Twinned so intimately with physics, over the past centuries mathematics has become superb—indeed, unreasonably effective—language for understanding planets moving in space, particles in a vacuum, the structure of spacetime, and so on. Yet, while Wigner speaks of the unreasonble effectiveness of mathematics in the natural sciences, equally eminent mathematicians, not least Gelfand, speak of the unreasonable *ineffectiveness* of mathematics in biology and related fields. Why such a difference?

A contrast between physics and biology is that while physical systems can often be studied in isolation—the particle in a vacuum—, biological systems are necessarily situated in their environment. A heart belongs in a body, an ant in a colony. One of the first to draw attention to this contrast was Ludwig on Bertalanffy, biologist and founder of general systems theory, who articulated the difference as one between closed and open systems:

> Conventional physics deals only with closed systems, i.e. systems which are considered to be isolated from their environment. . . . However, we find systems which by their very nature and definition are not closed systems. Every living organism is essentially an open system. It maintains itself in a continuous inflow and outflow, a building up and breaking down of components, never being, so long as it is alive, in a state of chemical and thermodynamic equilibrium but maintained in a so-called steady state which is distinct from the latter [Ber68].

While von Bertalanffy's general systems theory is still approaching its generalist ambition, his philosophy has had great impact in his home field of biology, leading to the modern field of systems biology. Half a century later, Dennis Noble, another great pioneer of systems biology and the originator of the first mathematical model of a working heart, describes the shift as one from reduction to integration.

> Systems biology... is about putting together rather than taking apart, integration rather than reduction. It requires that we develop ways of thinking about integration that are as rigorous as our reductionist programmes, but different [Nob06].

In this thesis we develop rigorous ways of thinking about integration or, as we refer to it, interconnection.

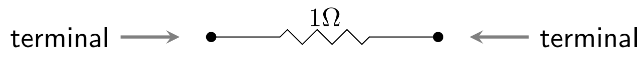
Interconnection and openness are tightly related. Indeed, openness implies that a system may be interconnected with its environment. But what is an environment but comprised of other systems? Thus the study of open systems becomes the study of how a system changes under interconnection with other systems.

To model this, we must begin by creating language to describe the interconnection of systems. While reductionism hopes that phenomena can be explained by reducing them to "elementary units investigable independently of each other" [Ber68], this philosophy of integration introduces as an additional and equal priority the investigation of the way these units are interconnected. As such, this thesis is predicated on the hope that the meaning of an expression in our new language is determined by the meanings of its constituent expressions together with the syntactic rules combining them. This is known as the principle of compositionality.

Also commonly known as Frege's principle, the principle of compositionality and its precise meaning and applicability both dates back to Ancient Greek and Vedic philosophy, and is still the subject of active research today [Jan86, Sza13]. More recently, through the work of Montague [Mon70] in natural language semantics and Strachey and Scott [SS71] in programming languages semantics, the principle of compositionality has found formal expression as the dictum that the interpretation of a language should be given by a homomorphism between an algebra of syntactic representations and an algebra of semantic objects. We too shall follow this route.

The question then arises: what do we mean by algebra? This mathematical question leads us back to our scientific objectives: what do we mean by system? Here we must narrow, or at least define, our scope. We give some examples. The investigations of thesis began with electrical circuits and their diagrams, and we will devote significant time to exploring their compositional formulation. We discussed biological systems above, and our notion of system includes these, modelled say in the form of chemical reaction networks and Markov processes, or the compartmental models of epidemiology, population biology, and ecology. From computer science, we consider Petri nets, automata, logic circuits, and similar. More abstractly, our notion of system encompasses matrices and systems of differential equations.

Drawing together these notions of system are well-developed diagrammatic representations based on network diagrams—that is, topological graphs. We call these network-style diagrammatic languages. In the abstract, by system we shall simply mean that which can be represented by box with a collection of terminals, perhaps of different types, through which it interfaces with the surroundings. Concretely, one might envision a circuit diagram with terminals, such as



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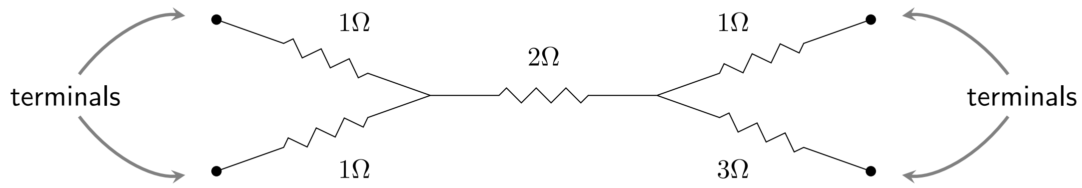

The algebraic structure of interconnection is then simply the structure that results from the ability to connect terminals of one system with terminals of another. This graphical approach motivates our language of interconnection: indeed, these diagrams will be the expressions of our language.

We claim that the existence of a network-style diagrammatic language to represent a system implies that interconnection is inherently important in understanding the system. Yet, while each of these example notions of system are well-studied in and of themselves, their compositional, or algebraic, structure has received scant attention. In this thesis, we study a type of monoidal category—first defined by Carboni and Walters [CW87, Car91] in the context of representation theory—called a hypergraph category, and argue that this is the relevant algebraic structure for modelling interconnection of open systems.

Given these pre-existing diagrammatic formalisms and our visual intuition, constructing algebras of syntactic representations is thus rather straightforward. The semantics and their algebraic structure are more subtle.

In some sense our semantics is already given to us too: in studying these systems as closed systems, scientists have already formalised the meaning of these diagrams. But we have shifted from a closed perspective to an open one, and we need our semantics to also account for points of interconnection.

Taking inspiration from Willems' behavioural approach [Wil07] and Deutsch's constructor theory [Deu0], in this thesis I advocate the following position. First, at each terminal of an open system we may make measurements appropriate to the type of terminal. Given a collection of terminals, the *universum* is then the set of all possible measurement outcomes. Each open system has a collection of terminals, and hence a universum. The semantics of an open system is the subset of measurement outcomes on the terminals that are permitted by the system. This is known as the *behaviour* of the system.

For example, consider a resistor of resistance r. This has two terminals—the two ends of the resistor—and at each terminal, we may measure the potential and the current. Thus the universum of this system is the set $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, where the summands represent respectively the potentials and currents at each of the two terminals. The resistor is governed by Kirchhoff's current law, or conservation of charge, and Ohm's law. Conservation of charge states that the current flowing into one terminal must equal the current flowing out of the other terminal, while Ohm's law states that this current will be proportional to the potential difference, with constant of proportionality $1/r$. Thus the behaviour of the resistor is the set

$$\big\{\big(\phi_1, \phi_2, -\tfrac{1}{r}(\phi_2 - \phi_1), \tfrac{1}{r}(\phi_2 - \phi_1)\big) \,\big|\, \phi_1, \phi_2 \in \mathbb{R}\big\}.$$

Note that in this perspective a law such as Ohm's law is a mechanism for partitioning *behaviours* into possible and impossible behaviours.[6]

Interconnection of terminals then asserts the identification of the variables at the identified terminals. Fixing some notion of open system and subsequently an algebra of syntactic representations for these systems, our approach, based on the principle of compositionality, requires this to define an algebra of semantic objects and a homomorphism from syntax to semantics. The first part of this thesis develops the mathematical tools necessary to pursue this vision for modelling open systems and their interconnection.

The next goal is to demonstrate the efficacy of this philosophy in applications. At core, this work is done in the faith that the right language allows deeper insight into the underlying structure. Indeed, after setting up such a language for open systems there are many questions to be asked: Can we find a sound and complete logic for determining when two syntactic expressions have the same semantics? Suppose

[6] That is, given a universum $\mathcal{U}$ of trajectories, a behaviour of a system is an element of the power set $\mathcal{P}(\mathcal{U})$ representing all possible measurements of this system, and a law or principle is an element of $\mathcal{P}(\mathcal{P}(\mathcal{U}))$ representing all possible behaviours of a class of systems.

we have systems have some property, like being controllable. In what ways can we interconnect controllable systems so that the combined system is also controllable? Can we compute the semantics of a large system quicker by computing the semantics of subsystems and then composing them? If I want a given system to acheive a specified trajectory, can I interconnect another system to make it do so? How do two notions of system relate to each other: can I find a functor between their algebraic representations? In the second part of this thesis we explore some applications in depth, providing answers to questions of the above sort.

The work here is underpinned not just by philosophical precedent, but by a rich tradition in mathematics, physics, and computer science. First and foremost, it draws from category theory.

in the work of Joyal and Street in the 1980s [JS91, JS93], it became clear that these developments were profoundly linked: monoidal categories have a precise graphical representation in terms of string diagrams, and conversely monoidal categories provide an algebraic foundation for the intuitions behind Feynman diagrams. The key insight is the use of categories where morphisms describe physical processes, rather than structure-preserving maps between mathematical objects [BS11, CP11]

compositional accounts of semantics associated to topological diagrams has long been a technique associated with topological quantum field theory, dating back to [Ati88].

These techniques have filtered into more immediate applications, particularly in computation and quantum computation [AC04, Bae06, Sel07].

Category-theoretic frameworks for general systems theory have been developed before. Notably Goguen and Rosen led efforts. Goguen from a more computer science perspective, Rosen more in biology.

Accessible, Kindergarten quantum mechanics.

Work on categorical approaches to control systems goes back at least to Goguen [Gog75] and Arbib and Manes [AM80].

Cospans are already familiar as a formalism for making entities with an arbitrarily designated 'input end' and 'output end' into the morphisms of a category. For example, in topological quantum field theory we use special cospans called 'cobordisms' to describe pieces of spacetime [BL11, BS11].

An advantage of the decorated cospan framework is that the resulting categories are hypergraph categories, and the resulting functors respect this structure. As dagger compact categories, hypergraph categories themselves have a rich diagrammatic nature [Sel11], and in cases when our decorated cospan categories are inspired by diagrammatic applications, the hypergraph structure provides language to describe natural operations on our diagrams, such as juxtaposing, rotating, and reflecting them.

At the present time,

it has most recently had significant influence in the nascent field of categorical network theory, with application to automata and computation [KSW00, Spi13], electrical circuits [BF], signal flow diagrams [BSZ14, BE15], Markov processes [BFP16, ASW11], and dynamical systems [VSL15], among others.

applying string diagrams to engineering, with the aim of giving diverse diagram languages a unified foundation based on category theory [KSW97, RSW05].

Baez and Erbele [BE15], Vagner, Spivak, and Lerman [VSL15], as well as Bonchi, Sobocínski, and Zanasi [BSZ14, BSZ16, BSZ15, Zan15].

Spivak operad approach

Most similar is the work of Walters, together with Sabadini and Rosebrugh. In particular, automata.

cospans and colimits

Outline

This thesis is divided into two parts. Part I, comprising Chapters 1 to 4, focusses on mathematical foundations. In it we develop the theory of hypergraph categories and a powerful tool for defining and manipulating them: decorated corelations. Part II, comprising Chapters 5 to 7, then discusses applications of this theory to examples of open systems.

Chapter 1 introduces hypergraph categories. These are symmetric monoidal categories in which every object is equipped with the structure of a special commutative Frobenius monoid in a way compatible with the monoidal product. As we will rely heavily on properties of monoidal categories, their functors, and their graphical calculus, we begin with a whirlwind review of these ideas. We then provide a definition of hypergraph categories, a strictification theorem, and an important example: the category of cospans in a category with finite colimits.

Cospans and their composition, using pushouts,

Chapter 2

Chapter 3

Chapter 4

The central message of decorated corelations is that hypergraph structure requires some sort of uniformity in the composition rule, and it is easier to work by acknowledging this structure, defining them as algebras over some theory.

Hypergraph categories are really just Set-valued lax symmetric monoidal functors: and these, being simply data structures, are often simpler to work with.

Then applications. Chapter 5 First to control theory from this behavioural perspective, demonstrating the ideas of black boxing and corelations, open systems and control by interconnection. We give a new characterisation of controllability. Second to passive linear networks, which speaks to the idea that we can find isomorphisms across disciplines, and explore the insights in new ways.

Chapter 6

Chapter 7

Statement of work

The Examination Schools make the following request:

> *Where some part of the thesis is not solely the work of the candidate or has been carried out in collaboration with one or more persons, the candidate shall submit a clear statement of the extent of his or her own contribution.*

I address this now.

The first four chapters are my own work. The applications chapters were developed with collaborators.

Chapter 5 arises from a weekly seminar with Paolo Rapisarda and Paweł Sobocínski at Southampton in the Spring of 2015. The text is a minor adaptation of that in the paper [FRS16]. For that paper I developed the corelation formalism, providing a first draft. Paweł provided much expertise in signal flow graphs, significantly revising the text and contributing the section on operational semantics. Paolo contributed comparisons to classical methods in control theory. A number of anonymous referees contributed helpful and detailed comments.

Chapter 6 is joint work with John Baez; the majority of the text is taken from our paper [BF]. For that paper John supplied writing on Dirichlet forms and the principle of minimum power that became the second section of Chapter 6, as well as parts of the next two sections. I produced a first draft of the rest of the paper. We collaboratively revised the text for publication.

Part I

Mathematical Foundations

Part II

Applications