The Algebra of Open and Interconnected Systems

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A thesis submitted for the degree of

Doctor of Philosophy in Computer Science

Trinity 2016

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Corelations: a tool for black boxing

Black boxing!

USE MATERIAL FROM INTRODUCTION OF CORELATIONS PAPER AND CORELATIONS SECTION IN BLACK BOX PAPER

3.1 Introduction

Consider a circuit diagram.

We often view such diagrams atomically, representing the complete physical system built as specified. Yet the very process of building such a system involves assembling it from its parts, each of which we might diagram in the same way. The goal of this paper is to develop formal category-theoretic tools for describing and interpreting this process of assembly.

Circuits are constructed to get at some behaviour, with two circuits equivalent if they specify the same behaviour. Here we focus on this 'black boxing' aspect.

Corelations [**?**].

Semantics live on boundaries only.

We then introduce a new framework for working with hypergraph categories: decorated corelations.

Decorated corelations adds compositional operations to network-diagram representations, and handles composition of semantics too.

Two main theorems:

Theorem 3.1. *Given a category $\mathcal{C}$ with finite colimits, factorisation system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ is stable under pushouts, and a symmetric lax monoidal functor*

$$\mathcal{C}; \mathcal{M}^{\mathrm{opp}} \longrightarrow \mathrm{Set},$$

we may define a hypergraph category of with morphisms decorated corelations.

Theorem 3.2. *Every hypergraph category can be constructed in this way.*

They apply to functors too.

3.2 Corelations

Given sets X, Y, a relation $X \rightsquigarrow Y$ is a subset of the product $X \times Y$. More abstractly, we might say a relation is a jointly-monic span in the category of sets (or an isomorphism class thereof). We generalise the dual concept.

3.2.1 Factorisation systems.

The relevant properties of jointly-monic spans come from the fact that monomorphisms form one half of a factorisation system. A factorisation system allows any morphism in a category to be factored into the composite of two morphisms in a coherent way.

Definition 3.3. A **factorisation system** $(\mathcal{E}, \mathcal{M})$ in a category $\mathcal{C}$ comprises subcategories $\mathcal{E}$, $\mathcal{M}$ of $\mathcal{C}$ such that

(i) $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms of $\mathcal{C}$.

(ii) every morphism $f \in \mathcal{C}$ admits a factorisation $f = m \circ e$, $e \in \mathcal{E}$, $m \in \mathcal{M}$.

(iii) given morphisms f, f', with factorisations $f = m \circ e$, $f' = m' \circ e'$ of the above sort, for every u, v such that the square

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "f'"'] & \cdot \end{tikzcd}$$

commutes, there exists a unique morphism s such that

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[r, "m"] \arrow[d, dotted, "\exists! s"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot \end{tikzcd}$$

commutes.

Examples 3.4. We introduce some factorisation systems of central importance in what follows.

- Write $\mathcal{I}_{\mathcal{C}}$ for the wide subcategory of $\mathcal{C}$ containing exactly the isomorphisms of $\mathcal{C}$. Then $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ are both factorisation systems in $\mathcal{C}$.

- The prototypical example of a factorisation system is the epi-mono factorisation system in Set. This follows from a more general fact, true in any category, that if every arrow can be factorised as an epi followed by a split mono, then this results in a factorisation system. The only non-trivial part to check is the uniqueness condition: given epis e_1, e_2, split monos m_1, m_2, and commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "e_1"] \arrow[d, "u"'] & \cdot \arrow[r, "m_1"] \arrow[d, dotted, "\exists! t"] & \cdot \arrow[d, "v"] \\ \cdot \arrow[r, "e_2"'] & \cdot \arrow[r, "m_2"'] & \cdot \end{tikzcd}$$

  we must show that there is a unique t that makes the diagram commute. Indeed let $t = m_2'vm_1$ where m_2' satisfies $m_2'm_2 = id$. To see that the right square commutes, observe

$$m_2te_1 = m_2m_2'vm_1e_1 = m_2m_2'm_2e_2u = m_2e_2u = vm_1e_1$$

  and since e_1 is epi we have $m_2t = vm_1$. For the left square,

$$te_1 = m_2'vm_1e_1 = m_2'm_2e_2u = e_2u.$$

  Uniqueness is immediate, since, e_1 is epi and m_2 is mono.

See [**?**, §14] for more details.

Definition 3.5. Call a factorisation system $(\mathcal{E}, \mathcal{M})$ in a monoidal category $(\mathcal{C}, \otimes)$ a **monoidal factorisation system** if $(\mathcal{E}, \otimes)$ is a monoidal category.

One might wonder why $\mathcal{M}$ does not appear in the above definition. To give a touch more intuition for this definition, we quote a theorem of Ambler. Recall a symmetric monoidal closed category is one in which each functor $-\otimes X$ has a specified right adjoint $[X,-]$. See Ambler for proof and further details [**?**, Lemma 5.2.2].

Proposition 3.6. *Let $(\mathcal{E},\mathcal{M})$ be a factorisation system in a symmetric monoidal closed category $(\mathcal{C},\otimes)$. Then the following are equivalent:*

(i) $(\mathcal{E},\mathcal{M})$ *is a monoidal factorisation system.*

(ii) $\mathcal{E}$ *is closed under* $-\otimes X$ *for all* $X\in\mathcal{C}$.

(iii) $\mathcal{M}$ *is closed under* $[X,-]$ *for all* $X\in\mathcal{C}$.

In a category with finite coproducts every factorisation system is a monoidal factorisation system for the coproduct.

Lemma 3.7. *Let $\mathcal{C}$ be a category with finite coproducts, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Then $(\mathcal{E},+)$ is a symmetric monoidal category.*

Proof. The only thing to check is that $\mathcal{E}$ is closed under $+$. That is, given $f\colon A\to B$ and $g\colon C\to D$ in $\mathcal{E}$, we wish to show that $f+g\colon A+C\to B+D$, defined in $\mathcal{C}$, is also a morphism in $\mathcal{E}$.

Let $f+g$ have factorisation $A+C\xrightarrow{e}\widetilde{B+D}\xrightarrow{m}B+D$, where $e\in\mathcal{E}$ and $m\in\mathcal{M}$. We will prove that m is an isomorphism. To construct an inverse, recall that by definition, as f and g lie in $\mathcal{E}$, there exist morphisms $x\colon B\to\widetilde{B+D}$ and $y\colon D\to\widetilde{B+D}$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, equal] \arrow[d, dotted, "x"] & B \arrow[d] \\ A+C \arrow[r, "e"'] & \widetilde{B+D} \arrow[r, "m"'] & B+D \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} C \arrow[r, "g"] \arrow[d] & D \arrow[r, equal] \arrow[d, dotted, "y"] & D \arrow[d] \\ A+C \arrow[r, "e"'] & \widetilde{B+D} \arrow[r, "m"'] & B+D \end{tikzcd} \tag{1}$$

The copairing $[x,y]$ is an inverse to m.

Indeed, taking the coproduct of the top rows of the two diagrams above and the copairings of the vertical maps gives the commutative diagram

$$\begin{tikzcd} A+C \arrow[r, "f+g"] \arrow[d, equal] & B+D \arrow[r, equal] \arrow[d, "{[x,y]}"'] & B+D \arrow[d, equal] \\ A+C \arrow[r, "e"] & \widetilde{B+D} \arrow[r, "m"] & B+D \end{tikzcd}$$

Reading the right-hand square immediately gives $m\circ[x,y]=1$.

Conversely, to see that $[x,y] \circ m = 1$, remember that by definition $f+g = m \circ e$. So the left-hand square above implies that

$$\begin{tikzcd} A+C \arrow[r, "e"] \arrow[d, equal] & \widetilde{B+D} \arrow[d, "{[x,y]\circ m}"] \\ A+C \arrow[r, "e"'] & \widetilde{B+D} \end{tikzcd}$$

commutes. But by the universal property of factorisation systems, there is a unique map $\widetilde{B+D} \to \widetilde{B+D}$ such that this diagram commutes, and clearly the identity map also suffices. Thus $[x,y] \circ m = 1$. □

3.2.2 Corelations

Relations, then, may be generalised as spans such that the span maps 'jointly' belong to some class $\mathcal{M}$ of an $(\mathcal{E},\mathcal{M})$-factorisation system. We define corelations in the dual manner.

Definition 3.8. Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. An **$(\mathcal{E},\mathcal{M})$-corelation** $X \rightsquigarrow Y$ is a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ such that the copairing $[i,o]\colon X+Y \to N$ lies in $\mathcal{E}$.

When the factorisation system is clear from context, we simply call $(\mathcal{E},\mathcal{M})$-corelations: 'corelations'.

We also say that a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ with the property that the copairing $[i,o]\colon X+Y \to N$ lies in $\mathcal{E}$ is **jointly $\mathcal{E}$-like**. Note that if a cospan is jointly $\mathcal{E}$-like then so are all isomorphic cospans. Thus the property of being a corelation is closed under isomorphism of cospans, and we again are often lazy with out language, referring to both jointly $\mathcal{E}$-like cospans and their isomorphism classes as corelations.

If $f\colon A \to N$ is a morphism with factorisation $f = m \circ e$, write $\widetilde{N}$ for the object such that $e\colon A \to \widetilde{N}$ and $m\colon \widetilde{N} \to N$. Now, given a cospan $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$, we may use the factorisation system to write the copairing $[i_X, o_Y]\colon X+Y \to N$ as

$$X+Y \xrightarrow{e} \widetilde{N} \xrightarrow{m} N.$$

From the universal property of the coproduct, we also have maps $\iota_X\colon X \to X+Y$ and $\iota_Y\colon Y \to X+Y$. We then call the corelation

$$X \xrightarrow{e\circ\iota_X} \widetilde{N} \xleftarrow{e\circ\iota_Y} Y$$

the **$\mathcal{E}$-part** of the above cospan. On occasion we will also write $e\colon X+Y \to \widetilde{N}$ for the same corelation.

Examples 3.9. Many examples of corelations are already familiar.

- For the morphism-isomorphism factorisation system $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, corelations are just cospans.

- For the isomorphism-morphism factorisation $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$, jointly $\mathcal{I}_{\mathcal{C}}$-like cospans $X \to Y$ are simply isomorphisms $X + Y \xrightarrow{\sim} N$. Thus there is a unique isomorphism class of corelation between any two objects.

- Note that the category Set has finite colimits and an epi-mono factorisation system. Epi-mono corelations from $X \to Y$ in Set surjective functions $X+Y \to N$; thus their isomorphism classes are partitions, or equivalence relations on $X + Y$.

3.2.3 Categories of corelations

We compose corelations by taking the $\mathcal{E}$-part of their composite cospan. That is, given corelations $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite is given by the cospan $X \to \widetilde{N +_Y} M \leftarrow Z$ in

$$\begin{tikzcd}
 & & N +_Y M & & \\
 & & \widetilde{N +_Y M} \arrow[u, "m"] & & \\
 & N \arrow[ur, "e\circ\iota_N"] & & M \arrow[ul, "e\circ\iota_M"'] & \\
X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z, \arrow[ul, "o_Z"']
\end{tikzcd}$$

where $m \circ e$ is the $(\mathcal{E}, \mathcal{M})$-factorisation of $[j_N, j_M]\colon N + M \to N +_Y M$.

It is well-known that this composite is unique up to isomorphism, and that when $\mathcal{M}$ is well-behaved it defines a category with morphisms isomorphism classes of corelations. For example, bicategorical version of the dual theorem, for spans and relations, can be found in [?]. Nonetheless, for the sake of completeness we sketch our own argument here.

Proposition 3.10. *Let $\mathcal{C}$ be a category with finite colimits and with a factorisation system $(\mathcal{E}, \mathcal{M})$. Then the above is a well-defined composition rule on isomorphism classes of corelations.*

Proof. Let $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y)$ and $(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ Y \xrightarrow{i'_Y} M' \xleftarrow{o'_Z} Z)$ be pairs of isomorphic jointly $\mathcal{E}$-like cospans. Their composites *as cospans* are isomorphic, so the factorisation system gives an isomorphism s such that the diagram

$$\begin{tikzcd}
X+Z \arrow[r, "e"] \arrow[d, equal] & \widetilde{N+_Y M} \arrow[r, "m"] \arrow[d, dotted, "\sim"', "s"] & N+_Y M \arrow[d, "\sim"] \\
X+Z \arrow[r, "e'"'] & \widetilde{N'+_Y M'} \arrow[r, "m'"'] & N'+_Y M'
\end{tikzcd}$$

commutes. This s is an isomorphism of the composite corelations. □

Composition of corelations is *not* associative in general. It is, however, associative when $\mathcal{M}$ is **stable under pushout**: that is, whenever

$$\begin{tikzcd}
\cdot \arrow[r, "j"] & \cdot \\
\cdot \arrow[u] \arrow[r, "m"] & \cdot \arrow[u]
\end{tikzcd}$$

is a pushout square such that $m \in \mathcal{M}$, we also have that $j \in \mathcal{M}$. For our proof sketch we rely on the following useful lemma.

Lemma 3.11. *Let $\mathcal{C}$ be a category with finite colimits, and let $\mathcal{M}$ be a subcategory of $\mathcal{C}$ stable under pushouts and containing all isomorphisms. Then $(\mathcal{M}, +)$ is a symmetric monoidal category.*

Proof. It is enough to show that for all morphisms $m, m' \in \mathcal{M}$ we have $m + m'$ in $\mathcal{M}$. Since $\mathcal{M}$ contains all isomorphisms, the coherence maps are inherited from $\mathcal{C}$. The required axioms—the functoriality of the tensor product, the naturality of the coherence maps, and the coherence laws—are also inherited as they hold in $\mathcal{C}$.

To see $m + m'$ is in $\mathcal{M}$, simply observe that we have the pushout square

$$\begin{tikzcd}
A+C \arrow[r, "m+1"] & B+C \\
A \arrow[u, "\iota"] \arrow[r, "m"] & B \arrow[u, "\iota"']
\end{tikzcd}$$

in $\mathcal{C}$. As $\mathcal{M}$ is stable under pushout, $m + 1 \in \mathcal{M}$. Similarly, $1 + m' \in \mathcal{M}$. Thus their composite $m + m'$ lies in $\mathcal{M}$, as required. □

Proposition 3.12. *Let $\mathcal{C}$ be a category with finite colimits and a factorisation system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{M}$ is stable under pushouts. Then composition of corelations is associative.*

Proof (sketch). The key concern is whether factorisation 'commutes' with taking pushouts. Using the above lemma and the fact that $\mathcal{M}$ is stable under pushouts, the pushout square

$$\begin{tikzcd} N +_Y \widetilde{M} \arrow[r, "m'"] & N +_Y M \\ N + \widetilde{M} \arrow[u] \arrow[r, "1+m"] & N + M \arrow[u] \end{tikzcd}$$

shows that the map $m' \colon N +_Y \widetilde{M} \to N +_Y M$ is in $\mathcal{M}$ whenever $m \colon \widetilde{N} \to N$ is. It can then be shown that the composite of any number of corelations can be given by taking the composite of them all as cospans, and then taking the jointly $\mathcal{E}$-like part. □

The identity axioms for corelations are self-evident. We thus have a category $\mathrm{Corel}(\mathcal{C})$.

Remark 3.13. An aside: proving associativity is not logically necessary in the development of this chapter. Associativity will follow from the existence of the functor from $\mathrm{Cospan}(\mathcal{C})$ to $\mathrm{Corel}(\mathcal{C})$, like the other necessary coherence results for hypergraph categories. Nonetheless, it is terminologically easier to establish that $\mathrm{Corel}(\mathcal{C})$ is indeed a category at this point, before we define hypergraph structure and functors on it.

3.2.4 Hypergraph structure

For modelling networks, we require not just a category, but a hypergraph category. Corelation categories come equipped with this extra structure.

Recall that we gave decorated cospan categories a hypergraph structure by defining a wide embedding $\mathrm{Cospan}(\mathcal{C}) \hookrightarrow F\mathrm{Cospan}$, via which $F\mathrm{Cospan}$ inherited the coherence and Frobenius maps (Theorem 2.4). We will argue similarly here, after showing that the 'quotient' map

$$\mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

taking each cospan to its jointly $\mathcal{E}$-like part is functorial. Indeed, we define the coherence and Frobenius maps of $\mathrm{Corel}(\mathcal{C})$ to be their image under this map. For the coproduct we again use the coproduct in $\mathcal{C}$ as monoidal product.

Theorem 3.14. *Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be factorisation system on $\mathcal{C}$ such that $\mathcal{M}$ is stable under pushout. Then there exists a hypergraph category* $\mathrm{Corel}_{(\mathcal{E},\mathcal{M})}(\mathcal{C})$ *with*

The hypergraph category $(\mathrm{Corel}_{(\mathcal{E},\mathcal{M})}(\mathcal{C}), +)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of* $(\mathcal{E}, \mathcal{M})$*-corelations in* $\mathcal{C}$
composition	*given by the* $\mathcal{E}$*-part of pushout*
monoidal product	*the coproduct in* $\mathcal{C}$
coherence maps	*inherited from* $\mathrm{Cospan}(\mathcal{C})$
hypergraph maps	*inherited from* $\mathrm{Cospan}(\mathcal{C})$

Again, we will drop explicit reference to the factorisation system when context allows, simply writing $\mathrm{Corel}(\mathcal{C})$.

Our proof of this theorem will have two parts. Here we check that the coproduct $\mathrm{Corel}(\mathcal{C}) + \mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ is functorial, so that we have specified the required data for a hypergraph category. In the next section we will check the axioms of a hypergraph category are satisfied.

Lemma 3.15. *The map* $(-) + (-)\colon \mathrm{Corel}(\mathcal{C}) \times \mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ *induced by the coproduct in* $\mathcal{C}$ *is functorial.*

Proof. As a preliminary, note that $+$ is a well-defined function: Lemma 3.7 implies that the coproduct of two corelations is again a corelation. Our task is to show that, given the four corelations

$$\begin{array}{cc} f = X \longrightarrow N \longleftarrow Y & \qquad g = Y \longrightarrow M \longleftarrow Z \\ h = X' \longrightarrow N' \longleftarrow Y' & \qquad k = Y' \longrightarrow M' \longleftarrow Z' \end{array}$$

we have $(g\circ f)+(k\circ h) = (g\circ k)+(f\circ h)$. The left and right side of this expression are respectively given by the first arrow in the upper and lower rows of the commutative diagram

$$\begin{tikzcd}
(X+Z)+(X'+Z') \arrow[r, "\mathcal{E}+\mathcal{E}"] \arrow[d, "\sim"] & (\widetilde{N+_Y M})+(\widetilde{N'+_{Y'} M'}) \arrow[r, "\mathcal{M}+\mathcal{M}"] \arrow[d, dotted] & (N+_Y M)+(N'+_{Y'} M') \arrow[d, "\sim"] \\
(X+X')+(Z+Z') \arrow[r, "\mathcal{E}"] & (N+N') \widetilde{+_{Y+Y'}} (M+M') \arrow[r, "\mathcal{M}"] & (N+N') +_{Y+Y'} (M+M').
\end{tikzcd}$$

The leftmost and rightmost vertical arrows are isomorphisms by properties of colimits. The upper row is an $(\mathcal{E}, \mathcal{M})$-factorisation as the first map is the coproduct of two maps in $\mathcal{E}$ and the second map is the coproduct of two maps in $\mathcal{M}$, both of which are monoidal with respect to the coproduct (Lemmas 3.7 and 3.11). The lower row is an $(\mathcal{E}, \mathcal{M})$-factorisation by definition. Thus, by the properties of factorisation systems, the dotted map s is an isomorphism, and hence $+$ is functorial. $\square$

To prove Theorem 3.14 it remains to show that our proposed data for $\mathrm{Corel}(\mathcal{C})$ obey the necessary axioms: the symmetric monoidal coherence laws, the special commutative Frobenius monoid laws. Our proof strategy will be a touch complicated. Again, recall that in Theorem 2.4 we proved $F\mathrm{Cospan}$ had hypergraph structure via a functor from $\mathrm{Cospan}(\mathcal{C})$. Instead of proving each axiom directly, we will again leverage the fact that we already know $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category, and show that $\mathrm{Corel}(\mathcal{C})$ is the image of $\mathrm{Cospan}(\mathcal{C})$ under a composition-preserving map that also respects (indeed, defines) the monoidal and hypergraph structure.

Note that $\mathcal{C}$ is trivially stable under pushout. By definition we have the equality of hypergraph categories $\mathrm{Corel}_{(\mathcal{C},\mathcal{I}_\mathcal{C})}(\mathcal{C}) = \mathrm{Cospan}(\mathcal{C})$. Thus the functor $\mathrm{Cospan}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ is a special case of functors between corelation categories. Taking advantage of this, we will discuss functors between corelation categories in general, before specialising to this case to prove that $\mathrm{Corel}(\mathcal{C})$ indeed is a well-defined hypergraph category.

This will be done in the next subsection. We conclude this subsection by returning to the example of a corelation category that we sketched in the introduction.

Example 3.16. In each factorisation system of Examples 3.9 the right factor $\mathcal{M}$ is stable under pushout. This gives the hypergraph categories of cospans in $\mathcal{C}$, the indiscrete category on the objects of $\mathcal{C}$, and equivalence relations between finite sets.

The last of these examples is perhaps the most instructive for black-boxing open systems.

An **extraspecial commutative Frobenius monoid** $(X, \mu, \eta, \delta, \epsilon)$ in a monoidal category $(\mathcal{C}, \otimes)$ is a special commutative Frobenius monoid that further obeys the extra law

 $=$

The extra law is a recent discovery, appearing first under this name in the work of Baez and Erbele [**?**], as the 'bone law' in [**?**, **?**], and as the 'irredundancy law' in [**?**].

Observe that each of these equations equate string diagrams that connect precisely the same elements of the domain and codomain. To wit, the associativity, coassociativity, and Frobenius laws show that the order in which we build a connected component through pairwise clustering is irrelevant, the special law shows that having multiple connections between points is irrelevant, and the extra law shows that 'extra' components not connected to the domain or codomain are irrelevant.

In fact the converse holds: two morphisms built from the generators of an extraspecial commutative Frobenius monoid are equal and if and only if their diagrams impose the same connectivity relations on the disjoint union of the domain and codomain. This is an extension of the spider theorem for special commutative Frobenius monoids.

3.2.5 Functors between corelation categories

We have seen that to construct a functor between cospan categories one may start with a colimit-preserving functor between the underlying categories. Corelation categories are similar, but we remove the part of each cospan that lies in $\mathcal{M}$. Hence for functors between corelation categories, we also require that the target category removes at least as much information as the source category.

Proposition 3.17. *Let $\mathcal{C}$, $\mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E},\mathcal{M})$, $(\mathcal{E}',\mathcal{M}')$, such that $\mathcal{M}$ and $\mathcal{M}'$ are stable under pushout. Further let $A\colon \mathcal{C} \to \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$.*

Then we may define a hypergraph functor $\square\colon \mathrm{Corel}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C}')$ sending each object X in $\mathrm{Corel}(\mathcal{C})$ to AX in $\mathrm{Corel}(\mathcal{C}')$ and each corelation

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$$

to the jointly-$\mathcal{E}'$-part

$$AX \xrightarrow{Ai_X} \widetilde{AN} \xleftarrow{Ao_Y} AY.$$

of the image cospan. The coherence maps are the isomorphisms $\kappa_{X,Y}\colon AX + AY \to A(X+Y)$ given as A preserves colimits.

Proof of Theorem 3.14 and Proposition 3.17. We caution that we still have not proved that $\mathrm{Corel}(\mathcal{C})$ is a category, let alone a hypergraph category. Thus to begin, all we can show is that the map $\square$ is composition-preserving, and then that $\square$ respects the monoidal and hypergraph structure. But this is enough to prove both results! Indeed, specialising to the case where A is the identity functor on $\mathcal{C}$ and $(\mathcal{E},\mathcal{M}) = (\mathcal{C},\mathcal{I}_{\mathcal{C}})$, observe

$$\square\colon \mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

is then the map taking each cospan to its jointly $\mathcal{E}$-like part. Note $\square$ maps fully (surjectively-on-morphisms) and bijectively-on-objects onto $\mathrm{Corel}(\mathcal{C})$, and by definition the coherence and hypergraph maps on $\mathrm{Corel}(\mathcal{C})$ are precisely the image of the corresponding maps of $\mathrm{Cospan}(\mathcal{C})$. As $\mathrm{Cospan}(\mathcal{C})$ is a hypergraph category and $\square$ is composition-preserving, we can consequently conclude that all corelation categories are indeed hypergraph categories, and hence that $\square$—in the general case—is a hypergraph functor.

First, that $\square$ preserves composition. Let

$$\alpha = (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y) \qquad \text{and} \qquad \beta = (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z)$$

be corelations in $\mathcal{C}$. By definition, the corelation $\square(\beta) \circ \square(\alpha)$ is given by the $\mathcal{E}$-part of the top row of the diagram below, while $\square(\beta \circ \alpha)$ is given by the $\mathcal{E}$-part of the bottom row:

$$\begin{tikzcd}
AX + AZ \arrow[r, "\mathcal{E}'"] \arrow[dd, equal] & \widetilde{\widetilde{AN} +_{AY} \widetilde{AM}} \arrow[rr, "\mathcal{M}'"] \arrow[dd, dotted, leftrightarrow, "n"] & & \widetilde{AN} +_{AY} \widetilde{AM} \arrow[d, "m'_{AN} +_{AY} m'_{AM}"] \\
& & & AN +_{AY} AM \arrow[d, leftrightarrow, "\sim"] \\
AX + AZ \arrow[r, "\mathcal{E}'"] & A(\widetilde{\widetilde{N +_Y M}}) \arrow[r, "\mathcal{M}'"] & A(\widetilde{N +_Y M}) \arrow[r, "Am_{N+_Y M}"] & A(N +_Y M)
\end{tikzcd} \tag{$*$}$$

In the above, the morphisms labelled $\mathcal{E}'$ lie in $\mathcal{E}'$, and similarly for $\mathcal{M}'$; these are given by the factorisation system on $\mathcal{C}'$. The remaining three maps in the lower right hand corner lie in $\mathcal{M}'$ too: $Am_{N+_Y M}$ as it is in the image of $\mathcal{M}$, $m'_{AN} +_{AY} m'_{AM}$ as $\mathcal{M}'$ is stable under pushout, and the morphism labelled $\sim$ as A preserves pushouts and $\mathcal{M}'$ contains all isomorphisms.

Moreover, the diagram commutes as both maps $AX+AZ \to AN+_{AY}AM$ compose to that given by the pushout of the images of α and β over AY. Thus the diagram represents two $(\mathcal{E}', \mathcal{M}')$ factorisations of the same morphism, and there exists an isomorphism n between the corelations $\square(\beta) \circ \square(\alpha)$ and $\square(\beta \circ \alpha)$. This proves that $\square$ preserves composition.

As A preserves colimits, it follows immediately that $\square$ respects the monoidal and hypergraph structure. The coherence maps are natural (WHY)

by the properties of coproducts, the symmetric monoidal coherence laws are each diagrams of isomorphisms of coproducts. and for example the Frobenius multiplication $[1_X, 1_X]$ (THIS IS NOT THE MULT) on an object X of $\mathcal{C}$ obeys $A[1_X, 1_X] \circ \kappa_{X,X} = [1_{AX}, 1_{AX}]$, as required. $\square$

Corollary 3.18. *There is a strict hypergraph functor*

$$\square \colon \mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

that takes each object of $\mathrm{Cospan}(\mathcal{C})$ *to itself as an object of* $\mathrm{Corel}(\mathcal{C})$ *and each cospan to its* $\mathcal{E}$*-part.*

Example 3.19. Note that if both $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ are epi-mono factorisations, then we always have that $F(\mathcal{E}) \subseteq \mathcal{E}'$ and $F(\mathcal{M}) \subseteq \mathcal{M}'$. Indeed, if an (one-sided) inverse exists in the domain category, it exists in the codomain category. Thus colimit-preserving functors between categories with finite colimits and epi-mono factorisation systems also induce a functor between the epi-mono corelation categories.

3.2.6 Example: linear relations as corelations in Vect

Recall that a linear relation $L\colon U \rightsquigarrow V$ is a subspace $L \subseteq U \oplus V$. We compose linear relations as we do relations, and vector spaces and linear relations form a category LinRel. This category can be constructed as the category of relations in the category Vect of vector spaces and linear maps with respect to epi-mono factorisations. We show that they may also be constructed as corelations in Vect with respect to epi-mono factorisations.

If we restrict to the full subcategory FinVect of finite dimensional vector spaces this is easy to see: after picking a basis for each vector space the transpose yields an equivalence of FinVect with its opposite category, so the category of $(\mathcal{E}, \mathcal{M})$-corelations (jointly-epic cospans) is isomorphic to the category of $(\mathcal{E}, \mathcal{M})$-relations (jointly-monic spans) in FinVect. This fact has been fundamental in work on finite dimensional linear systems and signal flow diagrams [**?**, **?**, **?**].

We prove the general case in detail. To begin, note Vect has an epi-mono factorisation system with monos stable under pushouts. This factorisation system is inherited from Set: the epimorphisms in Vect are precisely the surjective linear maps, the monomorphisms are the injective linear maps, and the image of a linear map is always a subspace of the codomain, and so itself a vector space. Monos are stable under pushout as the pushout of a diagram $V \xleftarrow{f} U \xrightarrow{m} W$ is $V \oplus W/\mathrm{Im}\,[f \ \ -g]$. The map $m'\colon V \to V \oplus W/\mathrm{Im}\,[f \ \ -g]$ into the pushout has kernel $f(\ker m)$. Thus when m is a monomorphism, m' is too.

Thus we have a category of corelations Corel(Vect). We show that the map Corel(Vect) $\to$ LinRel sending each vector space to itself and each corelation

$$U \xrightarrow{f} A \xleftarrow{g} V$$

to the linear subspace $\ker[f \ \ -g]$ is a full, faithful, and bijective-on-objects functor.

Indeed, corelations $U \xrightarrow{f} A \xleftarrow{g} V$ are in one-to-one correspondence with surjective linear maps $U \oplus V \to A$, which are in turn, by the isomorphism theorem, in one-to-one correspondence with subspaces of $U \oplus V$. These correspondences are described by the kernel construction above. Thus our map is evidently full, faithful, and bijective-on-objects. It also maps identities to identities. It remains to check that it preserves composition.

Suppose we have corelations $U \xrightarrow{f} A \xleftarrow{g} V$ and $V \xrightarrow{h} B \xleftarrow{k} W$. Then their pushout is given by $P = A \oplus B/\mathrm{Im}\,[g \ \ -h]$, and we may draw the pushout diagram

$$\begin{tikzcd}
U \arrow[dr, "f"'] & & V \arrow[dl, "g"] \arrow[dr, "h"'] & & W \arrow[dl, "k"] \\
& A \arrow[dr, "\iota_A"'] & & B \arrow[dl, "\iota_B"] & \\
& & P \arrow[u, phantom, "\wedge"] & &
\end{tikzcd}$$

We wish to show the equality of relations

$$\ker[f \ \ -g]; \ker[h \ \ -k] = \ker[\iota_A f \ \ -\iota_B g].$$

Now $(\mathbf{u}, \mathbf{w}) \in U \oplus W$ lies in the composite relation $\ker[f \ \ -g]; \ker[h \ \ -k]$ iff there exists $\mathbf{v} \in V$ such that $f\mathbf{u} = g\mathbf{v}$ and $h\mathbf{v} = k\mathbf{w}$. But as P is the pushout, this is true iff

$$\iota_A f\mathbf{u} = \iota_A g\mathbf{v} = \iota_B h\mathbf{v} = \iota_B k\mathbf{w}.$$

This in turn is true iff $(\mathbf{u}, \mathbf{w}) \in \ker[\iota_A f \ \ -\iota_B k]$, as required.

3.3 Decorated corelations

When enough structure is available to us, we may decorate corelations too. Furthermore, and key to the idea of 'black-boxing', we get a hypergraph functor from decorated cospans to decorated corelations.

3.3.1 Adjoining right adjoints

A subcategory stable under pushouts is a useful thing.

Proposition 3.20. *Let $\mathcal{C}$ be a category with finite colimits, and let $\mathcal{M}$ be a subcategory of $\mathcal{C}$ stable under pushouts. Then we define the category $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$ as follows*

The symmetric monoidal category $(\mathcal{C};\mathcal{M}^{\mathrm{opp}},+)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of cospans of the form* $\stackrel{c}{\rightarrow}\stackrel{m}{\leftarrow}$*, where* c *is a morphism in* $\mathcal{C}$ *and* m *a morphism in* $\mathcal{M}$
composition	*given by pushout*
monoidal product	*the coproduct in* $\mathcal{C}$
coherence maps	*the coherence maps in* $\mathcal{C}$

Proof. Composition is well-defined as $\mathcal{M}$ is stable under pushouts. Monoidal composition is well-defined by lemma. Necessary laws hold as they are inherited from $\mathcal{C}$. □

This category can be viewed as a bicategory, with 2-morphisms given by maps of cospans. In this bicategory every morphism of $\mathcal{M}$ has a right adjoint.

Examples 3.21.
- Note that $\mathcal{C};\mathcal{C}^{\mathrm{opp}}$ is by definition equal to Cospan($\mathcal{C}$).
- Writing $\mathcal{I}_{\mathcal{C}}$ for the wide subcategory of isomorphisms in $\mathcal{C}$, note that $\mathcal{C};\mathcal{I}_{\mathcal{C}}^{\mathrm{opp}}$ is naturally isomorphic to $\mathcal{C}$.

Lemma 3.22. *Let $\mathcal{C}$, $\mathcal{C}'$ be categories with finite colimits, and let $\mathcal{M}$, $\mathcal{M}'$ be subcategories each stable under pushouts. Let $A\colon \mathcal{C} \to \mathcal{C}'$ be functor that preserves colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$. Then A extends to a symmetric (strong) monoidal functor*

$$A\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \longrightarrow \mathcal{C}';\mathcal{M}'^{\mathrm{opp}}.$$

mapping X to AX and $\stackrel{c}{\rightarrow}\stackrel{m}{\leftarrow}$ to $\stackrel{Ac}{\rightarrow}\stackrel{Am}{\leftarrow}$.

Proof. Note $A(\mathcal{M}) \subseteq \mathcal{M}'$, so $\stackrel{Ac}{\rightarrow}\stackrel{Am}{\leftarrow}$ is indeed a morphism in $\mathcal{C}';\mathcal{M}'$. The functor A preserves colimits, so composition is preserved. □

Note this could be done more generally with any two isomorphism-containing wide subcategories stable under pushout.

3.3.2 Decorated corelations.

Definition 3.23. Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose that we also have a lax monoidal functor

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times).$$

Then we define an F-**decorated corelation** to be the isomorphism class of a pair

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd} \right)$$

where the cospan is jointly-$\mathcal{E}$-like.

Again, we will be lazy about the distinction between a decorated corelation and its isomorphism class.

Suppose we have decorated corelations

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd} \right) \quad \text{and} \quad \left(\begin{tikzcd} & M & \\ Y \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FM \\ 1 \arrow[u, "s"'] \end{tikzcd} \right).$$

Then their composite is given by the composite corelation

$$\begin{tikzcd} & \widetilde{N +_Y M} & \\ X \arrow[ur, "i"] & & Z \arrow[ul, "o"'] \end{tikzcd}$$

paired with the decoration

$$1 \longrightarrow F(N+M) \longrightarrow F(N +_Y M) \stackrel{F(m^{\mathrm{opp}})}{\longrightarrow} F(\widetilde{N +_Y M})$$

This is well-defined.

3.3.3 Categories of decorated corelations.

Theorem 3.24. *Let $\mathcal{C}$ be a category with finite colimits and factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{M}$ stable under pushout, and let*

$$F : (\mathcal{C}; \mathcal{M}^{\mathrm{opp}}, +) \longrightarrow (\mathrm{Set}, \times)$$

be a symmetric lax monoidal functor. Then we may define

The hypergraph category $(F\mathrm{Corel}, +)$	
objects	*the objects of* $\mathcal{C}$
morphisms	*isomorphism classes of F-decorated corelations in* $\mathcal{C}$
composition	*given by $\mathcal{E}$-part of pushout with restricted decoration*
monoidal product	*the coproduct in $\mathcal{C}$ on objects, coproduct of cospans and pair of decorations on morphisms.*
coherence maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*
hypergraph maps	*maps from* $\mathrm{Cospan}(\mathcal{C})$ *with restricted empty decoration*

Similar to the corelations theorem (Theorem 3.14), we prove this alongside the theorem in the next subsection.

Example 3.25. Note that decorated cospans are a special case of decorated corelations: we use an morphism–isomorphism factorisation system.

Example 3.26. Note that 'undecorated' corelations are a special case of decorated corelations: they are corelations decorated by the functor $1\colon \mathcal{C};\mathcal{M}^{\mathrm{opp}} \to \mathrm{Set}$ that maps each object to the one element set 1, and each morphism to the identity function on 1. This is a symmetric monoidal functor with the coherence maps all also the identity function on 1.

The key difference is to decorate cospans we need to know how to push decorations up. To decorate corelations we all need to know how to pull decorations back down. This is related to the existence of an extraspecial commutative Frobenius monoid in our main applications.

Associativity: To take a decoration on $A+B$ to one on $A+_C \widetilde{B}$ we may either reduce to the $\mathcal{E}$-part of B and then pushout over C, or pushout over C and then reduce to the $\mathcal{E}$ part of B. This lemma implies that both processes result in the same decoration.

3.3.4 Functors between decorated corelation categories

Proposition 3.27. *Let $\mathcal{C}$, $\mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E},\mathcal{M})$, $(\mathcal{E}',\mathcal{M}')$, such that $\mathcal{M}$ and $\mathcal{M}'$ are stable under pushout, and suppose that we have symmetric lax monoidal functors*

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times)$$

and

$$G : (\mathcal{C}';\mathcal{M}'^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times).$$

Further let $A\colon \mathcal{C}\to\mathcal{C}'$ be a functor that preserves finite colimits and such that the image of $\mathcal{M}$ lies in $\mathcal{M}'$. This functor A extends to a symmetric monoidal functor $\mathcal{C};\mathcal{M}^{\mathrm{opp}} \to \mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$.

Suppose we have a monoidal natural transformation θ:

$$\begin{tikzcd} \mathcal{C};\mathcal{M}^{\mathrm{opp}} \arrow[dr, "F"] \arrow[d, "A"'] & \\ \mathcal{C}';\mathcal{M}'^{\mathrm{opp}} \arrow[r, "G"'] \arrow[ur, phantom, "\Downarrow\theta"] & \mathrm{Set} \end{tikzcd}$$

Then we may define a hypergraph functor $T\colon F\mathrm{Corel} \to G\mathrm{Corel}$ *sending each object* $X \in F\mathrm{Corel}$ *to* $AX \in G\mathrm{Corel}$ *and each decorated corelation*

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN$$

to

$$AX \xrightarrow{Ai_X} \widetilde{AN} \xleftarrow{Ao_Y} AY, \quad 1 \xrightarrow{s} FN \xrightarrow{\theta_N} GAN \xrightarrow{Gm^{\mathrm{opp}}_{AN}} G\widetilde{AN}.$$

The coherence maps are given by

$$\kappa_{X,Y} = \left(\begin{tikzcd} & \widetilde{A(X+Y)} & \\ AX+AY \arrow[ur] & & A(X+Y) \arrow[ul] \end{tikzcd} \quad , \quad \begin{tikzcd} G(\widetilde{A(X+Y)}) \\ GA(X+Y) \arrow[u, "Gm^{\mathrm{opp}}_{AX+AY}"'] \\ G\varnothing \arrow[u, "G!"'] \\ 1 \arrow[u, "\gamma_1"'] \end{tikzcd} \right).$$

Proof. In the proof of Proposition 3.17 we used the existence of a bijective-on-objects, surjective-on-morphisms, composition preserving map $\mathrm{Cospan}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ to prove the associativity and other properties of $\mathrm{Corel}(\mathcal{C})$. Our proof strategy here is entirely analogous.

Similar to before, we still have not proved that decorated corelations form well-defined hypergraph categories. So we begin by merely showing that the map $\square$ is composition-preserving, and then that $\square$ respects the monoidal and hypergraph structure. We then specialise to the case where the domain forms a decorated cospan category that maps surjectively onto a generic decorated corelations codomain. Since we know decorated cospan categories are well-defined hypergraph categories, we can conclude the same for decorated corelations categories, and hence prove that $\square$ is a hypergraph functor.

We prove that $\square$ preserves composition. Suppose we have decorated corelations

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN \qquad \text{and} \qquad Y \xrightarrow{i_Y} M \xleftarrow{o_Y} Z, \quad 1 \xrightarrow{t} FM$$

We know the functor $\square$ preserves composition on the cospan part; this is precisely the content of Proposition 3.17. It remains to check that $\square(g \circ f)$ and $\square g \circ \square f$ have isomorphic decorations. This is expressed by the commutativity of the following diagram:

$$
\begin{tikzcd}
GA(\widetilde{\widetilde{N +_Y M}}) \arrow[rrrr, "Gn"] & & & & G(\widetilde{\widetilde{AN} +_{AY} \widetilde{AM}}) \\
GA(\widetilde{N +_Y M}) \arrow[u, "Gm^{\mathrm{opp}}_{A(\widetilde{N+_Y M})}"] & & (**) & & G(\widetilde{AN} +_{AY} \widetilde{AM}) \arrow[u, "Gm^{\mathrm{opp}}_{\widetilde{AN}+_{AY}\widetilde{AM}}"'] \\
F(\widetilde{N +_Y M}) \arrow[u, "\theta_{\widetilde{N+_Y M}}"] \arrow[dr, phantom, "(\textsc{tn})"] & GA(N +_Y M) \arrow[ul, "GAm^{\mathrm{opp}}_{N+_Y M}"'] \arrow[drr, phantom, "(\textsc{a})"] & & G(AN +_{AY} AM) \arrow[ll, "G\sim"'] \arrow[ur, "G(m^{\mathrm{opp}}_{AN}+_{AY}m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "(\textsc{c})"] & G(\widetilde{AN} + \widetilde{AM}) \arrow[u, "{G[j_{\widetilde{AN}},j_{\widetilde{AM}}]}"'] \\
F(N +_Y M) \arrow[u, "Fm^{\mathrm{opp}}_{N+_Y M}"] & GA(N + M) \arrow[u, "{GA[j_N,j_M]}"'] & & G(AN + AM) \arrow[ll, "G\alpha_{N,M}"] \arrow[u, "{G[j_{AN},j_{AM}]}"] \arrow[ur, "G(m^{\mathrm{opp}}_{AN}+m^{\mathrm{opp}}_{AM})"] \arrow[r, phantom, "(\textsc{gm})"] & G\widetilde{AN} \times G\widetilde{AM} \arrow[u, "\gamma_{\widetilde{AN},\widetilde{AM}}"'] \\
F(N + M) \arrow[u, "{F[j_N,j_M]}"] \arrow[ur, "\theta_{N+M}"'] & & (\textsc{tm}) & & GAN \times GAM \arrow[u, "Gm^{\mathrm{opp}}_{AN}\times Gm^{\mathrm{opp}}_{AM}"'] \arrow[ul, "\gamma_{AN,AM}"] \\
& & FN \times FM \arrow[ull, "\varphi_{N,M}"] \arrow[urr, "\theta_N\times\theta_M"'] & & \\
& & 1 \arrow[u, "\rho_1\circ(s\times t)"'] & &
\end{tikzcd}
$$

This diagram does indeed commute. To check this, first observe that (TM) commutes by the monoidality of θ, (GM) commutes by the monoidality of G, and (TN) commutes by the naturality of θ. The remaining three diagrams commute as they are G-images of diagrams that commute in $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$. Indeed, (A) commutes since A preserves colimits and G is functorial, (C) commutes as it is the G-image of a pushout square in $\mathcal{C}'$, so

$$\xleftarrow{m^{\mathrm{opp}}_{AN}+m^{\mathrm{opp}}_{AM}}\xrightarrow{[j_{\widetilde{AN}},j_{\widetilde{AM}}]} \quad \text{and} \quad \xrightarrow{[j_{AN},j_{AM}]}\xleftarrow{m^{\mathrm{opp}}_{AN}+_{AY}m^{\mathrm{opp}}_{AM}}$$

are equal as morphisms of $\mathcal{C}';\mathcal{M}'^{\mathrm{opp}}$, and $(**)$ commutes as it is the G-image of the right-hand subdiagram of $(*)$ used to define n.

It is evident that $\square$ is bijective-on-objects and surjective-on-morphisms. This proves the theorem. $\square$

Corollary 3.28. *Let $\mathcal{C}$ be a category with finite colimits, and let $(\mathcal{E},\mathcal{M})$ be a factorisation system on $\mathcal{C}$. Suppose that we also have a lax monoidal functor*

$$F : (\mathcal{C};\mathcal{M}^{\mathrm{opp}},+) \longrightarrow (\mathrm{Set},\times).$$

Then we may define a category $F\mathrm{Corel}$ *with objects the objects of* $\mathcal{C}$ *and morphisms isomorpism classes of* F*-decorated corelations.*

Write also F *for the restriction of* F *to the wide subcategory* $\mathcal{C}$ *of* $\mathcal{C};\mathcal{M}^{\mathrm{opp}}$*. We can thus also obtain the category* $F\mathrm{Cospan}$ *of* F*-decorated cospans. We moreover have a functor*

$$F\mathrm{Cospan} \to F\mathrm{Corel}$$

which takes each object of $F\mathrm{Cospan}$ *to itself as an object of* $F\mathrm{Corel}$*, and each decorated cospan*

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}, \quad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd}\right)$$

to its jointly-$\mathcal{E}$*-part*

$$\begin{tikzcd} & \widetilde{N} & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}$$

decorated by the composite

$$1 \xrightarrow{s} FN \xrightarrow{Fm_N^{\mathrm{opp}}} F\widetilde{N}.$$

3.4 All hypergraph categories are decorated corelation categories

structured categories are algebras over their graphical calculus operad [?]. These equivalences are 2-categorical.

Write $\int$ for the Grothendieck construction. The Grothendieck construction is

Theorem 3.29. *hypergraph categories are symmetric lax monoidal functors cospan to Set.*

$$\mathrm{HypCat} \cong \int^{\mathcal{O}\in\mathrm{Set}} \mathrm{SymLaxMon}(\mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}), \mathrm{Set})$$

[?]

Remark 3.30. Not all hypergraph categories are decorated *cospan* categories. To see this, we can count morphisms. The possible apices and decorations are the same for all morphisms. So for a decorated cospan category over the prop of finite sets, the number of morphisms $0 \to 1$ cannot be more than countably many times those $0 \to 0$ (we just get to choose an element of the apex). But the skeletal category of vector spaces over $\mathbb{R}$ with monoidal product the tensor product has $\mathbb{R}$ morphisms $0 \to 0$, and $\mathbb{R}^2$ morphisms $0 \to 1$.

Decorated corelation categories, however, are more powerful. We can recover all hypergraph categories by forcing the decorations to be on the coproduct of the domain and codomain itself. For this we use the isomorphism–morphism factorisation. Let $\mathcal{H}$ be a hypergraph category, and let $\mathcal{C}$ be the wide subcategory of all Frobenius morphisms. Then $\mathcal{H}$ can be recovered as the $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by global sections in $\mathcal{H}$.

3.4.1 The global sections construction.

Theorem 3.31. *All hypergraph categories are decorated corelation categories.*

Proof. Let $\mathcal{H}$ be a hypergraph category. Without loss of generality we can assume $\mathcal{H}$ has objects a free monoid under the tensor product; write $\mathcal{O}$ for a collection of generators for this free monoid, and $\mathrm{FinSet}_{\mathcal{O}}$ for $\mathcal{O}$ labelled finite sets (ie an object is a finite set X together with a function $X \to \mathcal{O}$). This is a finitely cocomplete category. Equivalent to finite lists of objects in $\mathcal{H}$.

Define the global sections functor

$$\begin{aligned} G\colon\ \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}}) &\longrightarrow \mathrm{Set} \\ A &\longmapsto \mathcal{H}(I, A) \\ \xrightarrow{f}\xleftarrow{g} &\longmapsto \text{action of Frobenius maps} \end{aligned}$$

This is symmetric lax monoidal functor. Note that $\mathcal{H}(I, A)$ depends on the order we choose to convert a multiset into an object A of $\mathcal{H}$. Nonetheless, from any two choices A, A' we get a canonical map $A \to A'$. This is really that clique in Set.

Consider the category $\mathrm{FinSet}_{\mathcal{O}}$ with an (isomorphism,morphism)-factorisation system. We get a decorated corelations category with objects multisets of generating objects of $\mathcal{H}$, and morphisms $A \to B$ trivial corelations $A \to A + B \leftarrow B$ decorated by some morphism $s \in \mathcal{H}(I, A + B)$. Recall that this decorated corelation is only specified up to isomorphism; in the following we always choose representatives such that the apex of the jointly-isomorphic cospan is always of the form $A + B$ for morphisms $A \to B$.

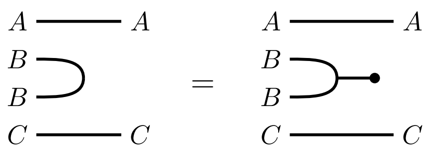
Given morphisms $s \in \mathcal{H}(I, A{+}B)$ and $t \in \mathcal{H}(I, B{+}C)$ of types $A \to B$ and $B \to C$ in $G\mathrm{Corel}$, composition is given by the map $H(I, A + B + B + C) \to H(I, A + C)$

arising as the G-image of the cospan $A+B+B+C \xrightarrow{[j,j]} A+B+C \xleftarrow{m} A+C$ where maps come from the pushout square

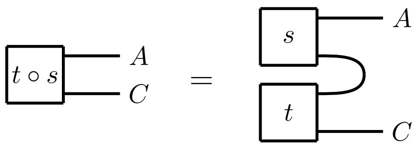
$$\begin{tikzcd}
 & & A+C \arrow[d] & & \\
 & & A+B+C & & \\
 & A+B \arrow[ur] & & B+C \arrow[ul] & \\
A \arrow[ur] & & B \arrow[ul] \arrow[ur] & & C \arrow[ul]
\end{tikzcd}$$

In terms of string diagrams in $\mathcal{H}$, this means composing the maps

with the Frobenius map

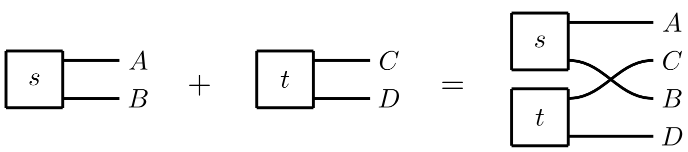


to get



in $\mathcal{H}(I, A+C)$.

The monoidal product is given by



recalling that we have chosen to represent the equivalence class of corelations $A+C \to B+D$ with the apex $A+C+B+D$.

Taking a hint from the compact closed structure, it is straightforward to construct a pair of inverse hypergraph functors between GCorel to $\mathcal{H}$. Indeed, GCorel is constructed to have the same collection of objects as $\mathcal{H}$; simply have the functors be the 'identity' on objects. On morphisms, we take $f : A \to B$ in $\mathcal{H}$ to its 'name' $\hat{f} : I \to A+B$ as a morphism of GCorel. This is a bijection.

To check it is composition and monoidal product preserving, we can easily use diagrammatic reasoning. For example

Therefore $\mathcal{H}$ is isomorphic as a hypergraph category to GCorel. □

Theorem 3.32. *All hypergraph functors are decorated corelation functors.*

Proof. Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraph categories, and $T\colon \mathcal{H} \to \mathcal{H}'$ be a hypergraph functor. By the above theorem, there exist symmetric lax monoidal functors

$$G\colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}}}) \to \mathrm{Set}$$

and

$$G'\colon \mathrm{Cospan}(\mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}'}}) \to \mathrm{Set}$$

such that $\mathcal{H} = G\mathrm{Corel}$ and $\mathcal{H}' = G'\mathrm{Corel}$. Furthermore, define a functor $A\colon \mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}}} \to \mathrm{FinSet}_{\mathcal{O}_{\mathcal{H}'}}$ taking $N \to \mathcal{O}_{\mathcal{H}}$ to $N \to \mathcal{O}_{\mathcal{H}} \to \mathcal{O}_{\mathcal{H}'}$, where the second map is that by the functor T on objects of $\mathcal{H}$. We claim this is a well-defined colimit-preserving functor and show that T can be constructed from a monoidal natural transformation between G and $G' \circ A$. □

Compare with Spivak Vagner construction.

3.4.2 Examples.

We give some examples reproducing hypergraph categories as decorated corelations categories.

Example 3.33. Example: empty decorations and equivalence relations.

Consider the hypergraph category Cospan(FinSet). This is the simplest hypergraph category: it is free hypergraph category on the one object discrete category. We show how to recover it as a decorated corelation category.

As per Example 3.26, Cospan(FinSet) is the hypergraph category of undecorated (morphism-isomorphism)-corelations in FinSet. It is also the partition-decorated (isomorphism-morphism)-corelations in FinSet.

First, the global sections functor $G\colon \mathrm{Cospan}(\mathrm{FinSet}) \to \mathrm{Set}$ takes each finite set X to the set of (equivalence classes of) cospans $0 \to D \leftarrow X$; that is, to the set of functions $X \to D$ where a unique D is chosen for each finite cardinality. Given a cospan $X \xrightarrow{f} N \xleftarrow{g} Y$, its image under the global sections functor maps a function $a\colon X \to D$, to the function $Y \to N +_Y D$ given by

$$\begin{tikzcd}
 & X \arrow[r, "d"] \arrow[d, "f"'] & D \arrow[d] \\
Y \arrow[r, "g"] & N \arrow[r] & N +_Y D
\end{tikzcd}$$

where the square is a pushout square.

The coherence maps $\gamma_1\colon 1 \to G\varnothing$ map the unique element of 1 to the unique function $!\colon \varnothing \to \varnothing$, and $\gamma_{X,Y}$ maps a pair of functions $a\colon X \to D$, $b\colon Y \to E$ to $a + b\colon X + Y \to D + E$. This is a symmetric lax monoidal functor. We use this functor to decorate cospans in FinSet.

A decorated cospan in FinSet with respect to this functor is a cospan $X \to N \leftarrow Y$ in FinSet together with a function $N \to D$ for some finite set. Using the isomorphism–morphism factorisation, a decorated corelation (a morphism in GCorel) is a cospan $X \to X + Y \leftarrow Y$, together with a function $X + Y \to D$. This is the same as a cospan.

The hypergraph structure is given by the decoration $X + X \to X$ etc, as the shift from cospans to corelations takes the 'factored out part' and puts it into the decoration. At this point the morphisms are specified entirely by their decoration.

It is straightforward to show the two categories are isomorphic. Note that the identity on FinSet maps $\mathcal{I}_{\mathrm{FinSet}}$ into FinSet, and so extends to morphism FinSet $\to$ Cospan(FinSet). We can define a monoidal natural transformation $1(X) = 1 \xrightarrow{\theta_X} GX = \{X \to D\}$ mapping the unique element to the identity function $1_X\colon X \to X$.

$$\begin{tikzcd}
\mathrm{FinSet} = \mathrm{FinSet}; \mathcal{I}_{\mathrm{FinSet}}^{\mathrm{opp}} \arrow[d, "\iota"'] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\Downarrow\theta"] & \\
\mathrm{Cospan}(\mathrm{FinSet}) = \mathrm{FinSet}; \mathrm{FinSet}^{\mathrm{opp}} \arrow[r, "G"'] & \mathrm{Set}
\end{tikzcd}$$

It is easy to verify that this is a monoidal natural transformation. This gives the hypergraph functor we expect, mapping the undecorated cospan $X \to N \leftarrow Y$ to $X \to X + Y \leftarrow Y$ decorated by $X + Y \to N$.

Like so many examples before, it is easy to verify this is a full, faithful, bijective-on-objects hypergraph functor.

Example 3.34. The previous example extends to any finitely cocomplete category $\mathcal{C}$: the hypergraph category Cospan($\mathcal{C}$) can always be constructed as (i) trivially decorated $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$-corelations, or (ii) $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms with domain the apex of the corelation, and moreover the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

More general still, a category of trivially decorated $(\mathcal{E}, \mathcal{M})$-corelations in $\mathcal{C}$ can always be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by equivalence classes of morphisms in $\mathcal{E}$ with domain the apex of the corelation, and the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

Most general, the theorem implies that any category of decorated corelations can be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$-corelations decorated by codomain decorated morphisms in $\mathcal{E}$. The latter are specified by a functor $\mathrm{Cospan}(\mathcal{C}) \to \mathrm{Set}$, as in the theorem.

The latter form is good for constructing functors that have image the corelation category.

Note that with decorated corelations we have a method of constructing hypergraph categories from other hypergraph categories. This is black boxing.

3.5 Examples

3.5.1 Path integrals and matrices

Let R be a commutative ring. Take functor

$$\begin{aligned} R^{(-)} : (\mathrm{FinSet}^{\mathrm{opp}}, +) &\longrightarrow (\mathrm{Set}, \times) \\ X &\longmapsto R^X \end{aligned}$$

Then $R^{(-)}\mathrm{Cospan}$ is path integrals, $R^{(-)}\mathrm{Corel}$ is matrices over R.

Many aspects of this example are ‘atypical’, regarding the intuition we have been working towards. Note that the monoidal product here is the tensor product of matrices, not the biproduct. Indeed, there is no special commutative Frobenius algebra in Vect if we use the biproduct, but if we use the tensor product then these correspond to orthonormal bases (Vicary). The comultiplication is the diagonal map, multiplication is codiagonal. unit produces basis.

We note that you could take decorations here in the category $R\mathrm{Mod}$ of R-modules. While Proposition 2.8 shows that the resulting decorated cospans category would be isomorphic, this hints at an enriched version of the theory.

3.5.2 Two constructions for linear relations

We saw earlier that linear relations are epi-mono corelations in Vect. The hypergraph structure is given by addition. We show how to recover this in another construction. We also get a hypergraph functor between them. This is very useful for compositional linear relations semantics of diagrams.

We can also construct linear relations in $\mathrm{Vect}^{\mathrm{opp}}$.

$$\begin{aligned} : \mathrm{Cospan}(\mathrm{FinSet}) &\longrightarrow \mathrm{Set} \\ X &\longmapsto \{\mathrm{subspaces of} k^X\} \\ f : X \to Y &\longmapsto L \mapsto \{v \mid v \circ f \in L\} \\ f^{\mathrm{opp}} : X \to Y &\longmapsto L \mapsto \{v = u \circ f \mid u \in L\} \end{aligned}$$

Then Cospan is cospans decorated by subspaces, and Corel is linear relations. This is important for circuits work [**?**, **?**].

3.5.3 Automata

This construction comes immediately from Walters et al. Automata are alphabet labelled graphs. There is a decorated cospan functor to categories enriched over languages, and this factors nicely to get a decorated corelation category with morphisms languages recognised between points in domain and codomain.

Part II

Applications

Bibliography

[Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page 19.)