The Algebra of Open and Interconnected Systems

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Part I

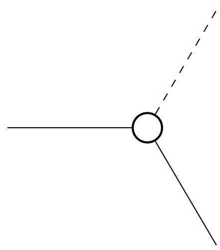
Mathematical Foundations

Chapter 1

Hypergraph categories: the algebra of interconnection

1.1 The algebra of interconnection

Our aim is to algebraicise network diagrams. A network diagram is built from pieces like so:



These represent open systems, concrete or abstract; for example a resistor, a chemical reaction, or a linear transformation. The essential feature, for openness and for networking, is that the system may have terminals, perhaps of different ‘types’, each one depicted by a line radiating from the central body. In the case of a resistor each terminal might represent a wire, for chemical reactions a chemical species, for linear transformations a variable in the domain or codomain. Network diagrams are formed by connecting terminals of systems to build larger systems.

A network-style diagrammatic language is a collection of network diagrams together with the stipulation that if we take some of these network diagrams, and connect terminals of the same type in any way we like, then we form another diagram in the collection. The point of this chapter is that hypergraph categories provide a precise formalisation of network-style diagrammatic languages.

In jargon, a hypergraph category is a symmetric monoidal category in which every object is equipped with a special commutative Frobenius monoid in a way compatible

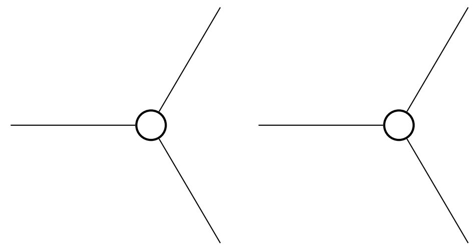


Figure 1.1: Interconnection of network diagrams. Note that we only connect terminals of the same type, but we can connect as many as we like.

with the monoidal product. We will walk through these terms in detail, illustrating them with examples and a few theorems.

The key data comprising a hypergraph category are its objects, morphisms, composition rule, monoidal product, and Frobenius maps. Each of these model a feature of network diagrams and their interconnection. The objects model the terminal types, while the morphisms model the network diagrams themselves. The composition, monoidal product, and Frobenius maps model different aspects of interconnection: composition models the interconnection of two terminals of the same type, the monoidal product models the network formed by taking two networks without interconnecting any terminals, while the Frobenius maps model multi-terminal interconnection.

These Frobenius maps are the distinguishing feature of hypergraph categories as compared to other structured monoidal categories, and are crucial for formalising the intuitive concept of network languages detailed above. In the case of electric circuits the Frobenius maps model the 'branching' of wires; in the case when diagrams simply model an abstract system of equations and terminals variables in these equations, the Frobenius maps allow variables to be shared between many systems of equations.

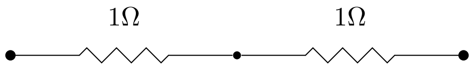
Examining these correspondences, it is worthwhile to ask whether hypergraph categories permit too much structure to be specified, given that the interconnection rule is now divided into three different aspects, and features such as domains and codomains of network diagrams, rather than just a collection of terminals, exist. The answer is given by examining the additional coherence laws that these data must obey. For example, in the case of the domain and codomain, we shall see that hypergraph categories are all compact closed categories, and so there is ultimately only a formal distinction between domain and codomain objects. One way to think of these data is as scaffolding. We could compare it to the use of matrices and bases to provide

Networks	Hypergraph categories
list of terminal types	object
network diagram	morphism
series connection	composition
juxtaposition	monoidal product
branching	Frobenius maps

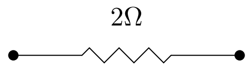
Figure 1.2: Corresponding features of networks and hypergraph categories.

language for talking about linear transformations and vector spaces. They are not part of the target structure, but nonetheless useful paraphenalia for constructing it.

Network languages are not only syntactic entities: as befitting the descriptor 'language', they typically have some associated semantics. Circuits diagrams, for instance, not only depict wire circuits that may be constructed, they also represent the electrical behaviour of that circuit. Such semantics considers the circuits



and



the same, even though as 'syntactic' diagrams they are distinct. A cornerstone of the utility of the hypergraph formalism is the ability to also realise the semantics of these diagrams as morphisms of another hypergraph category. This 'semantic' hypergraph category, as a hypergraph category, still permits the rich 'networking' interconnection structure, and a so-called hypergraph functor implies that the syntactic category provides a sound framework for depicting these morphisms. Network languages syntactically are often 'free' hypergraph categories, and much of the interesting structure lies in their functors to their semantic hypergraph categories.

In this chapter we introduce hypergraph categories, giving a definition, coherence theorem, and graphical language. We then explore a fundamental example of hypergraph categories: categories of cospans.

We assume basic familiarity with category theory and symmetric monoidal categories; although we give a sparse overview of the latter for reference. A proper introduction to both can be found in Mac Lane [?].

1.2 Symmetric monoidal categories

Hypergraph categories are first symmetric monoidal categories. Moreover, symmetric monoidal functors play a key role in our framework for defining and working with hypergraph categories: decorated cospans and corelations constructions. For this reason we provide, for reference, a definition of symmetric monoidal categories and their morphisms.

A symmetric monoidal category is a category with two notions of composition: ordinary categorical composition and monoidal composition, with the monoidal composition only associative and unital up to natural isomorphism. They are the algebra of processes that may occur simultaneously as well as sequentially. Relatedly, they have an associated graphical calculus, which strongly motivates their use here in the formalisation of network languages.

1.2.1 Monoidal categories

A **monoidal category** $(\mathcal{C}, \otimes)$ consists of a category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object I, and natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\rho_A : A \otimes I \to A$, and $\lambda_A : I \otimes A \to A$ such that for all A, B, C, D in $\mathcal{C}$ the following two diagrams commute:

$$\begin{tikzcd}
\big((A \otimes B) \otimes C\big) \otimes D \arrow[rr, "\alpha_{(A\otimes B),C,D}"] \arrow[d, "\alpha_{A,B,C} \otimes \mathrm{id}_D"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,(C\otimes D)}"] \\
\big(A \otimes (B \otimes C)\big) \otimes D \arrow[r, "\alpha_{A,(B\otimes C),D}"'] & A \otimes \big((B \otimes C) \otimes D\big) \arrow[r, "\mathrm{id}_A \otimes \alpha_{B,C,D}"'] & A \otimes \big(B \otimes (C \otimes D)\big)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

We call $\otimes$ the **monoidal product**, I the **monoidal unit**, α the **associator**, ρ and λ the **right** and **left unitor** respectively. The associator and unitors are known collectively as the **coherence maps**.

By Mac Lane's coherence theorem, these two axioms are equivalent to requiring that 'all formal diagrams'—that is, all diagrams in which the morphism are built from identity morphisms and the coherence maps using composition and the monoidal product—commute. Consequently, between any two products of the same ordered list of objects up to instances of the monoidal unit, such as $((A \otimes I) \otimes B) \otimes C$ and $A \otimes ((B \otimes C) \otimes (I \otimes I))$, there is a unique so-called **canonical** map. See Mac Lane [**?**, Corollary of Theorem VII.2.1] for a precise statement and proof.

A **lax monoidal functor** $(F, \varphi) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \boxtimes)$ between monoidal categories consists of a functor $F : \mathcal{C} \to \mathcal{C}'$, and natural transformations $\varphi_{A,B} : FA \boxtimes FB \to$

$F(A \otimes B)$ and $\varphi_1 : 1_{\mathcal{C}'} \to F1_{\mathcal{C}}$, such that for all $A, B, C \in \mathcal{C}$ the three diagrams

$$\begin{tikzcd}
(FA \otimes FB) \otimes FC \arrow[r, "\varphi_{A,B} \otimes \mathrm{id}_{FC}"] \arrow[d, "\alpha_{FA,FB,FC}"'] & F(A \otimes B) \otimes FC \arrow[r, "\varphi_{A \otimes B, C}"] & F((A \otimes B) \otimes C) \arrow[d, "F\alpha_{A,B,C}"] \\
FA \otimes (FB \otimes FC) \arrow[r, "\mathrm{id}_{FA} \otimes \varphi_{B,C}"'] & FA \otimes F(B \otimes C) \arrow[r, "\varphi_{A, B \otimes C}"'] & F(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
F(A) \otimes I' \arrow[r, "\rho"] \arrow[d, "\mathrm{id} \otimes \varphi_1"'] & F(A) \\
F(A) \otimes F(I) \arrow[r, "\varphi_{A,I}"'] & F(A \otimes I) \arrow[u, "F\rho"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I' \otimes F(A) \arrow[r, "\lambda"] \arrow[d, "\varphi_1 \otimes \mathrm{id}"'] & F(A) \\
F(I) \otimes F(A) \arrow[r, "\varphi_{I,A}"'] & F(I \otimes A) \arrow[u, "F\lambda"']
\end{tikzcd}$$

commute. We further say a monoidal functor is a **strong monoidal functor** if the φ are isomorphisms, and a **strict monoidal functor** if the φ are identities.

A **monoidal natural transformation** $\theta : (F, \varphi) \Rightarrow (G, \gamma)$ between two monoidal functors F and G is a natural transformation $\theta : F \Rightarrow G$ such that

$$\begin{tikzcd}
F1_{\mathcal{C}} \arrow[rr, "\theta_I"] & & G1_{\mathcal{C}} \\
& 1_{\mathcal{C}'} \arrow[ul, "\varphi_1"] \arrow[ur, "\gamma_1"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
FA \boxtimes FB \arrow[r, "\theta_A \otimes \theta_B"] \arrow[d, "\varphi_{A,B}"'] & GA \boxtimes GB \arrow[d, "\gamma_{A,B}"] \\
F(A \otimes B) \arrow[r, "\theta_{A \otimes B}"'] & G(A \otimes B)
\end{tikzcd}$$

commute for all objects A, B.

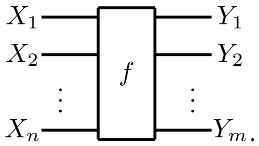
Two monoidal categories $\mathcal{C}, \mathcal{D}$ are **monoidally equivalent** if there exist strong monoidal functors $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ such that the composites FG and GF are monoidally naturally isomorphic to the identity functors. (Note that identity functors are immediately strict monoidal functors.)

1.2.2 Coherence

A **strict monoidal category** category is a monoidal category in which the associators and unitors are all identity maps. In this case then any two objects that can be related by associators and unitors are equal, and so we may write objects without parentheses and units without ambiguity. An equivalent statement of Mac Lane's coherence theorem is that every symmetric monoidal category is equivalent as a symmetric monoidal category to strict symmetric monoidal category.

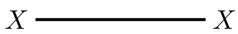
Yet another equivalent statement of the coherence theorem is the existence of a graphical calculus for morphisms in symmetric monoidal categories. Monoidal categories figure strongly in our current investigations precisely because of this. We leave the details to discussions elsewhere. The main point is that we shall be free to assume our symmetric monoidal categories are strict, writing $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ for objects in $(\mathcal{C}, \otimes)$ without a care for parentheses. We then depict a morphism $f\colon X_1 \otimes X_2 \otimes \cdots \otimes X_n \to Y_1 \otimes Y_2 \otimes \cdots \otimes Y_n$ with the diagram:

$f \quad =$



Identity morphisms are depicted by lines:

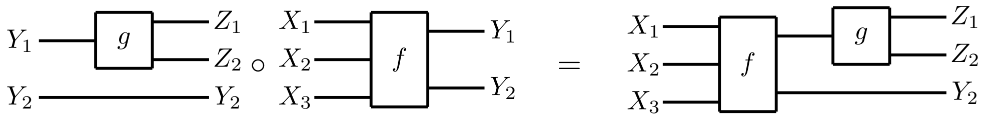
$\mathrm{id}_X \quad =$



and the monoidal unit is not depicted at all:

$$\mathrm{id}_I \quad =$$

Composition of morphisms is depicted by connecting the relevant 'wires'



while monoidal composition is just juxtaposition

$$.$$

For example, a diagram such as

$$.$$

reads as the equivalent algebraic expressions

$$.$$

and so on.

These two theorems show that the graphical calculi go beyond visualisations of the morphisms, having the ability to provide bona-fide proofs of equalities of morphisms. As a general principle, one which we shall demonstrate in this dissertation, this fact combined the intuitiveness of manipulations and the encoding of certain equalities and structural isomorphisms make the string diagrams better than the conventional algebraic language for understanding monoidal categories.

1.2.3 Symmetry

isomorphisms $\sigma_{A,B} : A \otimes B \to B \otimes A$ natural in A and B such that $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$ called the **braiding**

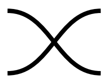
category (note we dropped associators)

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\sigma_{A,B} \otimes \mathrm{id}_C"] \arrow[d, "\alpha_{A,B,C}"'] & (B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"] & B \otimes (A \otimes C) \arrow[d, "\mathrm{id}_B \otimes \sigma_{A,C}"] \\
A \otimes (B \otimes C) \arrow[r, "\sigma_{A,B \otimes C}"'] & (B \otimes C) \otimes A \arrow[r, "\alpha_{B,C,A}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

functor

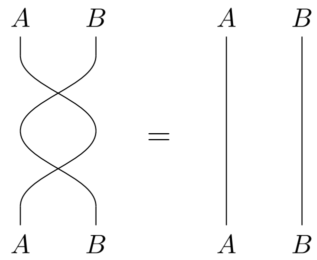
$$\begin{tikzcd}
FA \otimes FB \arrow[r, "\varphi_{A,B}"] \arrow[d, "\sigma'_{FA,FB}"'] & F(A \otimes B) \arrow[d, "F\sigma_{A,B}"] \\
FB \otimes FA \arrow[r, "\varphi_{B,A}"'] & F(B \otimes A)
\end{tikzcd}$$

In a symmetric monoidal category we represent the braiding with a special notation, the crossing of two wires:

$$\sigma_{A,B} =$$ 

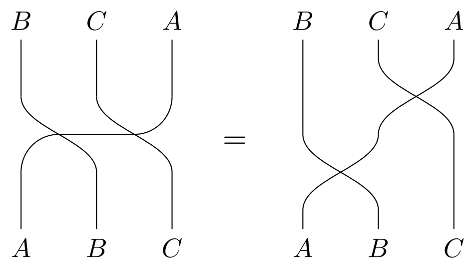
We will also later introduce similar special notations for the Frobenius maps in a hypergraph category.

The defining identities of the swap may then be written graphically as



(Sym1)

and



(Sym2)

Including these identity into our collection of allowable transformations of diagrams gives coherence theorem for symmetric monoidal categories.

Theorem 1.1 (Coherence of the graphical calculus for symmetric monoidal categories)**.** *Two morphisms in a symmetric monoidal category are equal with their equality following from the axioms of symmetric monoidal categories if and only if their diagrams are equal up to planar deformation and local applications of the identities Sym1 and Sym2.*

Proof. Joyal-Street [**?**, Theorem 2.3]. □

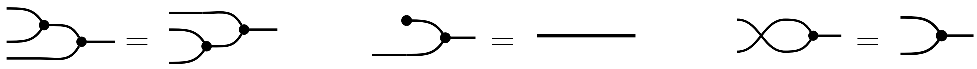
1.3 Hypergraph categories

1.3.1 Frobenius monoids

Definition 1.2. A **special commutative Frobenius monoid** $(X, \mu, \eta, \delta, \epsilon)$ in a monoidal category $(\mathcal{C}, \otimes)$ is an object X of $\mathcal{C}$ together with maps

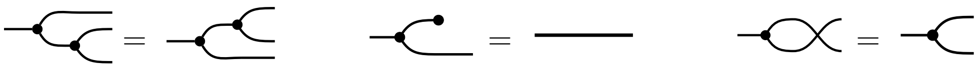
$\mu\colon X \otimes X \to X$ $\qquad$ $\eta\colon I \to X$ $\qquad$ $\delta\colon X \to X \otimes X$ $\qquad$ $\epsilon\colon X \to I$

obeying the commutative monoid axioms



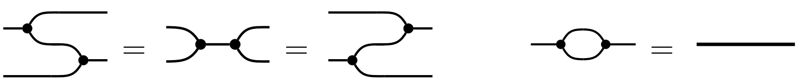
(associativity) (unitality) (commutativity)

the cocommutative comonoid axioms



(coassociativity) (counitality) (cocommutativity)

and the Frobenius and special axioms



(Frobenius) (special)

where  is the braiding on $X \otimes X$.

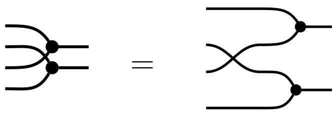
In addition to the 'upper' unitality law above, the mirror image 'lower' unitality law also holds, due to commutativity and the naturality of the braiding. While we write two equations for the Frobenius law, this is redundant: the equality of any two of the expressions implies the equality of all three. Note that a monoid and comonoid obeying the Frobenius law is commutative if and only if it is cocommutative. Thus while a commutative and cocommutative Frobenius monoid might more properly be called a bicommutative Frobenius monoid, there is no ambiguity if we only say commutative.

The Frobenius law and the special law go back to Carboni and Walters, under the names S=X law and the diamond=1 law respectively [?]. Special commutative Frobenius monoids is a more modern name; Carboni and Walters termed them commutative separable algebras [?],

Definition 1.3. A **hypergraph category** is a symmetric monoidal category in which each object X is equipped with a special commutative Frobenius structure $(X, \mu_X, \delta_X, \eta_X, \epsilon_X)$ such that

$$\mu_{X\otimes Y} = (\mu_X \otimes \mu_Y) \circ (1_X \otimes \sigma_{YX} \otimes 1_Y) \quad \eta_{X\otimes Y} = \eta_X \otimes \eta_Y$$
$$\delta_{X\otimes Y} = (1_X \otimes \sigma_{XY} \otimes 1_Y) \circ (\delta_X \otimes \delta_Y) \quad \epsilon_{X\otimes Y} = \epsilon_X \otimes \epsilon_Y.$$

In string diagrams these axioms become trivial:



where the left hand side of the equations are new notation to represent $\mu_{X\otimes Y}$ and $\eta_{X\otimes Y}$ respectively. The remaining two axioms are the mirror image of these.

Note that we do *not* require these Frobenius morphisms to be natural in X.

Note hypergraph structure is not seen by the morphisms in the category, but by the functors. In that sense it's more 2-categorical.

Related: Let $\mathcal{H} \to \mathcal{H}'$ be a fully faithful, essentially surjective, hypergraph functor. Then $\mathcal{H}$ and $\mathcal{H}'$ are not necessarily equivalent as hypergraph categories.

Definition 1.4. A functor (F, φ) of hypergraph categories, or **hypergraph functor**, is a strong symmetric monoidal functor (F, φ) such that for each object X the following diagrams commute:

$$\begin{tikzcd}
FX \boxtimes FX \arrow[rr, "\mu_{FX}"] \arrow[dr, "\varphi"'] & & FX \\
& F(X \otimes X) \arrow[ur, "F\mu_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathcal{D}} \arrow[rr, "\eta_{FX}"] \arrow[dr, "\varphi_1"'] & & FX \\
& F1_{\mathcal{C}} \arrow[ur, "F\eta_X"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
FX \arrow[rr, "\delta_{FX}"] \arrow[dr, "F\delta_X"'] & & FX \boxtimes FX \\
& F(X \otimes X) \arrow[ur, "\varphi^{-1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[rr, "\epsilon_{FX}"] \arrow[dr, "F\epsilon_X"'] & & 1_{\mathcal{D}} \\
& F1_{\mathcal{C}} \arrow[ur, "\varphi^{-1}"'] &
\end{tikzcd}$$

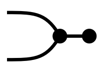
This means the special commutative Frobenius structure on FX is

$$(FX,\ \ F\mu_X \circ \varphi_{X,X},\ \ \varphi_{X,X}^{-1} \circ F\delta_X,\ \ F\eta_X \circ \varphi_1,\ \ \varphi_1^{-1} \circ F\epsilon_X).$$

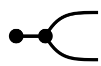
Just as monoidal natural transformations themselves are enough as morphisms between symmetric monoidal functors, so too they suffice as morphisms between hypergraph functors. Two hypergraph categories are **hypergraph equivalent** if there exist hypergraph functors with monoidal natural transformations to the identity functors. Write HypCat for the (2-)category of hypergraph categories.

Remark 1.5. This terminology was introduced recently [**?**], in reference to the fact that these special commutative Frobenius monoids provide precisely the structure required to draw graphs with 'hyperedges': wires connecting any number of inputs to any number of outputs. Hypergraph categories have held a number of names. They were first defined by Walters and Carboni with the name well-supported compact closed categories [**?**]. In recent years they have been rediscovered a number of times, also appearing under the names dungeon categories [**?**] and dgs-monoidal categories.

Remark 1.6. Hypergraph categories are self-dual compact closed. Note that if an object X is equipped with a Frobenius monoid structure then the maps

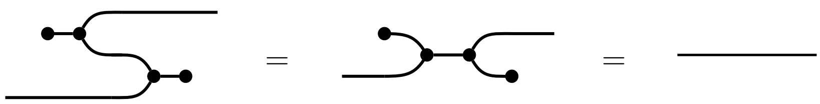


and



$\epsilon \circ \mu \colon X \otimes X \to 1$ $\qquad$ $\delta \circ \eta \colon 1 \to X \otimes X$

obey both



and the reflected equations. Thus if an object carries a Frobenius monoid it is also self-dual, and any hypergraph category is a fortiori self-dual compact closed.

As in any self-dual compact closed category, mapping each morphism $f \colon X \to Y$ to its dual morphism

$$\big((\epsilon_Y \circ \mu_Y) \otimes 1_X\big) \circ \big(1_Y \otimes f \otimes 1_X\big) \circ \big(1_Y \otimes (\delta_X \circ \eta_X)\big) \colon Y \longrightarrow X$$

further equips each hypergraph category with a so-called dagger functor—an involutive contravariant endofunctor that is the identity on objects—such that the category is a dagger compact category. Dagger compact categories were first introduced in the context of categorical quantum mechanics [**?**], under the name strongly compact closed category, and have been demonstrated to be a key structure in diagrammatic reasoning and the logic of quantum mechanics.

1.4 Coherence

The objects of a strict monoidal category form a monoid. The monoidal category is known as objectwise-free if this monoid is isomorphic to a free monoid.

Proposition 1.7. *Every hypergraph category is equivalent as a hypergraph category to a objectwise-free strict hypergraph category.*

Proof. Let $(\mathcal{H}, \otimes)$ be a hypergraph category. As $\mathcal{H}$ is a fortiori a symmetric monoidal category, Mac Lane's coherence theorem constructs an equivalent objectwise-free strict symmetric monoidal category $(\mathcal{H}_{\mathrm{str}}, \cdot)$ with objects finite lists $[x_1, \dots, x_n]$ of objects of $\mathcal{H}$ and morphisms $[x_1, \dots, x_n] \to [y_1, \dots, y_m]$ those morphisms from $((x_1 \otimes x_2) \otimes \dots) \otimes x_n \to ((y_1 \otimes y_2) \otimes \dots) \otimes y_m$ in $\mathcal{H}$. Composition is given by composition in $\mathcal{H}$.

The monoidal structure is given as follows. Given a list X of objects in $\mathcal{H}$, write PX for the corresponding monoidal product in $\mathcal{H}$ with all open parathesis at the front. The monoidal product of two objects is given by concatenation $\cdot$ of lists; the monoidal unit is the empty list. The monoidal product of two morphisms is given by their monoidal product in $\mathcal{H}$ pre- and post-composed with the necessary canonical maps: given $f\colon X \to Y$ and $g\colon Z \to W$, their product $f \cdot g\colon X \cdot Y \to Z \cdot W$ is

$$P(X \cdot Y) \longrightarrow PX \otimes PY \xrightarrow{f \otimes g} PZ \otimes PW \longrightarrow P(Z \cdot W).$$

By design, the associators and unitors are simply identity maps. The braiding $X \cdot Y \to Y \cdot X$ is given by the braiding $PX \otimes PY \to PY \otimes PX$ in $\mathcal{H}$, similarly pre- and post-composed with the necessary canonical maps. This defines a strict symmetric monoidal category [**?**].

To make $\mathcal{H}_{\mathrm{str}}$ into a hypergraph category, we equip each object X with a special commutative Frobenius monoid in the same way: we take the special commutative Frobenius monoid on PX and compose with the necessary canonical maps to get morphisms of the desired type. For example, the multiplication on X is given by

$$P(X \cdot X) \longrightarrow PX \otimes PX \xrightarrow{\mu_X} PX.$$

This hypergraph structure is well-defined. To check this, observe that the hypergraph structure on $[x_1, \dots, x_n] \cdot [y_1, \dots, y_m]$ is given by

We thus have a hypergraph category.

The strict hypergraph category $(\mathcal{H}_{\mathrm{str}}, \cdot)$	
objects	finite lists $[x_1, \dots, x_n]$ of objects of $\mathcal{H}$
morphisms	$\hom_{\mathcal{H}_{\mathrm{str}}}\big([x_1, \dots, x_n], [y_1, \dots, y_m]\big)$ $= \hom_{\mathcal{H}}\big(((x_1 \otimes x_2) \otimes \dots) \otimes x_n, ((y_1 \otimes y_2) \otimes \dots) \otimes y_m\big)$
composition	composition of corresponding maps in $\mathcal{H}$
monoidal product	concatenation of lists on objects, on morphisms inherited from $\mathcal{H}$
coherence maps	associators and unitors are strict; braiding is inherited from $\mathcal{H}$
hypergraph maps	inherited from $\mathcal{H}$

Mac Lane's equivalence is witnessed by strong symmetric monoidal functors $P\colon \mathcal{H}_{\mathrm{str}} \to \mathcal{H}$ extending the map P above, and $S\colon \mathcal{H} \to \mathcal{H}_{\mathrm{str}}$ sending $x \in \mathcal{H}$ to the string $[x]$ of length 1 in $\mathcal{H}_{\mathrm{str}}$. These extend to hypergraph functors.

In detail, the functor P is given on morphisms by taking a map in $\hom_{\mathcal{H}_{\mathrm{str}}}(X, Y)$ to the same map considered now as a map in $\hom_{\mathcal{H}}(PX, PY)$; its coherence maps are given by the canonical maps $PX \otimes PY \to P(X \cdot Y)$. The functor S is even easier to define: a morphism $x \to y$ in $\mathcal{H}$ is by definition a morphism $[x] \to [y]$ in $\mathcal{H}_{\mathrm{str}}$, so S is a monoidal embedding of $\mathcal{H}$ into $\mathcal{H}_{\mathrm{str}}$.

By Mac Lane's proof of the coherence theorem for monoidal categories these are both strong monoidal functors; by inspection they also preserve hypergraph structure, and so are hypergraph functors. As they already witness an equivalence of symmetric monoidal categories, thus $\mathcal{H}$ and $\mathcal{H}_{\mathrm{str}}$ are equivalent as hypergraph categories. $\square$

Note that two morphisms are equal if their string diagrams are equal via Frobenius laws, symmetry laws, and topological deformation.

Conjecture 1.8. *Graphical calculus for hypergraph categories: two morphisms are equal if and only if their string diagrams are equivalent via Frobenius laws and topological deformation.*

One might prove this using coherence theorem and spider theorem.

1.5 Example: cospan categories

A central example of a hypergraph category is the category $\mathrm{Cospan}(\mathcal{C})$ of cospans in any category $\mathcal{C}$ with finite colimits. We will later see that decorated cospan categories are a generalisation of such categories, and each inherits a hypergraph structure from such.

Let $\mathcal{C}$ be a category with finite colimits. Recall that a **cospan** $X \xrightarrow{i} N \xleftarrow{o} Y$ from X to Y in $\mathcal{C}$ is a pair of morphisms with common codomain. We refer to X and Y as the **feet**, and N as the **apex**. Given two cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ and $X \xrightarrow{i'} N' \xleftarrow{o'} Y$ with the same feet, a **map of cospans** is a morphism $n\colon N \to N'$ in $\mathcal{C}$ between the apices such that

$$\begin{tikzcd} & N \arrow[dd, "n"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd}$$

commutes.

Cospans may be composed, up to isomorphism, using the pushout from the common foot: given cospans $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite cospan is $X \xrightarrow{j_N \circ i_X} N +_Y M \xleftarrow{j_M \circ i_Z} Z$, where

$$\begin{tikzcd} & & N +_Y M & & \\ & N \arrow[ur, "j_N"] & & M \arrow[ul, "j_M"'] & \\ X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z \arrow[ul, "o_Z"'] \end{tikzcd}$$

is a pushout square.

We consider any category $\mathcal{C}$ as a symmetric monoidal category with monoidal product given by the coproduct, written $+$, and braiding given by the maps $A+B \to B+A$ by copairing identity maps.

Given maps $f\colon A \to C$, $g\colon B \to C$ with common codomain, the universal property of the coproduct gives a unique map $: A+B \to C$. We call this the **copairing** of f and g, and write it $[f, g]$.

Given a category $\mathcal{C}$ with pushouts, we may define a symmetric monoidal category $\mathrm{Cospan}(\mathcal{C})$ with objects the objects of $\mathcal{C}$ and morphisms isomorphism classes of cospans [**?**].

First, $\mathrm{Cospan}(\mathcal{C})$ inherits a symmetric monoidal structure from $\mathcal{C}$. We call a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ **wide** if $\mathcal{C}$ contains all objects of $\mathcal{D}$, and call a functor that is faithful and bijective-on-objects a **wide embedding**. Note then that we have a wide embedding

$$\mathcal{C} \hookrightarrow \mathrm{Cospan}(\mathcal{C})$$

that takes each object of $\mathcal{C}$ to itself as an object of $\mathrm{Cospan}(\mathcal{C})$, and each morphism $f\colon X \to Y$ in $\mathcal{C}$ to the cospan

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] & & Y, \arrow[ul, equals] \end{tikzcd}$$

where the extended 'equals' sign denotes an identity morphism. This allows us to view $\mathcal{C}$ as a wide subcategory of $\mathrm{Cospan}(\mathcal{C})$.

Now as $\mathcal{C}$ has finite colimits, it can be given a symmetric monoidal structure with the coproduct the monoidal product; we write this monoidal category $(\mathcal{C}, +)$, and write $\varnothing$ for the initial object, the monoidal unit of this category. Then $\mathrm{Cospan}(\mathcal{C})$ inherits the same symmetric monoidal structure: since the monoidal product $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is left adjoint to the diagram functor, it preserves colimits, and so extends to a functor $+\colon \mathrm{Cospan}(\mathcal{C}) \times \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{C})$. The remainder of the monoidal structure is inherited because $\mathcal{C}$ is a wide subcategory of $\mathrm{Cospan}(\mathcal{C})$.

Next, the Frobenius structure comes from copairings of identity morphisms. We call cospans

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & N & \\ Y \arrow[ur, "o"] & & X \arrow[ul, "i"'] \end{tikzcd}$$

that are reflections of each other **opposite** cospans. Given any object X in $\mathcal{C}$, the copairing $[1_X, 1_X]\colon X + X \to X$ of two identity maps on X, together with the unique map $!\colon \varnothing \to X$ from the initial object to X, define a monoid structure on X. Considering these maps as morphisms in $\mathrm{Cospan}(\mathcal{C})$, we may take them together with their opposites to give a special commutative Frobenius structure on X. In this way we consider each category $\mathrm{Cospan}(\mathcal{C})$ a hypergraph category.

Definition 1.9. Let $\mathcal{C}$ be a category with finite colimits. We define the hypergraph category $\mathrm{Cospan}(\mathcal{C})$ to comprise:

The hypergraph category $(\mathrm{Cospan}(\mathcal{C}), +)$	
objects	the objects of $\mathcal{C}$
morphisms	isomorphism classes of cospans in $\mathcal{C}$
composition	given by pushout
tensor product	the coproduct in $\mathcal{C}$.
coherence maps	inherited from $(\mathcal{C}, +)$ $\sigma_{X,Y} = [\iota_Y, \iota_X]\colon X + Y \to Y + X$
hypergraph maps	$\mu_X = [1_X, 1_X]$, $\eta_X = !$, $\delta_X = \mu_X^{\mathrm{opp}}$, $\epsilon_X = \eta_X^{\mathrm{opp}}$.

Notation 1.10. We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category $\mathrm{Cospan}(\mathcal{C})$; we of course refer instead to the isomorphism class of the said cospan.

Given $f\colon X \to Y$ in $\mathcal{C}$, we also abuse notation by writing $f \in \mathrm{Cospan}(\mathcal{C})$ for the cospan $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$, and f^{opp} for the cospan $Y \xrightarrow{1_Y} Y \xleftarrow{f} X$.

FinCocompleteCat faithfully embeds into HyperCat. ie any monoidal category of cospans has a hypergraph structure inherited from the identity morphisms.

Hypergraph categories are closely related to cospans. The free hypergraph category on a single object in the category of cospans in the category of finite sets. SpivakVagner?

Walters: cospan graph is the generic special commutative Frobenius monoid.

Later, also Vagner Spivak Schultz: hypergraph categories are algebras of cospan.

Part II

Applications

Bibliography

[Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page 17.)