

The Algebra of Open and Interconnected Systems



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Abstract

Herein we develop category-theoretic tools for understanding network-style diagrammatic languages. The archetypal network-style diagrammatic language is that of electric circuits; other examples include signal flow graphs, Markov processes, automata, Petri nets, chemical reaction networks, and so on. The key feature is that the language is comprised of a number of ‘components’ with multiple (input/output) terminals, each possibly labelled with some type, that may then be connected together along these terminals to form a larger ‘network’. The components form hyperedges between labelled vertices, and so a diagram in this language forms a hypergraph. We formalise the compositional structure by introducing the notion of a hypergraph category. Network-style diagrammatic languages and their semantics thus form hypergraph categories, and semantic interpretation gives a hypergraph functor.

The first part of this thesis develops the theory of hypergraph categories. In particular, we introduce the tools of decorated cospans and corelations. Decorated cospans allow straightforward construction of hypergraph categories from diagrammatic languages: the inputs, outputs, and their composition are modelled by the cospans, while the ‘decorations’ specify the components themselves. Not all hypergraph categories can be constructed, however, through decorated cospans. Decorated corelations are a more powerful version that permits construction of all hypergraph categories and hypergraph functors. These are often useful for constructing the semantic categories of diagrammatic languages and functors from diagrams to the semantics. To illustrate these principles, the second part of this thesis details applications to linear time-invariant dynamical systems and passive linear networks.

Contents

Introduction	v
0.1 Outline	viii
0.2 Related work	ix
0.3 Contributions of collaborators	ix
 I Mathematical Foundations	 1
1 Hypergraph categories: the algebra of interconnection	2
1.1 The algebra of interconnection	2
1.2 Symmetric monoidal categories	5
1.2.1 Monoidal categories	5
1.2.2 String diagrams	7
1.2.3 Symmetry	8
1.3 Hypergraph categories	9
1.3.1 Frobenius monoids	10
1.3.2 Hypergraph categories	11
1.3.3 Hypergraph categories are self-dual compact closed	13
1.3.4 Coherence	14
1.4 Example: cospan categories	16
 2 Decorated cospans: from closed systems to open	 20
2.1 From closed systems to open	20
2.2 Decorated cospan categories	23
2.2.1 Composing decorated cospans	23
2.2.2 The hypergraph structure	28
2.2.3 A bicategory of decorated cospans	33
2.3 Functors between decorated cospan categories	33
2.4 Decorations in Set are general	38

2.5	Examples	40
2.5.1	Labelled graphs	40
2.5.2	Linear relations	41
2.5.3	Electrical circuits	42
3	Corelations: a tool for black boxing	45
3.1	The idea of black boxing	45
3.2	Corelations	46
3.2.1	Factorisation systems.	46
3.2.2	Corelations	49
3.2.3	Categories of corelations	50
3.3	Functors between corelation categories	53
3.4	Examples	57
3.4.1	Equivalence relations as corelations in \mathbf{Set}	57
3.4.2	Linear relations as corelations in \mathbf{Vect}	59
4	Decorated corelations	61
4.1	Introduction	61
4.2	Decorated corelations	61
4.2.1	Adjoining right adjoints	62
4.2.2	Decorated corelations	63
4.2.3	Categories of decorated corelations	64
4.3	Functors between decorated corelation categories	65
4.4	All hypergraph categories are decorated corelation categories	69
4.4.1	The global sections construction.	70
4.4.2	Examples.	72
4.5	Examples	74
4.5.1	Path integrals and matrices	74
4.5.2	Two constructions for linear relations	75
4.5.3	Automata	75
II	Applications	76
5	Signal flow diagrams	77
5.1	Introduction	77
5.1.1	Behavioural control	77
5.1.2	Signal flow	80

5.2	Preliminaries	84
5.3	Linear time-invariant dynamical systems	85
5.4	Presentation of LTI	89
5.4.1	Syntax	89
5.4.2	Presentation of $\mathbf{Mat\,k}[s, s^{-1}]$	91
5.4.3	Presentation of $\mathbf{Cospan\,Mat\,k}[s, s^{-1}]$	92
5.4.4	Corelations	93
5.4.5	Presentation of $\mathbf{Corel\,Mat\,k}[s, s^{-1}]$	95
5.5	Controllability	98
5.5.1	A categorical characterisation	98
5.5.2	Control and interconnection	100
5.5.3	Comparison to matrix methods	102
6	Passive linear networks	104
6.1	Introduction	104
6.1.1	Finding your way through this paper	113
6.2	Circuits of linear resistors	114
6.2.1	Circuits as labelled graphs	114
6.2.2	Ohm’s law, Kirchhoff’s laws, and the principle of minimum power	116
6.2.3	A Dirichlet problem	119
6.2.4	Equivalent circuits	122
6.2.5	Dirichlet forms	125
6.3	Inductors and capacitors	127
6.3.1	The frequency domain and Ohm’s law revisited	128
6.3.2	The mechanical analogy	131
6.3.3	Generalized Dirichlet forms	132
6.3.4	A generalized minimizability result	134
6.3.5	Composition of Dirichlet forms	137
6.4	The category of passive linear circuits	139
6.4.1	A decorated cospan construction	140
6.4.2	A bicategory of circuits	141
6.5	Circuits as Lagrangian relations	142
6.5.1	Symplectic vector spaces	143
6.5.2	Lagrangian subspaces from quadratic forms	146
6.5.3	Lagrangian relations	147
6.5.4	The dagger compact category of Lagrangian relations	150

6.6	Ideal wires and corelations	151
6.6.1	Ideal wires	152
6.6.2	The category of corelations	153
6.6.3	Potentials on corelations	156
6.6.4	Currents on corelations	158
6.6.5	The functor from Corel to LagrRel	160
6.7	Defining the black box functor	163
6.7.1	Definition	164
6.7.2	Minimization via composition of relations	166
6.8	Proof of functoriality	167
6.8.1	From circuits to Dirichlet cospans	168
6.8.2	From Dirichlet cospans to Lagrangian cospans	171
6.8.3	From cospans to relations	173
7	Further directions	177
7.1	Open Markov processes	177
7.2	Flow networks	177
7.3	Chemical reaction networks	177
7.4	Bounded colimits	177
7.5	Other categorical structures	177
	Notation and conventions	178
	Bibliography	179

Introduction

This is a thesis in the mathematical sciences, with emphasis on the mathematics. But before we get to the category theory, I want to say a few, brief words about the scientific tradition this thesis draws from.

Mathematics is the language of science. New science, new scientific paradigms, requires new language.

This dissertation represents the beginnings of an attempt at a category theoretic framework for general systems theory. A loosely organised body of research dating back to biologist Ludwig von Bertalanffy in the mid-20th century [?], general systems theory represents the study of systems built from rich interconnections of simple component systems. Our central example in this dissertation will be passive linear circuits: systems built from linear resistors, inductors, and capacitors. While each component here is simple, networks built from such components are complex enough to form the foundation of modern electronics.

General system theory seeks to talk about composition

While in the past, science tried to explain observable phenomena by reducing them to an interplay of elementary units investigable independently of each other, conceptions appear in contemporary science that are concerned with what is somewhat vaguely termed ‘wholeness’, i.e. problems of organization, phenomena not resolvable into local events, dynamic interactions manifest in difference of behaviour of parts when isolated or in a higher configuration, etc.; in short, ‘systems’ of various order not understandable by investigation of their respective parts in isolation.

From closed systems to open

Conventional physics deals only with closed systems, i.e. systems which are considered to be isolated from their environment.

However, we find systems which by their very nature and definition are not closed systems. Every living organism is essentially an open system.

It maintains itself in a continuous inflow and outflow, a building up and breaking down of components, never being, so long as it is alive, in a state of chemical and thermodynamic equilibrium but maintained in a so-called steady state which is distinct from the latter.

It has always had a unification bent to it [?]

Not only are general aspects and viewpoints alike in different sciences; frequently we find formally identical or isomorphic laws in different fields. In many cases, isomorphic laws hold for certain classes or subclasses of 'systems', irrespective of the nature of the entities involved. There appear to exist general system laws which apply to any system of a certain type, irrespective of the particular properties of the system and of the elements involved.

We will return to these ideas about isomorphic laws in Chapter 5.

This vision extended into the social sciences and humanities, influencing economics, urban planning, sociology, psychology, management, philosophy. Eg Forrester. These aspects of system theory are beyond the scope of the present work.

Part of this vision, most notably in von Bertalanffy's home field of systems biology, has been realised through

"Systems biology...is about putting together rather than taking apart, integration rather than reduction. It requires that we develop ways of thinking about integration that are as rigorous as our reductionist programmes, but different....It means changing our philosophy, in the full sense of the term" (Denis Noble).

Feedback; Wiener's cybernetics was often viewed as identical in agenda,

More technical progress has been realised: cybernetics, catastrophe theory, chaos theory, complex systems, network analysis

The goal here is to investigate these ideas from an algebraic perspective, creating structural and compositional techniques for modelling interconnection, open systems, and formal analogies, or isomorphisms, between different fields.

A diagrammatic approach. In this program Odum, diagrams. universal language SBGN, UML

what is the algebra of network languages?

hypergraph categories.

graph example

black-boxing, information compression.

examples examples.

Example: trijunction

signal flow

circuits

automata, markov processes, flow networks.

The Yoneda lemma and representability.

Syntactically, by system we mean a ‘box’ with finitely many ‘ports’ through which it interfaces with the external world. These ports may be of different types. These systems may be connected together, along ports of the same type, to form larger systems. Examples of such systems abound; a motivating source of them is network-style diagrammatic languages, such as the aforementioned electrical circuit diagrams, but also including chemical reaction networks, Petri nets, automata, and Markov processes.

We capture this syntactic conception of a system is formally through the notion of a hypergraph category—a symmetric monoidal category in which every object is equipped with a special commutative Frobenius monoid in a way compatible with the monoidal product. This framework provides precise language for describing how we manipulate and interact with systems.

But what *is* a system, and what do these manipulations and interconnections mean? Taking inspiration from Willems [?] and Deutsch [?] among others, we consider a system to be merely the set of all possible different observations one might make by measuring all relevant variables at all ports of the ‘box’ we use to represent it. We call this set the *behaviour* of the system. This arises from a view of physical laws as a mechanism for simply partitioning the set of all trajectories of a system, the so-called *universum*, into possible and impossible trajectories.

For example, consider a resistor of resistance r . This has two ports—the two ends of the resistor—and at each port we may measure the potential, and the current flowing into the port. Now the resistor is governed by Kirchhoff’s current law, or conservation of charge, and Ohm’s law. Conservation of charge states that the current flowing into one port must equal the current flowing out of the other port, while Ohm’s law states that this current will be proportional to the potential difference, with constant of proportionality $1/r$. Thus the behaviour of the resistor is the set

$$\left\{ \left(\phi_1, \phi_2, -\frac{1}{r}(\phi_2 - \phi_1), \frac{1}{r}(\phi_2 - \phi_1) \right) \mid \phi_1, \phi_2 \in \mathbb{R} \right\}.$$

Here the universum is the set $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, where the summands represent respectively the potentials and currents at each of the two terminals.

The variables associated to each port are determined by the type of the port. Interconnection of ports then, simply asserts the identification of the variables at the connected ports. On the level of behaviours, this becomes a generalised version of composition of relations.

In this thesis I argue that this general framework has wide applicability to applied science and engineering, and with appropriate additional mathematical tools allows one to formalise certain diagrammatic languages and their relationships.

The central message of decorated corelations is that hypergraph structure requires some sort of uniformity in the composition rule, and it is easier to work by acknowledging this structure, defining them as algebras over some theory. But we can weaken this notion of theory.

Beyond category theoretic interest, the motivation for such a method lies in developing compositional accounts of semantics associated to topological diagrams. While this has long been a technique associated with topological quantum field theory, dating back to [?], it has most recently had significant influence in the nascent field of categorical network theory, with application to automata and computation [?, ?], electrical circuits [?], signal flow diagrams [?, ?], Markov processes [?, ?], and dynamical systems [?], among others.

An advantage of the decorated cospan framework is that the resulting categories are hypergraph categories, and the resulting functors respect this structure. As dagger compact categories, hypergraph categories themselves have a rich diagrammatic nature [?], and in cases when our decorated cospan categories are inspired by diagrammatic applications, the hypergraph structure provides language to describe natural operations on our diagrams, such as juxtaposing, rotating, and reflecting them.

Influence of computer science. syntax semantics. rewrite rules/local rather than global analysis.

Inspired by Coecke Abramsky CQM

0.1 Outline

Chapter 1 introduces hypergraph categories. Then decorated cospans and decorated corelations. Colimits?

Then applications. First to control theory from this behavioural perspective, demonstrating the ideas of black boxing and corelations, open systems and control

by interconnection. We give a new characterisation of controllability. Second to passive linear networks, which speaks to the idea that we can find isomorphisms across disciplines, and explore the insights in new ways.

0.2 Related work

Category-theoretic frameworks for general systems theory have been developed before. Notably Goguen and Rosen led efforts. Goguen from a more computer science perspective, Rosen more in biology.

Spivak

Zanasi, Sobociński, Bonchi

Most similar is the work of Walters, together with Sabadini and Rosebrugh. In particular, automata.

0.3 Contributions of collaborators

The first three chapters are my own work. The applications chapters were developed with collaborators. Chapter 4 arises from a weekly seminar with Paolo Rapisarda and Paweł Sobociński. I developed the corelation formalism and provided a first draft of the paper. Paweł provided much expertise in signal flow graphs, significantly revising the text and contributing the section on operational semantics. Paolo contributed comparisons to classical methods in control theory. Much of the text is taken from the paper [?].

Chapter 5 is joint work with John Baez. John proposed the topic of research, and supplied some notes on Ohm's law and the principle of minimum power. I was responsible for producing the first draft of the chapter from this start. John guided and assisted revisions of this text for publication [?].

I thank my collaborators.

Part I

Mathematical Foundations

Chapter 1

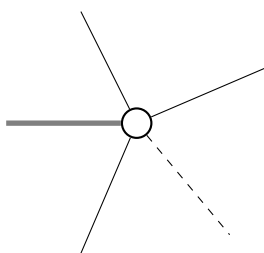
Hypergraph categories: the algebra of interconnection

In this chapter we introduce hypergraph categories, giving a definition, coherence theorem, and graphical language. We then explore a fundamental example of hypergraph categories: categories of cospans.

We assume basic familiarity with category theory and symmetric monoidal categories; although we give a sparse overview of the latter for reference. A proper introduction to both can be found in Mac Lane [?].

1.1 The algebra of interconnection

Our aim is to algebraicise network diagrams. A network diagram is built from pieces like so:



These represent open systems, concrete or abstract; for example a resistor, a chemical reaction, or a linear transformation. The essential feature, for openness and for networking, is that the system may have terminals, perhaps of different ‘types’, each one depicted by a line radiating from the central body. In the case of a resistor each terminal might represent a wire, for chemical reactions a chemical species, for linear transformations a variable in the domain or codomain. Network diagrams are formed by connecting terminals of systems to build larger systems.

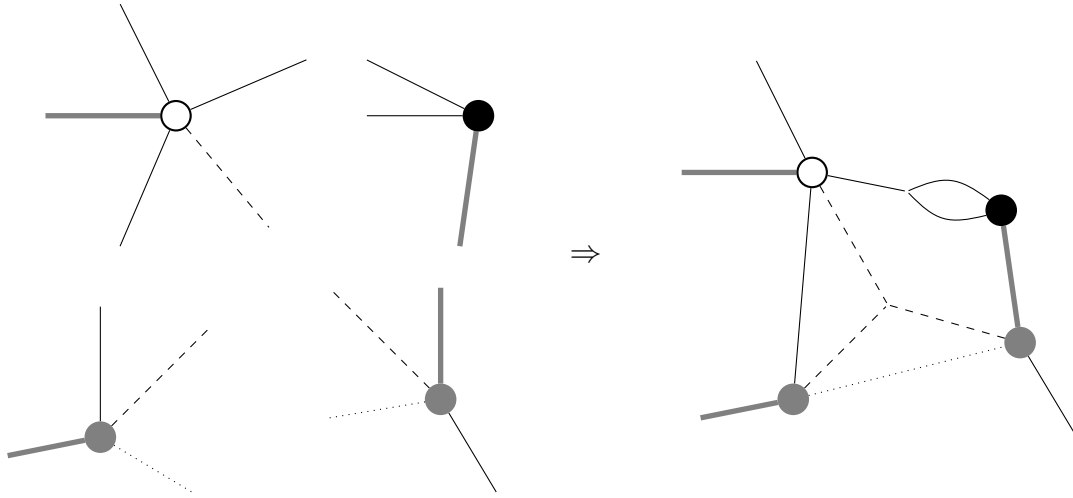


Figure 1.1: Interconnection of network diagrams. Note that we only connect terminals of the same type, but we can connect as many as we like.

A network-style diagrammatic language is a collection of network diagrams together with the stipulation that if we take some of these network diagrams, and connect terminals of the same type in any way we like, then we form another diagram in the collection. The point of this chapter is that hypergraph categories provide a precise formalisation of network-style diagrammatic languages.

In jargon, a hypergraph category is a symmetric monoidal category in which every object is equipped with a special commutative Frobenius monoid in a way compatible with the monoidal product. We will walk through these terms in detail, illustrating them with examples and a few theorems.

The key data comprising a hypergraph category are its objects, morphisms, composition rule, monoidal product, and Frobenius maps. Each of these model a feature of network diagrams and their interconnection. The objects model the terminal types, while the morphisms model the network diagrams themselves. The composition, monoidal product, and Frobenius maps model different aspects of interconnection: composition models the interconnection of two terminals of the same type, the monoidal product models the network formed by taking two networks without interconnecting any terminals, while the Frobenius maps model multi-terminal interconnection.

These Frobenius maps are the distinguishing feature of hypergraph categories as compared to other structured monoidal categories, and are crucial for formalising the intuitive concept of network languages detailed above. In the case of electric circuits

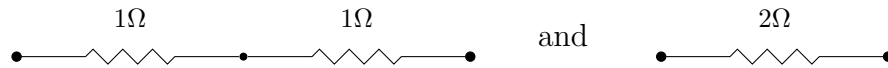
Networks	Hypergraph categories
list of terminal types	object
network diagram	morphism
series connection	composition
juxtaposition	monoidal product
branching	Frobenius maps

Figure 1.2: Corresponding features of networks and hypergraph categories.

the Frobenius maps model the ‘branching’ of wires; in the case when diagrams simply model an abstract system of equations and terminals variables in these equations, the Frobenius maps allow variables to be shared between many systems of equations.

Examining these correspondences, it is worthwhile to ask whether hypergraph categories permit too much structure to be specified, given that the interconnection rule is now divided into three different aspects, and features such as domains and codomains of network diagrams, rather than just a collection of terminals, exist. The answer is given by examining the additional coherence laws that these data must obey. For example, in the case of the domain and codomain, we shall see that hypergraph categories are all compact closed categories, and so there is ultimately only a formal distinction between domain and codomain objects. One way to think of these data is as scaffolding. We could compare it to the use of matrices and bases to provide language for talking about linear transformations and vector spaces. They are not part of the target structure, but nonetheless useful paraphernalia for constructing it.

Network languages are not only syntactic entities: as befitting the descriptor ‘language’, they typically have some associated semantics. Circuits diagrams, for instance, not only depict wire circuits that may be constructed, they also represent the electrical behaviour of that circuit. Such semantics considers the circuits

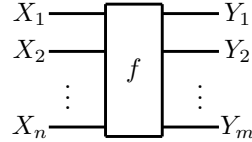


the same, even though as ‘syntactic’ diagrams they are distinct. A cornerstone of the utility of the hypergraph formalism is the ability to also realise the semantics of these diagrams as morphisms of another hypergraph category. This ‘semantic’ hypergraph category, as a hypergraph category, still permits the rich ‘networking’ interconnection structure, and a so-called hypergraph functor implies that the syntactic category provides a sound framework for depicting these morphisms. Network languages syn-

tactically are often ‘free’ hypergraph categories, and much of the interesting structure lies in their functors to their semantic hypergraph categories.

1.2 Symmetric monoidal categories

Suppose we have some tiles with inputs and outputs of various types like so:



These tiles may vary in height and width. We can place these tiles above and below each other, and to the left and right, so long as the inputs on the right tile match the outputs on the left. Suppose also that some arrangements of tiles are equal to other arrangements of tiles. How do we formalise this structure algebraically? The theory of monoidal categories provides an answer.

Hypergraph categories are first monoidal categories, indeed symmetric monoidal categories. A monoidal category is a category with two notions of composition: ordinary categorical composition and monoidal composition, with the monoidal composition only associative and unital up to natural isomorphism. They are the algebra of processes that may occur simultaneously as well as sequentially. First defined by Bénabou and Mac Lane in the 1960s [?, ?], their theory and their links with graphical representation are well explored. We bootstrap on this, using monoidal categories to define hypergraph categories, and so immediately arriving at an understanding of how hypergraph categories formalise our network languages.

Moreover, symmetric monoidal functors play a key role in our framework for defining and working with hypergraph categories: decorated cospans and corelations constructions. For this reason we provide, for quick reference, a definition of symmetric monoidal categories.

1.2.1 Monoidal categories

A **monoidal category** (\mathcal{C}, \otimes) consists of a category \mathcal{C} , a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, a distinguished object I , and natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C)$, $\rho_A : A \otimes I \rightarrow A$, and $\lambda_A : I \otimes A \rightarrow A$ such that for all A, B, C, D in \mathcal{C} the following

two diagrams commute:

$$\begin{array}{ccc}
 ((A \otimes B) \otimes C) \otimes D & \xrightarrow{\alpha_{(A \otimes B), C, D}} & (A \otimes B) \otimes (C \otimes D) \\
 \alpha_{A, B, C} \otimes \text{id}_D \downarrow & & \downarrow \alpha_{A, B, (C \otimes D)} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A, (B \otimes C), D}} A \otimes ((B \otimes C) \otimes D) \xrightarrow{\text{id}_A \otimes \alpha_{B, C, D}} & A \otimes (B \otimes (C \otimes D))
 \end{array}$$

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A, I, B}} & A \otimes (I \otimes B) \\
 \rho_A \otimes \text{id}_B \searrow & & \swarrow \text{id}_A \otimes \lambda_B \\
 & A \otimes B &
 \end{array}$$

We call \otimes the **monoidal product**, I the **monoidal unit**, α the **associator**, ρ and λ the **right** and **left unitor** respectively. The associator and unitors are known collectively as the **coherence maps**.

By Mac Lane's coherence theorem, these two axioms are equivalent to requiring that 'all formal diagrams'—that is, all diagrams in which the morphism are built from identity morphisms and the coherence maps using composition and the monoidal product—commute. Consequently, between any two products of the same ordered list of objects up to instances of the monoidal unit, such as $((A \otimes I) \otimes B) \otimes C$ and $A \otimes ((B \otimes C) \otimes (I \otimes I))$, there is a unique so-called **canonical** map. See Mac Lane [?, Corollary of Theorem VII.2.1] for a precise statement and proof.

A **lax monoidal functor** $(F, \varphi) : (\mathcal{C}, \otimes) \rightarrow (\mathcal{C}', \boxtimes)$ between monoidal categories consists of a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, and natural transformations $\varphi_{A, B} : FA \boxtimes FB \rightarrow F(A \otimes B)$ and $\varphi_1 : 1_{\mathcal{C}'} \rightarrow F1_{\mathcal{C}}$, such that for all $A, B, C \in \mathcal{C}$ the three diagrams

$$\begin{array}{ccc}
 (FA \otimes FB) \otimes FC & \xrightarrow{\varphi_{A, B} \otimes \text{id}_{FC}} F(A \otimes B) \otimes FC & \xrightarrow{\varphi_{A \otimes B, C}} F((A \otimes B) \otimes C) \\
 \alpha_{FA, FB, FC} \downarrow & & \downarrow F\alpha_{A, B, C} \\
 FA \otimes (FB \otimes FC) & \xrightarrow{\text{id}_{FA} \otimes \varphi_{B, C}} FA \otimes F(B \otimes C) & \xrightarrow{\varphi_{A, B \otimes C}} F(A \otimes (B \otimes C))
 \end{array}$$

$$\begin{array}{ccc}
 F(A) \otimes I' & \xrightarrow{\rho} & F(A) \\
 \text{id} \otimes \varphi_1 \downarrow & & \uparrow F\rho \\
 F(A) \otimes F(I) & \xrightarrow{\varphi_{A, I}} & F(A \otimes I)
 \end{array}
 \quad
 \begin{array}{ccc}
 I' \otimes F(A) & \xrightarrow{\lambda} & F(A) \\
 \varphi_1 \otimes \text{id} \downarrow & & \uparrow F\lambda \\
 F(I) \otimes F(A) & \xrightarrow{\varphi_{I, A}} & F(I \otimes A)
 \end{array}$$

commute. We further say a monoidal functor is a **strong monoidal functor** if the φ are isomorphisms, and a **strict monoidal functor** if the φ are identities.

A **monoidal natural transformation** $\theta : (F, \varphi) \Rightarrow (G, \gamma)$ between two monoidal functors F and G is a natural transformation $\theta : F \Rightarrow G$ such that

$$\begin{array}{ccc}
 F1_{\mathcal{C}} & \xrightarrow{\theta_I} & G1_{\mathcal{C}} \\
 \swarrow \varphi_1 & & \searrow \gamma_1 \\
 & 1_{\mathcal{C}'} &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 FA \boxtimes FB & \xrightarrow{\theta_A \otimes \theta_B} & GA \boxtimes GB \\
 \downarrow \varphi_{A,B} & & \downarrow \gamma_{A,B} \\
 F(A \otimes B) & \xrightarrow{\theta_{A \otimes B}} & G(A \otimes B)
 \end{array}$$

commute for all objects A, B .

Two monoidal categories \mathcal{C}, \mathcal{D} are **monoidally equivalent** if there exist strong monoidal functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ such that the composites FG and GF are monoidally naturally isomorphic to the identity functors. (Note that identity functors are immediately strict monoidal functors.)

1.2.2 String diagrams

A **strict monoidal category** is a monoidal category in which the associators and unitors are all identity maps. In this case then any two objects that can be related by associators and unitors are equal, and so we may write objects without parentheses and units without ambiguity. An equivalent statement of Mac Lane's coherence theorem is that every monoidal category is monoidally equivalent to strict monoidal category.

Yet another equivalent statement of the coherence theorem is the existence of a graphical calculus for monoidal categories. As discussed above, monoidal categories figure strongly in our current investigations precisely because of this. We leave the details to discussions elsewhere. The main point is that we shall be free to assume our monoidal categories are strict, writing $X_1 \otimes X_2 \otimes \cdots \otimes X_n$ for objects in (\mathcal{C}, \otimes) without a care for parentheses. We then depict a morphism $f: X_1 \otimes X_2 \otimes \cdots \otimes X_n \rightarrow Y_1 \otimes Y_2 \otimes \cdots \otimes Y_m$ with the diagram:

$$f = \begin{array}{ccc}
 X_1 & \text{---} & Y_1 \\
 X_2 & \text{---} & Y_2 \\
 \vdots & & \vdots \\
 X_n & \text{---} & Y_m
 \end{array}$$

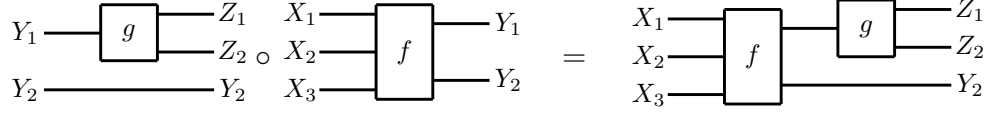
Identity morphisms are depicted by ‘wires’:

$$\text{id}_X = X \text{ ————— } X$$

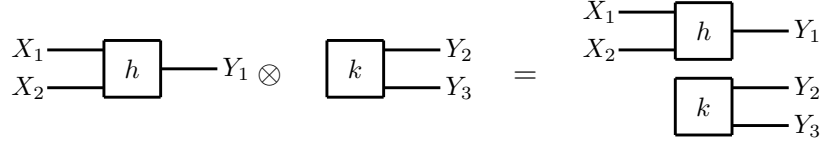
and the monoidal unit is not depicted at all:

$$\text{id}_I =$$

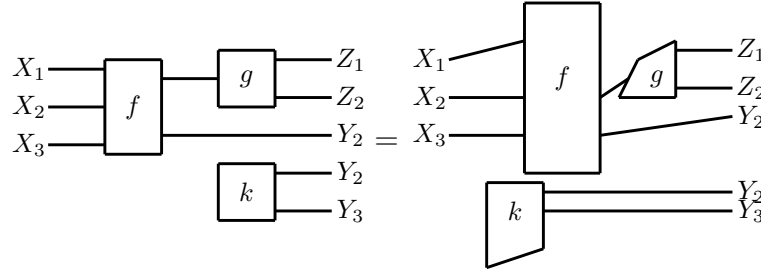
Composition of morphisms is depicted by connecting the relevant ‘wires’:



while monoidal composition is just juxtaposition:



Only the ‘topology’ of the diagrams matters: if two diagrams with the same domain and codomain are homotopy equivalent, they represent the same morphism. On the other hand, two algebraic expressions might have the same diagrammatic representation. For example, the equivalent diagrams



read as all of the equivalent algebraic expressions

$$((g \otimes \text{id}_{Y_2}) \otimes k) \circ f = (g \otimes ((\text{id}_{Y_2} \otimes k))) \circ \rho \circ (f \otimes \text{id}_I) = (g \otimes \text{id}_{Y_2}) \circ f \circ (\text{id}_{X_1 \otimes (X_2 \otimes X_3)} \otimes k)$$

and so on. The coherence theorem says that this does not matter: if two algebraic expressions have the same diagrammatic representation, then the algebraic expressions are equal. In more formal language, the graphical calculus is sound and complete for the axioms of monoidal categories.

The coherence theorem thus implies that the graphical calculi goes beyond visualisations of morphisms: it can provide provide bona-fide proofs of equalities of morphisms. As a general principle, string diagrams are more intuitive than the conventional algebraic language for understanding monoidal categories.

1.2.3 Symmetry

A symmetric braiding in a monoidal category provides the ability to permute objects or, equivalently, cross wires. We define symmetric monoidal categories making use of the graphical notation outlined above, but introducing a new, special symbol \bowtie .

A **symmetric monoidal category** is a monoidal category (\mathcal{C}, \otimes) together with natural isomorphisms

$$\begin{array}{c} A \quad B \\ \diagdown \quad \diagup \\ B \quad A \end{array}$$

$$\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$$

such that

$$\begin{array}{c} A \quad A \\ \diagdown \quad \diagup \\ B \quad B \end{array} = \begin{array}{c} A \quad A \\ \hline B \quad B \end{array}$$

and

$$\begin{array}{c} A \quad B \\ \diagdown \quad \diagup \\ B \quad C \\ \diagdown \quad \diagup \\ C \quad A \end{array} = \begin{array}{c} A \quad B \otimes C \\ \diagdown \quad \diagup \\ B \otimes C \quad A \end{array}$$

for all A, B, C in \mathcal{C} . We call σ the **braiding**. We will also talk, somewhat incidentally, of braided monoidal categories in the next chapter; a **braided monoidal category** is a monoidal category with a braiding that only obeys the latter axiom.

A **(lax/strong) symmetric monoidal functor** is a (lax/strong) monoidal functor that further obeys

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\varphi_{A,B}} & F(A \otimes B) \\ \sigma'_{FA,FB} \downarrow & & \downarrow F\sigma_{A,B} \\ FB \otimes FA & \xrightarrow{\varphi_{B,A}} & F(B \otimes A) \end{array}$$

Morphisms between symmetric monoidal functors are simply monoidal natural transformations. Thus two symmetric monoidal categories are **symmetric monoidally equivalent** if they are monoidally equivalent by strong *symmetric* monoidal functors. If our categories are merely braided, we refer to these functors as **braided monoidal functors**.

The coherence theorem for symmetric monoidal categories, with respect to string diagrams, states that two morphisms in a symmetric monoidal category are equal according to the axioms of symmetric monoidal categories if and only if their diagrams are equal up to homotopy equivalence and applications of the defining graphical identities above. See Joyal–Street [?, Theorem 2.3] for more precision and details.

1.3 Hypergraph categories

Just as symmetric monoidal categories equip monoidal categories with precisely enough extra structure to model crossing of strings in the graphical calculus, hypergraph categories equip symmetric monoidal categories with precisely enough extra structure to

model multi-input multi-output interconnections of strings of the same type. For this, we require each object to be equipped with a so-called special commutative Frobenius monoid, which provides chosen maps to model this interaction. These have a coherence result, known as the ‘spider theorem’, that says exactly how we use the maps to describe the connection of strings does not matter: all that matters is that the strings are connected.

1.3.1 Frobenius monoids

A Frobenius monoid comprises a monoid and comonoid on the same object that interact according to the so-called Frobenius law.

Definition 1.1. A **special commutative Frobenius monoid** $(X, \mu, \eta, \delta, \epsilon)$ in a symmetric monoidal category (\mathcal{C}, \otimes) is an object X of \mathcal{C} together with maps

$$\begin{array}{cccc}
 \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} & \begin{array}{c} \bullet \text{---} \end{array} & \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} & \begin{array}{c} \text{---} \bullet \end{array} \\
 \mu: X \otimes X \rightarrow X & \eta: I \rightarrow X & \delta: X \rightarrow X \otimes X & \epsilon: X \rightarrow I
 \end{array}$$

obeying the commutative monoid axioms

$$\begin{array}{ccc}
 \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} & \begin{array}{c} \bullet \text{---} \end{array} = \text{---} & \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \\
 \text{(associativity)} & \text{(unitality)} & \text{(commutativity)}
 \end{array}$$

the cocommutative comonoid axioms

$$\begin{array}{ccc}
 \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} & \begin{array}{c} \bullet \text{---} \end{array} = \text{---} & \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \\
 \text{(coassociativity)} & \text{(counitality)} & \text{(cocommutativity)}
 \end{array}$$

and the Frobenius and special axioms

$$\begin{array}{ccc}
 \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} & \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} = \text{---} \\
 \text{(Frobenius)} & \text{(special)}
 \end{array}$$

We call μ the **multiplication**, η the **unit**, δ the **comultiplication**, and ϵ the **counit**.

Special commutative Frobenius monoids were first formulated by Carboni and Walters, under the name commutative separable algebras. The Frobenius law and the special law were termed the S=X law and the diamond=1 law respectively [?, ?].

Alternate axiomatisations are possible. In addition to the ‘upper’ unitality law above, the mirror image ‘lower’ unitality law also holds, due to commutativity and the naturality of the braiding. While we write two equations for the Frobenius law, this is redundant: given the other axioms, the equality of any two of the diagrams implies the equality of all three. Further, note that a monoid and comonoid obeying the Frobenius law is commutative if and only if it is cocommutative. Thus while a commutative and cocommutative Frobenius monoid might more properly be called a bicommutative Frobenius monoid, there is no ambiguity if we only say commutative.

The common feature to these equations is that each side describes a different way of using the generators to connect some chosen set of inputs to some chosen set of outputs. This observation provides a ‘coherence’ type result for special commutative Frobenius monoids, known as the ‘spider theorem’.

Theorem 1.2. *Let $(X, \mu, \eta, \delta, \epsilon)$ be a special commutative Frobenius monoid, and let $f, g: X^{\otimes n} \rightarrow X^{\otimes m}$ be map constructed, using composition and the monoidal product, from $\mu, \eta, \delta, \epsilon$, the coherence maps and braiding, and the identity map on X . Then f and g are equal if and only if given their string diagrams in the above notation, there exists a bijection between the connected components of the two diagrams such that corresponding connected components connect the exact same sets of inputs and outputs.*

See [?, ?] [?, ?] for further details.

1.3.2 Hypergraph categories

Definition 1.3. A **hypergraph category** is a symmetric monoidal category in which each object X is equipped with a special commutative Frobenius structure $(X, \mu_X, \delta_X, \eta_X, \epsilon_X)$ such that

$$\begin{array}{ccc}
 \begin{array}{c} X \otimes Y \\ X \otimes Y \end{array} \curvearrowright X \otimes Y & = & \begin{array}{c} X \\ Y \\ X \\ Y \end{array} \begin{array}{c} \text{---} \bullet \text{---} X \\ \text{---} \bullet \text{---} Y \end{array} \\
 \\
 X \otimes Y \curvearrowright \begin{array}{c} X \otimes Y \\ X \otimes Y \end{array} & = & \begin{array}{c} X \\ Y \end{array} \begin{array}{c} \text{---} \bullet \text{---} X \\ \text{---} \bullet \text{---} Y \end{array} \\
 \\
 \bullet \text{---} X \otimes Y & = & \begin{array}{c} \bullet \text{---} X \\ \bullet \text{---} Y \end{array} \\
 \\
 X \otimes Y \text{---} \bullet & = & \begin{array}{c} X \text{---} \bullet \\ Y \text{---} \bullet \end{array}
 \end{array}$$

Note that we do *not* require these Frobenius morphisms to be natural in X . While morphisms in a hypergraph category need not interact with the Frobenius structure in any particular way, we do require functors between hypergraph categories to preserve it.

Definition 1.4. A functor (F, φ) of hypergraph categories, or **hypergraph functor**, is a strong symmetric monoidal functor (F, φ) such that for each object X the following diagrams commute:

$$\begin{array}{ccc}
 FX \boxtimes FX & \xrightarrow{\mu_{FX}} & FX \\
 & \searrow \varphi & \nearrow F\mu_X \\
 & F(X \otimes X) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 1_{\mathcal{D}} & \xrightarrow{\eta_{FX}} & FX \\
 & \searrow \varphi_1 & \nearrow F\eta_X \\
 & F1_{\mathcal{C}} &
 \end{array}$$

$$\begin{array}{ccc}
 FX & \xrightarrow{\delta_{FX}} & FX \boxtimes FX \\
 & \searrow F\delta_X & \nearrow \varphi^{-1} \\
 & F(X \otimes X) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 FX & \xrightarrow{\epsilon_{FX}} & 1_{\mathcal{D}} \\
 & \searrow F\epsilon_X & \nearrow \varphi^{-1} \\
 & F1_{\mathcal{C}} &
 \end{array}$$

Equivalently, a strong symmetric monoidal functor F is a hypergraph functor if for every X the special commutative Frobenius structure on FX is

$$(FX, F\mu_X \circ \varphi_{X,X}, \varphi_{X,X}^{-1} \circ F\delta_X, F\eta_X \circ \varphi_1, \varphi_1^{-1} \circ F\epsilon_X).$$

Just as monoidal natural transformations themselves are enough as morphisms between symmetric monoidal functors, so too they suffice as morphisms between hypergraph functors. Two hypergraph categories are **hypergraph equivalent** if there exist hypergraph functors with monoidal natural transformations to the identity functors.

The term hypergraph category was introduced recently [?, ?], in reference to the fact that these special commutative Frobenius monoids provide precisely the structure required to draw graphs with ‘hyperedges’: edges connecting any number of inputs to any number of outputs. Again first defined by Walters and Carboni [?], under the name well-supported compact closed categories, in recent years hypergraph categories have been rediscovered a number of times, also appearing under the names dungeon categories [?] and dgs-monoidal categories [?].

1.3.3 Hypergraph categories are self-dual compact closed

Note that if an object X is equipped with a Frobenius monoid structure then the maps

$$\begin{array}{ccc} \text{Diagram 1} & \text{and} & \text{Diagram 2} \\ \epsilon \circ \mu: X \otimes X \rightarrow 1 & & \delta \circ \eta: 1 \rightarrow X \otimes X \end{array}$$

obey both

$$\text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5}$$

and the reflected equations. Thus if an object carries a Frobenius monoid it is also self-dual, and any hypergraph category is a fortiori self-dual compact closed.

We introduce the notation

$$\text{Diagram 6} := \text{Diagram 7} \quad \text{Diagram 8} := \text{Diagram 9}$$

As in any self-dual compact closed category, mapping each morphism $X \text{---} \boxed{f} \text{---} Y$ to its dual morphism

$$\text{Diagram 10}$$

further equips each hypergraph category with a so-called dagger functor—an involutive contravariant endofunctor that is the identity on objects—such that the category is a dagger compact category. Dagger compact categories were first introduced in the context of categorical quantum mechanics [?], under the name strongly compact closed category, and have been demonstrated to be a key structure in diagrammatic reasoning and the logic of quantum mechanics.

Compactness allows us to blur the distinction between composition and the monoidal product of morphisms. Firstly, there is a one-to-one correspondence between morphisms $X \rightarrow Y$ and morphisms $1 \rightarrow X \otimes Y$ given by taking $X \text{---} \boxed{f} \text{---} Y$ to its so-called **name**

$$\text{Diagram 11}$$

By compactness, we have the equation

$$\text{Diagram 12} = \text{Diagram 13}$$

Here the right hand side is the name of the composite $f \circ g$, while the left hand side is the monoidal product post-composed with the map

$$\begin{array}{c} X \text{-----} X \\ Y \text{-----} \\ Y \text{-----} \\ Z \text{-----} Z \end{array}$$

Thus this morphism, the product of a cap and two identity maps, enacts the categorical composition on monoidal products of names. We will make liberal use of this fact.

1.3.4 Coherence

The lack of naturality of the Frobenius maps in hypergraph categories affects some common properties of structured categories. For example, it is not always possible to construct a skeletal hypergraph category hypergraph equivalent to a given hypergraph category: isomorphic objects may be equipped with ‘different’ Frobenius monoids. Similarly, a fully faithful, essentially surjective hypergraph functor does not necessarily define a hypergraph equivalence of categories.

Nonetheless, in this section we prove that every hypergraph category is hypergraph equivalent to a strict hypergraph category. This coherence result will be important in proving that every hypergraph category can be constructed using decorated corelations.

Theorem 1.5. *Every hypergraph category is hypergraph equivalent to a strict hypergraph category. Moreover, the objects of this strict hypergraph category form a free monoid.*

Proof. Let (\mathcal{H}, \otimes) be a hypergraph category. As \mathcal{H} is a fortiori a symmetric monoidal category, a standard construction (see Mac Lane [?, Theorem]) gives an equivalent strict symmetric monoidal category $(\mathcal{H}_{\text{str}}, \cdot)$ with objects finite lists $[x_1, \dots, x_n]$ of objects of \mathcal{H} and morphisms $[x_1, \dots, x_n] \rightarrow [y_1, \dots, y_m]$ those morphisms from $((x_1 \otimes x_2) \otimes \dots) \otimes x_n \rightarrow ((y_1 \otimes y_2) \otimes \dots) \otimes y_m$ in \mathcal{H} . Composition is given by composition in \mathcal{H} .

The monoidal structure is given as follows. Given a list X of objects in \mathcal{H} , write PX for the corresponding monoidal product in \mathcal{H} with all open parathesis at the front. The monoidal product of two objects is given by concatenation \cdot of lists; the monoidal unit is the empty list. The monoidal product of two morphisms is given by

their monoidal product in \mathcal{H} pre- and post-composed with the necessary canonical maps: given $f: X \rightarrow Y$ and $g: Z \rightarrow W$, their product $f \cdot g: X \cdot Y \rightarrow Z \cdot W$ is

$$P(X \cdot Y) \longrightarrow PX \otimes PY \xrightarrow{f \otimes g} PZ \otimes PW \longrightarrow P(Z \cdot W).$$

By design, the associators and unitors are simply identity maps. The braiding $X \cdot Y \rightarrow Y \cdot X$ is given by the braiding $PX \otimes PY \rightarrow PY \otimes PX$ in \mathcal{H} , similarly pre- and post-composed with the necessary canonical maps. This defines a strict symmetric monoidal category [?].

To make \mathcal{H}_{str} into a hypergraph category, we equip each object $[x_1, \dots, x_n]$ with a special commutative Frobenius monoid in a similar way. For example, the multiplication on $[x_1, \dots, x_n]$ is given by

$$\begin{aligned} P([x_1, \dots, x_n] \cdot [x_1, \dots, x_n]) &= ((((((x_1 \otimes x_2) \otimes \dots) \otimes x_n) \otimes x_1) \otimes x_2) \otimes \dots) \otimes x_n \\ &\quad \downarrow \\ &= (((x_1 \otimes x_1) \otimes (x_2 \otimes x_2)) \otimes \dots) \otimes (x_n \otimes x_n) \\ &\quad \downarrow ((\mu_{x_1} \otimes \mu_{x_2}) \otimes \dots) \otimes \mu_{x_n} \\ P([x_1, \dots, x_n]) &= ((x_1 \otimes x_2) \otimes \dots) \otimes x_n \end{aligned}$$

where the first map is the canonical map such that each pair of x_i 's remains in the same order. It is straightforward to check that this defines a hypergraph category.

The strict hypergraph category $(\mathcal{H}_{\text{str}}, \cdot)$	
objects	finite lists $[x_1, \dots, x_n]$ of objects of \mathcal{H}
morphisms	$\text{hom}_{\mathcal{H}_{\text{str}}}([x_1, \dots, x_n], [y_1, \dots, y_m])$ $= \text{hom}_{\mathcal{H}}(((x_1 \otimes x_2) \otimes \dots) \otimes x_n, ((y_1 \otimes y_2) \otimes \dots) \otimes y_m)$
composition	composition of corresponding maps in \mathcal{H}
monoidal product	concatenation of lists
coherence maps	associators and unitors are strict; braiding is inherited from \mathcal{H}
hypergraph maps	lists of hypergraph maps in \mathcal{H}

Our standard construction further gives strong symmetric monoidal functors $P: \mathcal{H}_{\text{str}} \rightarrow \mathcal{H}$ extending the map P above, and $S: \mathcal{H} \rightarrow \mathcal{H}_{\text{str}}$ sending $x \in \mathcal{H}$ to the string $[x]$ of length 1 in \mathcal{H}_{str} . These extend to hypergraph functors.

In detail, the functor P is given on morphisms by taking a map in $\text{hom}_{\mathcal{H}_{\text{str}}}(X, Y)$ to the same map considered now as a map in $\text{hom}_{\mathcal{H}}(PX, PY)$; its coherence maps are given by the canonical maps $PX \otimes PY \rightarrow P(X \cdot Y)$. The functor S is even easier

to define: a morphism $x \rightarrow y$ in \mathcal{H} is by definition a morphism $[x] \rightarrow [y]$ in \mathcal{H}_{str} , so S is a monoidal embedding of \mathcal{H} into \mathcal{H}_{str} .

By Mac Lane's proof of the coherence theorem for monoidal categories these are both strong monoidal functors; by inspection they also preserve hypergraph structure, and so are hypergraph functors. As they already witness an equivalence of symmetric monoidal categories, thus \mathcal{H} and \mathcal{H}_{str} are equivalent as hypergraph categories. \square

1.4 Example: cospan categories

A central example of a hypergraph category is the category $\text{Cospan}(\mathcal{C})$ of cospans in any category \mathcal{C} with finite colimits. We will later see that decorated cospan categories are a generalisation of such categories, and each inherits a hypergraph structure from such.

We first recall the basic definitions. Let \mathcal{C} be a category with finite colimits, writing the coproduct $+$. A **cospan**

$$\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}$$

from X to Y in \mathcal{C} is a pair of morphisms with common codomain. We refer to X and Y as the **feet**, and N as the **apex**. Given two cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ and $X \xrightarrow{i'} N' \xleftarrow{o'} Y$ with the same feet, a **map of cospans** is a morphism $n: N \rightarrow N'$ in \mathcal{C} between the apices such that

$$\begin{array}{ccccc} & & N & & \\ & i \nearrow & & \nwarrow o & \\ X & & & & Y \\ & i' \searrow & & \swarrow o' & \\ & & N' & & \end{array}$$

commutes.

Cospans may be composed using the pushout from the common foot: given cospans $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite cospan is $X \xrightarrow{j_N \circ i_X} N \rightarrow M \xleftarrow{o_Z} Z$.

$N +_Y M \xleftarrow{j_M \circ i_Z} Z$, where the top part of

$$\begin{array}{ccccc}
 & & N +_Y M & & \\
 & \nearrow j_N & & \nwarrow j_M & \\
 X & \xrightarrow{i_X} & N & \xleftarrow{o_Y} & M & \xleftarrow{i_Y} & M & \xleftarrow{o_Z} & Z \\
 & & Y & &
 \end{array}$$

is a pushout square. This composition rule is associative up to isomorphism, and so we may define a category, in fact a symmetric monoidal bicategory, $\text{Cospan}(\mathcal{C})$ with objects the objects of \mathcal{C} and morphisms isomorphism classes of cospans [?].

The symmetric monoidal structure is ‘inherited’ from \mathcal{C} . Indeed, we shall consider any category \mathcal{C} with finite colimits a symmetric monoidal category as follows. Given maps $f: A \rightarrow C$, $g: B \rightarrow C$ with common codomain, the universal property of the coproduct gives a unique map $A + B \rightarrow C$. We call this the **copairing** of f and g , and write it $[f, g]$. The monoidal product on \mathcal{C} is then given by the coproduct $+$, with monoidal unit the initial object \emptyset and coherence maps given by copairing the appropriate identity, inclusion, and initial object maps. For example, the braiding is given by $[\iota_X, \iota_Y]: X + Y \rightarrow Y + X$ where $\iota_X: X \rightarrow Y + X$ and $\iota_Y: Y \rightarrow Y + X$ are the inclusion maps into the coproduct $Y + X$.

The category $\text{Cospan}(\mathcal{C})$ inherits this symmetric monoidal structure from \mathcal{C} as follows. Call a subcategory \mathcal{C} of a category \mathcal{D} **wide** if \mathcal{C} contains all objects of \mathcal{D} , and call a functor that is faithful and bijective-on-objects a **wide embedding**. Note then that we have a wide embedding

$$\mathcal{C} \hookrightarrow \text{Cospan}(\mathcal{C})$$

that takes each object of \mathcal{C} to itself as an object of $\text{Cospan}(\mathcal{C})$, and each morphism $f: X \rightarrow Y$ in \mathcal{C} to the cospan

$$\begin{array}{ccc}
 & Y & \\
 f \nearrow & & \searrow \\
 X & & Y,
 \end{array}$$

where the extended ‘equals’ sign denotes an identity morphism. Now since the monoidal product $+: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is left adjoint to the diagram functor, it preserves colimits, and so extends to a functor $+: \text{Cospan}(\mathcal{C}) \times \text{Cospan}(\mathcal{C}) \rightarrow \text{Cospan}(\mathcal{C})$. The coherence maps are just the images of the coherence maps in \mathcal{C} under this wide

embedding; checking naturality is routine, and clearly they still obey the required axioms.

Write \mathbf{FinSet} for the category of finite sets and functions. Replacing \mathbf{FinSet} with its equivalent strict skeleton, it is well-known, due to Lack [?], that special commutative Frobenius monoids in a monoidal category \mathcal{C} are in one-to-one correspondence with strict monoidal functors $\mathbf{Cosp}(\mathbf{FinSet}) \rightarrow \mathcal{C}$. Over the next few chapters we will further explore this deep link between cospans and special commutative Frobenius monoids and hypergraph categories. To begin, we detail a natural hypergraph structure on $\mathbf{Cosp}(\mathcal{C})$.

This hypergraph structure also comes from copairings of identity morphisms. Call cospans

$$\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} & N & \\ o \nearrow & & \nwarrow i \\ Y & & X \end{array}$$

that are reflections of each other **opposite** cospans. Given any object X in \mathcal{C} , the copairing $[1_X, 1_X]: X + X \rightarrow X$ of two identity maps on X , together with the unique map $!: \emptyset \rightarrow X$ from the initial object to X , define a monoid structure on X . Considering these maps as morphisms in $\mathbf{Cosp}(\mathcal{C})$, we may take them together with their opposites to give a special commutative Frobenius structure on X . It is easily verified that this gives a hypergraph category.

Given $f: X \rightarrow Y$ in \mathcal{C} , abuse notation by writing $f \in \mathbf{Cosp}(\mathcal{C})$ for the cospan $X \xrightarrow{f} Y \xleftarrow{1_Y} Y$, and f^{opp} for the cospan $Y \xrightarrow{1_Y} Y \xleftarrow{f} X$. To summarise:

The hypergraph category $(\mathbf{Cosp}(\mathcal{C}), +)$	
objects	the objects of \mathcal{C}
morphisms	isomorphism classes of cospans in \mathcal{C}
composition	given by pushout
monoidal product	the coproduct in \mathcal{C} .
coherence maps	inherited from $(\mathcal{C}, +)$
hypergraph maps	$\mu = [1, 1]$, $\eta = !$, $\delta = [1, 1]^{\text{opp}}$, $\epsilon = !^{\text{opp}}$.

We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category $\mathbf{Cosp}(\mathcal{C})$; we of course refer instead to the isomorphism class of the said cospan.

It is not difficult to show that we in fact have a functor from the category with objects categories with finite colimits and morphisms colimit preserving functors to the category of hypergraph categories and hypergraph functors. Our next task will be

to show that this extends to a functor from the category of symmetric lax monoidal presheaves over categories with finite colimits. This is known as the decorated cospans construction.

The decorated cospans theorem then leads to the decorated corelations construction, which gives a way to build every hypergraph category using cospans. Write Mon_C for the category with objects functions $[n] = \{1, 2, \dots, n\} \rightarrow C$, where C is some set, and morphisms functions $[n] \rightarrow [m]$ that commute over C . We will prove the following.

Theorem 1.6. *The category of hypergraph categories is equivalent to the category of lax symmetric monoidal functors*

$$(\text{Cospan}(\text{Mon}_C), +) \rightarrow (\text{Set}, \times).$$

varying over sets C .

This theorem, couched in the language of algebras for operads, is also the subject of a forthcoming paper by Vagner, Spivak, and Schultz [?].

Chapter 2

Decorated cospans: from closed systems to open

In the previous chapter we discussed hypergraph categories as a model for network languages, and showed that cospans provide a method of constructing such categories. In many situations, however, we wish to model our systems not just with cospans, but ‘decorated’ cospans, where the apex of each cospan is equipped with some extra structure. In this chapter we detail a method for composing such decorated cospans.

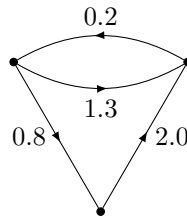
This chapter is based on [\[Fon15\]](#).

2.1 From closed systems to open

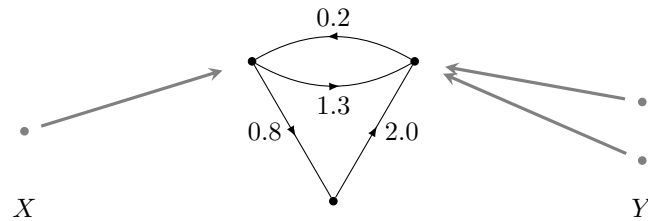
The key idea in this chapter is that we may take our network diagrams, mark points of interconnection, or terminals, using cospans, and connect these terminals with others using pushouts. This marking of terminals defines a boundary of the ‘system’, turning it from a closed system to an open system. This process ‘freely’ constructs a hypergraph category from a library of network pieces.

It has been recognised for some time that spans and cospans provide an intuitive framework for composing network diagrams [\[?\]](#), and the material we develop here is a variant on this theme. In the case of finite graphs, the intuition reflected is this: given two graphs, we may construct a third by gluing chosen vertices of the first with chosen vertices of the second. It is our goal in this article to view this process as composition of morphisms in a category, in a way that also facilitates the construction of a composition rule for any semantics associated to the diagrams, and a functor between these two resulting categories.

To see how this works, let us start with the following graph:

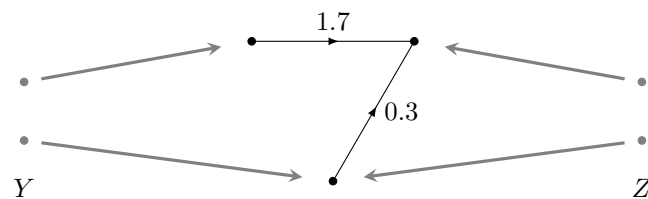


We shall work with labelled, directed graphs, as the additional data help highlight the relationships between diagrams. Now, for this graph to be a morphism, we must equip it with some notion of ‘input’ and ‘output’. We do this by marking vertices using functions from finite sets:

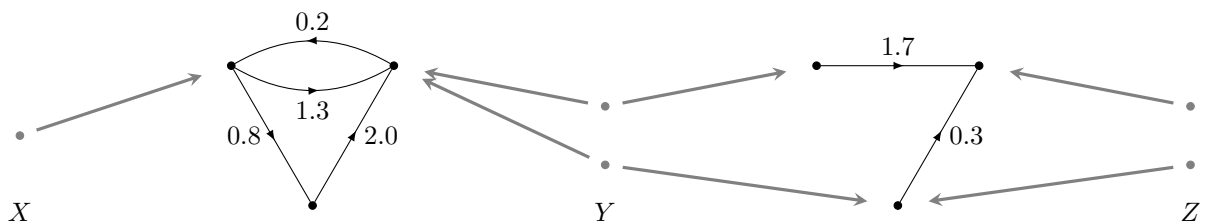


Let N be the set of vertices of the graph. Here the finite sets X , Y , and N comprise one, two, and three elements respectively, drawn as points, and the values of the functions $X \rightarrow N$ and $Y \rightarrow N$ are indicated by the grey arrows. This forms a cospan in the category of finite sets, one with the set at the apex decorated by our given graph.

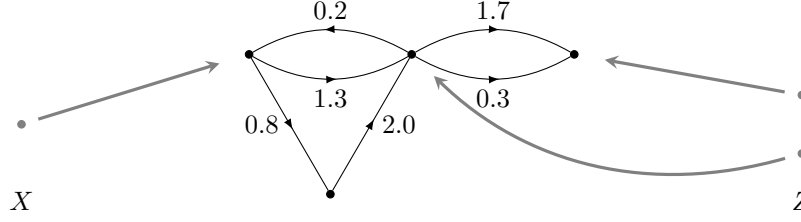
Given another such decorated cospan with input set equal to the output of the above cospan



composition involves gluing the graphs along the identifications



specified by the shared foot of the two cospans. This results in the decorated cospan



The decorated cospan framework generalises this intuitive construction.

More precisely: fix a set L . Then given a finite set N , we may talk of the collection of finite L -labelled directed multigraphs, to us just L -graphs or simply graphs, that have N as their set of vertices. Write such a graph (N, E, s, t, r) , where E is a finite set of edges, $s: E \rightarrow N$ and $t: E \rightarrow N$ are functions giving the source and target of each edge respectively, and $r: E \rightarrow L$ equips each edge with a label from the set L . Next, given a function $f: N \rightarrow M$, we may define a function from graphs on N to graphs on M mapping (N, E, s, t, r) to $(M, E, f \circ s, f \circ t, r)$. After dealing appropriately with size issues, this gives a lax monoidal functor from $(\text{FinSet}, +)$ to (Set, \times) .¹

Now, taking any lax monoidal functor $(F, \varphi): (\mathcal{C}, +) \rightarrow (\mathcal{D}, \otimes)$ with \mathcal{C} having finite colimits and coproduct written $+$, the decorated cospan category associated to F has as objects the objects of \mathcal{C} , and as morphisms pairs comprising a cospan in \mathcal{C} together with some morphism $1 \rightarrow FN$, where 1 is the unit in (\mathcal{D}, \otimes) and N is the apex of the cospan. In the case of our graph functor, this additional data is equivalent to equipping the apex N of the cospan with a graph. We thus think of our morphisms as having two distinct parts: an instance of our chosen structure on the apex, and a cospan describing interfaces to this structure. Our first theorem says that when (\mathcal{D}, \otimes) is braided monoidal and (F, φ) lax braided monoidal, we may further give this data a composition rule and monoidal product such that the resulting ‘decorated cospan category’ is hypergraph.

Suppose now we have two such lax monoidal functors; we then have two such decorated cospan categories. Our second theorem is that, given also a monoidal natural transformation between these functors, we may construct a strict hypergraph functor between their corresponding decorated cospan categories. These natural transformations can often be specified by some semantics associated to some type of topological

¹Here $(\text{FinSet}, +)$ is the monoidal category of finite sets and functions with disjoint union as monoidal product, and (Set, \times) is the category of sets and functions with cartesian product as monoidal product. One might ensure the collection of graphs forms a set in a number of ways. One such method is as follows: the categories of finite sets and finite graphs are essentially small; replace them with equivalent small categories. We then constrain the graphs (N, E, s, t, r) to be drawn only from the objects of our small category of finite graphs.

diagram. A trivial case of such is assigning to a finite graph its number of vertices, but richer examples abound, including assigning to a directed graph with edges labelled by rates its depicted Markov process, or assigning to an electrical circuit diagram the current–voltage relationship such a circuit would impose.

The structure of this chapter is straightforward: in the following section we review some basic background material, which then allows us to give the constructions of decorated cospan categories and their functors in Sections 2.2 and 2.3 respectively. We then explicate these definitions through some examples in Section 2.5.

2.2 Decorated cospan categories

Motivated by the previous section, we make the following definition.

Definition 2.1. Let \mathcal{C} be a category with finite colimits, and

$$(F, \varphi): (\mathcal{C}, +) \longrightarrow (\mathcal{D}, \otimes)$$

be a lax monoidal functor. We define a **decorated cospan**, or more precisely an F -decorated cospan, to be a pair

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right)$$

comprising a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ in \mathcal{C} together with an element $1 \xrightarrow{s} FN$ of the F -image FN of the apex of the cospan. We shall call the element $1 \xrightarrow{s} FN$ the **decoration** of the decorated cospan. A **morphism of decorated cospans**

$$n: (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN) \longrightarrow (X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y, 1 \xrightarrow{s'} FN')$$

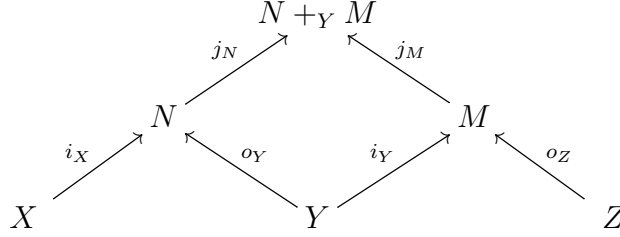
is a morphism $n: N \rightarrow N'$ of cospans such that $Fn \circ s = s'$.

In this section the task is to prove that decorated cospans can be used to construct hypergraph categories.

2.2.1 Composing decorated cospans

First, we show that isomorphism classes of decorated cospans form the morphisms of a category.

Proposition 2.2. *There is a category $FCospan$ of F -decorated cospans, with objects the objects of \mathcal{C} , and morphisms isomorphism classes of F -decorated cospans. On representatives of the isomorphism classes, composition in this category is given by pushout of cospans in \mathcal{C}*



paired with the composite

$$1 \xrightarrow{\lambda^{-1}} 1 \otimes 1 \xrightarrow{s \otimes t} FN \otimes FM \xrightarrow{\varphi_{N,M}} F(N + M) \xrightarrow{F[j_N, j_M]} F(N +_Y M)$$

of the tensor product of the decorations with the F -image of the copairing of the pushout maps.

Proof. The identity morphism on an object X in a decorated cospan category is simply the identity cospan decorated as follows:

$$\left(\begin{array}{c} X \\ \parallel \quad \parallel \\ X \quad X \end{array}, \quad \begin{array}{c} FX \\ \uparrow_{F!} \\ F\emptyset \\ \uparrow_{\varphi_1} \\ 1 \end{array} \right).$$

We must check that the composition defined is well-defined on isomorphism classes, is associative, and, with the above identity maps, obeys the unitality axiom. These are straightforward, but lengthy, exercises in using the available colimits and monoidal structure to show that the relevant diagrams of decorations commute.

Representation independence for composition of isomorphism classes of decorated cospans. Let

$$n: (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN) \longrightarrow (X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y, 1 \xrightarrow{s'} FN')$$

and

$$m: (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z, 1 \xrightarrow{t} FM) \longrightarrow (Y \xrightarrow{i'_Y} M' \xleftarrow{o'_Z} Z, 1 \xrightarrow{t'} FM')$$

be isomorphisms of decorated cospans. We wish to show that the composite of the decorated cospans on the left is isomorphic to the composite of the decorated cospans

on the right. As discussed, it is well-known that the composite cospans are isomorphic, and it remains to us to check the decorations agree too. Let $p: N +_Y M \rightarrow N' +_Y M'$ be the isomorphism given by the universal property of the pushout and the isomorphisms $n: N \rightarrow N'$ and $m: M \rightarrow M'$. Then the two decorations in question are given by the top and bottom rows of the following diagram.

$$\begin{array}{ccccccc}
 1 & \xrightarrow{\lambda^{-1}} & 1 \otimes 1 & \begin{array}{l} \nearrow s \otimes t \\ \searrow s' \otimes t' \end{array} & \begin{array}{c} FN \otimes FM \\ \sim \downarrow Fn \otimes Fm \end{array} & \xrightarrow{\varphi_{N,M}} & F(N + M) \xrightarrow{F[j_N, j_M]} F(N +_Y M) \\
 & & & \text{(I)} & \sim \downarrow Fn \otimes Fm \quad \text{(F)} & & \sim \downarrow F(n+m) \quad \text{(C)} & \sim \downarrow Fp \\
 & & & & FN' \otimes FM' & \xrightarrow{\varphi_{N',M'}} & F(N' + M') \xrightarrow{F[j_{N'}, j_{M'}]} F(N' +_Y M')
 \end{array}$$

The triangle (I) commutes as n and m are morphisms of decorated cospans and $- \otimes -$ is functorial, (F) commutes by the monoidality of F , and (C) commutes by properties of colimits in \mathcal{C} and the functoriality of F . This proves the claim.

Associativity. Suppose we have morphisms

$$\begin{aligned}
 (X &\xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN), \\
 (Y &\xrightarrow{i_Y} M \xleftarrow{o_Z} Z, \quad 1 \xrightarrow{t} FM), \\
 (Z &\xrightarrow{i_Z} P \xleftarrow{o_W} W, \quad 1 \xrightarrow{u} FP).
 \end{aligned}$$

It is well-known that composition of isomorphism classes of cospans via pushout of representatives is associative; this follows from the universal properties of the relevant colimit. We must check that the pushforward of the decorations is also an associative process. Write

$$\tilde{a}: (N +_Y M) +_Z P \longrightarrow N +_Y (M +_Z P)$$

for the unique isomorphism between the two pairwise pushouts constructions from the above three cospans. Consider then the following diagram, with leftmost column the decoration obtained by taking the composite of the first two morphisms first, and the rightmost column the decoration obtained by taking the composite of the last two

morphisms first.

$$\begin{array}{c}
 \begin{array}{ccc}
 F((N+Y M)+Z P) & \xrightarrow{F\tilde{a}} & F(N+Y(M+Z P)) \\
 \uparrow F[j_{N+Y M}, j_P] & & \uparrow F[j_N, j_{M+Z P}] \\
 F((N+Y M)+P) & & F(N+(M+Z P)) \\
 \uparrow F([j_N, j_M]+1_P) & & \uparrow F(1_N+[j_M, j_P]) \\
 F((N+M)+P) & \xrightarrow{Fa} & F(N+(M+P)) \\
 \uparrow \varphi_{N+Y M, P} & & \uparrow \varphi_{N, M+P} \\
 F(N+Y M) \otimes F P & & F N \otimes F(M+P) \\
 \uparrow F[j_N, j_M] \otimes 1_{F P} & & \uparrow 1_{F N} \otimes F[j_M, j_P] \\
 F(N+M) \otimes F P & & F N \otimes F(M+P) \\
 \uparrow \varphi_{N, M} \otimes 1_{F P} & & \uparrow 1_{F N} \otimes \varphi_{M, P} \\
 (F N \otimes F M) \otimes F P & \xrightarrow{a} & F N \otimes (F M \otimes F P)
 \end{array} \\
 \begin{array}{ccc}
 (s \otimes t) \otimes u & & s \otimes (t \otimes u) \\
 \uparrow & & \uparrow \\
 (1 \otimes 1) \otimes 1 & \xrightarrow{a} & 1 \otimes (1 \otimes 1) \\
 \uparrow \rho^{-1} \otimes 1 & & \uparrow 1 \otimes \lambda^{-1} \\
 1 \otimes 1 & & 1 \otimes 1 \\
 \uparrow \lambda^{-1} \\
 1
 \end{array}
 \end{array}$$

(D1) (D2) (F1) (F2) (F3) (C)

This diagram commutes as (D1) is the triangle coherence equation for the monoidal category (\mathcal{D}, \otimes) , (D2) is naturality for the associator a , (F1) is the associativity condition for the monoidal functor F , (F2) and (F3) commute by the naturality of φ , and (C) commutes as it is the F -image of a hexagon describing the associativity of the pushout. This shows that the two decorations obtained by the two different orders of composition of our three morphisms are equal up to the unique isomorphism \tilde{a} between the two different pushouts that may be obtained. Our composition rule is hence associative.

Identity morphisms. We shall show that the claimed identity morphism on Y , the decorated cospan

$$(Y \xrightarrow{1_Y} Y \xleftarrow{1_Y} Y, 1 \xrightarrow{F! \circ \varphi_1} FY),$$

is an identity for composition on the right; the case for composition on the left is similar. The cospan in this pair is known to be the identity cospan in $\text{Cospan}(\mathcal{C})$. We thus need to check that, given a morphism

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN),$$

the composite of the product $s \otimes (F! \circ \varphi_1)$ with the F -image of the copairing $[1_N, i_Y]: N + Y \rightarrow N$ of the pushout maps is again the same element s ; this composite being, by definition, the decoration of the composite of the given morphism and the claimed identity map. This is shown by the commutativity of the diagram below, with the path along the lower edge equal to the aforementioned pushforward.

This diagram commutes as each subdiagram commutes: (D1) commutes by the naturality of ρ , (D2) by the functoriality of the monoidal product in \mathcal{D} , (F1) by the unit axiom for the monoidal functor F , (F2) by the naturality of φ , and (C) due to the properties of colimits in \mathcal{C} and the functoriality of F .

We thus have a category. □

Remark 2.3. While at first glance it might seem surprising that we can construct a composition rule for decorations $s: 1 \rightarrow FN$ and $t: 1 \rightarrow FM$ just from monoidal structure, the copair $[j_N, j_M]: N + M \rightarrow N +_Y M$ of the pushout maps contains the data necessary to compose them. Indeed, this is the key insight of the decorated cospan construction. To wit, the coherence maps for the lax monoidal functor allow us to construct an element of $F(N + M)$ from the monoidal product $s \otimes t$ of the

decorations, and we may then post-compose with $F[j_N, j_M]$ to arrive at an element of $F(N +_Y M)$. The map $[j_N, j_M]$ encodes the identification of the image of Y in N with the image of the same in M , and so describes merging the ‘overlap’ of the two decorations.

2.2.2 The hypergraph structure

Our main theorem is that when *braided* monoidal structure is present, the category of decorated cospans is a hypergraph category, and moreover one into which the category of ‘undecorated’ cospans widely embeds. This embedding motivates the monoidal and hypergraph structures we put on $FCospan$.

Theorem 2.4. *Let \mathcal{C} be a category with finite colimits, (\mathcal{D}, \otimes) a braided monoidal category, and $(F, \varphi): (\mathcal{C}, +) \rightarrow (\mathcal{D}, \otimes)$ be a lax braided monoidal functor. Then we may give $FCospan$ a symmetric monoidal and hypergraph structure such that there is a wide embedding of hypergraph categories*

$$Cospan(\mathcal{C}) \hookrightarrow FCospan.$$

We first prove a lemma. Recall that the identity decorated cospan has apex decorated by $1 \xrightarrow{\varphi_1} F\emptyset \xrightarrow{F!} FX$. Given any cospan $X \rightarrow N \leftarrow Y$, we call the decoration $1 \xrightarrow{\varphi_1} F\emptyset \xrightarrow{F!} FN$ the **empty decoration** on N . Empty decorations are indeed empty: when composed with other decorations, they have no impact.

Lemma 2.5. *Let*

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN),$$

be a decorated cospan, and suppose we have an empty-decorated cospan

$$(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z, \quad 1 \xrightarrow{\varphi \circ F!} FM).$$

Then the composite of these decorated cospans is

$$(X \xrightarrow{j_N \circ i_X} N +_Y M \xleftarrow{j_M \circ o_Z} Z, \quad 1 \xrightarrow{Fj_N \circ s} F(N +_Y M)).$$

In particular, the decoration on the composite is the decoration s pushed forward along the F -image of the map $j_N: N \rightarrow N +_Y M$ to become an F -decoration on $N +_Y M$. The analogous statement also holds for composition with an empty-decorated cospan on the left.

Proof. As is now familiar, a statement of this sort is proved by a large commutative diagram:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 1 & \xrightarrow{s} & FN & \xrightarrow{Fj_N} & F(N +_Y M) \\
 \downarrow \rho_1^{-1} & & \uparrow \rho_{FN} & & \uparrow F[j_N, !] \\
 & (D1) & & (F1) & (C1) \\
 1 \otimes 1 & \xrightarrow{s \otimes 1} & FN \otimes 1 & \xrightarrow{1_{FN} \otimes \varphi_1} & FN \otimes F\emptyset & \xrightarrow{\varphi_{N, \emptyset}} & F(N + \emptyset) \\
 & & \downarrow s \otimes (F! \circ \varphi_1) & & \downarrow 1_{FN} \otimes F! & & \downarrow F(1_N + !) \\
 & & FN \otimes FM & \xrightarrow{\varphi_{N, M}} & F(N + M)
 \end{array} \\
 \begin{array}{c}
 \text{Additional arrows and labels:} \\
 - \text{From } 1 \otimes 1 \text{ to } FN \otimes 1: s \otimes 1 \\
 - \text{From } FN \otimes 1 \text{ to } FN \otimes F\emptyset: 1_{FN} \otimes \varphi_1 \\
 - \text{From } FN \otimes F\emptyset \text{ to } F(N + \emptyset): \varphi_{N, \emptyset} \\
 - \text{From } FN \otimes 1 \text{ to } FN \otimes FM: s \otimes (F! \circ \varphi_1) \\
 - \text{From } FN \otimes F\emptyset \text{ to } FN \otimes FM: 1_{FN} \otimes F! \\
 - \text{From } F(N + \emptyset) \text{ to } F(N + M): F(1_N + !) \\
 - \text{From } FN \otimes FM \text{ to } F(N + M): \varphi_{N, M} \\
 - \text{From } FN \text{ to } FN \otimes 1: \rho_{FN} \\
 - \text{From } FN \otimes F\emptyset \text{ to } FN: F\rho_N = F[1_N, !] \\
 - \text{From } F(N + \emptyset) \text{ to } FN: F[j_N, !] \\
 - \text{From } F(N + \emptyset) \text{ to } F(N + M): F[j_N, j_M] \\
 - \text{From } 1 \text{ to } FN: s \\
 - \text{From } FN \text{ to } F(N +_Y M): Fj_N
 \end{array}
 \end{array}$$

The subdiagrams in this diagram commute for the same reasons as their corresponding regions in the previous diagram for identity morphisms. \square

Proof of Theorem 2.4. We first prove that we have a wide embedding of categories, and then use this to transfer monoidal and hypergraph structure onto $FCospan$.

Embedding. We define a functor

$$Cospan(\mathcal{C}) \hookrightarrow FCospan.$$

mapping each object of $Cospan(\mathcal{C})$ to itself as an object of $FCospan$, and each cospan in \mathcal{C} to the same cospan decorated with the empty decoration on its apex.

To check this is well-defined, we must check that the composite of two empty-decorated cospans is again empty-decorated. This is an immediate consequence of Lemma 2.5: the composite of two empty-decorated cospans is again empty-decorated. This implies the functoriality of the wide embedding $Cospan(\mathcal{C}) \hookrightarrow FCospan$.

Monoidal structure. We define the monoidal product of objects X and Y of $FCospan$ to be their coproduct $X + Y$ in \mathcal{C} , and define the monoidal product of decorated cospans $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN)$ and $(X' \xrightarrow{i_{X'}} N' \xleftarrow{o_{Y'}} Y', 1 \xrightarrow{t} FN')$ to be

$$\left(\begin{array}{ccc} & N + N' & \\ i_X + i_{X'} \nearrow & & \nwarrow o_Y + o_{Y'} \\ X + X' & & Y + Y' \end{array} \right), \quad \begin{array}{c} F(N + N') \\ \uparrow \varphi_{N, N'} \\ FN \otimes FN' \\ \uparrow s \otimes t \\ 1 \otimes 1 \\ \uparrow \lambda^{-1} \\ 1 \end{array}$$

Using the braiding in \mathcal{D} , we can show that this proposed monoidal product is functorial.

Indeed, suppose we have decorated cospans

$$\begin{aligned} (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN), & \quad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z, 1 \xrightarrow{t} FM), \\ (U \xrightarrow{i_U} P \xleftarrow{o_V} V, 1 \xrightarrow{u} FP), & \quad (V \xrightarrow{i_V} Q \xleftarrow{o_W} W, 1 \xrightarrow{v} FQ). \end{aligned}$$

We must check the so-called interchange law: that the composite of the column-wise monoidal products is equal to the monoidal product of the row-wise composites.

Again, for the cospans we take this equality as familiar fact. Write

$$b: (N + P) +_{(Y+V)} (M + Q) \xrightarrow{\sim} (N +_Y M) + (P +_V Q).$$

for the isomorphism between the two resulting representatives of the isomorphism class of cospans. The two resulting decorations are then given by the leftmost and rightmost columns respectively of the diagram below.

$$\begin{array}{ccc} F((N+P)_{(Y+V)}(M+Q)) & \xrightarrow[\sim]{Fb} & F((N+_Y M) + (P+_V Q)) \\ \uparrow F[j_{N+P}, j_{M+Q}] & \nearrow (C) & \uparrow \varphi_{N+_Y M, P+_V Q} \\ F((N+P) + (M+Q)) & & F(N+_Y M) \otimes F(P+_V Q) \\ \uparrow \varphi_{N+P, M+Q} & \nwarrow (F2) & \uparrow F[j_N, j_M] \otimes F[j_P, j_Q] \\ F(N+P) \otimes F(M+Q) & & F(N+M) \otimes F(P+Q) \\ \uparrow \varphi_{N,P} \otimes \varphi_{M,Q} & \nwarrow (F1) & \uparrow \varphi_{N,M} \otimes \varphi_{P,Q} \\ (FN \otimes FP) \otimes (FM \otimes FQ) & \xrightarrow{\quad} & (FN \otimes FM) \otimes (FP \otimes FQ) \\ \nwarrow (s \otimes u) \otimes (t \otimes v) & (D) & \nearrow (s \otimes t) \otimes (u \otimes v) \\ & (1 \otimes 1) \otimes (1 \otimes 1) & \\ & \uparrow (\lambda^{-1} \otimes \lambda^{-1}) \circ \lambda^{-1} & \\ & 1 & \end{array}$$

These two decorations are related by the isomorphism b as the diagram commutes. We argue this more briefly than before, as the basic structure of these arguments is

now familiar to us. Briefly then, there exist dotted arrows of the above types such that the subdiagram (D) commutes by the naturality of the associators and braiding in \mathcal{D} , (F1) commutes by the coherence diagrams for the braided monoidal functor F , (F2) commutes by the naturality of the coherence map φ for F , and (C) commutes by the properties of colimits in \mathcal{C} and the functoriality of F .

Using now routine methods, it also is straightforward to show that the monoidal product of identity decorated cospans on objects X and Y is the identity decorated cospan on $X + Y$; for the decorations this amounts to the observation that the monoidal product of empty decorations is again an empty decoration.

Choosing associator, unitors, and braiding in $FCospan$ to be the images of those in $Cospan(\mathcal{C})$, we have a symmetric monoidal category. These transformations remain natural transformations when viewed in the category of F -decorated cospans as they have empty decorations.

We consider the case of the left unitor in detail; the naturality of the right unitor, associator, and braiding follows similarly, using the relevant axiom where here we use the left unitality axiom.

Given a decorated cospan $(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN)$, we must show that the diagram of decorated cospans

$$\begin{array}{ccccc} X + \emptyset & \xrightarrow{i+1} & N + \emptyset & \xleftarrow{i+1} & Y + \emptyset \\ \lambda_c \downarrow & & & & \downarrow \lambda_c \\ X & \xrightarrow{i} & N & \xleftarrow{o} & Y \end{array}$$

commutes, where the λ_c are the maps of the left unitor in \mathcal{C} considered as empty-decorated cospans, and where the top cospan has decoration

$$1 \xrightarrow{\lambda_{\mathcal{D}}^{-1}} 1 \otimes 1 \xrightarrow{s \otimes \varphi_1} FN \otimes F\emptyset \xrightarrow{\varphi_{N,\emptyset}} F(N + \emptyset),$$

and the lower cospan simply has decoration $1 \xrightarrow{s} FN$.

Now as the λ are isomorphisms in \mathcal{C} and have empty decorations, the composite through the upper right corner is isomorphic to the decorated cospan

$$(X + \emptyset \xrightarrow{i+1} N + \emptyset \xleftarrow{(o+1) \circ \lambda_c^{-1}} Y, 1 \xrightarrow{\varphi_{N,\emptyset} \circ (s \otimes \varphi_1) \circ \lambda_{\mathcal{D}}^{-1}} F(N + \emptyset)),$$

while the composite through the lower left corner is isomorphic to the decorated cospan

$$(X + \emptyset \xrightarrow{[i,!]} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN).$$

Furthermore, $\lambda_{\mathcal{C}}: N + \emptyset \rightarrow N$ gives an isomorphism between these two cospans, and the naturality of the left unitor and the left unitality axiom in \mathcal{D} imply that this is in fact an isomorphism of decorated cospans:

$$\begin{array}{c}
 \begin{array}{ccccc}
 1 & \xrightarrow{\quad s \quad} & FN \\
 \lambda_{\mathcal{D}}^{-1} \downarrow & & \nearrow \lambda_{\mathcal{D}} \\
 1 \otimes 1 & \xrightarrow{s \otimes 1} FN \otimes 1 & \xrightarrow{1 \otimes \varphi_1} FN \otimes F\emptyset & \xrightarrow{\varphi_{N, \emptyset}} & F(N + \emptyset) \\
 & & & & \uparrow F\lambda_{\mathcal{C}}
 \end{array}
 \end{array}$$

This the coherence maps are indeed natural transformations. Moreover, they obey the required coherence laws as they are images of maps that obey these laws in $\text{Cospan}(\mathcal{C})$.

Hypergraph structure. Similarly, to arrive at the hypergraph structure on $FCospan$, we simply equip each object X with the image of the special commutative Frobenius monoid specified by the hypergraph structure of $\text{Cospan}(\mathcal{C})$. It is evident that this choice of structures implies the above wide embedding is a hypergraph functor. \square

In summary, the hypergraph category $FCospan$ comprises:

The hypergraph category $(FCospan, +)$	
objects	the objects of \mathcal{C}
morphisms	isomorphism classes of F -decorated cospans in \mathcal{C}
composition	given by pushout
monoidal product	the coproduct in \mathcal{C}
coherence maps	empty decorated maps from $\text{Cospan}(\mathcal{C})$
hypergraph maps	empty decorated maps from $\text{Cospan}(\mathcal{C})$

Note that if the monoidal unit in (\mathcal{D}, \otimes) is the initial object, then each object only has one possible decoration: the empty decoration. This immediately implies the following corollary.

Corollary 2.6. *Let $1_{\mathcal{C}}: (\mathcal{C}, +) \rightarrow (\mathcal{C}, +)$ be the identity functor on a category \mathcal{C} with finite colimits. Then $\text{Cospan}(\mathcal{C})$ and $1_{\mathcal{C}}\text{Cospan}$ are isomorphic as hypergraph categories.*

Thus we see that there is always a hypergraph functor between decorated cospan categories $1_{\mathcal{C}}\text{Cospan} \rightarrow FCospan$. This provides an example of a more general way to construct hypergraph functors between decorated cospan categories. We detail this in the next section.

2.2.3 A bicategory of decorated cospans

Just as cospans form the morphisms of a symmetric monoidal bicategory, so too do decorated cospans. In this bicategory the 2-morphisms are given by morphisms of decorated cospans: morphisms n in \mathcal{C} such that the diagrams

$$\begin{array}{ccc}
 & N & \\
 i \nearrow & & \nwarrow o \\
 X & & Y \\
 i' \searrow & & \swarrow o' \\
 & N' &
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 & FN & \\
 s \nearrow & & \nwarrow \\
 1 & & \\
 s' \searrow & & \swarrow \\
 & FN' &
 \end{array}$$

commute. For more details see Courser [?]. These 2-morphisms could be employed to model transformations, coarse grainings, rewrite rules, and so on of open systems and their representations.

2.3 Functors between decorated cospan categories

Decorated cospans provide a setting for formulating various operations that we might wish to enact on the decorations, including the composition of these decorations, both sequential and monoidal, as well as dagger, dualising, and other operations afforded by the hypergraph structure. We now observe that these operations are formulated in a systematic way, so that transformations of the decorating structure—that is, monoidal transformations between the lax monoidal functors defining decorated cospan categories—respect these operations.

Theorem 2.7. *Let $\mathcal{C}, \mathcal{C}'$ be categories with finite colimits, abusing notation to write the coproduct in each category $+$, and $(\mathcal{D}, \otimes), (\mathcal{D}', \boxtimes)$ be braided monoidal categories. Further let*

$$(F, \varphi): (\mathcal{C}, +) \longrightarrow (\mathcal{D}, \otimes) \quad \text{and} \quad (G, \gamma): (\mathcal{C}', +) \longrightarrow (\mathcal{D}', \boxtimes)$$

be lax braided monoidal functors. This gives rise to decorated cospan categories $FCospan$ and $GCospan$.

Suppose then that we have a finite colimit-preserving functor $A: \mathcal{C} \rightarrow \mathcal{C}'$ with accompanying natural isomorphism $\alpha: A(-) + A(-) \Rightarrow A(- + -)$, a lax monoidal functor $(B, \beta): (\mathcal{D}, \otimes) \rightarrow (\mathcal{D}', \boxtimes)$, and a monoidal natural transformation $\theta: (B \circ$

$F, B\varphi \circ \beta) \Rightarrow (G \circ A, G\alpha \circ \gamma)$. This may be depicted by the diagram:

$$\begin{array}{ccc} (\mathcal{C}, +) & \xrightarrow{(F, \varphi)} & (\mathcal{D}, \otimes) \\ (A, \alpha) \downarrow & \not\cong_{\theta} & \downarrow (B, \beta) \\ (\mathcal{C}', +) & \xrightarrow{(G, \gamma)} & (\mathcal{D}', \boxtimes). \end{array}$$

Then we may construct a hypergraph functor

$$(T, \tau): FCospan \longrightarrow GCospan$$

mapping objects $X \in FCospan$ to $AX \in GCospan$, and morphisms

$$\left(\begin{array}{c} \begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1_{\mathcal{D}} \end{array} \end{array} \right) \quad to \quad \left(\begin{array}{c} \begin{array}{ccc} & AN & \\ Ai \nearrow & & \nwarrow Ao \\ AX & & AY \end{array}, \quad \begin{array}{c} GAN \\ \uparrow \theta_N \\ BFN \\ \uparrow B_s \\ B1_{\mathcal{D}} \\ \uparrow \beta_1 \\ 1_{\mathcal{D}'} \end{array} \end{array} \right).$$

Moreover, (T, τ) is a strict monoidal functor if and only if (A, α) is.

Proof. We must prove that (T, τ) is a functor, is strong symmetric monoidal, and that it preserves the special commutative Frobenius structure on each object.

Functoriality. Checking the functoriality of T is again an exercise in applying the properties of structure available—in this case the colimit-preserving nature of A and the monoidality of (\mathcal{D}, \boxtimes) , (B, β) , and θ —to show that the relevant diagrams of decorations commute.

In detail, let

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN) \quad \text{and} \quad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z, \quad 1 \xrightarrow{t} FM),$$

be morphisms in $FCospan$. As the composition of the cospan part is by pushout in \mathcal{C} in both cases, and as T acts as the colimit preserving functor A on these cospans, it is clear that T preserves composition of isomorphism classes of cospans. Write

$$c: AX +_{AY} AZ \xrightarrow{\sim} A(X +_Y Z)$$

for the isomorphism from the cospan obtained by composing the A -images of the above two decorated cospans to the cospan obtained by taking the A -image of their composite. To see that this extends to an isomorphism of decorated cospans, observe

that the decorations of these two cospans are given by the rightmost and leftmost columns respectively in the following diagram:

$$\begin{array}{ccccc}
 GA(N +_Y M) & \xleftarrow[\sim]{Gc} & & G(AN +_{AY} AM) & \\
 \uparrow \theta_{N+YM} & \swarrow GA[j_N, j_M] & (A) & \uparrow G[j_{AN}, j_{AM}] & \\
 BF(N +_Y M) & \xleftarrow[\sim]{G\alpha} & GA(N + M) & \xleftarrow[\sim]{G\alpha} & G(AN + AM) \\
 \uparrow BF[j_N, j_M] & \nearrow \theta_{N,M} & (T2) & \uparrow \gamma_{AN, AM} & \\
 BF(N + M) & & (T1) & GAN \boxtimes GAM & \\
 \uparrow B\varphi_{N,M} & & \beta_{FN, FM} & \uparrow \theta_N \boxtimes \theta_M & \\
 B(FN \otimes FM) & \xleftarrow[\sim]{\beta_{FN, FM}} & & BFN \boxtimes BFM & \\
 \uparrow B(s \otimes t) & & (B2) & \uparrow Bs \boxtimes Bt & \\
 B(1_{\mathcal{D}} \otimes 1_{\mathcal{D}}) & \xleftarrow[\sim]{\beta_{1,1}} & & B1_{\mathcal{D}} \boxtimes B1_{\mathcal{D}} & \\
 \uparrow \beta \lambda_1^{-1} & & (B1) & \nearrow \beta_1 \boxtimes 1 & \\
 B1_{\mathcal{D}} & \xrightarrow[\sim]{\lambda_{B1}^{-1}} & 1_{\mathcal{D}'} \boxtimes B1_{\mathcal{D}} & \xleftarrow[\sim]{1 \boxtimes \beta_1} & 1_{\mathcal{D}'} \boxtimes 1_{\mathcal{D}'} \\
 & \searrow \beta_1 & (D1) & \nearrow \lambda_1^{-1} & \\
 & & 1_{\mathcal{D}'} & &
 \end{array}$$

(D2)

From bottom to top, (D1) commutes by the naturality of λ , (D2) by the functoriality of the monoidal product $- \boxtimes -$, (B1) by the unit law for (B, β) , (B2) by the naturality of β , (T1) by the monoidality of the natural transformation θ , (T2) by the naturality of θ , and (A) by the colimit preserving property of A and the functoriality of G .

We must also show that identity morphisms are mapped to identity morphisms. Let

$$(X \xrightarrow{1_X} X \xleftarrow{1_X} X, 1 \xrightarrow{F! \circ \varphi_1} FX)$$

be the identity morphism on some object X in the category of F -decorated cospans. Now this morphism has T -image

$$(AX \xrightarrow{1_{AX}} AX \xleftarrow{1_{AX}} AX, 1 \xrightarrow{\theta_X \circ B(F! \circ \varphi_1) \circ \beta_1} GAX).$$

But we have the following diagram

$$\begin{array}{ccccccc}
 & & & BF\emptyset_{\mathcal{C}} & & & \\
 & \nearrow B\varphi_1 & & \downarrow \theta_{\emptyset} & \nwarrow BF! & & \\
 & B1_{\mathcal{D}} & & & & & BFX \\
 & \nearrow \beta_1 & & \text{(T1)} & & \text{(T2)} & \nearrow \theta_X \\
 1_{\mathcal{D}'} & \xrightarrow{\gamma_1} & G\emptyset_{\mathcal{C}'} & \xrightarrow{G\alpha_1} & GA\emptyset_{\mathcal{C}} & \xrightarrow{GA!} & GAX \\
 & \searrow \gamma_1 & & \text{(A1)} & & \text{(A2)} & \\
 & & & \sim & & & \\
 & & & G! & & & \\
 & & & G\emptyset_{\mathcal{C}'} & & & \\
 & & & \nearrow G! & & & \\
 & & & & & &
 \end{array}$$

Here (A1) and (A2) commute by the fact A preserves colimits, (T1) commutes by the unit law for the monoidal natural transformation θ , and (T2) commutes by the naturality of θ .

Thus we have the equality of decorations $\theta_X \circ B(F! \circ \varphi_1) \circ \beta_1 = G! \circ \gamma_1 : 1 \rightarrow GAX$, and so T sends identity morphisms to identity morphisms.

Monoidality. The coherence maps of the functor are given by the coherence maps for the monoidal functor A , viewed now as cospans with the empty decoration. That is, we define the coherence maps τ to be the collection of isomorphisms

$$\tau_1 = \left(\begin{array}{ccc} & A\emptyset_{\mathcal{C}} & \\ \nearrow \alpha_1 & & \\ \emptyset_{\mathcal{C}'} & & A\emptyset_{\mathcal{C}} \end{array}, \begin{array}{c} GA\emptyset_{\mathcal{C}} \\ \uparrow G! \\ G\emptyset_{\mathcal{C}'} \\ \uparrow \gamma_1 \\ 1_{\mathcal{D}'} \end{array} \right),$$

$$\tau_{X,Y} = \left(\begin{array}{ccc} & A(X+Y) & \\ \nearrow \alpha_{X,Y} & & \\ AX+AY & & A(X+Y) \end{array}, \begin{array}{c} GA(X+Y) \\ \uparrow G! \\ G\emptyset_{\mathcal{C}'} \\ \uparrow \gamma_1 \\ 1_{\mathcal{D}} \end{array} \right),$$

where X, Y are objects of $GCospan$. As (A, α) is already strong symmetric monoidal and τ merely views these maps in \mathcal{C} as empty-decorated cospans in $GCospan$, τ is natural in X and Y , and obeys the required coherence axioms for (T, τ) to also be strong symmetric monoidal.

Indeed, the monoidality of a functor (T, τ) has two aspects: the naturality of the transformation τ , and the coherence axioms. We discuss the former; since τ is just an empty-decorated version of α , the latter then immediately follow from the coherence of α .

The naturality of τ may be proved via the same method as that employed for the naturality of the coherence maps of decorated cospan categories: we first use the composition of empty decorations to compute the two paths around the naturality square, and then use the naturality of the coherence map α to show that these two decorated cospans are isomorphic.

In slightly more detail, suppose we have decorated cospans

$$(X \xrightarrow{i_X} N \xleftarrow{o_Z} Z, 1 \xrightarrow{s} FN) \quad \text{and} \quad (Y \xrightarrow{i_Y} M \xleftarrow{o_W} W, 1 \xrightarrow{t} FM).$$

Then naturality demands that the cospans

$$AX + AY \xrightarrow{Ai_X + Ai_Y} AN + AM \xleftarrow{(o_Z + o_W) \circ \alpha^{-1}} A(Z + W)$$

and

$$AX + AY \xrightarrow{A(i_X + i_Y) \circ \alpha} A(N + M) \xleftarrow{A(o_Z + o_W)} A(Z + W)$$

are isomorphic as decorated cospans, with decorations the top and bottom rows of the diagram below respectively.

$$\begin{array}{c} \nearrow \lambda^{-1} \\ 1 \\ \searrow \beta_1 \end{array} \begin{array}{l} 1 \otimes 1 \xrightarrow{\beta_1 \otimes \beta_1} B1 \otimes B1 \xrightarrow{Bs \otimes Bt} BFN \otimes BFM \xrightarrow{\theta_N \otimes \theta_M} GAN \otimes GAM \xrightarrow{\gamma_{AN, AM}} G(AN + AM) \\ B1 \xrightarrow{B\lambda^{-1}} B(1 \otimes 1) \xrightarrow{B(s \otimes t)} B(FN \otimes FM) \xrightarrow{B\varphi_{N, M}} B(F(N + M)) \xrightarrow{\theta_{N, M}} G(A(N + M)) \end{array} \downarrow G\alpha_{N, M}$$

As it is a subdiagram of the large functoriality commutative diagram, this diagram commutes. The diagrams required for $\alpha_{N, M}$ to be a morphism of cospans also commute, so our decorated cospans are indeed isomorphic. This proves τ is a natural tranfomation.

Moreover, as A is coproduct-preserving and the Frobenius structures on $FCospan$ and $GCospan$ are built using various copairings of the identity map, (T, τ) preserves the hypergraph structure.

Finally, it is straightforward to observe that the maps τ are identity maps if and only if the maps α are, so (T, τ) is a strict monoidal functor if and only (A, α) is. \square

When the decorating structure comprises some notion of topological diagram, such as a graph, these natural transformations θ might describe some semantic interpretation of the decorating structure. In this setting the above theorem constructs functorial semantics for the decorated cospan category of diagrams.

2.4 Decorations in Set are general

So far we have let decorations lie in any braided monoidal category. In conversations with the author, Sam Staton pointed out that it is general enough to let decorations lie in the symmetric monoidal category (Set, \times) of sets, functions, and the categorical product.

The key observation is that decorated cospans only make use of the sets of monoidal elements $x: I \rightarrow X$ in the decorating category \mathcal{D} . To extract this information, we define the monoidal global sections functor.

Proposition 2.8. *Let (\mathcal{D}, \otimes) be a braided monoidal category. Then there exists a braided monoidal functor, the **monoidal global sections functor** $\mathcal{D}(I, -): (\mathcal{D}, \otimes) \rightarrow (\text{Set}, \times)$ taking each object X in \mathcal{D} to the homset $\mathcal{D}(I, X)$, and each morphism $f: X \rightarrow Y$ to the function*

$$\begin{aligned} \mathcal{D}(I, f): \mathcal{D}(I, X) &\longrightarrow \mathcal{D}(I, Y); \\ x &\longmapsto f \circ x. \end{aligned}$$

The coherence maps are given by $d_1: 1 \rightarrow \mathcal{D}(I, I); * \mapsto \text{id}_I$ and

$$\begin{aligned} d_{X,Y}: \mathcal{D}(I, X) \times \mathcal{D}(I, Y) &\longrightarrow \mathcal{D}(I, X \otimes Y); \\ (x, y) &\longmapsto (I \xrightarrow{\rho^{-1}} I \otimes I \xrightarrow{x \otimes y} X \otimes Y). \end{aligned}$$

Proof. This functor is the covariant hom functor on the monoidal unit. The coherence axioms immediately follow from the coherence of the braided monoidal category \mathcal{D} . \square

Composing a decorating functor F with this global sections functor produces an isomorphic decorated cospans category.

Proposition 2.9. *Let $F: (\mathcal{C}, +) \rightarrow (\mathcal{D}, \otimes)$ be a braided lax monoidal functor. Then $FCospan$ and $(\mathcal{D}(I, -) \circ F)Cospan$ are isomorphic as hypergraph categories.*

Proof. Note that the global sections functor is always lax braided monoidal. We have the commutative-by-definition triangle of braided lax monoidal functors

$$\begin{array}{ccc} & & (\mathcal{D}, \otimes) \\ & \nearrow F & \downarrow \mathcal{D}(I, -) \\ (\mathcal{C}, +) & & (\text{Set}, \times) \\ & \searrow \mathcal{D}(I, -) \circ F & \end{array}$$

By Theorem 2.7, this gives a strict hypergraph functor $FCospan \rightarrow (\mathcal{D}(I, -) \circ F)Cospan$. It is easily checked that this functor is bijective-on-objects, full, and faithful. \square

Similarly, taking Set to be our decorating category does not impinge upon the functors that can be constructed using decorated cospans.

Proposition 2.10. *Let $T: FCospan \rightarrow GCospan$ be a decorated cospans functor. Then there is a decorated cospans functor U such that the square of hypergraph functors*

$$\begin{array}{ccc} FCospan & \xrightarrow{T} & GCospan \\ \sim \downarrow & & \downarrow \sim \\ (\mathcal{D}(I, -) \circ F)Cospan & \xrightarrow{U} & (\mathcal{D}'(I, -) \circ G)Cospan \end{array}$$

commutes, where the vertical maps are the isomorphisms given by the previous proposition.

Proof. Write

$$\begin{array}{ccc} (\mathcal{C}, +) & \xrightarrow{(F, \varphi)} & (\mathcal{D}, \otimes) \\ (A, \alpha) \downarrow & \not\Downarrow_{\theta} & \downarrow (B, \beta) \\ (\mathcal{C}', +) & \xrightarrow{(G, \gamma)} & (\mathcal{D}', \boxtimes). \end{array}$$

for the monoidal natural transformation yielding T , and write

$$\begin{aligned} \bar{\beta}_X: \mathcal{D}(I, X) &\longrightarrow \mathcal{D}'(I, BX); \\ (I_{\mathcal{D}} \xrightarrow{x} X) &\longmapsto (I_{\mathcal{D}'} \xrightarrow{\beta_I} BI_{\mathcal{D}} \xrightarrow{Bx} BX) \end{aligned}$$

for all $X \in \mathcal{D}$. This defines a monoidal natural transformation $\bar{\beta}: \mathcal{D}(I, -) \Rightarrow \mathcal{D}'(I, B-)$, with the monoidality following from the monoidality of the functor (B, β) .

Next, define U as the functor resulting from the decorated cospan construction applied to the horizontal composition of monoidal natural transformations

$$\begin{array}{ccccc} (\mathcal{C}, +) & \xrightarrow{(F, \varphi)} & (\mathcal{D}, \otimes) & \xrightarrow{\mathcal{D}(I, -)} & (Set, \times) \\ (A, \alpha) \downarrow & \not\Downarrow_{\theta} & \downarrow (B, \beta) & \not\Downarrow_{\bar{\beta}} & \parallel \\ (\mathcal{C}', +) & \xrightarrow{(G, \gamma)} & (\mathcal{D}', \boxtimes) & \xrightarrow{\mathcal{D}'(I, -)} & (Set, \times). \end{array}$$

To see that the required square of hypergraph functors commutes, observe that both functors $F\text{Cospan} \rightarrow (\mathcal{D}'(I, -) \circ G)\text{Cospan}$ take objects X to AX and morphisms

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN)$$

to

$$(AX \xrightarrow{Ai} AN \xleftarrow{Ao} AY, \theta_N \circ Bs \circ \beta_1 \in \mathcal{D}'(I, GAN)). \quad \square$$

Note that the braided monoidal categories $(\mathcal{C}, +)$ and (Set, \times) are both symmetric monoidal categories. Thus when working with the decorating category Set we may refer to the decorating functors F as symmetric lax monoidal, rather than merely braided lax monoidal.

2.5 Examples

In this final section we outline two constructions of decorated cospan categories, based on labelled graphs and linear subspaces respectively, and a functor between these two categories interpreting each graph as an electrical circuit. We shall see that the decorated cospan framework allows us to take a notion of closed system and construct a corresponding notion of open or composable system, together with functorial semantics for these systems.

This electrical circuits example is the motivating application for the decorated cospan construction, and its shortcomings motivate the further theoretical development of decorated cospans in the next chapter. Armed with these additional tools, we will return to the application for a full discussion in Chapter 6.

2.5.1 Labelled graphs

Recall that a **$(0, \infty)$ -graph** (N, E, s, t, r) comprises a finite set N of vertices (or nodes), a finite set E of edges, functions $s, t: E \rightarrow N$ describing the source and target of each edge, and a function $r: E \rightarrow (0, \infty)$ labelling each edge. The decorated cospan framework allows us to construct a category with, roughly speaking, these graphs as morphisms. More precisely, our morphisms will consist of these graphs, together with subsets of the nodes marked, with multiplicity, as ‘input’ and ‘output’ connection points.

Pick small categories equivalent to the category of $(0, \infty)$ -graphs such that we may talk about the set of all $(0, \infty)$ -graphs on each finite set N . Then we may consider the functor

$$\text{Graph}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

taking a finite set N to the set $\text{Graph}(N)$ of $(0, \infty)$ -graphs (N, E, s, t, r) with set of nodes N . On morphisms let it take a function $f: N \rightarrow M$ to the function that pushes labelled graph structures on a set N forward onto the set M :

$$\begin{aligned} \text{Graph}(f): \text{Graph}(N) &\longrightarrow \text{Graph}(M); \\ (N, E, s, t, r) &\longmapsto (M, E, f \circ s, f \circ t, r). \end{aligned}$$

As this map simply acts by post-composition, our map Graph is indeed functorial.

We then arrive at a lax braided monoidal functor (Graph, ζ) by equipping this functor with the natural transformation

$$\begin{aligned} \zeta_{N,M}: \text{Graph}(N) \times \text{Graph}(M) &\longrightarrow \text{Graph}(N + M); \\ ((N, E, s, t, r), (M, F, s', t', r')) &\longmapsto (N + M, E + F, s + s', t + t', [r, r']), \end{aligned}$$

together with the unit map

$$\begin{aligned} \zeta_1: 1 = \{\bullet\} &\longrightarrow \text{Graph}(\emptyset); \\ \bullet &\longmapsto (\emptyset, \emptyset, !, !, !), \end{aligned}$$

where we remind ourselves that we write $[r, r']$ for the copairing of the functions r and r' . The naturality of this collection of morphisms, as well as the coherence laws for lax braided monoidal functors, follow from the universal property of the coproduct.

Theorem 2.4 thus allows us to construct a hypergraph category GraphCospan . For an intuitive visual understanding of the morphisms of this category and its composition rule, see this paper's introduction.

2.5.2 Linear relations

Another example of a decorated cospan category arising from a functor $(\text{FinSet}, +) \rightarrow (\text{Set}, \times)$ is closely related to the category of linear relations. Here we decorate each cospan in Set with a linear subspace of $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$, the sum of the vector space generated by the apex N over \mathbb{R} and its vector space dual.

First let us recall some facts about relations. Let $R \subseteq X \times Y$ be a relation; we write this also as $R: X \rightarrow Y$. The opposite relation $R^{\text{opp}}: Y \rightarrow X$, is the subset $R^{\text{opp}} \subseteq Y \times X$ such that $(y, x) \in R^{\text{opp}}$ if and only if $(x, y) \in R$. We say that the image of a subset $S \subseteq X$ under a relation $R: X \rightarrow Y$ is the subset of all elements of the codomain Y related by R to an element of S . Note that if X and Y are vector spaces and S and R are both linear subspaces, then the image $R(S)$ of S under R is again a linear subspace.

Now any function $f: N \rightarrow M$ induces a linear map $f^*: \mathbb{R}^M \rightarrow \mathbb{R}^N$ by precomposition. This linear map f^* itself induces a dual map $f_*: (\mathbb{R}^N)^* \rightarrow (\mathbb{R}^M)^*$ by precomposition. Furthermore f^* has, as a linear relation $f^* \subseteq \mathbb{R}^M \oplus \mathbb{R}^N$, an opposite linear relation $(f^*)^{\text{opp}}: \mathbb{R}^N \rightarrow \mathbb{R}^M$. Define the functor

$$\text{LinSub}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

taking a finite set N to set of linear subspaces of the vector space $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$, and taking a function $f: N \rightarrow M$ to the function $\text{LinSub}(N) \rightarrow \text{LinSub}(M)$ induced by the sum of these two relations:

$$\begin{aligned} \text{LinSub}(f): \text{LinSub}(N) &\longrightarrow \text{LinSub}(M); \\ L &\longmapsto ((f^*)^{\text{opp}} \oplus f_*)(L). \end{aligned}$$

The above operations on f used in the construction of this map are functorial, and so it is readily observed that LinSub is indeed a functor.

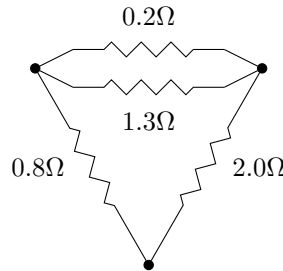
It is moreover lax braided monoidal as the sum of linear subspace of $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$ and a linear subspace of $\mathbb{R}^M \oplus (\mathbb{R}^M)^*$ may be viewed as a subspace of $\mathbb{R}^N \oplus (\mathbb{R}^N)^* \oplus \mathbb{R}^M \oplus (\mathbb{R}^M)^* \cong \mathbb{R}^{N+M} \oplus (\mathbb{R}^{N+M})^* \cong \mathbb{R}^{M+N} \oplus (\mathbb{R}^{M+N})^*$, and the empty subspace is a linear subspace of each $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$.

We thus have a hypergraph category LinSubCospan .

2.5.3 Electrical circuits

Electrical circuits and their diagrams are the motivating application for the decorated cospan construction. Specialising to the case of networks of linear resistors, we detail here how we may use the category LinSubCospan to provide semantics for the morphisms of GraphCospan as diagrams of networks of linear resistors.

Intuitively, after choosing a unit of resistance, say ohms (Ω), each $(0, \infty)$ -graph can be viewed as a network of linear resistors, with the $(0, \infty)$ -graph of the introduction now more commonly depicted as



GraphCospan may then be viewed as a category with morphisms circuits of linear resistors equipped with chosen input and output terminals.

The suitability of this language is seen in the way the different categorical structures of GraphCospan capture different operations that can be performed with circuits. To wit, the sequential composition expresses the fact that we can connect the outputs of one circuit to the inputs of the next, while the monoidal composition models the placement of circuits side-by-side. Furthermore, the symmetric monoidal structure allows us reorder input and output wires, the compactness captures the interchangeability between input and output terminals of circuits—that is, the fact that we can choose any input terminal to our circuit and consider it instead as an output terminal, and vice versa—and the Frobenius structure expresses the fact that we may wire any node of the circuit to as many additional components as we like.

Moreover, Theorem 2.7 provides semantics. Each node in a network of resistors can be assigned an electric potential and a net current outflow at that node, and so the set N of vertices of a $(0, \infty)$ -graph can be seen as generating a space $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$ of electrical states of the network. We define a natural transformation

$$\text{Res}: \text{Graph} \Longrightarrow \text{LinSub}$$

mapping each $(0, \infty)$ -graph on N , viewed as a network of resistors, to the linear subspace of $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$ of electrical states permitted by Ohm's law.² In detail, let $\psi \in \mathbb{R}^N$. We define the power $Q: \mathbb{R}^N \rightarrow \mathbb{R}$ corresponding to a $(0, \infty)$ -graph (N, E, s, t, r) to be the function

$$Q(\psi) = \sum_{e \in E} \frac{1}{r(e)} \left(\psi(t(e)) - \psi(s(e)) \right)^2.$$

Then the states of a network of resistors are given by a potential ϕ on the nodes and the gradient of the power at this potential:

$$\begin{aligned} \text{Res}_N: \text{Graph}(N) &\longrightarrow \text{LinSub}(N) \\ (N, E, s, t, r) &\longmapsto \{(\phi, \nabla Q_\phi) \mid \phi \in \mathbb{R}^N\}. \end{aligned}$$

This defines a monoidal natural transformation. Hence, by Theorem 2.7, we obtain a hypergraph functor $\text{GraphCospan} \rightarrow \text{LinSubCospan}$.

The semantics provided by this functor match the standard interpretation of networks of linear resistors. The maps of the Frobenius monoid take on the interpretation of perfectly conductive wires, forcing the potentials at all nodes they connect to be equal, and the sum of incoming currents to equal the sum of outgoing

²Note that these states need not obey Kirchhoff's current law.

currents—precisely the behaviour implied by Kirchhoff’s laws. More generally, let $(X \xrightarrow{i} N \xleftarrow{o} Y, (N, E, s, t, r))$ be a morphism of GraphCospan , with Q the power function corresponding to the graph $\Gamma = (N, E, s, t, r)$. The image of this decorated cospan in LinSubCospan is the decorated cospan $(X \xrightarrow{i} N \xleftarrow{o} Y, \{(\phi, \nabla Q_\phi) \mid \phi \in \mathbb{R}^N\})$. Then it is straightforward to check that the subspace

$$\text{LinSub}[i, o](\text{Res}_N(\Gamma)) \subseteq \mathbb{R}^{X+Y} \oplus (\mathbb{R}^{X+Y})^*$$

is the subspace of electrical states on the terminals $X + Y$ such that currents and potentials can be chosen across the network of resistors (N, E, s, t, r) that obey Ohm’s and, on its interior, Kirchhoff’s laws. In particular, after passing to a subspace of the terminals in this way, composition in LinSubCospan corresponds to enforcing Kirchhoff’s laws on the shared terminals of the two networks.

This behaviour at the terminals $X + Y$ is often all we are interested in: for example, in a large electrical network, substituting a subcircuit for a different subcircuit with the same terminal behaviour will not affect the rest of the network. Yet this terminal behaviour is not yet encoded directly in the categorical structure. The images of circuits in GraphCospan in LinSubCospan are cospans decorated by a subspace of $\mathbb{R}^N \oplus (\mathbb{R}^N)^*$, keeping track of *all* the internal behaviour as well. This can be undesirable, for reasons of information compression as well as reasoning about equivalence. The next chapter addresses this issue by introducing correlations.

Chapter 3

Corelations: a tool for black boxing

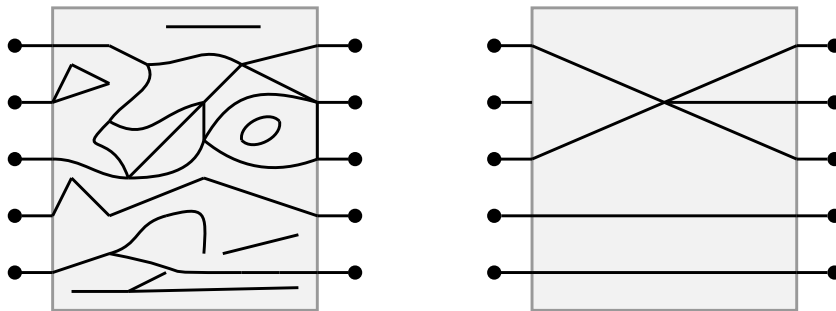
In this chapter we develop the theory of corelations.

3.1 The idea of black boxing

Thus far we have argued that network languages should be modelled using hypergraph categories, and shown that cospans provide a good language for talking about interconnection. We then developed the theory of decorated cospans, which allows us to take diagrams, mark ‘inputs’ and ‘outputs’ using cospans, and then compose these diagrams using pushouts. This turns a notion of closed system into a notion of open one.

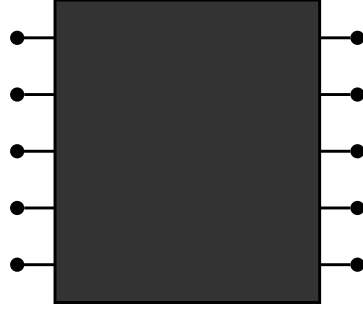
A serious limitation of using cospans alone, however, is that cospans indiscriminately accumulate information. If we compose twenty graphs in the category GraphCospan , we create an open graph containing all twenty of those graphs as subgraphs, disjoint but for a finite collection of vertices. This is often unnecessary and inefficient.

For example, consider the following two circuits, comprised only of perfectly conductive wires:



We interact with these circuits by connecting electric circuits and other devices to the terminals. This allows us to probe, for example, whether one terminal is connected

to another. These circuits are equivalent in the following sense: if these circuits were encased, but for their terminals, in a black box



we would be unable to distinguish them through our electrical investigations. In this sense these circuits are equivalent.

In this chapter and the next we pursue a notion of composition that reflects this, discarding extraneous information as we compose our systems. Next chapter we will extend these ideas to allow decorations and so a richer notion of system; for now we return to an undecorated situation, so that we may focus on the compositional aspects.

3.2 Corelations

Given sets X, Y , a relation $X \rightsquigarrow Y$ is a subset of the product $X \times Y$. More abstractly, we might say a relation is a jointly-monic span in the category of sets (or an isomorphism class thereof). We generalise the dual concept.

3.2.1 Factorisation systems.

The relevant properties of jointly-monic spans come from the fact that monomorphisms form one half of a factorisation system. A factorisation system allows any morphism in a category to be factored into the composite of two morphisms in a coherent way.

Definition 3.1. A **factorisation system** $(\mathcal{E}, \mathcal{M})$ in a category \mathcal{C} comprises subcategories \mathcal{E}, \mathcal{M} of \mathcal{C} such that

- (i) \mathcal{E} and \mathcal{M} contain all isomorphisms of \mathcal{C} .
- (ii) every morphism $f \in \mathcal{C}$ admits a factorisation $f = m \circ e$, $e \in \mathcal{E}$, $m \in \mathcal{M}$.

- (iii) given morphisms f, f' , with factorisations $f = m \circ e$, $f' = m' \circ e'$ of the above sort, for every u, v such that the square

$$\begin{array}{ccc} & \xrightarrow{f} & \\ u \downarrow & & \downarrow v \\ & \xrightarrow{f'} & \end{array}$$

commutes, there exists a unique morphism s such that

$$\begin{array}{ccc} & \xrightarrow{e} & \xrightarrow{m} \\ u \downarrow & & \downarrow v \\ & \xrightarrow{e'} & \xrightarrow{m'} \\ & & \exists! s \end{array}$$

commutes.

Examples 3.2. We introduce some factorisation systems of central importance in what follows.

- Write $\mathcal{I}_{\mathcal{C}}$ for the wide subcategory of \mathcal{C} containing exactly the isomorphisms of \mathcal{C} . Then $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ and $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ are both factorisation systems in \mathcal{C} .
- The prototypical example of a factorisation system is the epi-mono factorisation system (Sur, Inj) in Set . This follows from a more general fact, true in any category, that if every arrow can be factorised as an epi followed by a split mono, then this results in a factorisation system. The only non-trivial part to check is the uniqueness condition: given epis e_1, e_2 , split monos m_1, m_2 , and commutative diagram

$$\begin{array}{ccc} & \xrightarrow{e_1} & \xrightarrow{m_1} \\ u \downarrow & & \downarrow v \\ & \xrightarrow{e_2} & \xrightarrow{m_2} \\ & & \exists! t \end{array}$$

we must show that there is a unique t that makes the diagram commute. Indeed let $t = m'_2 v m_1$ where m'_2 satisfies $m'_2 m_2 = \text{id}$. To see that the right square commutes, observe

$$m_2 t e_1 = m_2 m'_2 v m_1 e_1 = m_2 m'_2 m_2 e_2 u = m_2 e_2 u = v m_1 e_1$$

and since e_1 is epi we have $m_2 t = v m_1$. For the left square,

$$t e_1 = m'_2 v m_1 e_1 = m'_2 m_2 e_2 u = e_2 u.$$

Uniqueness is immediate, since, e_1 is epi and m_2 is mono.

See [?, §14] for more details.

Definition 3.3. Call a factorisation system $(\mathcal{E}, \mathcal{M})$ in a monoidal category (\mathcal{C}, \otimes) a **monoidal factorisation system** if (\mathcal{E}, \otimes) is a monoidal category.

One might wonder why \mathcal{M} does not appear in the above definition. To give a touch more intuition for this definition, we quote a theorem of Ambler. Recall a symmetric monoidal closed category is one in which each functor $- \otimes X$ has a specified right adjoint $[X, -]$. See Ambler for proof and further details [?, Lemma 5.2.2].

Proposition 3.4. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system in a symmetric monoidal closed category (\mathcal{C}, \otimes) . Then the following are equivalent:*

- (i) $(\mathcal{E}, \mathcal{M})$ is a monoidal factorisation system.
- (ii) \mathcal{E} is closed under $- \otimes X$ for all $X \in \mathcal{C}$.
- (iii) \mathcal{M} is closed under $[X, -]$ for all $X \in \mathcal{C}$.

In a category with finite coproducts every factorisation system is a monoidal factorisation system for the coproduct.

Lemma 3.5. *Let \mathcal{C} be a category with finite coproducts, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . Then $(\mathcal{E}, +)$ is a symmetric monoidal category.*

Proof. The only thing to check is that \mathcal{E} is closed under $+$. That is, given $f: A \rightarrow B$ and $g: C \rightarrow D$ in \mathcal{E} , we wish to show that $f + g: A + C \rightarrow B + D$, defined in \mathcal{C} , is also a morphism in \mathcal{E} .

Let $f + g$ have factorisation $A + C \xrightarrow{e} \overline{B + D} \xrightarrow{m} B + D$, where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. We will prove that m is an isomorphism. To construct an inverse, recall that by definition, as f and g lie in \mathcal{E} , there exist morphisms $x: B \rightarrow \overline{B + D}$ and $y: D \rightarrow \overline{B + D}$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \vdots x \\ A + C & \xrightarrow{e} & \overline{B + D} \end{array} \quad \text{and} \quad \begin{array}{ccc} C & \xrightarrow{g} & D \\ \downarrow & & \vdots y \\ A + C & \xrightarrow{e} & \overline{B + D} \end{array} \quad \begin{array}{ccc} B & \xlongequal{\quad} & B \\ \downarrow & & \downarrow \\ B + D & & B + D \end{array} \quad (1)$$

The copairing $[x, y]$ is an inverse to m .

Indeed, taking the coproduct of the top rows of the two diagrams above and the copairings of the vertical maps gives the commutative diagram

$$\begin{array}{ccccc} A + C & \xrightarrow{f+g} & B + D & \xlongequal{\quad} & B + D \\ \parallel & & \downarrow [x,y] & & \parallel \\ A + C & \xrightarrow{e} & \overline{B + D} & \xrightarrow{m} & B + D \end{array}$$

Reading the right-hand square immediately gives $m \circ [x, y] = 1$.

Conversely, to see that $[x, y] \circ m = 1$, remember that by definition $f + g = m \circ e$. So the left-hand square above implies that

$$\begin{array}{ccc} A + C & \xrightarrow{e} & \overline{B + D} \\ \parallel & & \downarrow [x,y] \circ m \\ A + C & \xrightarrow{e} & \overline{B + D} \end{array}$$

commutes. But by the universal property of factorisation systems, there is a unique map $\overline{B + D} \rightarrow \overline{B + D}$ such that this diagram commutes, and clearly the identity map also suffices. Thus $[x, y] \circ m = 1$. \square

3.2.2 Corelations

Relations, then, may be generalised as spans such that the span maps ‘jointly’ belong to some class \mathcal{M} of an $(\mathcal{E}, \mathcal{M})$ -factorisation system. We define corelations in the dual manner.

Definition 3.6. Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . An **$(\mathcal{E}, \mathcal{M})$ -corelation** $X \rightsquigarrow Y$ is a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ such that the copairing $[i, o]: X + Y \rightarrow N$ lies in \mathcal{E} .

When the factorisation system is clear from context, we simply call $(\mathcal{E}, \mathcal{M})$ -corelations: ‘corelations’.

We also say that a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ with the property that the copairing $[i, o]: X + Y \rightarrow N$ lies in \mathcal{E} is **jointly \mathcal{E} -like**. Note that if a cospan is jointly \mathcal{E} -like then so are all isomorphic cospans. Thus the property of being a corelation is closed under isomorphism of cospans, and we again are often lazy with our language, referring to both jointly \mathcal{E} -like cospans and their isomorphism classes as corelations.

If $f: A \rightarrow N$ is a morphism with factorisation $f = m \circ e$, write \overline{N} for the object such that $e: A \rightarrow \overline{N}$ and $m: \overline{N} \rightarrow N$. Now, given a cospan $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$, we may use the factorisation system to write the copairing $[i_X, o_Y]: X + Y \rightarrow N$ as

$$X + Y \xrightarrow{e} \overline{N} \xrightarrow{m} N.$$

From the universal property of the coproduct, we also have maps $\iota_X: X \rightarrow X + Y$ and $\iota_Y: Y \rightarrow X + Y$. We then call the corelation

$$X \xrightarrow{e\iota_X} \overline{N} \xleftarrow{e\iota_Y} Y$$

the **\mathcal{E} -part** of the above cospan. On occasion we will also write $e: X + Y \rightarrow \overline{N}$ for the same corelation.

Examples 3.7. Many examples of corelations are already familiar.

- For the morphism-isomorphism factorisation system $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, corelations are just cospans.
- For the isomorphism-morphism factorisation $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$, jointly $\mathcal{I}_{\mathcal{C}}$ -like cospans $X \rightarrow Y$ are simply isomorphisms $X + Y \xrightarrow{\sim} N$. Thus there is a unique isomorphism class of corelation between any two objects.
- Note that the category \mathbf{Set} has finite colimits and an epi-mono factorisation system. Epi-mono corelations from $X \rightarrow Y$ in \mathbf{Set} surjective functions $X + Y \rightarrow N$; thus their isomorphism classes are partitions, or equivalence relations on $X + Y$.

3.2.3 Categories of corelations

We compose corelations by taking the \mathcal{E} -part of their composite cospan. That is, given corelations $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$, their composite is given by the cospan $X \rightarrow \overline{N +_Y M} \leftarrow Z$ in

$$\begin{array}{ccccc}
 & & N +_Y M & & \\
 & & \uparrow m & & \\
 & & \overline{N +_Y M} & & \\
 & \nearrow e\iota_N & & \nwarrow e\iota_M & \\
 X & \xrightarrow{i_X} & N & & M \xleftarrow{o_Z} Z, \\
 & \nwarrow o_Y & & \nearrow i_Y & \\
 & & Y & &
 \end{array}$$

where $m \circ e$ is the $(\mathcal{E}, \mathcal{M})$ -factorisation of $[j_N, j_M]: N + M \rightarrow N +_Y M$.

It is well-known that this composite is unique up to isomorphism, and that when \mathcal{M} is well-behaved it defines a category with morphisms isomorphism classes of corelations. For example, bicategorical version of the dual theorem, for spans and relations,

can be found in [?]. Nonetheless, for the sake of completeness we sketch our own argument here.

Proposition 3.8. *Let \mathcal{C} be a category with finite colimits and with a factorisation system $(\mathcal{E}, \mathcal{M})$. Then the above is a well-defined composition rule on isomorphism classes of corelations.*

Proof. Let $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y)$ and $(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z, Y \xrightarrow{i'_Y} M' \xleftarrow{o'_Z} Z)$ be pairs of isomorphic jointly \mathcal{E} -like cospans. Their composites *as cospans* are isomorphic, so the factorisation system gives an isomorphism s such that the diagram

$$\begin{array}{ccccc} X + Z & \xrightarrow{e} & \overline{N +_Y M} & \xrightarrow{m} & N +_Y M \\ \parallel & & \downarrow \scriptstyle \sim s & & \downarrow \scriptstyle \sim \\ X + Z & \xrightarrow{e'} & \overline{N' +_Y M'} & \xrightarrow{m'} & N' +_Y M' \end{array}$$

commutes. This s is an isomorphism of the composite corelations. \square

Composition of corelations is *not* associative in general. It is, however, associative when \mathcal{M} is **stable under pushout**: that is, whenever

$$\begin{array}{ccc} & j & \\ \uparrow & \longrightarrow & \uparrow \\ & m & \end{array}$$

is a pushout square such that $m \in \mathcal{M}$, we also have that $j \in \mathcal{M}$.

Stability under pushout is a powerful property. A key corollary, both for associativity and in general, is that it implies \mathcal{M} is also closed under $+$.

Lemma 3.9. *Let \mathcal{C} be a category with finite colimits, and let \mathcal{M} be a subcategory of \mathcal{C} stable under pushouts and containing all isomorphisms. Then $(\mathcal{M}, +)$ is a symmetric monoidal category.*

Proof. It is enough to show that for all morphisms $m, m' \in \mathcal{M}$ we have $m + m'$ in \mathcal{M} . Since \mathcal{M} contains all isomorphisms, the coherence maps are inherited from \mathcal{C} . The required axioms—the functoriality of the tensor product, the naturality of the coherence maps, and the coherence laws—are also inherited as they hold in \mathcal{C} .

To see $m + m'$ is in \mathcal{M} , simply observe that we have the pushout square

$$\begin{array}{ccc} A + C & \xrightarrow{m+1} & B + C \\ \uparrow \scriptstyle \iota & & \uparrow \scriptstyle \iota \\ A & \xrightarrow{m} & B \end{array}$$

in \mathcal{C} . As \mathcal{M} is stable under pushout, $m + 1 \in \mathcal{M}$. Similarly, $1 + m' \in \mathcal{M}$. Thus their composite $m + m'$ lies in \mathcal{M} , as required. \square

An analogous argument shows that pushouts of maps $m +_Y m'$ also lie in \mathcal{M} . Using this lemma it is not difficult to show associativity—the key point is that factorisation ‘commutes’ with pushouts, and that we have a category $\text{Corel}_{(\mathcal{E}, \mathcal{M})}(\mathcal{C})$. Again, this is all well-known, and can be found in [?]. We will incidentally reprove these facts in the following, while pursuing richer structure.

Indeed, for modelling networks, we require not just a category, but a hypergraph category. Correlation categories come equipped with this extra structure. Recall that we gave decorated cospan categories a hypergraph structure by defining a wide embedding $\text{Cospans}(\mathcal{C}) \hookrightarrow F\text{Cospans}$, via which $F\text{Cospans}$ inherited the coherence and Frobenius maps (Theorem 2.4). We will argue similarly here, after showing that the ‘quotient’ map

$$\text{Cospans}(\mathcal{C}) \longrightarrow \text{Corel}(\mathcal{C})$$

taking each cospan to its jointly \mathcal{E} -like part is functorial. Indeed, we define the coherence and Frobenius maps of $\text{Corel}(\mathcal{C})$ to be their image under this map. For the monoidal product we again use the coproduct in \mathcal{C} ; the monoidal product of two correlations is their monoidal product as cospans.

Theorem 3.10. *Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be factorisation system on \mathcal{C} such that \mathcal{M} is stable under pushout. Then there exists a hypergraph category $\text{Corel}_{(\mathcal{E}, \mathcal{M})}(\mathcal{C})$ with*

<i>The hypergraph category $(\text{Corel}_{(\mathcal{E}, \mathcal{M})}(\mathcal{C}), +)$</i>	
objects	<i>the objects of \mathcal{C}</i>
morphisms	<i>isomorphism classes of $(\mathcal{E}, \mathcal{M})$-corelations in \mathcal{C}</i>
composition	<i>given by the \mathcal{E}-part of pushout</i>
monoidal product	<i>the coproduct in \mathcal{C}</i>
coherence maps	<i>inherited from $\text{Cospans}(\mathcal{C})$</i>
hypergraph maps	<i>inherited from $\text{Cospans}(\mathcal{C})$</i>

Again, we will drop explicit reference to the factorisation system when context allows, simply writing $\text{Corel}(\mathcal{C})$.

Proposition 3.8 shows that our composition rule is a well-defined function, Lemma 3.5 shows likewise for the monoidal product $+: \text{Corel}(\mathcal{C}) \times \text{Corel}(\mathcal{C}) \rightarrow \text{Corel}(\mathcal{C})$. Thus we have the required data for a hypergraph category. It just remains to check a number of axioms: associativity and unitality of the categorical composition, functoriality

of the monoidal product, naturality of the coherence maps, the coherence axioms for symmetric monoidal categories, the Frobenius laws.

Our strategy for this will simply be to show that the (surjective) function taking a cospan to its jointly \mathcal{E} -part preserves composition and has natural strict monoidal coherence maps. This implies that to evaluate an expression in the monoidal category of corelations, we may simply evaluate it in the monoidal category of cospans, and then take the \mathcal{E} -part. Thus if an equation is true for cospans, it is true for corelations.

Instead of proving just this, however, we will prove a generalisation regarding the analogous ‘reduction’ map between any two corelation ‘categories’. This will reduce to the desired special case by taking the domain to be the ‘trivial’ $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ -corelations, which we know is equal to the hypergraph category of cospans. But the generality is not spurious: it has the advantage of proving the existence of a class of hypergraph functors between corelation categories in the same fell swoop.

Although a touch convoluted, this strategy is worth the pause for thought. We will use it once again for *decorated corelations*, to great economy.

3.3 Functors between corelation categories

We have seen that to construct a functor between cospan categories one may start with a colimit-preserving functor between the underlying categories. Corelation categories are similar, but we remove the part of each cospan that lies in \mathcal{M} . Hence for functors between corelation categories, we also require that the target category removes at least as much information as the source category.

Proposition 3.11. *Let $\mathcal{C}, \mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M}), (\mathcal{E}', \mathcal{M}')$, such that \mathcal{M} and \mathcal{M}' are stable under pushout. Further let $A: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of \mathcal{M} lies in \mathcal{M}' .*

Then we may define a hypergraph functor $\square: \text{Corel}(\mathcal{C}) \rightarrow \text{Corel}(\mathcal{C}')$ sending each object X in $\text{Corel}(\mathcal{C})$ to AX in $\text{Corel}(\mathcal{C}')$ and each corelation

$$X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$$

to the \mathcal{E}' -part

$$AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY.$$

of the image cospan. The coherence maps are the \mathcal{E}' -part $\overline{\kappa_{X,Y}}$ of the isomorphisms $\kappa_{X,Y}: AX + AY \rightarrow A(X + Y)$ given as A preserves colimits.

As discussed, we are still yet to prove that $\text{Corel}(\mathcal{C})$ is a hypergraph category. We address this first with two lemmas regarding these proposed functors.

Lemma 3.12. *The above function $\square: \text{Corel}(\mathcal{C}) \rightarrow \text{Corel}(\mathcal{C}')$ preserves composition.*

Proof. Let $f = (X \rightarrow N \leftarrow Y)$ and $g = (Y \rightarrow M \leftarrow Z)$ be corelations in \mathcal{C} . By definition, the corelations $\square(g) \circ \square(f)$ and $\square(g \circ f)$ are given by the first arrows in the top and bottom row respectively of the diagram:

$$\begin{array}{ccccccc}
 AX + AZ & \xrightarrow{\mathcal{E}'} & \overline{AN} +_{AY} \overline{AM} & \xrightarrow{\mathcal{M}'} & \overline{AN} +_{AY} \overline{AM} & \xrightarrow{m'_{AN} +_{AY} m'_{AM}} & AN +_{AY} AM \\
 \parallel & & \uparrow \scriptstyle n & & & & \downarrow \scriptstyle \sim \\
 AX + AZ & \xrightarrow{\mathcal{E}'} & A(\overline{N +_Y M}) & \xrightarrow{\mathcal{M}'} & A(\overline{N +_Y M}) & \xrightarrow{Am_{N+Y M}} & A(N +_Y M) \\
 & & & & & & (*)
 \end{array}$$

The morphisms labelled \mathcal{E}' lie in \mathcal{E}' , and similarly for \mathcal{M}' ; these are given by the factorisation system on \mathcal{C}' . The maps $Am_{N+Y M}$ and $m'_{AN} +_{AY} m'_{AM}$ lie in \mathcal{M}' too: $Am_{N+Y M}$ as it is in the image of \mathcal{M} , and $m'_{AN} +_{AY} m'_{AM}$ as \mathcal{M}' is stable under pushout.

Moreover, the diagram commutes as both maps $AX + AZ \rightarrow AN +_{AY} AM$ compose to that given by the pushout of the images of f and g over AY . Thus the diagram represents two $(\mathcal{E}', \mathcal{M}')$ factorisations of the same morphism, and there exists an isomorphism n between the corelations $\square(g) \circ \square(f)$ and $\square(g \circ f)$. This proves that \square preserves composition. \square

This first lemma allows us to verify the associativity and unit laws for $\text{Corel}(\mathcal{C})$.

Corollary 3.13. *$\text{Corel}(\mathcal{C})$ is well-defined as a category.*

Proof. Consider the case of Proposition 3.11 with $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, and $A = 1_{\mathcal{C}}$. Then the domain of \square is $\text{Cospan}(\mathcal{C})$ by definition. In this case, the function $\square: \text{Cospan}(\mathcal{C}) \rightarrow \text{Corel}(\mathcal{C})$ is bijective-on-objects and surjective-on-morphisms. Thus to compute the composite of any two corelations, we may consider them as cospans, compute their composite *as cospans*, and then take the \mathcal{E} -part of the result. Since composition of cospans is associative and unital, so is composition of corelations, with the identity corelation just the image of the identity cospan. \square

Note that the identity in $\text{Corel}(\mathcal{C})$ may not be the identity cospan itself. For example, with the factorisation system $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ the $\mathcal{I}_{\mathcal{C}}$ -part of the identity cospan is simply $X \xrightarrow{\iota_{X_1}} X + X \xleftarrow{\iota_{X_2}} X$, where ι_{X_i} is the inclusion of X into the X_i factor of the coproduct $X + X$.

This first lemma is also useful in proving the second important lemma: the naturality of $\bar{\kappa}$.

Lemma 3.14. *The maps $\bar{\kappa}_{X,Y}$ above are natural.*

Proof. Let $f = (X \rightarrow N \leftarrow Y)$, $g = (Z \rightarrow M \leftarrow W)$ be corelations in \mathcal{C} . We wish to show that

$$\begin{array}{ccc} AX + AY & \xrightarrow{\square(f) + \square(g)} & AZ + AW \\ \bar{\kappa}_{X,Y} \downarrow & & \downarrow \bar{\kappa}_{Z,W} \\ A(X + Y) & \xrightarrow{\square(f+g)} & A(Z + W) \end{array}$$

commutes in $\text{Corel}(\mathcal{C}')$.

Consider the following commutative diagram in \mathcal{C}' , with the outside square equivalent to the naturality square for the coherence maps of the monoidal functor $\text{Cospan}(\mathcal{C}) \rightarrow \text{Cospan}(\mathcal{C}')$:

$$\begin{array}{ccccc} (AX + AY) + (AZ + AW) & \xrightarrow{\mathcal{E}' + \mathcal{E}'} & \overline{AN} + \overline{AM} & \xrightarrow{\mathcal{M}' + \mathcal{M}'} & AN + AM \\ \downarrow \kappa_{X,Y} + \kappa_{Z,W} & & \downarrow p & & \downarrow \kappa_{N,M} \\ A(X + Y) + A(Z + W) & \xrightarrow{\mathcal{E}'} & \overline{A(N + M)} & \xrightarrow{\mathcal{M}'} & A(N + M) \end{array} \quad (\#)$$

We have factored the top edge as the coproduct of the respective factorisations of f and g , and the bottom edge simply as the factorisation of the coproduct $f + g$.

Note that by Lemma 3.5 the coproduct of two maps in \mathcal{E}' is again in \mathcal{E}' , while Lemma 3.9 implies the same for \mathcal{M}' . Thus the top edge is an $(\mathcal{E}', \mathcal{M}')$ -factorisation, and the uniqueness of factorisations gives the isomorphism n . Given that the map reducing cospans to corelations is functorial, the commutative square

$$\begin{array}{ccc} (AX + AY) + A(Z + W) & \xrightarrow{1 + \kappa_{Z,W}^{-1}} & (AX + AY) + (AZ + AW) \xrightarrow{\mathcal{E}' + \mathcal{E}'} \overline{AN} + \overline{AM} \\ \parallel & & \downarrow n \\ (AX + AY) + A(Z + W) & \xrightarrow{\kappa_{X,Y} + 1} & A(X + Y) + A(Z + W) \xrightarrow{\mathcal{E}'} \overline{A(N + M)} \end{array}$$

then implies the naturality of the maps $\bar{\kappa}$. \square

These lemmas now imply that $\text{Corel}(\mathcal{C})$ is a well-defined hypergraph category.

Proof of Theorem 3.10. To complete the proof then, again consider the case of Proposition 3.11 with $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}, \mathcal{I}_{\mathcal{C}})$, and $A = 1_{\mathcal{C}}$. Note that by definition this function maps the coherence and hypergraph maps of $\text{Cospan}(\mathcal{C})$ onto the corresponding maps of $\text{Corel}(\mathcal{C})$. As $\text{Cospan}(\mathcal{C})$ is a hypergraph, and \square preserves composition

and respects the monoidal and hypergraph structure, $\text{Corel}(\mathcal{C})$ is also a hypergraph category.

For instance, suppose we want to check the functoriality of the monoidal product $+$. We then wish to show $(g \circ f) + (k \circ h) = (g + k) \circ (f + h)$ for corelations of the appropriate types. But \square preserves composition, and the naturality of κ , here the identity map, implies that for any two cospans the \mathcal{E} -part of their coproduct is equal to the coproduct of their \mathcal{E} -parts. Thus we may compute these two expressions by viewing f, g, h , and k as cospans, evaluating them in the category of cospans, and then taking their \mathcal{E} -parts. Since the equality holds in the category of cospans, it holds in the category of corelations. \square

Corollary 3.15. *There is a strict hypergraph functor*

$$\square: \text{Cospan}(\mathcal{C}) \longrightarrow \text{Corel}(\mathcal{C})$$

that takes each object of $\text{Cospan}(\mathcal{C})$ to itself as an object of $\text{Corel}(\mathcal{C})$ and each cospan to its \mathcal{E} -part.

Finally, we complete the proof that \square is always a hypergraph functor.

Proof of Proposition 3.11. We show \square is a functor, a symmetric monoidal functor, and then finally a hypergraph functor.

Functoriality. First, recall that \square preserves composition (Lemma 3.12). Thus to prove \square is a functor it remains to show identities are mapped to identities. The general idea for this and for similar axioms is to recall that the special maps are given by reduced versions of particular colimits, and that $(\mathcal{E}', \mathcal{M}')$ reduces maps more than $(\mathcal{E}, \mathcal{M})$.

In this case, recall the identity corelation is given by the \mathcal{E} -part $X + X \rightarrow \overline{X}$ of $[1, 1]: X + X \rightarrow X$. Thus the image of the identity on X and the identity on AX are given by the top and bottom rows of the commuting square

$$\begin{array}{ccccccc} A(X + X) & \xrightarrow{\mathcal{E}'} & \overline{AX} & \xrightarrow{\mathcal{M}'} & A\overline{X} & \xrightarrow{AM} & AX \\ \sim \downarrow \kappa^{-1} & & \downarrow n & & & & \parallel \\ AX + AX & \xrightarrow{\mathcal{E}'} & \overline{AX} & \xrightarrow{\mathcal{M}'} & & & AX \end{array}$$

The outside square commutes as we know A maps the identity cospan of \mathcal{C} to the identity cospan of \mathcal{C}' . The top row is the image under A of the identity cospan in \mathcal{C} , factored first in \mathcal{C} , and then in \mathcal{C}' . The bottom row is just the factored identity

cospan on AX in \mathcal{C}' . As A maps \mathcal{M} into \mathcal{M}' , the map marked $A\mathcal{M}$ lies in \mathcal{M}' . Thus both rows are $(\mathcal{E}', \mathcal{M}')$ -factorisations, and so we have the isomorphism n . Thus \square preserves identities.

Strong monoidality. We proved in Lemma 3.14 that our proposed coherence maps are natural. The rest of the properties follow from the composition preserving map $\text{Cospan}(\mathcal{C}') \rightarrow \text{Corel}(\mathcal{C}')$. Since the κ obey all the required axioms as cospans, they obey them as corelations too.

Hypergraph structure. The proof of preservation of the hypergraph structure follows the same pattern as the identity maps. \square

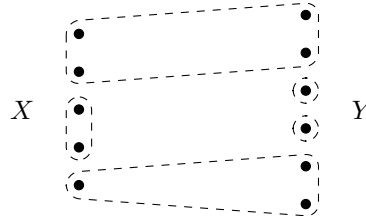
Example 3.16. Note that if both $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ are epi-split mono factorisations, then we always have that $F(\mathcal{M}) \subseteq \mathcal{M}'$. Indeed, if an (one-sided) inverse exists in the domain category, it exists in the codomain category. Thus colimit-preserving functors between categories with finite colimits and epi-split mono factorisation systems also induce a functor between the epi-split mono corelation categories. We will use this in Chapter ref.

3.4 Examples

3.4.1 Equivalence relations as corelations in Set

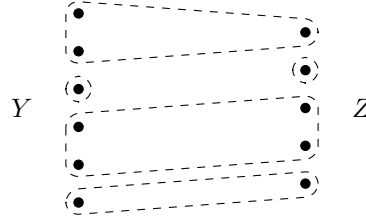
In each factorisation system of Examples 3.7 the right factor \mathcal{M} is stable under pushout. This gives the hypergraph categories of cospans in \mathcal{C} , the indiscrete category on the objects of \mathcal{C} , and equivalence relations between finite sets. The last of these examples is perhaps the most instructive for black-boxing open systems.

Epi-mono corelations in Set are isomorphism classes of jointly epic cospans. Thus a corelation $X \rightarrow Y$ is an equivalence relation on $X + Y$. We might depict these as follows

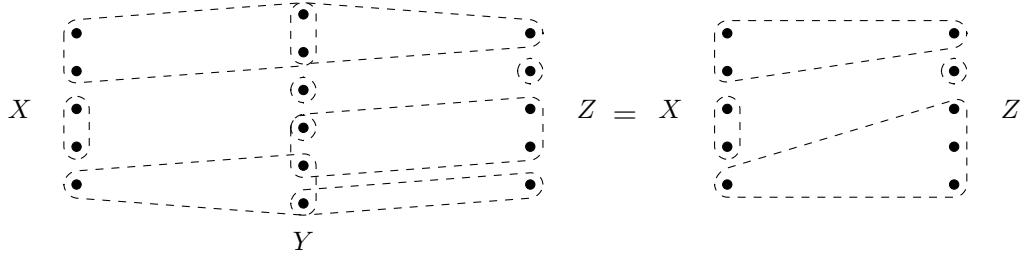


Here we have a corelation from a set X of five elements to a set Y of six elements. Elements belonging to the same equivalence class of $X + Y$ are grouped ('connected') by a dashed line.

Composition of corelations first takes the transitive closure of the two partitions (the pushout in \mathbf{Set}), before restricting the partition to the new domain and codomain (restricting to the jointly epic part). For example, suppose in addition to the corelation $\alpha: X \rightarrow Y$ above we have another corelation $\beta: Y \rightarrow Z$

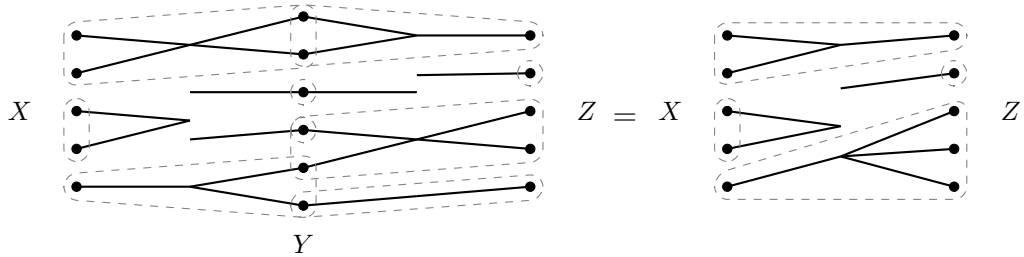


Then the composite $\beta \circ \alpha$ of our two corelations is given by



Informally, this captures the idea that two elements of $X + Z$ are ‘connected’ if we may travel from one to the other staying within connected components of α and β .

Epi-mono corelations in \mathbf{Set} can also be visualised as terminals connected by junctions of ideal wires. We draw these by marking each equivalence class with a point (the ‘junction’), and then connecting each element of the domain and codomain to their equivalence class with a ‘wire’. Composition then involves collapsing connected junctions down to a point.



The composition law captures the idea that connectivity is all that matters: as long as the wires are ‘ideal’, the exact path does not matter.

In Coya–Fong we formalise this idea by saying that corelations are the prop for extraspecial commutative Frobenius monoids [?]. An **extraspecial commutative**

Frobenius monoid $(X, \mu, \eta, \delta, \epsilon)$ in a monoidal category (\mathcal{C}, \otimes) is a special commutative Frobenius monoid that further obeys the extra law

$$\bullet \text{---} \bullet =$$

Two morphisms built from the generators of an extraspecial commutative Frobenius monoid are equal and if and only if their diagrams impose the same connectivity relations on the disjoint union of the domain and codomain. This is an extension of the spider theorem for special commutative Frobenius monoids.

3.4.2 Linear relations as corelations in Vect

Recall that a linear relation $L: U \rightsquigarrow V$ is a subspace $L \subseteq U \oplus V$. We compose linear relations as we do relations, and vector spaces and linear relations form a category LinRel . It is well-known that this category can be constructed as the category of relations in the category Vect of vector spaces and linear maps with respect to epi-mono factorisations. We show that they may also be constructed as corelations in Vect with respect to epi-mono factorisations.

If we restrict to the full subcategory FinVect of finite dimensional vector spaces this is easy to see: after picking a basis for each vector space the transpose yields an equivalence of FinVect with its opposite category, so the category of $(\mathcal{E}, \mathcal{M})$ -corelations (jointly-epic cospans) is isomorphic to the category of $(\mathcal{E}, \mathcal{M})$ -relations (jointly-monic spans) in FinVect . This fact has been fundamental in work on finite dimensional linear systems and signal flow diagrams [?, ?, ?].

We prove the general case in detail. To begin, note Vect has an epi-mono factorisation system with monos stable under pushouts. This factorisation system is inherited from Set : the epimorphisms in Vect are precisely the surjective linear maps, the monomorphisms are the injective linear maps, and the image of a linear map is always a subspace of the codomain, and so itself a vector space. Monos are stable under pushout as the pushout of a diagram $V \xleftarrow{f} U \xrightarrow{m} W$ is $V \oplus W / \text{Im}[f - g]$. The map $m': V \rightarrow V \oplus W / \text{Im}[f - g]$ into the pushout has kernel $f(\ker m)$. Thus when m is a monomorphism, m' is too.

Thus we have a category of corelations $\text{Corel}(\text{Vect})$. We show that the map $\text{Corel}(\text{Vect}) \rightarrow \text{LinRel}$ sending each vector space to itself and each corelation

$$U \xrightarrow{f} A \xleftarrow{g} V$$

to the linear subspace $\ker[f - g]$ is a full, faithful, and bijective-on-objects functor.

Indeed, corelations $U \xrightarrow{f} A \xleftarrow{g} V$ are in one-to-one correspondence with surjective linear maps $U \oplus V \rightarrow A$, which are in turn, by the isomorphism theorem, in one-to-one correspondence with subspaces of $U \oplus V$. These correspondences are described by the kernel construction above. Thus our map is evidently full, faithful, and bijective-on-objects. It also maps identities to identities. It remains to check that it preserves composition.

Suppose we have corelations $U \xrightarrow{f} A \xleftarrow{g} V$ and $V \xrightarrow{h} B \xleftarrow{k} W$. Then their pushout is given by $P = A \oplus B / \text{Im}[g - h]$, and we may draw the pushout diagram

$$\begin{array}{ccccc}
 U & & V & & W \\
 & \searrow f & & \swarrow h & \\
 & A & & B & \\
 & \swarrow \iota_A & \wedge & \searrow \iota_B & \\
 & P & & &
 \end{array}$$

We wish to show the equality of relations

$$\ker[f - g]; \ker[h - k] = \ker[\iota_A f - \iota_B g].$$

Now $(\mathbf{u}, \mathbf{w}) \in U \oplus W$ lies in the composite relation $\ker[f - g]; \ker[h - k]$ iff there exists $\mathbf{v} \in V$ such that $f\mathbf{u} = g\mathbf{v}$ and $h\mathbf{v} = k\mathbf{w}$. But as P is the pushout, this is true iff

$$\iota_A f\mathbf{u} = \iota_A g\mathbf{v} = \iota_B h\mathbf{v} = \iota_B k\mathbf{w}.$$

This in turn is true iff $(\mathbf{u}, \mathbf{w}) \in \ker[\iota_A f - \iota_B k]$, as required.

This corelations perspective is important as it fits the relational picture into our philosophy of black boxing. In Chapter ref we will see it is the corelation construction of LinRel that correctly generalises to the case of non-controllable systems.

Chapter 4

Decorated corelations

When enough structure is available to us, we may decorate corelations too. Furthermore, and key to the idea of ‘black-boxing’, we get a hypergraph functor from decorated cospans to decorated corelations.

4.1 Introduction

Semantics live on boundaries only.

We then introduce a new framework for working with hypergraph categories: decorated corelations.

Decorated corelations adds compositional operations to network-diagram representations, and handles composition of semantics too.

Two main theorems:

Theorem 4.1. *Given a category \mathcal{C} with finite colimits, factorisation system $(\mathcal{E}, \mathcal{M})$ such that \mathcal{M} is stable under pushouts, and a symmetric lax monoidal functor*

$$\mathcal{C}; \mathcal{M}^{\text{opp}} \longrightarrow \text{Set},$$

we may define a hypergraph category of with morphisms decorated corelations.

Theorem 4.2. *Every hypergraph category can be constructed in this way.*

They apply to functors too.

4.2 Decorated corelations

The key difference is to decorate cospans we need to know how to push decorations up. To decorate corelations we all need to know how to pull decorations back down. This is related to the existence of an extraspecial commutative Frobenius monoid in our main applications.

4.2.1 Adjoining right adjoints

Suppose we have a cospan $X + Y \rightarrow N$ with a decoration on N . Reducing this to a corelation requires us to factor this to $X + Y \xrightarrow{c} \overline{N} \xrightarrow{m} N$. To define a category of decorated corelations, then, we must specify how to take decoration on N and ‘pull it back’ along m to a decoration on \overline{N} .

For decorated cospans, it is enough to have a functor F from a category \mathcal{C} with finite colimits; the image Ff of morphisms f in \mathcal{C} describes how to move decorations forward along f . In this subsection we explain how to expand \mathcal{C} to include morphisms m^{opp} for each $m \in \mathcal{C}$, so that the image of m^{opp} describes how to move morphisms backwards along m .

Proposition 4.3. *Let \mathcal{C} be a category with finite colimits, and let \mathcal{M} be a subcategory of \mathcal{C} stable under pushouts. Then we define the category $\mathcal{C}; \mathcal{M}^{\text{opp}}$ as follows*

The symmetric monoidal category $(\mathcal{C}; \mathcal{M}^{\text{opp}}, +)$	
objects	the objects of \mathcal{C}
morphisms	isomorphism classes of cospans of the form $\xrightarrow{c} \xleftarrow{m}$, where c lies in \mathcal{C} and m in \mathcal{M}
composition	given by pushout
monoidal product	the coproduct in \mathcal{C}
coherence maps	the coherence maps in \mathcal{C}

Proof. This is a symmetric monoidal subcategory of $\text{Cospan}(\mathcal{C})$. Our data is well-defined: composition because \mathcal{M} is stable under pushouts, and monoidal composition by Lemma 3.9. \square

This category can be viewed as a bicategory, with 2-morphisms given by maps of cospans. In this bicategory every morphism of \mathcal{M} has a right adjoint.

Examples 4.4. Our familiar examples:

- $\mathcal{C}; \mathcal{C}^{\text{opp}}$ is by definition equal to $\text{Cospan}(\mathcal{C})$.
- $\mathcal{C}; \mathcal{I}_{\mathcal{C}}^{\text{opp}}$ is naturally isomorphic to \mathcal{C} .
- $\text{Set}; \text{Inj}^{\text{opp}}$ is the category of partial functions.

Lemma 4.5. *Let $\mathcal{C}, \mathcal{C}'$ be categories with finite colimits, and let $\mathcal{M}, \mathcal{M}'$ be subcategories each stable under pushouts. Let $A: \mathcal{C} \rightarrow \mathcal{C}'$ be functor that preserves colimits*

and such that the image of \mathcal{M} lies in \mathcal{M}' . Then A extends to a symmetric strong monoidal functor

$$A: \mathcal{C}; \mathcal{M}^{\text{opp}} \longrightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}.$$

mapping X to AX and $\xrightarrow{c} \xleftarrow{m}$ to $\xrightarrow{Ac} \xleftarrow{Am}$.

Proof. Note $A(\mathcal{M}) \subseteq \mathcal{M}'$, so $\xrightarrow{Ac} \xleftarrow{Am}$ is indeed a morphism in $\mathcal{C}'; \mathcal{M}'^{\text{opp}}$. This is then a restriction and corestriction of the usual functor $\text{Cospan}(\mathcal{C}) \rightarrow \text{Cospan}(\mathcal{C}')$ to the above domain and codomain. \square

That this ‘subcospan category’ construction could be defined more generally using any two isomorphism-containing wide subcategories stable under pushout, but the above suffices for decorated corelations.

4.2.2 Decorated corelations

Decorated corelations are constructed from a lax monoidal functor from $\mathcal{C}; \mathcal{M}^{\text{opp}}$ to Set .

Definition 4.6. Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . Suppose we also have a lax monoidal functor

$$F: (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

We define an F -**decorated corelation** to a pair

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right)$$

where the cospan is jointly \mathcal{E} -like. A morphism of decorated corelations is a morphism of decorated cospans between two decorated corelations.

Suppose we have decorated corelations

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right) \quad \text{and} \quad \left(\begin{array}{ccc} & M & \\ i \nearrow & & \nwarrow o \\ Y & & Z \end{array}, \quad \begin{array}{c} FM \\ \uparrow s \\ 1 \end{array} \right).$$

Then their composite is given by the composite corelation

$$\begin{array}{ccc} & \overline{N +_Y M} & \\ i \nearrow & & \nwarrow o \\ X & & Z \end{array}$$

paired with the decoration

$$1 \longrightarrow F(N + M) \longrightarrow F(N +_Y M) \xrightarrow{F(m^{\text{opp}})} F(\overline{N +_Y M})$$

As composition of corelations and decorated cospans are both well-defined up to isomorphism, this too is well-defined up to isomorphism. Again, we will be lazy about the distinction between a decorated corelation and its isomorphism class.

4.2.3 Categories of decorated corelations

In this subsection we give a definition of the hypergraph category $F\text{Corel}$ of decorated corelations. In analogy with how we defined the hypergraph category of corelations, we leverage the fact that decorated cospans form a hypergraph category, this time using a structure preserving map

$$\square: F\text{Cospan} \longrightarrow F\text{Corel}.$$

Here $F\text{Cospan}$ denotes the decorated cospan category constructed from the restriction of the functor $F: \mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \text{Set}$ to the domain \mathcal{C} .

Given a cospan $X \rightarrow N \leftarrow Y$, write $m: \overline{N} \rightarrow N$ for the \mathcal{M} factor of the copairing $X + Y \rightarrow N$. The functor \square takes a decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN)$$

to the decorated corelation

$$(X \xrightarrow{\bar{i}} \overline{N} \xleftarrow{\bar{o}} Y, 1 \xrightarrow{Fm^{\text{opp}} \circ s} F\overline{N}),$$

where the corelation is given by the jointly \mathcal{E} -part of the cospan, and the decoration is given by composing s with the F -image $Fm^{\text{opp}}: FN \rightarrow F\overline{N}$ of the map $N \xrightarrow{1_N} N \xleftarrow{m} \overline{N}$ in $\mathcal{C}; \mathcal{M}^{\text{opp}}$. We call $Fm^{\text{opp}} \circ s$ the **restricted decoration** of the decorated cospan $(X \rightarrow N \leftarrow Y, 1 \xrightarrow{s} FN)$.

The monoidal product of two decorated corelations is their monoidal product as decorated cospans.

Theorem 4.7. *Let \mathcal{C} be a category with finite colimits and factorisation system $(\mathcal{E}, \mathcal{M})$ with \mathcal{M} stable under pushout, and let*

$$F: (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times)$$

be a symmetric lax monoidal functor. Then we may define

<i>The hypergraph category $(F\text{Corel}, +)$</i>	
objects	<i>the objects of \mathcal{C}</i>
morphisms	<i>isomorphism classes of F-decorated corelations in \mathcal{C}</i>
composition	<i>given by \mathcal{E}-part of pushout with restricted decoration</i>
monoidal product	<i>the coproduct in \mathcal{C}</i>
coherence maps	<i>maps from $\text{Cospan}(\mathcal{C})$ with restricted empty decoration</i>
hypergraph maps	<i>maps from $\text{Cospan}(\mathcal{C})$ with restricted empty decoration</i>

Similar to the corelations theorem (Theorem 3.10), we have specified well-defined data and now need to check a collection of coherence axioms. As before, we prove this alongside the theorem about decorated corelation functors in the next section.

Example 4.8. Note that decorated cospans are a special case of decorated corelations: simply use an morphism–isomorphism factorisation system.

Example 4.9. Note that ‘undecorated’ corelations are a special case of decorated corelations: they are corelations decorated by the functor $1: \mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \text{Set}$ that maps each object to the one element set 1, and each morphism to the identity function on 1. This is a symmetric monoidal functor with the coherence maps all also the identity function on 1.

4.3 Functors between decorated corelation categories

Functors between decorated corelation categories hold no surprises: their requirements combine the requirements of corelations and decorated cospans. Recall that Lemma 4.5 says that we can extend a colimit-preserving functor $\mathcal{C} \rightarrow \mathcal{C}'$ to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}$.

Proposition 4.10. *Let $\mathcal{C}, \mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M}), (\mathcal{E}', \mathcal{M}')$, such that \mathcal{M} and \mathcal{M}' are stable under pushout, and suppose that we have symmetric lax monoidal functors*

$$F : (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times)$$

and

$$G : (\mathcal{C}'; \mathcal{M}'^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

Further let $A: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of \mathcal{M} lies in \mathcal{M}' . This functor A extends to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}$.

Suppose we have a monoidal natural transformation θ :

$$\begin{array}{ccc} \mathcal{C}; \mathcal{M}^{\text{opp}} & & \\ \downarrow A & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \theta \\ \xrightarrow{G} \end{array} & \text{Set} \\ \mathcal{C}'; \mathcal{M}'^{\text{opp}} & & \end{array}$$

Then we may define a hypergraph functor $T: F\text{Corel} \rightarrow G\text{Corel}$ sending each object $X \in F\text{Corel}$ to $AX \in G\text{Corel}$ and each decorated corelation

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN)$$

to

$$(AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY, \quad 1 \xrightarrow{s} FN \xrightarrow{\theta_N} GAN \xrightarrow{Gm_{AN}^{\text{opp}}} \overline{GAN}).$$

The coherence maps $\overline{\kappa_{X,Y}}$ are given by the coherence maps of A with the restricted empty decoration.

Proof of Theorem 4.7 and Proposition 4.10. In the proof of Theorem 3.10 and Proposition 3.11 we proved that the map

$$\square: \text{Corel}(\mathcal{C}) \longrightarrow \text{Corel}(\mathcal{C}')$$

preserved composition and had natural coherence maps. Specialising to the case when $\text{Corel}(\mathcal{C}) = \text{Cospan}(\mathcal{C}')$, we saw that this bijective-on-objects, surjective-on-morphisms, composition and monoidal product preserving map proved $\text{Corel}(\mathcal{C}')$ is a hypergraph category, and it immediately followed that \square is a hypergraph functor.

The analogous argument holds here: we simply need to prove

$$\square: F\text{Corel} \longrightarrow G\text{Corel}$$

preserves composition and has natural coherence maps. Theorem 4.7 then follows from examining the map $F\text{Cospan} \rightarrow F\text{Corel}$ obtained by choosing $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}', \mathcal{I}_{\mathcal{C}'})$, F the restriction of G to \mathcal{C}' , A the identity functor on \mathcal{C}' , and θ the identity natural transformation. Subsequently Proposition 4.10 follows from noting that all the axioms hold for the corresponding maps in $G\text{Cospan}$.

□ **preserves composition.** Suppose we have decorated corelations

$$f = (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN) \quad \text{and} \quad g = (Y \xrightarrow{i_Y} M \xleftarrow{o_Y} Z, 1 \xrightarrow{t} FM)$$

We know the functor □ preserves composition on the cospan part; this is precisely the content of Proposition 3.11. It remains to check that □($g \circ f$) and □ $g \circ$ □ f have isomorphic decorations. This is expressed by the commutativity of the following diagram:

$$\begin{array}{ccccc}
 \overline{GA(N+Y M)} & \xrightarrow{Gn} & \overline{G(\overline{AN} +_{AY} \overline{AM})} & & \\
 \uparrow Gm_{A(N+Y M)}^{\text{opp}} & & \uparrow Gm_{\overline{AN} +_{AY} \overline{AM}}^{\text{opp}} & & \\
 \overline{GA(N+Y M)} & & \overline{G(\overline{AN} +_{AY} \overline{AM})} & & \\
 \uparrow \theta_{N+Y M} & \swarrow GAm_{N+Y M}^{\text{opp}} & & \searrow G(m_{AN}^{\text{opp}} +_{AY} m_{AM}^{\text{opp}}) & \\
 F(\overline{N+Y M}) & \xleftarrow{G\sim} & G(\overline{AN} +_{AY} \overline{AM}) & \xrightarrow{(c)} & G(\overline{AN} + \overline{AM}) \\
 \uparrow Fm_{N+Y M}^{\text{opp}} & & \uparrow GA[j_N, j_M] & & \uparrow G[j_{AN}, j_{AM}] \\
 F(N+Y M) & \xleftarrow{(TN)} & GA(N+Y M) & \xleftarrow{(A)} & G(AN+AM) \\
 \uparrow F[j_N, j_M] & & \uparrow G\alpha_{N,M} & & \uparrow G(m_{AN}^{\text{opp}} + m_{AM}^{\text{opp}}) \\
 F(N+M) & \xleftarrow{\theta_{N+M}} & GA(N+M) & \xleftarrow{(GM)} & G\overline{AN} \times G\overline{AM} \\
 & & & & \uparrow \gamma_{\overline{AN}, \overline{AM}} \\
 & & & & Gm_{AN}^{\text{opp}} \times Gm_{AM}^{\text{opp}} \\
 & & & & \uparrow \gamma_{AN, AM} \\
 & & & & GAN \times GAM \\
 & \swarrow \varphi_{N,M} & & \searrow \theta_N \times \theta_M & \\
 & FN \times FM & & & \\
 & \uparrow \rho_1 \circ (s \times t) & & & \\
 & 1 & & &
 \end{array}$$

This diagram does indeed commute. To check this, first observe that (TM) commutes by the monoidality of θ , (GM) commutes by the monoidality of G , and (TN) commutes by the naturality of θ . The remaining three diagrams commute as they are G -images of diagrams that commute in \mathcal{C}' ; $\mathcal{M}'^{\text{opp}}$. Indeed, (A) commutes since A preserves colimits and G is functorial, (C) commutes as it is the G -image of a pushout square in \mathcal{C}' , so

$$m_{AN}^{\text{opp}} + m_{AM}^{\text{opp}} [j_{AN}, j_{AM}] \quad \text{and} \quad [j_{AN}, j_{AM}] m_{AN}^{\text{opp}} +_{AY} m_{AM}^{\text{opp}}$$

are equal as morphisms of \mathcal{C}' ; $\mathcal{M}'^{\text{opp}}$, and (**) commutes as it is the G -image of the right-hand subdiagram of (*) used to define n in the proof of Lemma 3.12.

Coherence maps are natural. Let $f = (X \longrightarrow N \longleftarrow Y, 1 \rightarrow FN)$, $g = (Z \longrightarrow M \longleftarrow W, 1 \rightarrow FM)$ be F -decorated corelations in \mathcal{C} . We wish to show that

$$\begin{array}{ccc} AX + AY & \xrightarrow{\square f + \square g} & AZ + AW \\ \overline{\kappa_{X,Y}} \downarrow & & \downarrow \overline{\kappa_{Z,W}} \\ A(X + Y) & \xrightarrow{\square(f+g)} & A(Z + W) \end{array}$$

commutes in $G\text{Corel}$, where the coherence maps are given by

$$\overline{\kappa_{X,Y}} = \left(\begin{array}{ccc} & \overline{A(X + Y)} & \\ \nearrow & & \nwarrow \\ AX + AY & & A(X + Y) \end{array}, \begin{array}{c} G(\overline{A(X + Y)}) \\ \uparrow Gm_{AX+AY}^{\text{opp}} \\ GA(X + Y) \\ \uparrow G! \\ G\emptyset \\ \uparrow \gamma_1 \\ 1 \end{array} \right).$$

Lemma 3.14 shows that the composites of corelations agree. It remains to check that the decorations also agree.

Here Lemma 2.5 is helpful. Since \square is composition preserving, we can replace the $\overline{\kappa}$ with the empty decorated coherence maps κ of $G\text{Cospan}$, and compute these composites in $G\text{Cospan}$, before restricting to the \mathcal{E}' -parts. Lemma 2.5 then implies that the restricted empty decorations on the isomorphisms $\overline{\kappa}$ play no role in determining the composite decorations. It is thus enough to prove that the decorations of $\square f + \square g$ and $\square(f + g)$ are the same up to the isomorphism $p: G(\overline{AN} + \overline{AM}) \rightarrow G\overline{A(N + M)}$ between their apices, as defined in the diagram (#) in the proof of Lemma 3.14.

This comes down to proving the following diagram commutes:

$$\begin{array}{ccccc} & GAN \times GAM & \xrightarrow{Gm \times Gm} & G\overline{AN} \times G\overline{AM} & \xrightarrow{\gamma} & G(\overline{AN} + \overline{AM}) \\ & \uparrow \theta & & \downarrow \gamma & & \uparrow G(m+m) \\ 1 & \xrightarrow{\langle s, t \rangle} & FN \times FM & \xrightarrow{\varphi} & F(N + M) & \xrightarrow{\theta} & GA(N + M) & \xrightarrow{Gm} & G\overline{A(N + M)} \\ & & & & & & \downarrow G\kappa & & \downarrow Gp \\ & & & & & & & & \downarrow \sim \end{array}$$

(T) (G) (##)

It is straightforward to check this commutes: (T) by the monoidality of θ , (G) by the monoidality of G , and (##) as it is the G -image of the rightmost square in (#). \square

Corollary 4.11. Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . Suppose that we also have a lax monoidal functor

$$F : (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

Then we may define a category $F\text{Corel}$ with objects the objects of \mathcal{C} and morphisms isomorphism classes of F -decorated correlations.

Write also F for the restriction of F to the wide subcategory \mathcal{C} of $\mathcal{C}; \mathcal{M}^{\text{opp}}$. We can thus also obtain the category $F\text{Cospan}$ of F -decorated cospan. We moreover have a functor

$$F\text{Cospan} \rightarrow F\text{Corel}$$

which takes each object of $F\text{Cospan}$ to itself as an object of $F\text{Corel}$, and each decorated cospan

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array} , \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right)$$

to its jointly- \mathcal{E} -part

$$\begin{array}{ccc} & \overline{N} & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}$$

decorated by the composite

$$1 \xrightarrow{s} FN \xrightarrow{Fm_N^{\text{opp}}} F\overline{N}.$$

4.4 All hypergraph categories are decorated correlation categories

structured categories are algebras over their graphical calculus operad [?]. These equivalences are 2-categorical.

Write \int for the Grothendieck construction. The Grothendieck construction is

Theorem 4.12. *hypergraph categories are symmetric lax monoidal functors cospan to Set.*

$$\text{HypCat} \cong \int^{\mathcal{O} \in \text{Set}} \text{SymLaxMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}}), \text{Set})$$

[?]

Remark 4.13. Not all hypergraph categories are decorated *cospan* categories. To see this, we can count morphisms. The possible apices and decorations are the same for all morphisms. So for a decorated cospan category over the prop of finite sets, the number of morphisms $0 \rightarrow 1$ cannot be more than countably many times those $0 \rightarrow 0$ (we just get to choose an element of the apex). But the skeletal category of vector

spaces over \mathbb{R} with monoidal product the tensor product has \mathbb{R} morphisms $0 \rightarrow 0$, and \mathbb{R}^2 morphisms $0 \rightarrow 1$.

Decorated corelation categories, however, are more powerful. We can recover all hypergraph categories by forcing the decorations to be on the coproduct of the domain and codomain itself. For this we use the isomorphism–morphism factorisation. Let \mathcal{H} be a hypergraph category, and let \mathcal{C} be the wide subcategory of all Frobenius morphisms. Then \mathcal{H} can be recovered as the $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by global sections in \mathcal{H} .

4.4.1 The global sections construction.

Theorem 4.14. *All hypergraph categories are decorated corelation categories.*

Proof. Let \mathcal{H} be a hypergraph category. Without loss of generality we can assume \mathcal{H} has objects a free monoid under the tensor product; write \mathcal{O} for a collection of generators for this free monoid, and $\text{FinSet}_{\mathcal{O}}$ for \mathcal{O} labelled finite sets (ie an object is a finite set X together with a function $X \rightarrow \mathcal{O}$). This is a finitely cocomplete category. Equivalent to finite lists of objects in \mathcal{H} .

Define the global sections functor

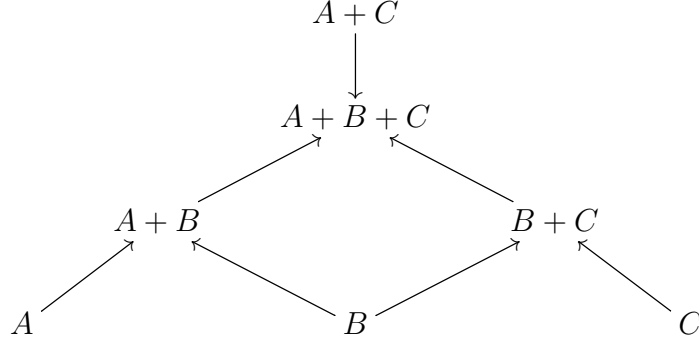
$$\begin{aligned} G: \text{Cospan}(\text{FinSet}_{\mathcal{O}}) &\longrightarrow \text{Set} \\ A &\longmapsto \mathcal{H}(I, A) \\ \xrightarrow{f} \xleftarrow{g} &\longmapsto \text{action of Frobenius maps} \end{aligned}$$

This is symmetric lax monoidal functor. Note that $\mathcal{H}(I, A)$ depends on the order we choose to convert a multiset into an object A of \mathcal{H} . Nonetheless, from any two choices A, A' we get a canonical map $A \rightarrow A'$. This is really that clique in Set .

Consider the category $\text{FinSet}_{\mathcal{O}}$ with an (isomorphism, morphism)-factorisation system. We get a decorated corelations category with objects multisets of generating objects of \mathcal{H} , and morphisms $A \rightarrow B$ trivial corelations $A \rightarrow A + B \leftarrow B$ decorated by some morphism $s \in \mathcal{H}(I, A + B)$. Recall that this decorated corelation is only specified up to isomorphism; in the following we always choose representatives such that the apex of the jointly-isomorphic cospan is always of the form $A + B$ for morphisms $A \rightarrow B$.

Given morphisms $s \in \mathcal{H}(I, A + B)$ and $t \in \mathcal{H}(I, B + C)$ of types $A \rightarrow B$ and $B \rightarrow C$ in $G\text{Corel}$, composition is given by the map $H(I, A + B + B + C) \rightarrow H(I, A + C)$

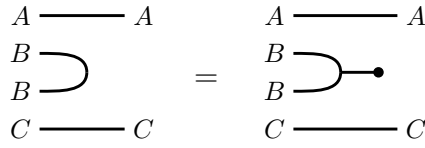
arising as the G -image of the cospan $A + B + B + C \xrightarrow{[j,i]} A + B + C \xleftarrow{m} A + C$ where maps come from the pushout square



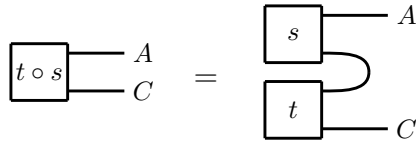
In terms of string diagrams in \mathcal{H} , this means composing the maps



with the Frobenius map

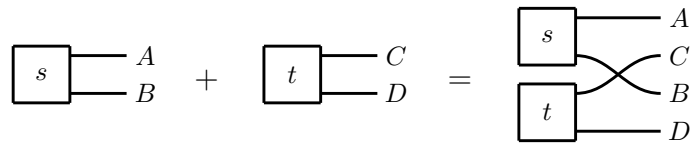


to get



in $\mathcal{H}(I, A + C)$.

The monoidal product is given by



recalling that we have chosen to represent the equivalence class of corelations $A + C \rightarrow B + D$ with the apex $A + C + B + D$.

Taking a hint from the compact closed structure, it is straightforward to construct a pair of inverse hypergraph functors between $G\text{Corel}$ to \mathcal{H} . Indeed, $G\text{Corel}$ is constructed to have the same collection of objects as \mathcal{H} ; simply have the functors be the ‘identity’ on objects. On morphisms, we take $f : A \rightarrow B$ in \mathcal{H} to its ‘name’ $\hat{f} : I \rightarrow A + B$ as a morphism of $G\text{Corel}$. This is a bijection.

To check it is composition and monoidal product preserving, we can easily use diagrammatic reasoning. For example

Therefore \mathcal{H} is isomorphic as a hypergraph category to $G\text{Corel}$. \square

Theorem 4.15. *All hypergraph functors are decorated corelation functors.*

Proof. Let \mathcal{H} and \mathcal{H}' be hypergraph categories, and $T: \mathcal{H} \rightarrow \mathcal{H}'$ be a hypergraph functor. By the above theorem, there exist symmetric lax monoidal functors

$$G: \text{Cospan}(\text{FinSet}_{\mathcal{O}_{\mathcal{H}}}) \rightarrow \text{Set}$$

and

$$G': \text{Cospan}(\text{FinSet}_{\mathcal{O}_{\mathcal{H}'}}) \rightarrow \text{Set}$$

such that $\mathcal{H} = G\text{Corel}$ and $\mathcal{H}' = G'\text{Corel}$. Furthermore, define a functor $A: \text{FinSet}_{\mathcal{O}_{\mathcal{H}}} \rightarrow \text{FinSet}_{\mathcal{O}_{\mathcal{H}'}}$ taking $N \rightarrow \mathcal{O}_{\mathcal{H}}$ to $N \rightarrow \mathcal{O}_{\mathcal{H}} \rightarrow \mathcal{O}_{\mathcal{H}'}$, where the second map is that by the functor T on objects of \mathcal{H} . We claim this is a well-defined colimit-preserving functor and show that T can be constructed from a monoidal natural transformation between G and $G' \circ A$. \square

Compare with Spivak Vagner construction.

4.4.2 Examples.

We give some examples reproducing hypergraph categories as decorated corelations categories.

Example 4.16. Example: empty decorations and equivalence relations.

Consider the hypergraph category $\text{Cospan}(\text{FinSet})$. This is the simplest hypergraph category: it is free hypergraph category on the one object discrete category. We show how to recover it as a decorated corelation category.

As per Example 4.9, $\text{Cospan}(\text{FinSet})$ is the hypergraph category of undecorated (morphism-isomorphism)-corelations in FinSet . It is also the partition-decorated (isomorphism-morphism)-corelations in FinSet .

First, the global sections functor $G: \text{Cospan}(\text{FinSet}) \rightarrow \text{Set}$ takes each finite set X to the set of (equivalence classes of) cospans $0 \rightarrow D \leftarrow X$; that is, to the set of functions $X \rightarrow D$ where a unique D is chosen for each finite cardinality. Given a

cospan $X \xrightarrow{f} N \xleftarrow{g} Y$, its image under the global sections functor maps a function $a: X \rightarrow D$, to the function $Y \rightarrow N +_Y D$ given by

$$\begin{array}{ccc} X & \xrightarrow{d} & D \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{g} N \longrightarrow & N +_Y D \end{array}$$

where the square is a pushout square.

The coherence maps $\gamma_1: 1 \rightarrow G\emptyset$ map the unique element of 1 to the unique function $!: \emptyset \rightarrow \emptyset$, and $\gamma_{X,Y}$ maps a pair of functions $a: X \rightarrow D$, $b: Y \rightarrow E$ to $a + b: X + Y \rightarrow D + E$. This is a symmetric lax monoidal functor. We use this functor to decorate cospans in \mathbf{FinSet} .

A decorated cospan in \mathbf{FinSet} with respect to this functor is a cospan $X \rightarrow N \leftarrow Y$ in \mathbf{FinSet} together with a function $N \rightarrow D$ for some finite set. Using the isomorphism-morphism factorisation, a decorated corelation (a morphism in $G\mathbf{Corel}$) is a cospan $X \rightarrow X + Y \leftarrow Y$, together with a function $X + Y \rightarrow D$. This is the same as a cospan.

The hypergraph structure is given by the decoration $X + X \rightarrow X$ etc, as the shift from cospans to corelations takes the ‘factored out part’ and puts it into the decoration. At this point the morphisms are specified entirely by their decoration.

It is straightforward to show the two categories are isomorphic. Note that the identity on \mathbf{FinSet} maps $\mathcal{I}_{\mathbf{FinSet}}$ into \mathbf{FinSet} , and so extends to morphism $\mathbf{FinSet} \rightarrow \mathbf{Cospan}(\mathbf{FinSet})$. We can define a monoidal natural transformation $1(X) = 1 \xrightarrow{\theta_X} GX = \{X \rightarrow D\}$ mapping the unique element to the identity function $1_X: X \rightarrow X$.

$$\begin{array}{ccc} \mathbf{FinSet} = \mathbf{FinSet}; \mathcal{I}_{\mathbf{FinSet}}^{\text{opp}} & \xrightarrow{1} & \mathbf{Set} \\ \downarrow \iota & \searrow \Downarrow \theta & \\ \mathbf{Cospan}(\mathbf{FinSet}) = \mathbf{FinSet}; \mathbf{FinSet}^{\text{Gpp}} & & \end{array}$$

It is easy to verify that this is a monoidal natural transformation. This gives the hypergraph functor we expect, mapping the undecorated cospan $X \rightarrow N \leftarrow Y$ to $X \rightarrow X + Y \leftarrow Y$ decorated by $X + Y \rightarrow N$.

Like so many examples before, it is easy to verify this is a full, faithful, bijective-on-objects hypergraph functor.

Example 4.17. The previous example extends to any finitely cocomplete category \mathcal{C} : the hypergraph category $\mathbf{Cospan}(\mathcal{C})$ can always be constructed as (i) trivially decorated $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ -corelations, or (ii) $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by equivalence classes

of morphisms with domain the apex of the corelation, and moreover the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

More general still, a category of trivially decorated $(\mathcal{E}, \mathcal{M})$ -corelations in \mathcal{C} can always be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by equivalence classes of morphisms in \mathcal{E} with domain the apex of the corelation, and the isomorphism of these hypergraph categories is given by a monoidal natural transformation between the decorating functors.

Most general, the theorem implies that any category of decorated corelations can be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by codomain decorated morphisms in \mathcal{E} . The latter are specified by a functor $\text{Cospan}(\mathcal{C}) \rightarrow \text{Set}$, as in the theorem.

The latter form is good for constructing functors that have image the corelation category.

Note that with decorated corelations we have a method of constructing hypergraph categories from other hypergraph categories. This is black boxing.

4.5 Examples

4.5.1 Path integrals and matrices

Let R be a commutative ring. Take functor

$$\begin{aligned} R^{(-)} : (\text{FinSet}^{\text{opp}}, +) &\longrightarrow (\text{Set}, \times) \\ X &\longmapsto R^X \end{aligned}$$

Then $R^{(-)}\text{Cospan}$ is path integrals, $R^{(-)}\text{Corel}$ is matrices over R .

Many aspects of this example are ‘atypical’, regarding the intuition we have been working towards. Note that the monoidal product here is the tensor product of matrices, not the biproduct. Indeed, there is no special commutative Frobenius algebra in Vect if we use the biproduct, but if we use the tensor product then these correspond to orthonormal bases (Vicary). The comultiplication is the diagonal map, multiplication is codiagonal. unit produces basis.

We note that you could take decorations here in the category $R\text{Mod}$ of R -modules. While Proposition 2.9 shows that the resulting decorated cospans category would be isomorphic, this hints at an enriched version of the theory.

4.5.2 Two constructions for linear relations

We saw earlier that linear relations are epi-mono corelations in \mathbf{Vect} . The hypergraph structure is given by addition. We show how to recover this in another construction. We also get a hypergraph functor between them. This is very useful for compositional linear relations semantics of diagrams.

We can also construct linear relations in $\mathbf{Vect}^{\text{opp}}$.

$$\begin{aligned}
 & : \mathbf{Cospan}(\mathbf{FinSet}) \longrightarrow \mathbf{Set} \\
 & X \longmapsto \{\text{subspaces of } k^X\} \\
 & f : X \rightarrow Y \longmapsto L \mapsto \{v \mid v \circ f \in L\} \\
 & f^{\text{opp}} : X \rightarrow Y \longmapsto L \mapsto \{v = u \circ f \mid u \in L\}
 \end{aligned}$$

Then \mathbf{Cospan} is cospans decorated by subspaces, and \mathbf{Corel} is linear relations. This is important for circuits work [?, ?].

4.5.3 Automata

This construction comes immediately from Walters et al. Automata are alphabet labelled graphs. There is a decorated cospan functor to categories enriched over languages, and this factors nicely to get a decorated corelation category with morphisms languages recognised between points in domain and codomain.

Part II

Applications

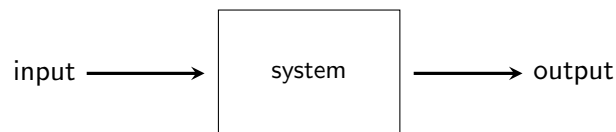
Chapter 5

Signal flow diagrams

5.1 Introduction

5.1.1 Behavioural control

Control theory begins with the following picture:



In this picture we have an object under study, referred to as a *system*, which when fed certain inputs produces certain outputs. These outputs need not be uniquely determined by the inputs; in general the relationship may be stochastic or non-deterministic, or depend also on internal states of the system. Although we take it as given that the system is an *open* system—so it interacts with its environment, and so we can observe its inputs and outputs—we assume no access to the details of these internal states or the inner workings of the system, and this can add considerable complexity to our models. The end goal of control theory is then to control the system: to understand how to influence its behaviour in order to achieve some desired goal. Two key questions arise:

- **Analysis:** Given a system, what is the relationship it induces between input and output?
- **Synthesis:** Given a target relationship between input and output, how do we build a system that produces this relationship?

The first, the question of system analysis, provides the basic understanding required to find the inputs that lead to the desired outputs. The second question, the question

of synthesis, is the central question of feedback control theory, which aims to design controllers to regulate other systems. We give brief overview of the field, illustrated with some questions of these kinds.

Control has a long history. Indeed, many systems found in the biology and chemistry of living organisms have interesting interpretations from a control theoretic viewpoint, such as those that are responsible for body temperature regulation or bipedal balance [?]. Human understanding of control developed alongside engineering, and so dates back at least as far as antiquity, with for example the ancient Romans devising elaborate systems to maintain desired water levels in aqueducts [?]. The origin of formal mathematical control theory, however, is more recent, and in general taken to be James Clerk Maxwell's seminal analysis of centrifugal governors, presented to the Royal Society of London in 1868 [?].

A good example of a basic, but characteristic, system studied by control theory, centrifugal governors were developed to address the problem of speed regulation for steam engines. A centrifugal governor consists of masses attached via lever arms to a vertical spindle, as depicted in Figure . This spindle is then connected to the output wheel of an engine, and the height of the masses is inversely related to the aperture of the steam valve into the engine. Higher speeds therefore cause the spindle to rotate faster, which in turn causes the masses to move upward and decrease the flow of steam into the engine. This moderates or prevents acceleration, and hence overspeeding of the engine. Maxwell's paper classifies the possible trajectories over time of the speed of a governed steam engine.

With respect to the framework laid out above, there are three systems under investigation here. The first is the steam engine, with input steam flow and output engine speed. The second is the centrifugal governor, with input engine speed and output steam flow. The third is the combined governor-steam engine, which forms the main system under investigation, with again input steam flow and output engine speed.

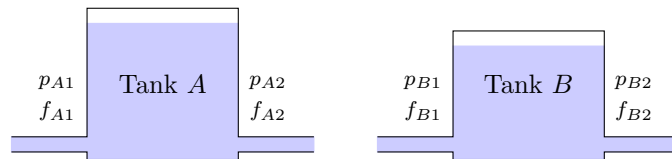
The techniques used and developed from Maxwell's paper, in particular by Rayleigh and Heaviside, are in general known as *transfer function* techniques. A transfer function is a linear map from the set of inputs to the set of outputs of a linear time-invariant system with zero initial conditions. Transfer functions are commonly used in the analysis of single-input single-output linear systems, but become unwieldy or inapplicable for more general systems.

In the 1960s, led by Wiener and Kalman, so-called *state space* techniques were developed to address multiple-input multiple-output, time-varying, nonlinear systems

[?]. These methods are characterised by defining a system as a collection of input, output, and internal state variables, with the state changing as a function of the input and state variables over time, and the output a function of the input and state. These functions are often implicitly specified by differential equations.

In general, however, classical control theory remains grounded in a paradigm that defines and analyses systems in terms of inputs and outputs or, from another perspective, causes and effects. In recent years Willems, among others, has argued that this input-output perspective is limiting, as in the case of many systems studied through control theory there is no clear distinction between input and output variables [?]. For example, given a circuit component for which the relevant variables are the voltages and currents, different contexts may call for the voltage to be viewed as the input and current output, or vice versa. It is useful to have a single framework capable of discussing the behaviour of the component without making this choice.

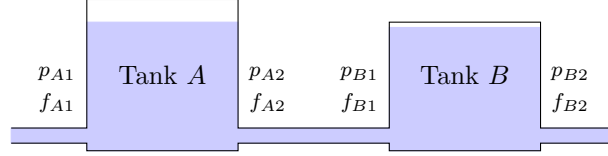
Moreover, in the drive to understand larger and more complex systems, increasing emphasis has been put on understanding the way systems can be broken down into composite subsystems, and conversely how systems link together to form larger systems [?, ?]. Indeed, interconnection or composition of systems has always played a central role in systems engineering, as in the combination of the centrifugal governor and steam engine of Maxwell's research, and the difficulty of discussing how systems compose within an input-output framework lends support to Willems' call for a more nuanced definition of control system. To illustrate these difficulties, consider the simple example of two water tanks each with two access pipes:



For each tank, the relevant variables are the pressure p and the flow f . A typical control theoretic analysis might view the water pressure as inducing flow through the tank, and hence take pressure as the input variable, flow as the output variable, and describe each tank as a transfer function $H_\bullet : P_\bullet \rightarrow F_\bullet$ between the sets P_\bullet and F_\bullet these vary over.

Ideally then, the composite system below, connecting pipe 2 of Tank A to pipe 1 of Tank B, would be described by the composite of the transfer functions H_A and

H_B of these tanks.



This is rarely the case, and indeed makes little sense: the output of transfer function H_A describes the flow through Tank A , while the domain of the transfer function H_B describes pressures through Tank B , so taking the output of H_A as the input of H_B runs into data type issues. Instead, here the relationship between the transfer functions H_A and H_B , and the transfer function H_{AB} of the composite system can be understood through the fact that connection requires the pressure at pipe 1 of Tank A must be equal to the pressure at pipe 2 of Tank B , and that the flow out of the former pipe must equal the flow into the latter—that is, by imposing the relations

$$\begin{aligned} p_{A2} &= p_{B1} \\ f_{A2} &= -f_{B1} \end{aligned}$$

on the variables. Indeed in many contexts, hydraulics and electronics among them, connections between systems are characterised not by the ‘output’ of one system forming the ‘input’ of the next, but by *variable sharing* between the systems. Such relations are often difficult to describe using the language of transfer functions and state space methods.

5.1.2 Signal flow

The remit of this paper is the development of a sound and fully complete equational theory of linear time-invariant (LTI) dynamical systems. This theory is *graphical*, with its terms—modelling the LTI systems themselves—best represented as diagrams closely resembling the signal flow graphs of Shannon [?].

To acquaint ourselves with signal flow graphs, we begin with the example below, rendered in traditional, directed notation.

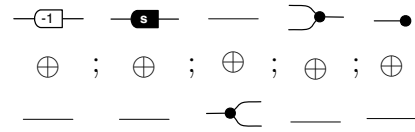
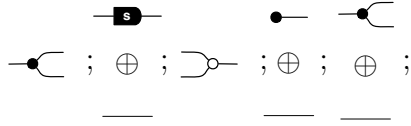


This system takes, as input on the left, a stream of values from a field \mathbf{k} , e.g. the rational numbers, and on the right outputs a processed stream of values. The white circles are adders, the black circles are duplicators, the s gates are 1-step delays and

the -1 gate is an instance of an amplifier that outputs -1 times its input. Processing is done synchronously according to a global clock.

For instance, assume that at time 0 the left s gate ‘stores’ the value 1 and the right s gate stores 2. Given an input of -1 , the flow graph first adds the left stored value 1, and then adds -1×2 , for an output of -2 . Immediately after this time step the s gates, acting as delays, now store -1 and -2 respectively, and we repeat the process with the next input. Thus from this time 0 an input stream of $-1, 1, -1, 1 \dots$ results in an output stream of $-2, 2, -2, 2, \dots$

We can express (5.1) as a string diagram, a notation for the arrows of monoidal categories made popular by Joyal and Street [?], by forgetting the directionality of wires and composing the following basic building blocks using the operations of monoidal categories.



The building blocks come from the signature of an algebraic theory—a *symmetric monoidal theory* to be exact. The terms of this theory comprise the morphisms of a *prop*, a symmetric monoidal category in which the objects are the natural numbers. With an operational semantics suggested by the above example, the terms can also be considered as a process algebra for signal flow graphs. The idea of understanding complex systems by “tearing” them into more basic components, “zooming” to understand their individual behaviour and “linking” to obtain a composite system is at the core of the behavioural approach in control, originated by Willems [?]. The algebra of symmetric monoidal categories thus seems a good fit for a formal account of these compositional principles.

This paper is the first to make this link between monoidal categories and the behavioural approach to control explicit. Moreover, it is the first to endow signal flow graphs with their standard systems theoretic semantics in which the registers—the ‘ s ’ gates—are permitted to hold *arbitrary* values at the beginning of a computation (in previous work [?, ?] they were initialised with 0). This extended notion of behaviour is not merely a theoretical curiosity: it gives the class of *complete LTI discrete dynamical systems* [?], which is practically the lingua franca of control theory. The interest of

systems theorists is due to practical considerations: physical systems seldom evolve from zero initial conditions.

Technically, the semantics of diagrams are sets of *biinfinite* streams: those sequences of elements of \mathbf{k} that are infinite in the past *as well* as in the future—that is, elements of $\mathbf{k}^{\mathbb{Z}}$. Starting with the operational description, one obtains a biinfinite trajectory by executing circuits forwards and backwards in time, for some initialisation of the registers. The dynamical system defined by a signal flow diagram is the set of trajectories obtained by considering all possible executions from all possible initialisations.

An equational theory also requires equations between the terms. We obtain the equations in two steps. First, we show there is a full, but not faithful, morphism from the prop $\text{Cospan Mat } \mathbf{k}[s, s^{-1}]$ of cospans of matrices over the ring $\mathbf{k}[s, s^{-1}]$ to the prop LTI of complete LTI discrete dynamical systems. Using the presentation of $\text{Cospan Mat } \mathbf{k}[s, s^{-1}]$ in [?, ?], the result is a sound, but not complete, proof system. The second ingredient is restricting our attention from cospans to jointly-epic cospans, or *corelations*. This gives a faithful morphism, allowing us to present the prop of corelations as a symmetric monoidal theory, and hence giving a sound and complete proof system for reasoning about LTIs (Theorem 5.16).

The advantages of the string diagram calculus over the traditional matrix calculus are manifold. The operational semantics make the notation intuitive, as does the compositional aspect: it is cumbersome to describe connection of systems using matrices, whereas with string diagrams you just connect the right terminals. Moreover, the calculus unifies the variety of distinct methods for representing LTI systems with matrix equations—built from kernel and image representations [?, ?]—into a single framework, heading off possibilities for ambiguity and confusion.

We hope, however, the greatest advantage will be the way these properties can be leveraged in analysis of controllability. In Theorem 5.23, we show that in our setting controllability has an elegant structural characterisation. Compositionality pays off here, with our proof system giving a new technique for reasoning about control of compound systems (Prop. 5.27). From the systems theoretic point of view, these results are promising since the compositional, diagrammatic techniques we bring to the subject seem well-suited to problems such as controllability of interconnections, of primary interest for multiagent and spatially interconnected systems [?].

Summing up, our original technical contributions are:

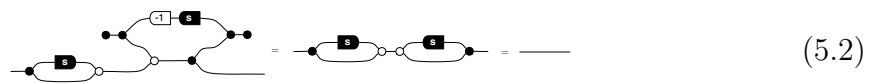
- a characterisation of the class of LTI systems as a category of corelations of matrices

- a presentation of this category of corelations of matrices as a symmetric monoidal theory
- an operational semantics that agrees with the standard systems theoretic semantics of signal flow graphs
- a characterisation of controllability

Our work lies in the intersection of computer science, mathematics, and systems theory. From computer science, we use concepts of formal semantics of programming languages, with an emphasis on compositionality and a firm denotational foundation for operational definitions. From a mathematical perspective, we obtain presentations of several relevant domains, and identify the rich underlying algebraic structures. For systems theory, our insight is that mere matrices are not optimised for discussing behaviour; instead it is profitable to use signal flow graphs, which treat linear subspaces rather than linear transformations as the primitive concept and are thus closer to the idea of system as a set of trajectories. At the core is the maxim—perhaps best understood by computer scientists—that the right language allows deeper insights into the underlying structure.

Related work. Work on categorical approaches to control systems goes back at least to Goguen [?] and Arbib and Manes [?]. In recent years, there has been a resurgence of interest in the topic, including work by Baez and Erbele [?], Vagner, Spivak, and Lerman [?], as well as Bonchi, Zanasi, and the second author [?, ?, ?, ?].

Although previous work [?, ?, ?] made the connection between signal flow graphs and string diagrams, their operational semantics is more restrictive than that considered here, considering only trajectories with finite past and demanding that, initially, all the registers contain the value 0. Indeed, with this restriction, it is not difficult to see that the trajectories of (5.1) are those where the output is *the same* as the input. The input/output behaviour is thus that of a stateless wire. The equational presentation in this case is the theory $\mathbb{HH}_{k[s]}$ of interacting Hopf algebras [?], and indeed, in $\mathbb{HH}_{k[s]}$:



$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} \quad (5.2)$$

Note that (5.2) is *not sound* for circuits with our more liberal, operational semantics. Indeed, recall that when the registers of (5.1) initially hold values 1 and 2, the input $-1, 1, -1, 1, \dots$ results in the output $-2, 2, -2, 2, \dots$. This trajectory is not

permitted by a stateless wire, so $\mathbb{H}_{k[s]}$ is not sound for reasoning about LTI systems in general. The contribution of this paper is to provide sound and complete theory to do just that.

In terms of the algebraic semantics, the difference from previous work [?, ?] is that where there streams were handled with Laurent (formal power) series, here we use the aforementioned biinfinite streams. Indeed, this is the very extension that allows us to discuss non-controllable behaviours; in [?, ?, ?, ?] all definable behaviours were controllable.

Structure of the paper. In §5.3 we develop a categorical account of complete LTI discrete dynamical systems. This serves as a denotational semantics for the graphical language, introduced in §5.4, where we also derive the equational characterisation. In we relate this to the operational semantics. We conclude in §5.5 with a structural account of controllability.

5.2 Preliminaries

We assume familiarity with basic concepts of linear algebra and category theory.

A **prop** is a strict symmetric monoidal category where the set of objects is the natural numbers \mathbb{N} , and monoidal product (\oplus) on objects is addition. Homomorphism of props are identity-on-objects strict symmetric monoidal functors.

A **symmetric monoidal theory** (SMT) is a presentation of a prop: a pair (Σ, E) where Σ is a set of **generators** $\sigma: m \rightarrow n$, where m is the **arity** and n the **coarity**. A Σ -term is a obtained from Σ , identity $\text{id}: 1 \rightarrow 1$ and symmetry $\text{tw}: 2 \rightarrow 2$ by composition and monoidal product, according to the grammar

$$\tau ::= \sigma \mid \text{id} \mid \text{tw} \mid \tau ; \tau \mid \tau \oplus \tau$$

where $;$ and \oplus satisfy the standard typing discipline that keeps track of the domains (arities) and codomains (coarities)

$$\frac{\tau : m \rightarrow d \quad \tau' : d \rightarrow n}{\tau ; \tau' : m \rightarrow n} \quad \frac{\tau : m \rightarrow n \quad \tau' : m' \rightarrow n'}{\tau \oplus \tau' : m + m' \rightarrow n + n'}$$

The second component E of an SMT is a set of **equations**, where an equation is a pair (τ, μ) of Σ -terms with compatible types, i.e. $\tau, \mu: m \rightarrow n$ for some $m, n \in \mathbb{N}$.

Given an SMT (Σ, E) , the prop $\mathbf{S}_{(\Sigma, E)}$ has as arrows the Σ -terms quotiented by the smallest congruence that includes the laws of symmetric monoidal categories and

equations E . We sometimes abuse notation by referring to $\mathbf{S}_{(\Sigma, E)}$ as an SMT. Given an arbitrary prop \mathbb{X} , a **presentation** of \mathbb{X} is an SMT (Σ, E) s.t. $\mathbb{X} \cong \mathbf{S}_{(\Sigma, E)}$.

String diagrams play an important role in our work. Given generators Σ , we consider string diagrams to be arrows of $\mathbf{S}_{(\Sigma, \emptyset)}$, that is, syntactic objects constructed by composition from the generators, quotiented by the laws of symmetric monoidal categories. For example, consider generators $2 \rightarrow 1$ and $0 \rightarrow 1$, which we draw below, with ‘dangling wires’ accounting for arities and coarities:

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array}$$

Armed with the graphical notation, we can present sets of equations as equations between diagrams. For example, the SMT of commutative monoids consists of these generators together with equations

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \quad \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} \quad \text{---} = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array}$$

that respectively are commutativity, associativity, and the unit law.

A **split mono** is a morphism $m: X \rightarrow Y$ such that there exists $m': Y \rightarrow X$ with $m'm = \text{id}_X$. For the remainder of the paper \mathbf{k} is a field: for concreteness one may take this to be the rationals \mathbb{Q} , the reals \mathbb{R} or the booleans \mathbb{Z}_2 .

5.3 Linear time-invariant dynamical systems

This section focusses on the mathematical domains of interest for the remainder of the paper. We rely on definitions of the *behavioural approach* in control, which is informed by compositional considerations [?]. The concepts are standard in systems theory. Our categorical insights are, to the best of our knowledge, original.

Following Willems [?], a **dynamical system** (T, W, \mathcal{B}) is: a **time axis** T , a **signal space** W , and a **behaviour** $\mathcal{B} \subseteq W^T$. We refer to $w \in \mathcal{B}$ as **trajectories**. In this paper we are interested in discrete trajectories that are **biinfinite**: infinite in past and future. Our time axis is thus the integers \mathbb{Z} . The signal space is \mathbf{k}^d , where d is the number of **terminals** of a system. These, in engineering terms, are the interconnection variables that enable interaction with an environment.

The dynamical systems of concern to us are thus specified by some natural number d and a subset \mathcal{B} of $(\mathbf{k}^d)^{\mathbb{Z}}$. The sets $(\mathbf{k}^d)^{\mathbb{Z}}$ are \mathbf{k} -vector spaces, with pointwise addition and scalar multiplication. We restrict attention to *linear* systems, meaning that \mathcal{B} is required to be a *\mathbf{k} -linear subspace*—i.e. closed under addition and multiplication by \mathbf{k} -scalars—of $(\mathbf{k}^d)^{\mathbb{Z}}$.

We partition terminals into a *domain* and *codomain* of m and n terminals respectively, writing $\mathcal{B} \subseteq (\mathbf{k}^m)^\mathbb{Z} \oplus (\mathbf{k}^n)^\mathbb{Z} \cong (\mathbf{k}^d)^\mathbb{Z}$. This may seem artificial, in the sense that the assignment is arbitrary. In particular, it is crucial not to confuse the domains (codomains) with inputs (outputs). In spite of the apparent contrivedness of choosing such a partition, Willems and others have argued that it is vital for a sound theory of system *decomposition*; indeed, it enables the “tearing” of Willems’ tearing, zooming and linking [?].

Once the domains and codomains have been chosen, systems are linked by connecting terminals. In models of physical systems this means variable coupling or sharing [?]; in our discrete setting where behaviours are subsets of a cartesian product—i.e. relations—it amounts to relational composition. Since behaviours are both relations and linear subspaces, a central underlying mathematical notion—as in previous work [?, ?]—is a linear relation.

Definition 5.1. The monoidal category $\text{LinRel}_\mathbf{k}$ of \mathbf{k} -linear relations has \mathbf{k} -vector spaces as objects, and as arrows from V to W , linear subspaces of $V \oplus W$, considered as \mathbf{k} -vector spaces. Composition is relational: given $A: U \rightarrow V$, $B: V \rightarrow W$, $A ; B: U \rightarrow W$ is the relation

$$\{(u, w) \mid \exists v \in V \text{ s.t. } (u, v) \in A \text{ and } (v, w) \in B\}$$

that is easily checked to be a linear subspace. Finally, the monoidal product on both objects and morphisms is direct sum.

A behaviour is **time-invariant** when for every trajectory $w \in \mathcal{B}$ and any fixed $i \in \mathbb{Z}$, the trajectory whose value at every time $t \in \mathbb{Z}$ is $w(t + i)$ is also in \mathcal{B} . Time-invariance brings with it a connection with the algebra of polynomials. Following the standard approach in control theory, going back to Rosenbrock [?], we work with polynomials over an indeterminate s as well as its formal inverse s^{-1} —i.e. the elements of the ring $\mathbf{k}[s, s^{-1}]$.¹

The indeterminate s acts on a given biinfinite stream $w \in \mathbf{k}^\mathbb{Z}$ as a one-step delay, and s^{-1} as its inverse, a one step acceleration:

$$(s \cdot w)(t) \stackrel{\text{def}}{=} w(t - 1), \quad (s^{-1} \cdot w)(t) \stackrel{\text{def}}{=} w(t + 1).$$

¹The introduction of the formal inverse s^{-1} is a departure from previous work [?, ?] that dealt with Laurent streams (finite in the past, infinite in the future), and algebraically with the field of polynomial fractions. As we will see below, there is a natural action of $\mathbf{k}[s, s^{-1}]$ on biinfinite streams, but it does not make sense, in general, to define the action of a polynomial fraction on a biinfinite stream.

We can extend this, in the obvious linear, pointwise manner, to an action of any polynomial $p \in \mathbf{k}[s, s^{-1}]$ on w . Since $\mathbf{k}^{\mathbb{Z}}$ is a \mathbf{k} -vector space, any such p defines a \mathbf{k} -linear map $\mathbf{k}^{\mathbb{Z}} \rightarrow \mathbf{k}^{\mathbb{Z}}$ ($w \mapsto p \cdot w$).

Given this, we can view $n \times m$ matrices over $\mathbf{k}[s, s^{-1}]$ as \mathbf{k} -linear maps from $(\mathbf{k}^m)^{\mathbb{Z}}$ to $(\mathbf{k}^n)^{\mathbb{Z}}$. This viewpoint can be explained succinctly as a functor from the prop $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$, defined below, to the category of \mathbf{k} -vector spaces and linear transformations $\mathbf{Vect}_{\mathbf{k}}$.

Definition 5.2. The prop $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ has as arrows $m \rightarrow n$ the $n \times m$ -matrices over $\mathbf{k}[s, s^{-1}]$. Composition is matrix multiplication, and the monoidal product of A and B is $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. The symmetries are permutation matrices.

The functor of interest

$$\theta: \mathbf{Mat} \mathbf{k}[s, s^{-1}] \longrightarrow \mathbf{Vect}_{\mathbf{k}}$$

takes a natural number n to $(\mathbf{k}^n)^{\mathbb{Z}}$, and an $n \times m$ matrix to the induced linear transformation $(\mathbf{k}^m)^{\mathbb{Z}} \rightarrow (\mathbf{k}^n)^{\mathbb{Z}}$. Note that θ is faithful.

The final restriction on the set of behaviours is called *completeness*, and is a touch more involved. For $t_0, t_1 \in \mathbb{Z}$, $t_0 \leq t_1$, write $w|_{[t_0, t_1]}$ for the restriction of $w: \mathbb{Z} \rightarrow \mathbf{k}^n$ to the set $[t_0, t_1] = \{t_0, t_0 + 1, \dots, t_1\}$. Write $\mathcal{B}|_{[t_0, t_1]}$ for the set of the restrictions of all trajectories $w \in \mathcal{B}$ to $[t_0, t_1]$. Then \mathcal{B} is **complete** when $w|_{[t_0, t_1]} \in \mathcal{B}|_{[t_0, t_1]}$ for all $t_0, t_1 \in \mathbb{Z}$ implies $w \in \mathcal{B}$. This topological condition is important as it characterises the linear time-invariant behaviours that are kernels of the action of $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$; see Theorem 5.5.

Definition 5.3. A **linear time-invariant (LTI) behaviour** comprises a domain $(\mathbf{k}^m)^{\mathbb{Z}}$, a codomain $(\mathbf{k}^n)^{\mathbb{Z}}$, and a subset $\mathcal{B} \subseteq (\mathbf{k}^m)^{\mathbb{Z}} \oplus (\mathbf{k}^n)^{\mathbb{Z}}$ such that $(\mathbb{Z}, \mathbf{k}^m \oplus \mathbf{k}^n, \mathcal{B})$ is a complete, linear, time-invariant dynamical system.

The algebra of LTI behaviours is captured concisely as a prop.

Proposition 5.4. *There exists a prop LTI with morphisms $m \rightarrow n$ the LTI behaviours with domain $(\mathbf{k}^m)^{\mathbb{Z}}$ and codomain $(\mathbf{k}^n)^{\mathbb{Z}}$. Composition is relational. The monoidal product is direct sum.*

The proof of Proposition 5.4 relies on *kernel representations* of LTI systems. The following result lets us pass between behaviours and polynomial matrix algebra.

Theorem 5.5 (Willems [?, Th. 5]). *Let \mathcal{B} be a subset of $(\mathbf{k}^n)^{\mathbb{Z}}$ for some $n \in \mathbb{N}$. Then \mathcal{B} is an LTI behaviour iff there exists $M \in \mathbf{Mat} \mathbf{k}[s, s^{-1}]$ such that $\mathcal{B} = \ker(\theta M)$.*

The prop $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ is equivalent to the category $\mathbf{FMod} \mathbf{k}[s, s^{-1}]$ of finite-dimensional free $\mathbf{k}[s, s^{-1}]$ -modules. Since $\mathbf{FMod} R$ over a principal ideal domain (PID) R has finite colimits [?], and $\mathbf{k}[s, s^{-1}]$ is a PID, $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ has finite colimits, and thus it has pushouts.

We can therefore define the prop $\mathbf{Cospan} \mathbf{Mat} \mathbf{k}[s, s^{-1}]$ where arrows are (isomorphism classes of) cospans of matrices: arrows $m \rightarrow n$ comprise a natural number d together with a $d \times m$ matrix A and a $d \times n$ matrix B ; we write this $m \xrightarrow{A} d \xleftarrow{B} n$. Composition is given by pushout, and isomorphic cospans are identified; for details see Bénabou [?].

We can then extend θ to the functor

$$\Theta: \mathbf{Cospan} \mathbf{Mat} \mathbf{k}[s, s^{-1}] \longrightarrow \mathbf{LinRel}_{\mathbf{k}}$$

where on objects $\Theta(n) = \theta(n) = (\mathbf{k}^n)^{\mathbb{Z}}$, and on arrows

$$m \xrightarrow{A} d \xleftarrow{B} n$$

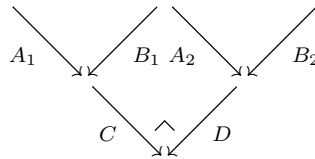
maps to

$$\{(\mathbf{x}, \mathbf{y}) \mid (\theta A)\mathbf{x} = (\theta B)\mathbf{y}\} \subseteq (\mathbf{k}^m)^{\mathbb{Z}} \oplus (\mathbf{k}^n)^{\mathbb{Z}}. \quad (5.3)$$

It is straightforward to prove that this is well-defined.

Proposition 5.6. *Θ is a functor.*

Proof. Identities are clearly preserved; it suffices to show that composition is too. Consider the diagram below, where the pushout is calculated in $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$.



To show that Θ preserves composition we must verify that

$$\{(\mathbf{x}, \mathbf{y}) \mid \theta C A_1 \mathbf{x} = \theta D B_2 \mathbf{y}\} = \{(\mathbf{x}, \mathbf{z}) \mid \theta A_1 \mathbf{x} = \theta B_1 \mathbf{z}\}; \{(\mathbf{z}, \mathbf{y}) \mid \theta A_2 \mathbf{z} = \theta B_2 \mathbf{y}\}.$$

The inclusion \subseteq follows from properties of pushouts in $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ (see [?, Prop. 5.7]). To see \supseteq , we need to show that if there exists \mathbf{z} such that $\theta A_1 \mathbf{x} = \theta B_1 \mathbf{z}$ and $\theta A_2 \mathbf{z} = \theta B_2 \mathbf{y}$, then $\theta C A_1 \mathbf{x} = \theta D B_2 \mathbf{y}$. But $\theta C A_1 \mathbf{x} = \theta C B_1 \mathbf{z} = \theta D A_2 \mathbf{z} = \theta D B_2 \mathbf{y}$. \square

$$\theta \begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{0},$$
$$\Theta(\overset{A}{\rightarrow}\overset{B}{\leftarrow}) = \ker \theta \begin{bmatrix} A & -B \end{bmatrix}.$$

5.4 Presentation of LTI

5.4.1 Syntax

$$\begin{aligned} & \{ \text{---} \cup \text{---} \text{---}, \text{---} \circ \text{---}, \text{---} \bullet \text{---} \cup \text{---}, \text{---} \bullet \text{---}, \text{---} \boxed{\text{s}} \text{---}, \\ & \text{---} \cup \text{---} \text{---}, \text{---} \circ \text{---}, \text{---} \cup \text{---} \bullet \text{---}, \text{---} \bullet \text{---}, \text{---} \boxed{\text{s}} \text{---} \} \\ & \cup \{ \text{---} \boxed{\text{a}} \text{---} \mid a \in \mathbf{k} \} \cup \{ \text{---} \boxed{\text{a}} \text{---} \mid a \in \mathbf{k} \} \quad (5.4) \end{aligned}$$
$$\text{---} \circ \text{---} \mapsto \{ ((\begin{smallmatrix} \tau \\ v \end{smallmatrix}), \tau + v) \mid \tau, v \in \mathbb{k}^{\mathbb{Z}} \} : 2 \rightarrow 1$$

$$\begin{aligned}
\circ\text{---} &\mapsto \{ ((), 0) \} \subseteq \mathbf{k}^{\mathbb{Z}}: 0 \rightarrow 1 \\
\text{---}\bullet &\mapsto \{ (\tau, (\tau)) \mid \tau \in \mathbf{k}^{\mathbb{Z}} \}: 1 \rightarrow 2 \\
\text{---}\bullet &\mapsto \{ (\tau, ()) \mid \tau \in \mathbf{k}^{\mathbb{Z}} \}: 1 \rightarrow 0 \\
\text{---}\boxed{a}\text{---} &\mapsto \{ (\tau, a \cdot \tau) \mid \tau \in \mathbf{k}^{\mathbb{Z}} \}: 1 \rightarrow 1 \quad (a \in \mathbf{k}) \\
\text{---}\boxed{s}\text{---} &\mapsto \{ (\tau, s \cdot \tau) \mid \tau \in \mathbf{k}^{\mathbb{Z}} \}: 1 \rightarrow 1
\end{aligned}$$

The denotations of the mirror image generators are the opposite relations. Parenthetically, we note that a finite set of generators is possible over a finite field, or the field \mathbb{Q} of rationals, *cf.* §5.4.2.

The following result guarantees that the syntax is fit for purpose: every behaviour in LTI has a syntactic representation in \mathbb{S} .

Proposition 5.8. $\llbracket - \rrbracket : \mathbb{S} \rightarrow \text{LTI}$ is full.

Proof. The fact that $\llbracket - \rrbracket$ is a prop morphism is immediate since \mathbb{S} is free on the SMT (Σ, \emptyset) with no equations. Fullness follows from the fact that $\llbracket - \rrbracket$ factors as the composite of two full functors:

$$\begin{array}{ccc}
\mathbb{S} & & \\
\downarrow & \searrow \llbracket - \rrbracket & \\
\text{Cospan } \mathbf{Mat} \, \mathbf{k}[s, s^{-1}] & \xrightarrow{\bar{\Theta}} & \text{LTI}
\end{array}$$

The functor $\bar{\Theta}$ is full by definition. The existence and fullness of the functor $\mathbb{S} \rightarrow \text{Cospan } \mathbf{k}[s, s^{-1}]$ follows from [?, Th. 3.41]. We give details in the next two subsections. \square

Having defined the syntactic prop \mathbb{S} to represent arrows in LTI, our task for this section is to identify an equational theory that *characterises* LTI: i.e. one that is sound and fully complete. The first step is to use the existence of $\bar{\Theta}$: with the results of [?, ?] we can obtain a presentation for $\text{Cospan } \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$. This is explained in the next two subsections: in §5.4.2 we present $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$ and in §5.4.3, the equations for cospans of matrices.

5.4.2 Presentation of $\text{Mat } k[s, s^{-1}]$

To obtain a presentation of $\text{Mat } k[s, s^{-1}]$ as an SMT we only require some of the generators:

$$\{ \text{ } \cup \{ \text{ } \mid a \in k \} \}$$

and the following equations. First, the white and the black structure forms a (bicommutative) bimonoid:

Next, the formal indeterminate s is compatible with the bimonoid structure and has its mirror image as a formal inverse.

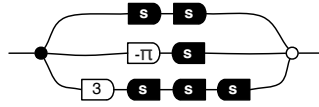
Finally, we insist that the algebra of k be compatible with the bimonoid structure and commute with s .

The three sets of equations above form the theory of Hopf algebras. Write $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$ for the prop induced by the SMT consisting of the equations above. The following follows from [?, Prop. 3.9].

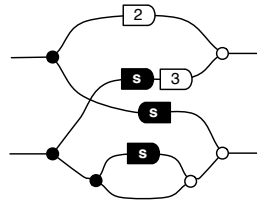
Proposition 5.9. $\mathbf{Mat} \mathbf{k}[s, s^{-1}] \cong \mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$.

Arrows of $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ are matrices with polynomial entries, but it may not be obvious to the reader how polynomials arise with the string diagrammatic syntax. We illustrate this below.

Example 5.10. Any polynomial $p = \sum_{i=u}^v a_i s^i$, where $u \leq v \in \mathbb{Z}$ and with coefficients $a_i \in \mathbf{k}$, can be written graphically using the building blocks of $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$. Rather than giving a tedious formal construction, we illustrate this with an example for $\mathbf{k} = \mathbb{R}$. A term for $3s^{-3} - \pi s^{-1} + s^2$ is:



As an arrow $1 \rightarrow 1$ in $\mathbf{Mat} \mathbb{R}[s, s^{-1}]$, the above term represents a 1×1 -matrix over $\mathbb{R}[s, s^{-1}]$. To demonstrate how higher-dimensional matrices can be written, we also give a term for the 2×2 -matrix $\begin{bmatrix} 2 & 3s \\ s^{-1} & s+1 \end{bmatrix}$:

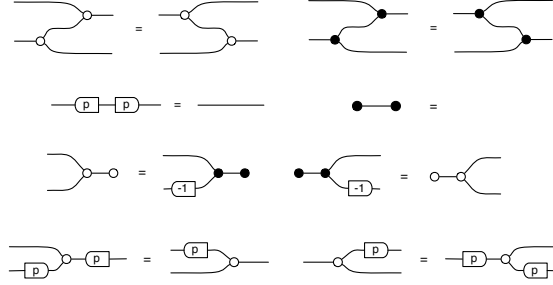


The above examples are intended to be suggestive of a normal form for terms in $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$; for further details see [?].

5.4.3 Presentation of Cospan $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$

To obtain the equational theory of Cospan $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ we need the full set of generators (5.4), along with the equations of $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$, their mirror images, and the

following



where p ranges over the nonzero elements of $\mathbf{k}[s, s^{-1}]$ (see Example 5.10). Note that in the second equation on the right-hand side, we use the so-called ‘empty diagram’, or blank space, to represent the identity map on the monoidal unit, 0.

The equations of $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$ ensure the generators of $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$ behave as morphisms in $\mathbf{Mat}\, \mathbf{k}[s, s^{-1}]$, while their mirror images ensure the remaining generators behave as morphisms in the opposite category $\mathbf{Mat}\, \mathbf{k}[s, s^{-1}]^{\text{op}}$. The additional equations above govern the interaction between these two sets of generators, axiomatising pushouts in $\mathbf{Mat}\, \mathbf{k}[s, s^{-1}]$. Let $\mathbb{H}\mathbb{H}^{\text{Csp}}$ denote the resulting SMT. The procedure for obtaining the equations from a distributive law of props is explained in [?, §3.3].

Proposition 5.11 (Zanasi [?, Th. 3.41]).

$$\text{Cospan } \mathbf{Mat}\, \mathbf{k}[s, s^{-1}] \cong \mathbb{H}\mathbb{H}^{\text{Csp}}.$$

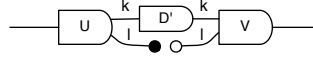
Using Prop. 5.11 and the existence of $\overline{\Theta}$, the equational theory of $\mathbb{H}\mathbb{H}^{\text{Csp}}$ is a sound proof system for reasoning about LTI. Due to the fact that $\overline{\Theta}$ is not faithful (see Remark 5.7), however, the system is not complete. Achieving completeness is our task for the remainder of this section.

5.4.4 Corelations

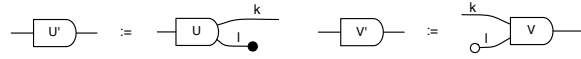
In the category of sets, relations can be identified with jointly-monic spans of functions; that is, those spans $X \xleftarrow{p} R \xrightarrow{q} Y$ where the induced map $R \xrightarrow{\langle p, q \rangle} X \times Y$ is injective. Corelations are a dual concept: we consider cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ where the induced map $X + Y \xrightarrow{[i, j]} S$ is surjective. To make sense of this more generally, one needs a category with a factorisation system. In this subsection we identify a factorisation system in $\mathbf{Mat}\, \mathbf{k}[s, s^{-1}]$, and show that the induced prop $\text{Corel } \mathbf{Mat}\, \mathbf{k}[s, s^{-1}]$ of corelations is isomorphic to LTI. We then give a presentation of $\text{Corel } \mathbf{Mat}\, \mathbf{k}[s, s^{-1}]$ and arrive at a sound and fully complete equational theory for LTI.

Proposition 5.12. *Every morphism $R \in \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$ can be factored as $R = BA$, where A is an epi and B is a split mono.*

Proof. Given any matrix R , the Smith normal form [?, Section 6.3] gives us $R = VDU$, where U and V are invertible, and D is diagonal. In graphical notation we can write it thus:



This implies we may write it as $R = U'; D'; V'$, where U' is a split epimorphism, D' diagonal of full rank, and V' a split monomorphism. Explicitly, the construction is given by



Recall that $\mathbf{k}[s, s^{-1}]$ is a PID, so the full rank diagonal matrix D' is epi. It can be checked that $R = V'(D'U')$ is an epi-split mono factorisation. \square

A more careful examination of Proposition 5.12 yields:

Corollary 5.13. *Let \mathcal{E} be the subcategory of epis and \mathcal{M} the subcategory of split monos. The pair $(\mathcal{E}, \mathcal{M})$ is a factorisation system on $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$, with \mathcal{M} stable under pushout.*

Given finite colimits and an $(\mathcal{E}, \mathcal{M})$ -factorisation system with \mathcal{M} stable under pushouts, we may define a category of corelations. The morphisms are isomorphism classes of cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ where the copairing $[i, j]: X + Y \rightarrow S$ is in \mathcal{E} . Composition is given by pushout, as in the category of cospans, followed by factorising the copairing of the resulting cospan. This is a dualisation of the well-known construction of relations from spans [?]; further details can be found in [?].

Definition 5.14. The prop $\mathbf{Corel} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$ has as morphisms equivalence classes of jointly-epic cospans in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$.

We have a full morphism

$$F: \mathbf{Cospan} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}] \longrightarrow \mathbf{Corel} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$$

mapping a cospan to its jointly-epic counterpart given by the factorisation system. Then $\bar{\Theta}$ factors through F as follows:

$$\begin{array}{ccc} \mathbf{Cospan} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}] & & \\ \downarrow F & \searrow \bar{\Theta} & \\ \mathbf{Corel} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}] & \xrightarrow{\Phi} & \mathbf{LTI} \end{array}$$

The morphism Φ along the base of this triangle is an isomorphism of props, and this is our main technical result, Theorem 5.16. The proof relies on the following beautiful result of systems theory.

Proposition 5.15 (Willems [?, p.565]). *Let M, N be matrices over $\mathbf{k}[s, s^{-1}]$. Then $\ker \theta M \subseteq \ker \theta N$ iff \exists a matrix X s.t. $XM = N$.*

Further details and a brief history of the above proposition can be found in Schumacher [?, pp.7–9].

Theorem 5.16. *There is an isomorphism of props*

$$\Phi: \text{Corel Mat } \mathbf{k}[s, s^{-1}] \longrightarrow \text{LTI}$$

taking a corelation $\xrightarrow{A} \xleftarrow{B}$ to $\overline{\Theta}(\xrightarrow{A} \xleftarrow{B}) = \ker \theta[A \quad -B]$.

Proof. For functoriality, start from $\theta: \text{Mat } \mathbf{k}[s, s^{-1}] \rightarrow \text{Vect}_{\mathbf{k}}$. Now (i) $\text{Vect}_{\mathbf{k}}$ has an epi-mono factorisation system, (ii) θ maps epis to epis and (iii) split monos to monos, so θ preserves factorisations. Since it is a corollary of Prop. 5.6 that θ preserves colimits, it follows that θ extends to $\Psi: \text{Corel Mat } \mathbf{k}[s, s^{-1}] \rightarrow \text{Corel Vect}_{\mathbf{k}}$. But $\text{Corel Vect}_{\mathbf{k}}$ is isomorphic to $\text{LinRel}_{\mathbf{k}}$ (see [?]). By Theorem 5.5, the image of Ψ is LTI, and taking the corestriction to gives us precisely Φ , which is therefore a full morphism of props.

As corelations $n \rightarrow m$ are in one-to-one correspondence with epis out of $n + m$, to prove faithfulness it suffices to prove that if two epis R and S with the same domain have the same kernel, then there exists an invertible matrix U such that $UR = S$. This is immediate from Proposition 5.15: if $\ker R = \ker S$, then we can find U, V such that $UR = S$ and $VS = R$. Since R is an epimorphism, and since $VUR = VS = R$, we have that $VU = 1$ and similarly $UV = 1$. This proves that any two corelations with the same image are isomorphic, and so Φ is full and faithful. \square

5.4.5 Presentation of $\text{Corel Mat } \mathbf{k}[s, s^{-1}]$

Thanks to Theorem 5.16, the task of obtaining a presentation of LTI is that of obtaining one for $\text{Corel Mat } \mathbf{k}[s, s^{-1}]$. To do this, we start with the presentation IIIH^{Csp} for $\text{Cospan Mat } \mathbf{k}[s, s^{-1}]$ of §5.4.3; the task of this section is to identify the additional equations that equate exactly those cospans that map via F to the same corelation.

In fact, only one new equation is required, the “white bone law”:

$$\bigcirc \text{---} \bigcirc = \quad (5.5)$$

where we have carefully drawn the empty diagram to the right of the equality symbol. Expressed in terms of cospans, (5.5) asserts that $0 \rightarrow 1 \leftarrow 0$ and $0 \rightarrow 0 \leftarrow 0$ are identified: indeed, the two clearly yield the same corelation. The intuition here is that cospans $X \xrightarrow{i} S \xleftarrow{j} Y$ map to the same corelation if their respective copairings $[i, j]: X + Y \rightarrow S$ have the same jointly-epic parts. More colloquially, this allows us to ‘discard’ any part of the cospan that is not connected to the terminals. This is precisely what (5.5) represents. Further details on this viewpoint can be found in [?].

Let \mathbb{HH}^{Cor} be the SMT obtained from the equations of \mathbb{HH}^{Csp} together with equation (5.5).

Theorem 5.17. $\text{Corel Mat } k[s, s^{-1}] \cong \mathbb{HH}^{\text{Cor}}$.

Proof. Since equation (5.5) holds in $\text{Corel Mat } k[s, s^{-1}]$, we have a full morphism $\mathbb{HH}^{\text{Cor}} \rightarrow \text{Corel Mat } k[s, s^{-1}]$; it remains to show that it is faithful. It clearly suffices to show that in the equational theory \mathbb{HH}^{Cor} one can prove that every cospan is equal to its corelation. Suppose then that $m \xrightarrow{A} k \xleftarrow{B} n$ is a cospan and $m \xrightarrow{A'} k' \xleftarrow{B'} n$ is its corelation. Then, by definition, there exists a split mono $M: k' \rightarrow k$ such that $MA' = A$ and $MB' = B$. Moreover, by the construction of the epi-split mono factorisation in $\text{Mat } k[s, s^{-1}]$, M is of the form $\frac{k'}{k-k'} \text{---} M' \text{---} k$ where $M': k \rightarrow k$ is invertible. We can now give the derivation in \mathbb{HH}^{Cor} :

$$\begin{array}{c}
 \frac{m}{\text{---} A \text{---} k \text{---} B \text{---} n} \stackrel{\mathbb{HH}^{\text{Csp}}}{=} \frac{m}{\text{---} A' \text{---} \frac{k'}{k-k'} \text{---} M' \text{---} k \text{---} M' \text{---} \frac{k'}{k-k'} \text{---} B' \text{---} n} \\
 \stackrel{\mathbb{HH}^{\text{Csp}}}{=} \frac{m}{\text{---} A' \text{---} \frac{k'}{k-k'} \text{---} B' \text{---} n} \\
 \stackrel{(5.5)}{=} \frac{m}{\text{---} A' \text{---} k' \text{---} B' \text{---} n}
 \end{array}$$

□

We therefore have a sound and fully complete equational theory for LTI systems, and also a normal form for each LTI system: every such system can be written, in an essentially unique way, as a jointly-epic cospan of terms in $\mathbb{HIA}_{k[s, s^{-1}]}$ in normal form.

Remark 5.18. \mathbb{HH}^{Cor} can also be described as having the equations of $\mathbb{HH}_{k[s, s^{-1}]}$ [?, ?], but *without* $\text{---} \boxed{p} \text{---} \boxed{p} \text{---} = \text{---}$, and related $\text{---} \boxed{p} \text{---} \bullet \text{---} \boxed{p} \text{---} = \text{---} \boxed{p} \text{---} \bullet \text{---}$ and $\text{---} \bullet \text{---} \boxed{p} \text{---} = \text{---} \boxed{p} \text{---} \bullet \text{---} \boxed{p} \text{---}$. Our results generalise; given any PID R we have (informally speaking):

$$\begin{aligned}
 \mathbb{HH}_R^{\text{Cor}} &= \mathbb{HH}_R - \left\{ \begin{array}{l} \text{---} \boxed{r} \text{---} \boxed{r} \text{---} = \text{---} , \\ \text{---} \boxed{r} \text{---} \bullet \text{---} \boxed{r} \text{---} = \text{---} \boxed{r} \text{---} \bullet \text{---} , \\ \text{---} \bullet \text{---} \boxed{r} \text{---} = \text{---} \boxed{r} \text{---} \bullet \text{---} \boxed{r} \text{---} \end{array} \right. \left| r \neq 0 \in R \right\} \\
 &\cong \text{Corel Mat } R
 \end{aligned}$$

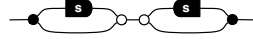
and, because of the transpose duality of matrices:

$$\mathbb{I}\mathbb{H}_R^{\text{Rel}} = \mathbb{I}\mathbb{H}_R - \left\{ \begin{array}{l} \text{---} \boxed{r} \text{---} \boxed{r} \text{---} = \text{---} , \\ \text{---} \boxed{r} \text{---} \boxed{r} \text{---} = \text{---} \boxed{r} \text{---} \boxed{r} \text{---} , \\ \text{---} \boxed{r} \text{---} \boxed{r} \text{---} = \text{---} \boxed{r} \text{---} \boxed{r} \text{---} \end{array} \right\} \left| r \neq 0 \in R \right.$$

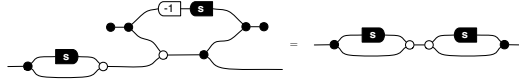
$$\cong \text{Rel Mat } R.$$

Remark 5.19. The omitted equations each associate a cospan with a span that, in terms of behaviour, has the effect of passing to a sub-behaviour². Often this is a strict sub-behaviour, hence the failure of soundness of $\mathbb{I}\mathbb{H}$ identified in the introduction.

For example, consider the system \mathcal{B} represented by the cospan



We met this system in the introduction; indeed the following derivation can be performed in $\mathbb{I}\mathbb{H}^{\text{Cor}}$:



The trajectories are $w = (w_1, w_2) \in \mathbf{k}^{\mathbb{Z}} \oplus \mathbf{k}^{\mathbb{Z}}$ where $(s+1) \cdot w_1 = (s+1) \cdot w_2$; that is, they satisfy the difference equation

$$w_1(t-1) + w_1(t) - w_2(t-1) - w_2(t) = 0.$$

As we saw in the introduction, however, an equation of $\mathbb{I}\mathbb{H}_{\mathbf{k}[s]}$ omitted from $\mathbb{I}\mathbb{H}_{\mathbf{k}[s, s-1]}^{\text{Cor}}$ equates this to the identity system --- , with the identity behaviour those sequences of the form $w = (w_1, w_1) \in \mathbf{k}^{\mathbb{Z}} \oplus \mathbf{k}^{\mathbb{Z}}$. The identity behaviour is strictly smaller than \mathcal{B} ; e.g., \mathcal{B} contains (w_1, w_2) with $w_1(t) = (-1)^t$ and $w_2(t) = 0$.

The similarity between our equational presentations of $\mathbb{I}\mathbb{H}_{\mathbf{k}[s, s-1]}^{\text{Cor}}$ and that of $\mathbb{I}\mathbb{H}_{\mathbf{k}[s]}$ given in [?, ?] is remarkable, considering the differences between the intended semantics of signal flow graphs that string diagrams in those theories represent, as well as the underlying mathematics of streams, which for us are elements of the \mathbf{k} vector space $\mathbf{k}^{\mathbb{Z}}$ and in [?, ?] are Laurent series. We contend that this is evidence of the *robustness* of the algebraic approach: the equational account of how various components of signal flow graphs interact is, in a sense, a higher-level specification than the technical details of the underlying mathematical formalisations.

²In fact the ‘largest controllable sub-behaviour’ of the system. We explore controllability in Section 5.5.

5.5 Controllability

Suppose we are given a current and a target trajectory for a system. Is it always possible, in finite time, to steer the system onto the target trajectory? If so, the system is deemed controllable, and the problem of controllability of systems is at the core of control theory. The following definition is due to Willems [?].

Definition 5.20. A system (T, W, \mathcal{B}) (or simply the behaviour \mathcal{B}) is **controllable** if for all $w, w' \in \mathcal{B}$ and all times $t_0 \in \mathbb{Z}$, there exists $w'' \in \mathcal{B}$ and $t'_0 \in \mathbb{Z}$ such that $t'_0 > t_0$ and w'' obeys

$$w''(t) = \begin{cases} w(t) & t \leq t_0 \\ w'(t - t'_0) & t \geq t'_0. \end{cases}$$

As mentioned previously, a novel feature of our graphical calculus is that it allows us to consider non-controllable behaviours.

Example 5.21. Consider the system in the introduction, further elaborated in Remark 5.19. As noted previously, the trajectories of this system are precisely those sequences $w = (w_1, w_2) \in \mathbf{k}^{\mathbb{Z}} \oplus \mathbf{k}^{\mathbb{Z}}$ that satisfy the difference equation

$$w_1(t-1) + w_1(t) - w_2(t-1) - w_2(t) = 0.$$

To see that the system is non-controllable, note that

$$w_1(t-1) - w_2(t-1) = -(w_1(t) - w_2(t)),$$

so $(w_1 - w_2)(t) = (-1)^t c_w$ for some $c_w \in \mathbf{k}$. This c_w is a time-invariant property of any trajectory. Thus if w and w' are trajectories such that $c_w \neq c_{w'}$, then it is not possible to transition from the past of w to the future of w' along some trajectory in \mathcal{B} .

Explicitly, taking $w(t) = ((-1)^t, 0)$ and $w'(t) = ((-1)^{t+2}, 0)$ suffices to show \mathcal{B} is not controllable.

5.5.1 A categorical characterisation

We now show that controllable systems are precisely those representable as *spans* of matrices. This novel characterisation leads to new ways of reasoning about controllability of composite systems.

Among the various equivalent conditions for controllability, the existence of *image representations* is most useful for our purposes.

Proposition 5.22 (Willems [?, p.86]). *An LTI behaviour \mathcal{B} is controllable iff $\exists M \in \text{Mat } \mathbf{k}[s, s^{-1}]$ such that $\mathcal{B} = \text{Im } \theta M$.*

Restated in our language, Prop. 5.22 states that controllable systems are precisely those representable as *spans* of matrices.

Theorem 5.23. *Let $m \xrightarrow{A} d \xleftarrow{B} n$ be a corelation in $\text{Corel } \text{Mat } \mathbf{k}[s, s^{-1}]$. Then $\Phi(\xrightarrow{A} \xleftarrow{B})$ is controllable iff $\exists R : e \rightarrow m, S : e \rightarrow n$ s.t.*

$$m \xleftarrow{R} e \xrightarrow{S} n = m \xrightarrow{A} d \xleftarrow{B} n$$

as morphisms in $\text{Corel } \text{Mat } \mathbf{k}[s, s^{-1}]$.

Proof. To begin, note that the behaviour of a span is its joint image. That is, $\Phi(\xleftarrow{R} \xrightarrow{S})$ is the composite of linear relations $\ker \theta[\text{id}_m - R]$ and $\ker \theta[S - \text{id}_n]$, which comprises all $(\mathbf{x}, \mathbf{y}) \in (\mathbf{k}^m)^{\mathbb{Z}} \oplus (\mathbf{k}^n)^{\mathbb{Z}}$ s.t. $\exists \mathbf{z} \in (\mathbf{k}^e)^{\mathbb{Z}}$ with $\mathbf{x} = \theta R \mathbf{z}$ and $\mathbf{y} = \theta S \mathbf{z}$. Thus

$$\Phi(\xleftarrow{R} \xrightarrow{S}) = \text{Im } \theta \begin{bmatrix} R \\ S \end{bmatrix}.$$

The result then follows immediately from Prop. 5.22. \square

In terms of the graphical theory, this means that a term in the form $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}; \mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}^{op}$ ('cospan form') is controllable iff we can find a derivation, using the rules of $\mathbb{H}\mathbb{H}^{\text{Cor}}$, that puts it in the form $\mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}^{op}; \mathbb{H}\mathbb{A}_{\mathbf{k}[s, s^{-1}]}$ ('span form'). This provides a general, easily recognisable representation for controllable systems.

Span representations also lead to a test for controllability: take the pullback of the cospan and check whether the system described by it coincides with the original one. Indeed, note that as $\mathbf{k}[s, s^{-1}]$ is a PID, the category $\text{Mat } \mathbf{k}[s, s^{-1}]$ has pullbacks. A further consequence of Th. 5.23, together with Prop. 5.15, is the following.

Proposition 5.24. *Let $m \xrightarrow{A} d \xleftarrow{B} n$ be a cospan in $\text{Mat } \mathbf{k}[s, s^{-1}]$, and write the pullback of this cospan $m \xleftarrow{R} e \xrightarrow{S} n$. Then the behaviour of the pullback span $\Phi(\xleftarrow{R} \xrightarrow{S})$ is the maximal controllable sub-behaviour of $\Phi(\xrightarrow{A} \xleftarrow{B})$.*

Proof. Suppose we have another controllable behaviour \mathcal{C} contained in $\ker \theta[A - B]$. Then this behaviour is the Φ -image of some span $m \xleftarrow{R'} e' \xrightarrow{S'} n$. As $\text{Im } \theta \begin{bmatrix} R' \\ S' \end{bmatrix}$ lies in $\ker \theta[A - B]$, the universal property of the pullback gives a map $e' \rightarrow e$ such that the relevant diagram commutes. This implies that the controllable behaviour $\mathcal{C} = \text{Im } \theta \begin{bmatrix} R' \\ S' \end{bmatrix}$ is contained in $\text{Im } \theta \begin{bmatrix} R \\ S \end{bmatrix}$, as required. \square

Corollary 5.25. *Suppose that an LTI behaviour \mathcal{B} has cospan representation*

$$m \xrightarrow{A} d \xleftarrow{B} n.$$

Then \mathcal{B} is controllable iff the Φ -image of the pullback of this cospan in $\mathbf{Mat} \mathbf{k}[s, s^{-1}]$ is equal to \mathcal{B} .

Moreover, taking the pushout of this pullback span gives another cospan. The morphism from the pushout to the original cospan, given by the universal property of the pushout, describes the way in which the system fails to be controllable.

Graphically, the pullback may be computed by using the axioms of the theory of interacting Hopf algebras $\mathbb{HH}_{\mathbf{k}[s, s^{-1}]}$ [?, ?]. For example, the pullback span of the system of Ex. 5.21 is simply the identity span, as derived in equation (5.2) of the introduction. In the traditional matrix calculus for control theory, one derives this by noting the system has kernel representation $\ker \theta \begin{bmatrix} s+1 & -(s+1) \end{bmatrix}$, and eliminating the common factor $s+1$ between the entries. Either way, we conclude that the maximally controllable subsystem of $1 \xrightarrow{[s+1]} 1 \xleftarrow{[s+1]} 1$ is simply the identity system $1 \xrightarrow{[1]} 1 \xleftarrow{[1]} 1$.

5.5.2 Control and interconnection

From this vantage point we can make useful observations about controllable systems and their composites: we simply need to ask whether we can rewrite them as spans.

Example 5.26. Suppose that \mathcal{B} has cospan representation $m \xrightarrow{A} d \xleftarrow{B} n$. Then \mathcal{B} is easily seen to be controllable when A or B is invertible. Indeed, if A is invertible, then $m \xleftarrow{A^{-1}B} n \xrightarrow{\text{id}_n} n$ is an equivalent span; if B is invertible, then $m \xleftarrow{\text{id}_m} m \xrightarrow{B^{-1}A} n$.

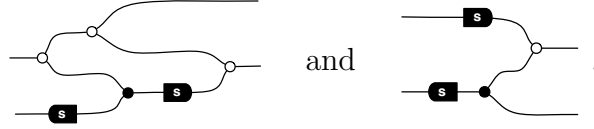
More significantly, the compositionality of our framework aids understanding of how controllability behaves under the interconnection of systems—an active field of investigation in current control theory. We give an example application of our result.

First, consider the following proposition.

Proposition 5.27. *Let \mathcal{B}, \mathcal{C} be controllable systems, given by the respective Φ -images of the spans $m \xleftarrow{B_1} d \xrightarrow{B_2} n$ and $n \xleftarrow{C_1} e \xrightarrow{C_2} l$. Then the composite $\mathcal{C} \circ \mathcal{B} : m \rightarrow l$ is controllable if $\Phi(\xrightarrow{B_2} \xleftarrow{C_1})$ is controllable.*

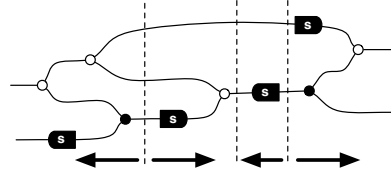
Proof. Replacing $\xrightarrow{B_2} \xleftarrow{C_1}$ with an equivalent span gives a span representation for $\mathcal{C} \circ \mathcal{B}$. □

Example 5.28. Consider LTI systems

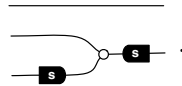


These systems are controllable because each is represented by a span in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$. Indeed, recall that each generator of $\mathbf{LTI} = \mathbf{Corel} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$ arises as the image of a generator in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$ or $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]^{\text{op}}$; for example, the white monoid map $\text{---} \circ \text{---}$ represents a morphism in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$, while the black monoid map $\text{---} \bullet \text{---}$ represents a morphism in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]^{\text{op}}$. The above diagrams are spans as we may partition the diagrams above so that each generator in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]^{\text{op}}$ lies to the left of each generator in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$.

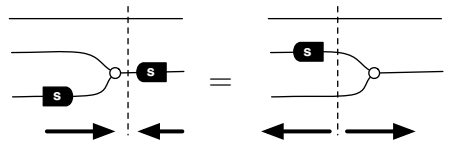
To determine controllability of the interconnected system



Prop. 5.27 states that it is enough to consider the controllability of the subsystem

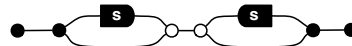


The above diagram gives a representation of the subsystem as a cospan in $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$. We can prove it is controllable by rewriting it as a span using an equation of LTI:



Thus the composite system is controllable.

Remark 5.29. Note the converse of Proposition 5.27 fails. For a simple example, consider the system



This is equivalent to the empty system, and so trivially controllable. The central span, however, is not controllable (Example 5.21).

5.5.3 Comparison to matrix methods

The facility with which the graphical calculus formalises and solves such controllability issues is especially appealing in view of potential applications in the analysis of controllability of *systems over networks* (see [?]). To make the reader fully appreciate such potential, we sketch how complicated such analysis is using standard algebraic methods and dynamical system theory even for the highly restrictive case of two systems that compose to make a single-input, single-output system. See also pp. 513–516 of [?], where a generalization of the result of Prop. 5.27 is given in a polynomial- and operator-theoretic setting.

In the following we abuse notation by writing a matrix for its image under the functor θ . The following is a useful result for analysing the controllability of kernel representations applying only to the single-input single-output case.

Proposition 5.30 (Willems [?, p.75]). *Let $\mathcal{B} \subseteq (\mathbf{k}^2)^{\mathbb{Z}}$ be a behaviour given by the kernel of the matrix $\theta \begin{bmatrix} A & B \end{bmatrix}$, where A and B are column vectors with entries in $\mathbf{k}[s, s^{-1}]$. Then \mathcal{B} is controllable if and only if the greatest common divisor $\gcd(A, B)$ of A and B is 1.*

Using the notation of Prop. 5.27, the trajectories of \mathcal{B} and \mathcal{C} respectively are those $(w_1, w_2) \in (\mathbf{k}^n)^{\mathbb{Z}} \oplus (\mathbf{k}^m)^{\mathbb{Z}}$ and $(w'_2, w_3) \in (\mathbf{k}^m)^{\mathbb{Z}} \oplus (\mathbf{k}^p)^{\mathbb{Z}}$ such that

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \ell_1 \quad \text{and} \quad \begin{bmatrix} w'_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \ell_2 \quad (5.6)$$

for some $\ell_1 \in (\mathbf{k}^b)^{\mathbb{Z}}$, $\ell_2 \in (\mathbf{k}^c)^{\mathbb{Z}}$. These are the explicit image representations of the two systems. We assume without loss of generality that the representations (5.6) are *observable* (see [?]); this is equivalent to $\gcd(B_1, B_2) = \gcd(C_1, C_2) = 1$. Augmenting (5.6) with the interconnection constraint $w_2 = B_2 \ell_1 = C_1 \ell_2 = w'_2$ we obtain the representation of the interconnection:

$$\begin{bmatrix} w_1 \\ w_2 \\ w'_2 \\ w_3 \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ B_2 & 0 \\ 0 & C_1 \\ 0 & C_2 \\ B_2 & -C_1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}. \quad (5.7)$$

Prop. 5.27 concerns the controllability of the set $\mathcal{C} \circ \mathcal{B}$ of trajectories (w_1, w_3) for which there exist trajectories $w_2, w'_2, \ell_1, \ell_2$ such that (5.7) holds.

To obtain a representation of such behavior the variables ℓ_1, ℓ_2, w_2 and w'_2 must be eliminated from (5.7) via algebraic manipulations (see the discussion on p. 237 of [?]). Denote $G = \gcd(B_1, C_2)$, and write $C_2 = GC'_2$ and $B_1 = GB'_1$, where $\gcd(B'_1, C'_2) = 1$. Without entering in the algebraic details, it can be shown that a kernel representation of the projection of the behavior of (5.7) on the variables w_1 and w_3 is

$$\begin{bmatrix} C'_2 B_2 & -B'_1 C_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix} = 0 . \quad (5.8)$$

We now restrict to the single-input single-output case. Recalling Prop. 5.30, the behavior represented by (5.8) is controllable if and only if $\gcd(C'_2 B_2, B'_1 C_1) = 1$.

Finally then, to complete our alternate proof of the single-input single-output case of Prop. 5.27, note that $\overline{\Theta}(\xrightarrow{B_2} \xleftarrow{C_1})$ is controllable if $\gcd(B_2, C_1) = 1$. Given the observability assumption, this implies $\gcd(C'_2 B_2, B'_1 C_1) = 1$, and so the interconnected behaviour $\mathcal{C} \circ \mathcal{B}$ represented by (5.8) is controllable.

In the multi-input, multi-output case stating explicit conditions on the controllability of the interconnection given properties of the representations of the individual systems and their interconnection is rather complicated. This makes the simplicity of Prop. 5.27 and the straightforward nature of its proof all the more appealing.

Chapter 6

Passive linear networks

6.1 Introduction

In late 1940s, just as Feynman was developing his diagrams for processes in particle physics, Eilenberg and Mac Lane initiated their work on category theory. Over the subsequent decades, and especially in the work of Joyal and Street in the 1980s [?, ?], it became clear that these developments were profoundly linked: monoidal categories have a precise graphical representation in terms of string diagrams, and conversely monoidal categories provide an algebraic foundation for the intuitions behind Feynman diagrams. The key insight is the use of categories where morphisms describe physical processes, rather than structure-preserving maps between mathematical objects [?, ?].

In work on fundamental physics, the cutting edge has moved from categories to higher categories [?]. But the same techniques have filtered into more immediate applications, particularly in computation and quantum computation [?, ?, ?]. This paper is part of a still nascent program of applying string diagrams to engineering, with the aim of giving diverse diagram languages a unified foundation based on category theory [?, ?, ?, ?, ?].

Indeed, even before physicists began using Feynman diagrams, various branches of engineering were using diagrams that in retrospect are closely related. Foremost among these are the ubiquitous electrical circuit diagrams. Although less well-known, similar diagrams are used to describe networks consisting of mechanical, hydraulic, thermodynamic and chemical systems. Further work, pioneered in particular by Forrester [?] and Odum [?], applies similar diagrammatic methods to biology, ecology, and economics.

As discussed in detail by Olsen [?], Paynter [?] and others, there are mathematically precise analogies between these different systems. In each case, the system's state

is described by variables that come in pairs, with one variable in each pair playing the role of ‘displacement’ and the other playing the role of ‘momentum’. In engineering, the time derivatives of these variables are sometimes called ‘flow’ and ‘effort’. In classical mechanics, this pairing of variables is well understood using symplectic geometry. Thus, any mathematical formulation of the diagrams used to describe networks in engineering needs to take symplectic geometry as well as category theory into account.

	displacement q	flow \dot{q}	momentum p	effort \dot{p}
Electronics	charge	current	flux linkage	voltage
Mechanics (translation)	position	velocity	momentum	force
Mechanics (rotation)	angle	angular velocity	angular momentum	torque
Hydraulics	volume	flow	pressure momentum	pressure
Thermodynamics	entropy	entropy flow	temperature momentum	temperature
Chemistry	moles	molar flow	chemical momentum	chemical potent

While diagrams of networks have been independently introduced in many disciplines, we do not expect formalizing these diagrams to immediately help the practitioners of these disciplines. At first the flow of information will mainly go in the other direction: by translating ideas from these disciplines into the language of modern mathematics, we can provide mathematicians with food for thought and interesting new problems to solve. We hope that in the long run mathematicians can return the favor by bringing new insights to the table.

Although we shall keep the broad applicability of network diagrams in the back of our minds, we couch our discussion in terms of electrical circuits, for the sake of familiarity. In this paper our goal is somewhat limited. We only study circuits built from ‘passive’ components: that is, those that do not produce energy. Thus, we exclude batteries and current sources. We only consider components that respond linearly to an applied voltage. Thus, we exclude components such as nonlinear resistors or diodes. Finally, we only consider components with one input and one output, so that a circuit can be described as a graph with edges labeled by components. Thus, we also exclude transformers. The most familiar components our framework covers are linear resistors, capacitors and inductors.

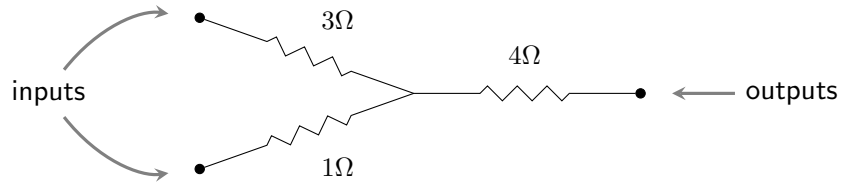
While we hope to expand our scope in future work, the class of circuits made from these components has appealing mathematical properties, and is worthy of deep study. Indeed, this class has been studied intensively for many decades by electrical

engineers [?, ?, ?]. Even circuits made exclusively of resistors have inspired work by mathematicians of the caliber of Weyl [?] and Smale [?].

Our work relies on this research. All we are adding is an emphasis on symplectic geometry and an explicitly ‘compositional’ framework, which clarifies the way a larger circuit can be built from smaller pieces. This is where monoidal categories become important: the main operations for building circuits from pieces are composition and tensoring.

Our strategy is most easily illustrated for circuits made of linear resistors. Such a resistor dissipates power, turning useful energy into heat at a rate determined by the voltage across the resistor. However, a remarkable fact is that a circuit made of these resistors always acts to *minimize* the power dissipated this way. This ‘principle of minimum power’ can be seen as the reason symplectic geometry becomes important in understanding circuits made of resistors, just as the principle of least action leads to the role of symplectic geometry in classical mechanics.

Here is a circuit made of linear resistors:



The wiggly lines are resistors, and their resistances are written beside them: for example, 3Ω means 3 ohms, an ‘ohm’ being a unit of resistance. To formalize this, define a circuit of linear resistors to consist of:

- a set N of nodes,
- a set E of edges,
- maps $s, t: E \rightarrow N$ sending each edge to its source and target node,
- a map $r: E \rightarrow (0, \infty)$ specifying the resistance of the resistor labelling each edge,
- maps $i: X \rightarrow N, o: Y \rightarrow N$ specifying the inputs and outputs of the circuit.

When we run electric current through such a circuit, each node $n \in N$ gets a ‘potential’ $\phi(n)$. The ‘voltage’ across an edge $e \in E$ is defined as the change in

potential as we move from the source of e to its target, $\phi(t(e)) - \phi(s(e))$, and the power dissipated by the resistor on this edge equals

$$\frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2.$$

The total power dissipated by the circuit is therefore twice

$$P(\phi) = \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2.$$

The factor of $\frac{1}{2}$ is convenient in some later calculations. Note that P is a nonnegative quadratic form on the vector space \mathbb{R}^N . However, not every nonnegative definite quadratic form on \mathbb{R}^N arises in this way from some circuit of linear resistors with N as its set of nodes. The quadratic forms that do arise are called ‘Dirichlet forms’. They have been extensively investigated [?, ?, ?, ?], and they play a major role in our work.

We write $\partial N = i(X) \cup o(Y)$ for the set of ‘terminals’: that is, nodes corresponding to inputs and outputs. The principle of minimum power says that if we fix the potential at the terminals, the circuit will choose the potential at other nodes to minimize the total power dissipated. An element ψ of the vector space $\mathbb{R}^{\partial N}$ assigns a potential to each terminal. Thus, if we fix ψ , the total power dissipated will be twice

$$Q(\psi) = \min_{\substack{\phi \in \mathbb{R}^N \\ \phi|_{\partial N} = \psi}} P(\phi)$$

The function $Q: \mathbb{R}^{\partial N} \rightarrow \mathbb{R}$ is again a Dirichlet form. We call it the ‘power functional’ of the circuit.

Now, suppose we are unable to see the internal workings of a circuit, and can only observe its ‘external behavior’: that is, the potentials at its terminals and the currents flowing into or out of these terminals. Remarkably, this behavior is completely determined by the power functional Q . The reason is that the current at any terminal can be obtained by differentiating Q with respect to the potential at this terminal, and relations of this form are *all* the relations that hold between potentials and currents at the terminals.

The Laplace transform allows us to generalize this immediately to circuits that can also contain linear inductors and capacitors, simply by changing the field we work over, replacing \mathbb{R} by the field $\mathbb{R}(s)$ of rational functions of a single real variable, and talking of ‘impedance’ where we previously talked of resistance. We obtain a category Circ where an object is a finite set, a morphism $f: X \rightarrow Y$ is a circuit with input set

X and output set Y , and composition is given by identifying the outputs of one circuit with the inputs of the next, and taking the resulting union of labelled graphs. Each such circuit gives rise to a Dirichlet form, now defined over $\mathbb{R}(s)$, and this Dirichlet form completely describes the externally observable behavior of the circuit.

We can take equivalence classes of circuits, where two circuits count as the same if they have the same Dirichlet form. We wish for these equivalence classes of circuits to form a category. Although there is a notion of composition for Dirichlet forms, we find that it lacks identity morphisms or, equivalently, it lacks morphisms representing ideal wires of zero impedance. To address this we turn to Lagrangian subspaces of symplectic vector spaces. These generalize quadratic forms via the map

$$\left(Q: \mathbb{F}^{\partial N} \rightarrow \mathbb{F}\right) \mapsto \left(\text{Graph}(dQ) = \{(\psi, dQ_\psi) \mid \psi \in \mathbb{F}^{\partial N}\} \subseteq \mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*\right)$$

taking a quadratic form Q on the vector space $\mathbb{F}^{\partial N}$ over the field \mathbb{F} to the graph of its differential dQ . Here we think of the symplectic vector space $\mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*$ as the state space of the circuit, and the subspace $\text{Graph}(dQ)$ as the subspace of attainable states, with $\psi \in \mathbb{F}^{\partial N}$ describing the potentials at the terminals, and $dQ_\psi \in (\mathbb{F}^{\partial N})^*$ the currents.

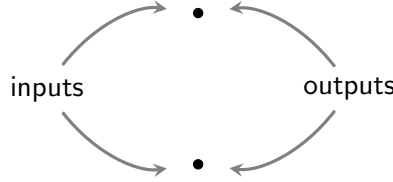
This construction is well-known in classical mechanics [?], where the principle of least action plays a role analogous to that of the principle of minimum power here. The set of Lagrangian subspaces is actually an algebraic variety, the ‘Lagrangian Grassmannian’, which serves as a compactification of the space of quadratic forms. The Lagrangian Grassmannian has already played a role in Sabot’s work on circuits made of resistors [?, ?]. For us, its importance is that we can find identity morphisms for the composition of Dirichlet forms by taking circuits made of parallel resistors and letting their resistances tend to zero: the limit is not a Dirichlet form, but it exists in the Lagrangian Grassmannian. Indeed, there exists a category LagrRel with finite dimensional symplectic vector spaces as objects and ‘Lagrangian relations’ as morphisms: that is, linear relations from V to W that are given by Lagrangian subspaces of $\bar{V} \oplus W$, where \bar{V} is the symplectic vector space conjugate to V .

To move from the Lagrangian subspace defined by the graph of the differential of the power functional to a morphism in the category LagrRel —that is, to a Lagrangian relation—we must treat seriously the input and output functions of the circuit. These express the circuit as built upon a cospan

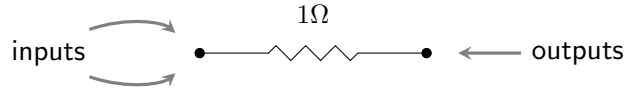
$$\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y. \end{array}$$

Applicable far more broadly than this present formalization of circuits, cospans model systems with two ‘ends’, an input and output end, albeit without any connotation of directionality: we might just as well exchange the role of the inputs and outputs by taking the mirror image of the above diagram. The role of the input and output functions, as we have discussed, is to mark the terminals we may glue onto the terminals of another circuit, and the pushout of cospans gives formal precision to this gluing construction.

One upshot of this cospan framework is that we may consider circuits with elements of N that are both inputs and outputs, such as this one:



This corresponds to the identity morphism on the finite set with two elements. Another is that some points may be considered an input or output multiple times; we draw this:



This allows us to connect two distinct outputs to the above double input.

Given a set X of inputs or outputs, we understand the electrical behavior on this set by considering the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$, the direct sum of the space \mathbb{F}^X of potentials and the space $(\mathbb{F}^X)^*$ of currents at these points. A Lagrangian relation specifies which states of the output space $\mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$ are allowed for each state of the input space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$. Turning the Lagrangian subspace $\text{Graph}(dQ)$ of a circuit into this information requires that we understand the ‘symplectification’

$$Sf: \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \mathbb{F}^A \oplus (\mathbb{F}^A)^*$$

and ‘twisted symplectification’

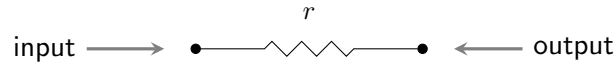
$$S^t f: \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \overline{\mathbb{F}^A \oplus (\mathbb{F}^A)^*}$$

of a function $f: A \rightarrow B$ between finite sets. In particular we need to understand how these apply to the input and output functions with codomain restricted to ∂N ; abusing notation, we also write these $i: X \rightarrow \partial N$ and $o: Y \rightarrow \partial N$.

The symplectification is a Lagrangian relation, and the catch phrase is that it ‘copies voltages’ and ‘splits currents’. More precisely, for any given potential-current

pair (ψ, ι) in $\mathbb{F}^B \oplus (\mathbb{F}^B)^*$, its image under Sf comprises all elements of $(\psi', \iota') \in \mathbb{F}^A \oplus (\mathbb{F}^A)^*$ such that the potential at $a \in A$ is equal to the potential at $f(a) \in B$, and such that, for each fixed $b \in B$, collectively the currents at the $a \in f^{-1}(b)$ sum to the current at b . We use the symplectification So of the output function to relate the state on ∂N to that on the outputs Y . As our current framework is set up to report the current *out* of each node, to describe input currents we define the twisted symplectification $S^t f: \mathbb{F}^B \oplus (\mathbb{F}^B)^* \rightarrow \overline{\mathbb{F}^A \oplus (\mathbb{F}^A)^*}$ almost identically to the above, except that we flip the sign of the currents $\iota' \in (\mathbb{F}^A)^*$. We use the twisted symplectification $S^t i$ of the input function to relate the state on ∂N to that on the inputs.

The Lagrangian relation corresponding to a circuit is then the set of all potential–current pairs that are possible at the inputs and outputs of that circuit. For instance, consider a resistor of resistance r , with one end considered as an input and the other as an output:



To obtain the corresponding Lagrangian relation, we must first specify domain and codomain symplectic vector spaces. In this case, as the input and output sets each consist of a single point, these vector spaces are both $\mathbb{F} \oplus \mathbb{F}^*$, where the first summand is understood as the space of potentials, and the second the space of currents.

Now, the resistor has power functional $Q: \mathbb{F}^2 \rightarrow \mathbb{F}$ given by

$$Q(\psi_1, \psi_2) = \frac{1}{2r}(\psi_2 - \psi_1)^2,$$

and the graph of the differential of Q is

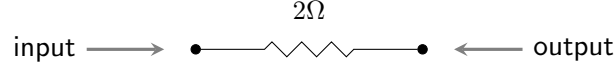
$$\text{Graph}(dQ) = \left\{ \left(\psi_1, \psi_2, \frac{1}{r}(\psi_1 - \psi_2), \frac{1}{r}(\psi_2 - \psi_1) \right) \mid \psi_1, \psi_2 \in \mathbb{F} \right\} \subseteq \mathbb{F}^2 \oplus (\mathbb{F}^2)^*.$$

In this example the input and output functions i, o are simply the identity functions on a one element set, so the symplectification of the output function is simply the identity linear transformation, and the twisted symplectification of the input function is the isomorphism between conjugate symplectic vector spaces $\mathbb{F} \oplus \mathbb{F}^* \rightarrow \overline{\mathbb{F} \oplus \mathbb{F}^*}$ mapping (ϕ, i) to $(\phi, -i)$. This implies that the behavior associated to this circuit is the Lagrangian relation

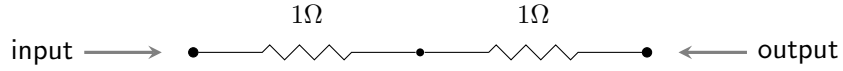
$$\left\{ (\psi_1, i, \psi_2, i) \mid \psi_1, \psi_2 \in \mathbb{F}, i = \frac{1}{r}(\psi_2 - \psi_1) \right\} \subseteq \overline{\mathbb{F} \oplus \mathbb{F}^*} \oplus \mathbb{F} \oplus \mathbb{F}^*.$$

This is precisely the set of potential–current pairs that are allowed at the input and output of a resistor of resistance r . In particular, the relation $i = \frac{1}{r}(\psi_2 - \psi_1)$ is well-known in electrical engineering: it is ‘Ohm’s law’.

A crucial fact is that the process of mapping a circuit to its corresponding Lagrangian relation identifies distinct circuits. For example, a single 2-ohm resistor:



has the same Lagrangian relation as two 1-ohm resistors in series:



The Lagrangian relation does not shed any light on the internal workings of a circuit. Thus, we call the process of computing this relation ‘black boxing’: it is like encasing the circuit in an opaque box, leaving only its terminals accessible. Fortunately, the Lagrangian relation of a circuit is enough to completely characterize its external behavior, including how it interacts when connected with other circuits.

Put more precisely, the black boxing process is *functorial*: we can compute the black boxed version of a circuit made of parts by computing the black boxed versions of the parts and then composing them. In fact we shall prove that Circ and LagrRel are dagger compact categories, and the black box functor preserves all this extra structure:

Theorem 6.1. *There exists a symmetric monoidal dagger functor, the **black box functor***

$$\blacksquare: \text{Circ} \rightarrow \text{LagrRel},$$

mapping a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^$ it generates, and a circuit $((N, E, s, t, r), i, o)$ to the Lagrangian relation*

$$\bigcup_{v \in \text{Graph}(dQ)} S^t i(v) \times S^o(v) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*,$$

where Q is the circuit’s power functional.

The goal of this paper is to prove and explain this result. The proof itself is more tricky than one might first expect, but our approach introduces various concepts that should be useful throughout the study of networks, such as ‘decorated cospans’ and ‘corelations’. These provide a general framework for discussing open networked systems—and not only the passive linear systems discussed here, but also others, such as Markov processes [?].

Cospans are already familiar as a formalism for making entities with an arbitrarily designated ‘input end’ and ‘output end’ into the morphisms of a category. For example, in topological quantum field theory we use special cospans called ‘cobordisms’ to describe pieces of spacetime [?, ?]. Earlier we introduced ‘decorated cospans’ to describe circuits with specified inputs and outputs. Later, with the help of machinery developed in a companion paper [?], we use decorated cospans to set up several functors that appear as factors of the black box functor, as steps in proving its functoriality.

Just as a relation is an isomorphism class of spans of a special sort, namely jointly monic spans, a ‘corelation’ is an isomorphism class of jointly epic cospans. While corelations are less widely used than relations, they turn out to be perfectly suited for describing circuits of ideal perfectly conductive wires—precisely the class of circuits that cannot be modeled simply as Dirichlet forms, since the power functional becomes infinite when it includes terms where the resistance is zero. We introduce corelations in Section 6.6, and they too play a key role in constructing the black box functor.

With these tools in hand, the black box functor turns out to rely on a tight relationship between Kirchhoff’s laws, the minimization of Dirichlet forms, and the ‘symplectification’ of corelations. It is well-known that away from the terminals, a circuit must obey two rules known as Kirchhoff’s laws. We have already noted that the principle of minimum power states that a circuit will ‘choose’ potentials on its interior that minimize the power functional. We clarify the relation between these points in Theorems 6.20 and 6.22, which together show that minimizing a Dirichlet form over some subset amounts to assuming that the corresponding circuit obeys Kirchhoff’s laws on that subset.

We have also mentioned the symplectification of functions above. Extending this to allow symplectification of corelations, this process gives a map sending corelations to Lagrangian relations that describe the behavior of ideal perfectly conductive wires. We prove that these symplectified corelations simultaneously impose Kirchhoff’s laws (Proposition 6.48 and Example 6.49) and accomplish the minimization of Dirichlet forms (Theorem 6.53).

Together, our results show that these three concepts—Kirchhoff’s laws from circuit theory, the analytic idea of minimizing power dissipation, and the algebraic idea of symplectification of corelations—are merely different faces of one law: the law of composition of circuits.

6.1.1 Finding your way through this paper

This paper is split into three parts, addressing in turn the questions:

- I. What do circuit diagrams mean?
- II. How do we interact with circuit diagrams?
- III. How is meaning preserved under these interactions?

We begin Part I, on the semantics of circuit diagrams, with a discussion of circuits of linear resistors, developing the intuition for the governing laws of passive linear circuits—Ohm’s law, Kirchhoff’s voltage law, and Kirchhoff’s current law—in a time-independent setting (Section 6.2). This allows us to develop the concept of Dirichlet form as a representation of power consumption, and understand their composition as minimizing power, an expression of the current law. In Section 6.3, the Laplace transform then allows us to recapitulate these ideas after introducing inductors and capacitors, speaking of impedance where we formerly spoke of resistance, and generalizing Dirichlet forms from the field \mathbb{R} to the field $\mathbb{R}(s)$ of real rational functions. While in this setting the principle of minimum power is replaced by a variational principle, the intuitions gained from circuits of resistors still remain useful.

At the end of this part, we show that Dirichlet forms alone do not provide the flexibility to construct a category representing the semantics of circuit diagrams. The fundamental reason is that they do not allow us to describe circuits involving ideal perfectly conductive wires. This motivates the development of more powerful machinery.

Part II, on the syntax of circuit diagrams, contains the main technical contributions of the paper. It begins with Section 6.4, which develops machinery to construct what we call categories of decorated cospans. These are categories where the objects are finite sets and the morphisms are cospans in the category of finite sets together with some extra structure on the apex. Circuits, as defined above, are naturally an example of such a construction, and Section 6.4 lays out the details of this. In Section 6.5 we then review the basic theory of linear Lagrangian relations, giving details to the correspondence we have defined between Dirichlet forms, and hence passive linear circuits, and Lagrangian relations. Section 6.6 then takes immediate advantage of the added flexibility of Lagrangian relations, discussing the ‘trivial’ circuits comprising only perfectly conductive wires, which mediate the notion of composition of circuits.

Having introduced these prerequisites, we get to the point in Part III. In Section 6.7 we introduce the black box functor, and in Section 6.8 we prove our main result.

6.2 Circuits of linear resistors

In this part we review the meaning of circuit diagrams comprising resistors, inductors, and capacitors, giving an answer to the question “What do circuit diagrams mean?”.

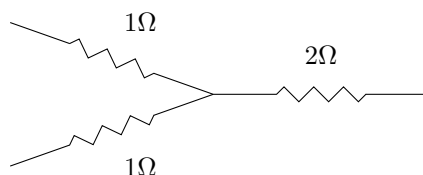
To elaborate, while circuit diagrams model electric circuits according to their physical form, another, often more relevant, way to understand a circuit is by its external behavior. This means the following. To an electric circuit we associate two quantities to each edge: voltage and current. We are not free, however, to choose these quantities as we like; circuits are subject to governing laws that imply voltages and currents must obey certain relationships. From the perspective of control theory we are particularly interested in the values these quantities take at the so-called terminals, and how altering one value will affect the other values. We call two circuits equivalent when they determine the same relationship. Our main task in this first part is to explore when two circuits are equivalent.

In order to let physical intuition lead the way, we begin by specialising to the case of linear resistors. In this section we describe how to find the function of a circuit from its form, advocating in particular the perspective of the principle of minimum power. This allows us to identify the external behavior of a circuit with a so-called Dirichlet form representing the dependence of its power consumption on potentials at its terminals.

6.2.1 Circuits as labelled graphs

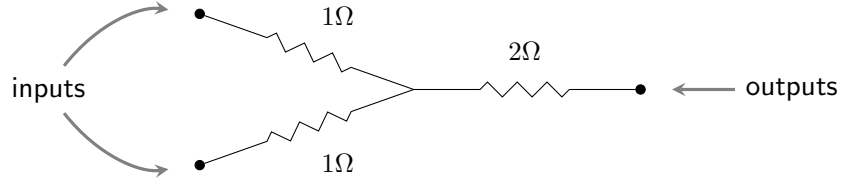
The concept of an abstract open electrical circuit made of linear resistors is well-known in electrical engineering, but we shall need to formalize it with more precision than usual. The basic idea is that a circuit of linear resistors is a graph whose edges are labelled by positive real numbers called ‘resistances’, and whose sets of vertices is equipped with two subsets: the ‘inputs’ and ‘outputs’. This unfolds as follows.

A (closed) circuit of resistors looks like this:

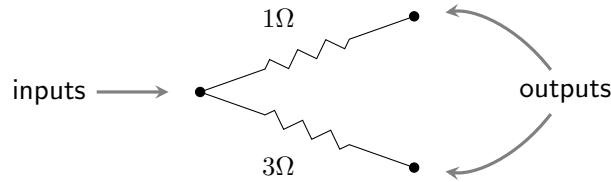


We can consider this a labelled graph, with each resistor an edge of the graph, its resistance its label, and the vertices of the graph the points at which resistors are connected.

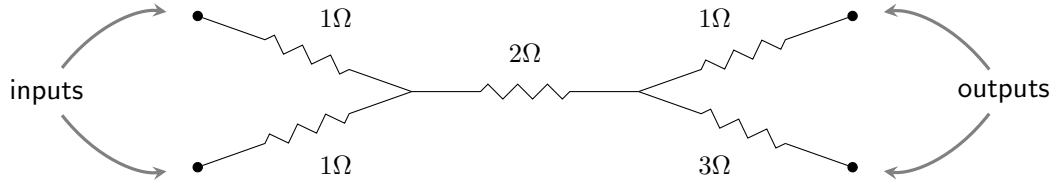
A circuit is ‘open’ if it can be connected to other circuits. To do this we first mark points at which connections can be made by denoting some vertices as input and output terminals:



Then, given a second circuit, we may choose a relation between the output set of the first and the input set of this second circuit, such as the simple relation of the single output vertex of the circuit above with the single input vertex of the circuit below.



We connect the two circuits by identifying output and input vertices according to this relation, giving in this case the composite circuit:



More formally, we define a **graph**¹ to be a pair of functions $s, t: E \rightarrow N$ where E and N are finite sets. We call elements of E **edges** and elements of N **vertices** or **nodes**. We say that the edge $e \in E$ has **source** $s(e)$ and **target** $t(e)$, and also say that e is an edge **from** $s(e)$ **to** $t(e)$.

To study circuits we need graphs with labelled edges:

Definition 6.2. Given a set L of **labels**, an **L -graph** is a graph equipped with a function $r: E \rightarrow L$:

$$L \xleftarrow{r} E \xrightleftharpoons[t]{s} N.$$

¹In this paper we refer to directed multigraphs simply as graphs.

For circuits made of resistors we take $L = (0, \infty)$, but later we shall take L to be a set of positive elements in some more general field. In either case, a circuit will be an L -graph with some extra structure:

Definition 6.3. Given a set L , a **circuit over L** is an L -graph $L \xleftarrow{r} E \xrightleftharpoons[t]{s} N$ together with finite sets X, Y , and functions $i: X \rightarrow N$ and $o: Y \rightarrow N$. We call the sets $i(X)$, $o(Y)$, and $\partial N = i(X) \cup o(Y)$ the **inputs**, **outputs**, and **terminals** or **boundary** of the circuit, respectively.

We will later make use of the notion of connectedness in graphs. Recall that given two vertices $v, w \in N$ of a graph, a **path from v to w** is a finite sequence of vertices $v = v_0, v_1, \dots, v_n = w$ and edges e_1, \dots, e_n such that for each $1 \leq i \leq n$, either e_i is an edge from v_i to v_{i+1} , or an edge from v_{i+1} to v_i . A subset S of the vertices of a graph is **connected** if, for each pair of vertices in S , there is a path from one to the other. A **connected component** of a graph is a maximal connected subset of its vertices.²

In the rest of this section we take $L = (0, \infty) \subseteq \mathbb{R}$ and fix a circuit over $(0, \infty)$. The edges of this circuit should be thought of as ‘wires’. The label $r_e \in (0, \infty)$ stands for the **resistance** of the resistor on the wire e . There will also be a voltage and current on each wire. In this section, these will be specified by functions $V \in \mathbb{R}^E$ and $I \in \mathbb{R}^E$. Here, as customary in engineering, we use I for ‘intensity of current’, following Ampère.

6.2.2 Ohm’s law, Kirchhoff’s laws, and the principle of minimum power

In 1827, Georg Ohm published a book which included a relation between the voltage and current for circuits made of resistors [?]. At the time, the critical reception was harsh: one contemporary called Ohm’s work “a web of naked fancies, which can never find the semblance of support from even the most superficial of observations”, and the German Minister of Education said that a professor who preached such heresies was unworthy to teach science [?, ?]. However, a simplified version of his relation is now widely used under the name of ‘Ohm’s law’. We say that **Ohm’s law** holds if for all edges $e \in E$ the voltage and current functions of a circuit obey:

$$V(e) = r(e)I(e).$$

²In the theory of directed graphs the qualifier ‘weakly’ is commonly used before the word ‘connected’ in these two definitions, in distinction from a stronger notion of connectedness requiring paths to respect edge directions. As we never consider any other sort of connectedness, we omit this qualifier.

Kirchhoff's laws date to Gustav Kirchhoff in 1845, generalising Ohm's work. They were in turn generalized into Maxwell's equations a few decades later. We say **Kirchhoff's voltage law** holds if there exists $\phi \in \mathbb{R}^N$ such that

$$V(e) = \phi(t(e)) - \phi(s(e)).$$

We call the function ϕ a **potential**, and think of it as assigning an electrical potential to each node in the circuit. The voltage then arises as the differences in potentials between adjacent nodes. If Kirchhoff's voltage law holds for some voltage V , the potential ϕ is unique only in the trivial case of the empty circuit: when the set of nodes N is empty. Indeed, two potentials define the same voltage function if and only if their difference is constant on each connected component of the graph Γ .

We say **Kirchhoff's current law** holds if for all nonterminal nodes $n \in N \setminus \partial N$ we have

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e).$$

This is an expression of conservation of charge within the circuit; it says that the total current flowing in or out of any nonterminal node is zero. Even when Kirchhoff's current law is obeyed, terminals need not be sites of zero net current; we call the function $\iota \in \mathbb{R}^{\partial N}$ that takes a terminal to the difference between the outward and inward flowing currents,

$$\begin{aligned} \iota : \partial N &\longrightarrow \mathbb{R} \\ n &\longmapsto \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e), \end{aligned}$$

the **boundary current** for I .

A **boundary potential** is also a function in $\mathbb{R}^{\partial N}$, but instead thought of as specifying potentials on the terminals of a circuit. As we think of our circuits as open circuits, with the terminals points of interaction with the external world, we shall think of these potentials as variables that are free for us to choose. Using the above three principles—Ohm's law, Kirchhoff's voltage law, and Kirchhoff's current law—it is possible to show that choosing a boundary potential determines unique voltage and current functions on that circuit.

The so-called 'principle of minimum power' gives some insight into how this occurs, by describing a way potentials on the terminals might determine potentials at all nodes. From this, Kirchhoff's voltage law then gives rise to a voltage function on the edges, and Ohm's law gives us a current function too. We shall show, in fact, that a

potential satisfies the principle of minimum power for a given boundary potential if and only if this current obeys Kirchhoff's current law.

A circuit with current I and voltage V dissipates energy at a rate equal to

$$\sum_{e \in E} I(e)V(e).$$

Ohm's law allows us to rewrite I as V/r , while Kirchhoff's voltage law gives us a potential ϕ such that $V(e)$ can be written as $\phi(t(e)) - \phi(s(e))$, so for a circuit obeying these two laws the power can also be expressed in terms of this potential. We thus arrive at a functional mapping potentials ϕ to the power dissipated by the circuit when Ohm's law and Kirchhoff's voltage law are obeyed for ϕ .

Definition 6.4. The **extended power functional** $P: \mathbb{R}^N \rightarrow \mathbb{R}$ of a circuit is defined by

$$P(\phi) = \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2.$$

The factor of $\frac{1}{2}$ is inserted to cancel the factor of 2 that appears when we differentiate this expression. We call P the *extended* power functional as we shall see that it is defined even on potentials that are not compatible with the three governing laws of electric circuits. We shall later restrict the domain of this functional so that it is defined precisely on those potentials that *are* compatible with the governing laws. Note that this functional does not depend on the directions chosen for the edges of the circuit.

This expression lets us formulate the 'principle of minimum power', which gives us information about the potential ϕ given its restriction to the boundary of Γ . Call a potential $\phi \in \mathbb{R}^N$ an **extension** of a boundary potential $\psi \in \mathbb{R}^{\partial N}$ if ϕ is equal to ψ when restricted to $\mathbb{R}^{\partial N}$ —that is, if $\phi|_{\partial N} = \psi$.

Definition 6.5. We say a potential $\phi \in \mathbb{R}^N$ **obeys the principle of minimum power** for a boundary potential $\psi \in \mathbb{R}^{\partial N}$ if ϕ minimizes the extended power functional P subject to the constraint that ϕ is an extension of ψ .

As promised, in the presence of Ohm's law and Kirchhoff's voltage law, the principle of minimum power is equivalent to Kirchhoff's current law.

Proposition 6.6. *Let ϕ be a potential extending some boundary potential ψ . Then ϕ obeys the principle of minimum power for ψ if and only if the current*

$$I(e) = \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))$$

obeys Kirchhoff's current law.

Proof. Fixing the potentials at the terminals to be those given by the boundary potential ψ , the power is a nonnegative quadratic function of the potentials at the nonterminals. This implies that an extension ϕ of ψ minimizes P precisely when

$$\left. \frac{\partial P(\varphi)}{\partial \varphi(n)} \right|_{\varphi=\phi} = 0$$

for all nonterminals $n \in N \setminus \partial N$. Note that the partial derivative of the power with respect to the potential at n is given by

$$\begin{aligned} \left. \frac{\partial P}{\partial \varphi(n)} \right|_{\varphi=\phi} &= \sum_{t(e)=n} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e))) - \sum_{s(e)=n} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e))) \\ &= \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e). \end{aligned}$$

Thus ϕ obeys the principle of minimum power for ψ if and only if

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e)$$

for all $n \in N \setminus \partial N$, and so if and only if Kirchhoff's current law holds. \square

6.2.3 A Dirichlet problem

We remind ourselves that we are in the midst of understanding circuits as objects that define relationships between boundary potentials and boundary currents. This relationship is defined by the stipulation that voltage–current pairs on a circuit must obey Ohm's law and Kirchhoff's laws—or equivalently, Ohm's law, Kirchhoff's voltage law, and the principle of minimum power. In this subsection we show these conditions imply that for each boundary potential ψ on the circuit there exists a potential ϕ on the circuit extending ψ , unique up to what may be interpreted as a choice of reference potential on each connected component of the circuit. From this potential ϕ we can then compute the unique voltage, current, and boundary current functions compatible with the given boundary potential.

Fix again a circuit with extended power functional $P: \mathbb{R}^N \rightarrow \mathbb{R}$. Let $\nabla: \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the operator that maps a potential $\phi \in \mathbb{R}^N$ to the function from N to \mathbb{R} given by

$$n \mapsto \left. \frac{\partial P}{\partial \varphi(n)} \right|_{\varphi=\phi}.$$

As we have seen, this function takes potentials to twice the pointwise currents that they induce. We have also seen that a potential ϕ is compatible with the governing laws of circuits if and only if

$$\nabla \phi|_{\mathbb{R}^{\partial N}} = 0. \tag{6.1}$$

The operator ∇ acts as a discrete analogue of the Laplacian for the graph Γ , so we call this operator the **Laplacian** of Γ , and say that equation (6.1) is a version of Laplace's equation. We then say that the problem of finding an extension ϕ of some fixed boundary potential ψ that solves this Laplace's equation—or, equivalently, the problem of finding a ϕ that obeys the principle of minimum power for ψ —is a discrete version of the **Dirichlet problem**.

As we shall see, this version of the Dirichlet problem always has a solution. However, the solution is not necessarily unique. If we take a solution ϕ and some $\alpha \in \mathbb{R}^N$ that is constant on each connected component and vanishes on the boundary of Γ , it is clear that $\phi + \alpha$ is still an extension of ψ and that

$$\left. \frac{\partial P(\varphi)}{\partial \varphi(n)} \right|_{\varphi=\phi} = \left. \frac{\partial P(\varphi)}{\partial \varphi(n)} \right|_{\varphi=\phi+\alpha},$$

so $\phi + \alpha$ is another solution. We say that a connected component of a circuit **touches the boundary** if it contains a vertex in ∂N . Note that such an α must vanish on all connected components touching the boundary.

With these preliminaries in hand, we can solve the Dirichlet problem:

Proposition 6.7. *For any boundary potential $\psi \in \mathbb{R}^{\partial N}$ there exists a potential ϕ obeying the principle of minimum power for ψ . If we also demand that ϕ vanish on every connected component of Γ not touching the boundary, then ϕ is unique.*

Proof. For existence, observe that the power is a nonnegative quadratic form, the extensions of ψ form an affine subspace of \mathbb{R}^N , and a nonnegative quadratic form restricted to an affine subspace of a real vector space must reach a minimum somewhere on this subspace.

For uniqueness, suppose that both ϕ and ϕ' obey the principle of minimum power for ψ . Let

$$\alpha = \phi' - \phi.$$

Then

$$\alpha|_{\partial N} = \phi'|_{\partial N} - \phi|_{\partial N} = \psi - \psi = 0,$$

so $\phi + \lambda\alpha$ is an extension of ψ for all $\lambda \in \mathbb{R}$. This implies that

$$f(\lambda) := P(\phi + \lambda\alpha)$$

is a smooth function attaining its minimum value at both $t = 0$ and $t = 1$. In particular, this implies that $f'(0) = 0$. But this means that when writing f as a

quadratic, the coefficient of λ must be 0, so we can write

$$\begin{aligned}
 2f(\lambda) &= \sum_{e \in E} \frac{1}{r(e)} ((\phi + \lambda\alpha)(t(e)) - (\phi + \lambda\alpha)(s(e)))^2 \\
 &= \sum_{e \in E} \frac{1}{r(e)} \left((\phi(t(e)) - \phi(s(e))) + \lambda(\alpha(t(e)) - \alpha(s(e))) \right)^2 \\
 &= \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2 + \lambda\text{-term} + \lambda^2 \sum_{e \in E} \frac{1}{r(e)} (\alpha(t(e)) - \alpha(s(e)))^2 \\
 &= \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2 + \lambda^2 \sum_{e \in E} \frac{1}{r(e)} (\alpha(t(e)) - \alpha(s(e)))^2.
 \end{aligned}$$

Then

$$f(1) - f(0) = \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\alpha(t(e)) - \alpha(s(e)))^2 = 0,$$

so $\alpha(t(e)) = \alpha(s(e))$ for every edge $e \in E$. This implies that α is constant on each connected component of the graph Γ of our circuit.

Note that as $\alpha|_{\partial N} = 0$, α vanishes on every connected component of Γ touching the boundary. Thus, if we also require that ϕ and ϕ' vanish on every connected component of Γ not touching the boundary, then $\alpha = \phi' - \phi$ vanishes on all connected components of Γ , and hence is identically zero. Thus $\phi' = \phi$, and this extra condition ensures a unique solution to the Dirichlet problem. \square

We have also shown the following:

Proposition 6.8. *Suppose $\psi \in \mathbb{R}^{\partial N}$ and ϕ is a potential obeying the principle of minimum power for ψ . Then ϕ' obeys the principle of minimum power for ψ if and only if the difference $\phi' - \phi$ is constant on every connected component of Γ and vanishes on every connected component touching the boundary of Γ .*

Furthermore, ϕ depends linearly on ψ :

Proposition 6.9. *Fix $\psi \in \mathbb{R}^{\partial N}$, and suppose $\phi \in \mathbb{R}^N$ is the unique potential obeying the principle of minimum power for ψ that vanishes on all connected components of Γ not touching the boundary. Then ϕ depends linearly on ψ .*

Proof. Fix $\psi, \psi' \in \mathbb{R}^{\partial N}$, and suppose $\phi, \phi' \in \mathbb{R}^N$ obey the principle of minimum power for ψ, ψ' respectively, and that both ϕ and ϕ' vanish on all connected components of Γ not touching the boundary.

Then, for all $\lambda \in \mathbb{R}$,

$$(\phi + \lambda\phi')|_{\mathbb{R}^{\partial N}} = \phi|_{\mathbb{R}^{\partial N}} + \lambda\phi'|_{\mathbb{R}^{\partial N}} = \psi + \lambda\psi'$$

and

$$(\nabla(\phi + \lambda\phi'))|_{\mathbb{R}\partial N} = (\nabla\phi)|_{\mathbb{R}\partial N} + \lambda(\nabla\phi')|_{\mathbb{R}\partial N} = 0.$$

Thus $\phi + \lambda\phi'$ solves the Dirichlet problem for $\psi + \lambda\psi'$, and thus ϕ depends linearly on ψ . \square

Bamberg and Sternberg [?] describe another way to solve the Dirichlet problem, going back to Weyl [?].

6.2.4 Equivalent circuits

We have seen that boundary potentials determine, essentially uniquely, the value of all the electric properties across the entire circuit. But from the perspective of control theory, this internal structure is irrelevant: we can only access the circuit at its terminals, and hence only need concern ourselves with the relationship between boundary potentials and boundary currents. In this section we streamline our investigations above to state the precise way in which boundary currents depend on boundary potentials. In particular, we shall see that the relationship is completely captured by the functional taking boundary potentials to the minimum power used by any extension of that boundary potential. Furthermore, each such power functional determines a different boundary potential–boundary current relationship, and so we can conclude that two circuits are equivalent if and only if they have the same power functional.

An ‘external behavior’, or **behavior** for short, is an equivalence class of circuits, where two are considered equivalent when the boundary current is the same function of the boundary potential. The idea is that the boundary current and boundary potential are all that can be observed ‘from outside’, i.e. by making measurements at the terminals. Restricting our attention to what can be observed by making measurements at the terminals amounts to treating a circuit as a ‘black box’: that is, treating its interior as hidden from view. So, two circuits give the same behavior when they behave the same as ‘black boxes’.

First let us check that the boundary current is a function of the boundary potential. For this we introduce an important quadratic form on the space of boundary potentials:

Definition 6.10. The **power functional** $Q: \mathbb{R}^{\partial N} \rightarrow \mathbb{R}$ of a circuit with extended power functional P is given by

$$Q(\psi) = \min_{\phi|_{\mathbb{R}\partial N} = \psi} P(\phi).$$

Proposition 6.7 shows the minimum above exists, so the power functional is well-defined. Thanks to the principle of minimum power, $Q(\psi)$ equals $\frac{1}{2}$ times the power dissipated by the circuit when the boundary voltage is ψ . We will later see that in fact $Q(\psi)$ is a nonnegative quadratic form on $\mathbb{R}^{\partial N}$.

Since Q is a smooth real-valued function on $\mathbb{R}^{\partial N}$, its differential dQ at any given point $\psi \in \mathbb{R}^{\partial N}$ defines an element of the dual space $(\mathbb{R}^{\partial N})^*$, which we denote by dQ_ψ . In fact, this element is equal to the boundary current ι corresponding to the boundary voltage ψ :

Proposition 6.11. *Suppose $\psi \in \mathbb{R}^{\partial N}$. Suppose ϕ is any extension of ψ minimizing the power. Then $dQ_\psi \in (\mathbb{R}^{\partial N})^* \cong \mathbb{R}^{\partial N}$ gives the boundary current of the current induced by the potential ϕ .*

Proof. Note first that while there may be several choices of ϕ minimizing the power subject to the constraint that $\phi|_{\mathbb{R}^{\partial N}} = \psi$, Proposition 6.8 says that the difference between any two choices vanishes on all components touching the boundary of Γ . Thus, these two choices give the same value for the boundary current $\iota: \partial N \rightarrow \mathbb{R}$. So, with no loss of generality we may assume ϕ is the unique choice that vanishes on all components not touching the boundary. Write $\bar{\iota}: N \rightarrow \mathbb{R}$ for the extension of $\iota: \partial N \rightarrow \mathbb{R}$ to N taking value 0 on $N \setminus \partial N$.

By Proposition 6.9, there is a linear operator

$$f: \mathbb{R}^{\partial N} \longrightarrow \mathbb{R}^N$$

sending $\psi \in \mathbb{R}^{\partial N}$ to this choice of ϕ , and then

$$Q(\psi) = P(f\psi).$$

Given any $\psi' \in \mathbb{R}^{\partial N}$, we thus have

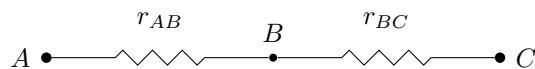
$$\begin{aligned}
dQ_\psi(\psi') &= \left. \frac{d}{d\lambda} Q(\phi + \lambda\psi') \right|_{\lambda=0} \\
&= \left. \frac{d}{d\lambda} P(f(\psi + \lambda\psi')) \right|_{\lambda=0} \\
&= \left. \frac{1}{2} \frac{d}{d\lambda} \sum_{e \in E} \frac{1}{r(e)} \left(f(\psi + \lambda\psi')(t(e)) - f(\psi + \lambda\psi')(s(e)) \right)^2 \right|_{\lambda=0} \\
&= \left. \frac{1}{2} \frac{d}{d\lambda} \sum_{e \in E} \frac{1}{r(e)} \left((f\psi(t(e)) - f\psi(s(e))) + \lambda(f\psi'(t(e)) - f\psi'(s(e))) \right)^2 \right|_{\lambda=0} \\
&= \sum_{e \in E} \frac{1}{r(e)} (f\psi(t(e)) - f\psi(s(e))) (f\psi'(t(e)) - f\psi'(s(e))) \\
&= \sum_{e \in E} I(e) (f\psi'(t(e)) - f\psi'(s(e))) \\
&= \sum_{n \in N} \left(\sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e) \right) f\psi'(n) \\
&= \sum_{n \in N} \bar{\iota}(n) f\psi'(n) \\
&= \sum_{n \in \partial N} \iota(n) \psi'(n).
\end{aligned}$$

This shows that $dQ_\psi^* = \iota$, as claimed. Note that this calculation explains why we inserted a factor of $\frac{1}{2}$ in the definition of P : it cancels the factor of 2 obtained from differentiating a square. \square

Note this only depends on Q , which makes no mention of the potentials at nonterminals. This is amazing: the way power depends on boundary potentials completely characterizes the way boundary currents depend on boundary potentials. In particular, in Part we shall see that this allows us to define a composition rule for behaviors of circuits.

To demonstrate these notions, we give a basic example of equivalent circuits.

Example 6.12 (Resistors in series). Resistors are said to be placed in **series** if they are placed end to end or, more precisely, if they form a path with no self-intersections. It is well known that resistors in series are equivalent to a single resistor with resistance equal to the sum of their resistances. To prove this, consider the following circuit comprising two resistors in series, with input A and output C :



Now, the extended power functional $P: \mathbb{R}^{\{A,B,C\}} \rightarrow \mathbb{R}$ for this circuit is

$$P(\phi) = \frac{1}{2} \left(\frac{1}{r_{AB}} (\phi(A) - \phi(B))^2 + \frac{1}{r_{BC}} (\phi(B) - \phi(C))^2 \right),$$

while the power functional $Q: \mathbb{R}^{\{A,C\}} \rightarrow \mathbb{R}$ is given by minimization over values of $\phi(B) = x$:

$$Q(\psi) = \min_{x \in \mathbb{R}} \frac{1}{2} \left(\frac{1}{r_{AB}} (\psi(A) - x)^2 + \frac{1}{r_{BC}} (x - \psi(C))^2 \right).$$

Differentiating with respect to x , we see that this minimum occurs when

$$\frac{1}{r_{AB}} (x - \psi(A)) + \frac{1}{r_{BC}} (x - \psi(C)) = 0,$$

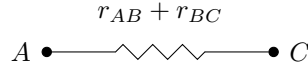
and hence when x is the r -weighted average of $\psi(A)$ and $\psi(C)$:

$$x = \frac{r_{BC}\psi(A) + r_{AB}\psi(C)}{r_{BC} + r_{AB}}.$$

Substituting this value for x into the expression for Q above and simplifying gives

$$Q(\psi) = \frac{1}{2} \cdot \frac{1}{r_{AB} + r_{BC}} (\psi(A) - \psi(C))^2.$$

This is also the power functional of the circuit



and so the circuits are equivalent.

6.2.5 Dirichlet forms

In the previous subsection we claimed that power functionals are quadratic forms on the boundary of the circuit whose behavior they represent. They comprise, in fact, precisely those quadratic forms known as Dirichlet forms.

Definition 6.13. Given a finite set S , a **Dirichlet form** on S is a quadratic form $Q: \mathbb{R}^S \rightarrow \mathbb{R}$ given by the formula

$$Q(\psi) = \sum_{i,j} c_{ij} (\psi_i - \psi_j)^2$$

for some nonnegative real numbers c_{ij} .

Note that we may assume without loss of generality that $c_{ii} = 0$ and $c_{ij} = c_{ji}$; we do this henceforth. Any Dirichlet form is nonnegative: $Q(\psi) \geq 0$ for all $\psi \in \mathbb{R}^S$. However, not all nonnegative quadratic forms are Dirichlet forms. For example, if $S = \{1, 2\}$, the nonnegative quadratic form $Q(\psi) = (\psi_1 + \psi_2)^2$ is not a Dirichlet form. That said, the concept of Dirichlet form is vastly more general than the above definition: such quadratic forms are studied not just on finite-dimensional vector spaces \mathbb{R}^S but on L^2 of any measure space. When this measure space is just a finite set, the concept of Dirichlet form reduces to the definition above. For a thorough introduction to Dirichlet forms, see the text by Fukushima [?]. For a fun tour of the underlying ideas, see the paper by Doyle and Snell [?].

The following characterizations of Dirichlet forms help illuminate the concept:

Proposition 6.14. *Given a finite set S and a quadratic form $Q: \mathbb{R}^S \rightarrow \mathbb{R}$, the following are equivalent:*

- (i) Q is a Dirichlet form.
- (ii) $Q(\phi) \leq Q(\psi)$ whenever $|\phi_i - \phi_j| \leq |\psi_i - \psi_j|$ for all i, j .
- (iii) $Q(\phi) = 0$ whenever ϕ_i is independent of i , and Q obeys the **Markov property**:
 $Q(\phi) \leq Q(\psi)$ when $\phi_i = \min(\psi_i, 1)$.

Proof. See Fukushima [?]. □

While the extended power functionals of circuits are evidently Dirichlet forms, it is not immediate that all power functionals are. For this it is crucial that the property of being a Dirichlet form is preserved under minimising over linear subspaces of the domain that are generated by subsets of the given finite set.

Proposition 6.15. *If $Q: \mathbb{R}^{S+T} \rightarrow \mathbb{R}$ is a Dirichlet form, then*

$$\min_{\nu \in \mathbb{R}^T} Q(-, \nu): \mathbb{R}^S \rightarrow \mathbb{R}$$

is Dirichlet.

Proof. We first note that $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is a quadratic form. Again, $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is well-defined as a nonnegative quadratic form also attains its minimum on an affine subspace of its domain. Furthermore $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is itself a quadratic form, as the partial derivatives of Q are linear, and hence the points at which these minima are attained depend linearly on the argument of $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$.

Now by Proposition 6.14, $Q(\phi) \leq Q(\phi')$ whenever $|\phi_i - \phi_j| \leq |\phi'_i - \phi'_j|$ for all $i, j \in S + T$. In particular, this implies $\min_{\nu \in \mathbb{R}^T} Q(\psi, \nu) \leq \min_{\nu \in \mathbb{R}^T} Q(\psi', \nu)$ whenever $|\psi_i - \psi_j| \leq |\psi'_i - \psi'_j|$ for all $i, j \in S$. Using Proposition 6.14 again then implies that $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is a Dirichlet form. \square

Corollary 6.16. *Let $Q: \mathbb{R}^{\partial N} \rightarrow \mathbb{R}$ be the power functional for some circuit. Then Q is a Dirichlet form.*

Proof. The extended power functional P is a Dirichlet form, and writing $\mathbb{R}^N = \mathbb{R}^{\partial N} \oplus \mathbb{R}^{N \setminus \partial N}$ allows us to write

$$Q(-) = \min_{\phi \in \mathbb{R}^{N \setminus \partial N}} P(-, \phi). \quad \square$$

The converse is also true: simply construct the circuit with set of vertices ∂N and an edge of resistance $\frac{1}{2c_{ij}}$ between any $i, j \in \partial N$ such that the term $c_{ij}(\psi_i - \psi_j)$ appears in the Dirichlet form. This gives:

Proposition 6.17. *A function Q is the power functional for some circuit if and only if Q is a Dirichlet form.*

This is an expression of the ‘star-mesh transform’, a well-known fact of electrical engineering stating that every circuit of linear resistors is equivalent to some complete graph of resistors between its terminals. For more details see [?]. We may interpret the proof of Proposition 6.15 as showing that intermediate potentials at minima depend linearly on boundary potentials, in fact a weighted average, and that substituting these into a quadratic form still gives quadratic form.

In summary, in this section we have shown the existence of a surjective function

$$\left\{ \begin{array}{c} \text{circuits of linear resistors} \\ \text{with boundary } \partial N \end{array} \right\} \longrightarrow \left\{ \text{Dirichlet forms on } \partial N \right\}$$

mapping two circuits to the same Dirichlet form if and only if they have the same external behavior. In the next section we extend this result to encompass inductors and capacitors too.

6.3 Inductors and capacitors

The intuition gleaned from the study of resistors carries over to inductors and capacitors too, to provide a framework for studying what are known as passive linear

networks. To understand inductors and capacitors in this way, however, we must introduce a notion of time dependency and subsequently the Laplace transform, which allows us to work in the so-called frequency domain. Here, like resistors, inductors and capacitors simply impose a relationship of proportionality between the voltages and currents that run across them. The constant of proportionality is known as the impedance of the component.

As for resistors, the interconnection of such components may be understood, at least formally, as a minimization of some quantity, and we may represent the behaviors of this class of circuits with a more general idea of Dirichlet form. We conclude this section by noting an obstruction to building a composition rule for Dirichlet forms, motivating our work in Part .

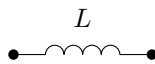
6.3.1 The frequency domain and Ohm's law revisited

In broadening the class of electrical circuit components under examination, we find ourselves dealing with components whose behaviors depend on the rates of change of current and voltage with respect to time. We thus now consider time-varying voltages $v: [0, \infty) \rightarrow \mathbb{R}$ and currents $i: [0, \infty) \rightarrow \mathbb{R}$, where $t \in [0, \infty)$ is a real variable representing time. For mathematical reasons, we restrict these voltages and currents to only those with (i) zero initial conditions (that is, $f(0) = 0$) and (ii) Laplace transform lying in the field

$$\mathbb{R}(s) = \left\{ Z(s) = \frac{P(s)}{Q(s)} \mid P, Q \text{ polynomials over } \mathbb{R} \text{ in } s, Q \neq 0 \right\}$$

of real rational functions of one variable. While it is possible that physical voltages and currents might vary with time in a more general way, we restrict to these cases as the rational functions are, crucially, well-behaved enough to form a field, and yet still general enough to provide arbitrarily close approximations to currents and voltages found in standard applications.

An **inductor** is a two-terminal circuit component across which the voltage is proportional to the rate of change of the current. By convention we draw this as follows, with the inductance L the constant of proportionality:³

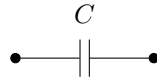


³We follow the standard convention of denoting inductance by the letter L , after the work of Heinrich Lenz and to avoid confusion with the I used for current.

Writing $v_L(t)$ and $i_L(t)$ for the voltage and current over time t across this component respectively, and using a dot to denote the derivative with respect to time t , we thus have the relationship

$$v_L(t) = L \dot{i}_L(t).$$

Permuting the roles of current and voltage, a **capacitor** is a two-terminal circuit component across which the current is proportional to the rate of change of the voltage. We draw this as follows, with the capacitance C the constant of proportionality:



Writing $v_C(t)$, $i_C(t)$ for the voltage and current across the capacitor, this gives the equation

$$i_C(t) = C \dot{v}_C(t).$$

We assume here that inductances L and capacitances C are positive real numbers.

Although inductors and capacitors impose a linear relationship if we involve the derivatives of current and voltage, to mimic the above work on resistors we wish to have a constant of proportionality between functions representing the current and voltage themselves. Various integral transforms perform just this role; electrical engineers typically use the Laplace transform. This lets us write a function of time t instead as a function of frequencies s , and in doing so turns differentiation with respect to t into multiplication by s , and integration with respect to t into division by s .

In detail, given a function $f(t): [0, \infty) \rightarrow \mathbb{R}$, we define the **Laplace transform** of f

$$\mathfrak{L}\{f\}(s) = \int_0^{\infty} f(t)e^{-st} dt.$$

We also use the notation $\mathfrak{L}\{f\}(s) = F(s)$, denoting the Laplace transform of a function in upper case, and refer to the Laplace transforms as lying in the **frequency domain** or **s-domain**. For us, the three crucial properties of the Laplace transform are then:

- (i) linearity: $\mathfrak{L}\{af + bg\}(s) = aF(s) + bG(s)$ for $a, b \in \mathbb{R}$;
- (ii) differentiation: $\mathfrak{L}\{\dot{f}\}(s) = sF(s) - f(0)$;
- (iii) integration: if $g(t) = \int_0^t f(\tau)d\tau$ then $G(s) = \frac{1}{s}F(s)$.

Writing $V(s)$ and $I(s)$ for the Laplace transform of the voltage $v(t)$ and current $i(t)$ across a component respectively, and recalling that by assumption $v(t) = i(t) = 0$ for $t \leq 0$, the s -domain behaviors of components become, for a resistor of resistance R :

$$V(s) = RI(s),$$

for an inductor of inductance L :

$$V(s) = sLI(s),$$

and for a capacitor of capacitance C :

$$V(s) = \frac{1}{sC}I(s).$$

Note that for each component the voltage equals the current times a rational function of the real variable s , called the **impedance** and in general denoted by Z . Note also that the impedance is a **positive real function**, meaning that it lies in the set

$$\mathbb{R}(s)^+ = \{Z \in \mathbb{R}(s) : \forall s \in \mathbb{C} \text{ Re}(s) > 0 \implies \text{Re}(Z(s)) > 0\}.$$

While Z is a quotient of polynomials with real coefficients, in this definition we are applying it to complex values of s , and demanding that its real part be positive in the open left half-plane. Positive real functions were introduced by Otto Brune in 1931, and they play a basic role in circuit theory [?].

Indeed, Brune convincingly argued that for any conceivable passive linear component we have this generalization of Ohm's law:

$$V(s) = Z(s)I(s)$$

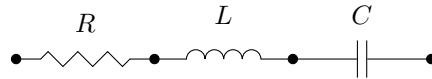
where $I \in \mathbb{R}(s)$ is the **current**, $V \in \mathbb{R}(s)$ is the **voltage** and $Z \in \mathbb{R}(s)^+$ is the **impedance** of the component. As we shall see, generalizing from circuits of linear resistors to arbitrary passive linear circuits is just a matter of formally replacing resistances by impedances. This amounts to replacing the field \mathbb{R} by the larger field $\mathbb{R}(s)$, and replacing the set of positive reals, $\mathbb{R}^+ = (0, \infty)$, by the set of positive real functions, $\mathbb{R}(s)^+$. From a mathematical perspective we might as well work with any field with a mildly well-behaved notion of 'positive element', and we do this in the next section.

6.3.2 The mechanical analogy

Now that we have introduced inductors and capacitors, it is worth taking another glance at the analogy chart in Section 6.1. What are the analogues of resistance, inductance and capacitance in mechanics? If we restrict attention to systems with translational degrees of freedom, the answer is given in the following chart.

Electronics	Mechanics (translation)
charge Q	position q
current $i = \dot{Q}$	velocity $v = \dot{q}$
flux linkage λ	momentum p
voltage $v = \dot{\lambda}$	force $F = \dot{p}$
resistance R	damping coefficient c
inductance L	mass m
inverse capacitance C^{-1}	spring constant k

A famous example concerns an electric circuit with a resistor of resistance R , an inductor of inductance L , and a capacitor of capacitance C , all in series:



We saw in Example 6.12 that for resistors in series, the resistances add. The same fact holds more generally for passive linear circuits, so the impedance of this circuit is the sum

$$Z = sL + R + (sC)^{-1}.$$

Thus, the voltage across this circuit is related to the current through the circuit by

$$V(s) = (sL + R + (sC)^{-1})I(s)$$

If $v(t)$ and $i(t)$ are the voltage and current as functions of time, we conclude that

$$v(t) = L \frac{d}{dt} i(t) + Ri(t) + C^{-1} \int_0^t i(s) ds$$

It follows that

$$L\ddot{Q} + R\dot{Q} + C^{-1}Q = v$$

where $Q(t) = \int_0^t i(s) ds$ has units of charge. As the chart above suggests, this equation is analogous to that of a damped harmonic oscillator:

$$m\ddot{q} + c\dot{q} + kq = F$$

where m is the mass of the oscillator, c is the damping coefficient, k is the spring constant and F is a time-dependent external force.

For details, and many more analogies of this sort, see the book by Karnopp, Margolis and Rosenberg [?] or Brown's enormous text [?]. While it would be a distraction to discuss them further here, these analogies mean that our work applies to a wide class of networked systems, not just electrical circuits.

6.3.3 Generalized Dirichlet forms

To understand the behavior of passive linear circuits we need to understand how the behaviors of individual components, governed by Ohm's law, fit together give the behavior of an entire network. Kirchhoff's laws still hold, and so does a version of the principle of minimum power. To see this, we generalize our remarks so far to any field with a set of 'positive elements'.

Definition 6.18. Given a field \mathbb{F} , we define a **set of positive elements** for \mathbb{F} to be any subset $\mathbb{F}^+ \subset \mathbb{F}$ containing 1 but not 0, and closed under addition, multiplication and division.

Our first motivating example arises from circuits made of resistors. Here $\mathbb{F} = \mathbb{R}$ is the field of real numbers and we take $\mathbb{F}^+ = (0, \infty)$. Our second motivating example arises from general passive linear circuits. Here $\mathbb{F} = \mathbb{R}(s)$ is the field of rational functions in one real variable, and we take $\mathbb{F}^+ = \mathbb{R}(s)^+$ to be the positive real functions, as defined in the last section.

In all that follows, we fix a field \mathbb{F} equipped with a set of positive elements \mathbb{F}^+ . By a 'circuit', we shall henceforth mean a circuit over \mathbb{F}^+ , as explained in Definition 6.3. To fix the notation:

Definition 6.19. A **(passive linear) circuit** is a graph $s, t: E \rightarrow N$ with E as its set of **edges** and N as its set of **nodes**, equipped with function $Z: E \rightarrow \mathbb{F}^+$ assigning each edge an **impedance**, together with finite sets X, Y , and functions $i: X \rightarrow N$ and $o: Y \rightarrow N$. We call the sets $i(X)$, $o(Y)$, and $\partial N = i(X) \cup o(Y)$ the **inputs**, **outputs**, and **terminals** or **boundary** of the circuit, respectively.

Generalizing from circuits of resistors, we define the **extended power functional** $P: \mathbb{F}^N \rightarrow \mathbb{F}$ of any circuit by

$$P(\varphi) = \frac{1}{2} \sum_{e \in E} \frac{1}{Z(e)} (\varphi(t(e)) - \varphi(s(e)))^2.$$

and we call $\varphi \in \mathbb{F}^N$ a **potential**. Note that a field of characteristic 2 cannot be given a set of positive elements, so dividing by 2 is allowed. Note also that the extended power functional is a Dirichlet form on N .

Although it is not clear what it means to minimize over the field \mathbb{F} , we can use formal derivatives to formulate an analogue of the principle of minimum power. This will actually be a ‘variational principle’, saying the derivative of the power functional vanishes with respect to certain variations in the potential. As before we shall see that given Ohm’s law, this principle is equivalent to Kirchhoff’s current law.

Indeed, the extended power functional $P(\varphi)$ can be considered an element of the polynomial ring $\mathbb{F}[\{\varphi(n)\}_{n \in N}]$ generated by formal variables $\varphi(n)$ corresponding to potentials at the nodes $n \in N$. We may thus take formal derivatives of the extended power functional with respect to the $\varphi(n)$. We then call $\phi \in \mathbb{F}^N$ a **realizable potential** for the given circuit if for each nonterminal node, $n \in N \setminus \partial N$, the formal partial derivative of the extended power functional with respect to $\varphi(n)$ equals zero when evaluated at ϕ :

$$\left. \frac{\partial P}{\partial \varphi(n)} \right|_{\varphi=\phi} = 0$$

This terminology arises from the following fact, a generalization of Proposition 6.6:

Theorem 6.20. *The potential $\phi \in \mathbb{F}^N$ is a realizable potential for a given circuit if and only if the induced current*

$$I(e) = \frac{1}{Z(e)}(\phi(t(e)) - \phi(s(e)))$$

obeys Kirchhoff’s current law:

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e)$$

for all $n \in N \setminus \partial N$.

Proof. The proof of this statement is exactly that for Proposition 6.6. □

A corollary of Theorem 6.20 is that the set of states—that is, potential–current pairs—that are compatible with the governing laws of a circuit is given by the set of realizable potentials together with their induced currents.

We now generalize the theory of Dirichlet forms:

Definition 6.21. Given a field \mathbb{F} with a set of positive elements \mathbb{F}^+ and a finite set S , a **Dirichlet form over \mathbb{F}** on S is a quadratic form $Q: \mathbb{F}^S \rightarrow \mathbb{F}$ given by the formula

$$Q(\psi) = \sum_{i,j \in S} c_{ij}(\psi_i - \psi_j)^2$$

for some choice of $c_{ij} \in \mathbb{F}^+ \cup \{0\}$.

Note that the extended power functional of any circuit is a Dirichlet form, since $Z(e) \in \mathbb{F}^+$ implies $\frac{1}{2} \frac{1}{Z(e)} \in \mathbb{F}^+$.

6.3.4 A generalized minimizability result

We begin to move from a discussion of the intrinsic behaviors of circuits to a discussion of their behaviors under composition. The key fact for composition of generalized Dirichlet forms is that, in analogy with Proposition 6.15, we may speak of a formal version of minimization of Dirichlet forms. We detail this here. In what follows, fix a field \mathbb{F} with a set of positive elements \mathbb{F}^+ , and let P be a Dirichlet form over \mathbb{F} on some finite set S .

Recall that given $R \subseteq S$, we call $\tilde{\psi} \in \mathbb{F}^S$ an **extension** of $\psi \in \mathbb{F}^R$ if $\tilde{\psi}$ restricted to R equals ψ . We call such an extension **realizable** if

$$\left. \frac{\partial P}{\partial \varphi(s)} \right|_{\varphi=\tilde{\psi}} = 0$$

for all $s \in S \setminus R$. Note that over the real numbers \mathbb{R} this means that among all the extensions of ψ , $\tilde{\psi}$ minimizes the function P .

Theorem 6.22. *Let P be a Dirichlet form over \mathbb{F} on S , and let $R \subseteq S$ be an inclusion of finite sets. Then we may uniquely define a Dirichlet form*

$$\min_{S \setminus R} P : \mathbb{F}^R \rightarrow \mathbb{F}$$

on R by sending each $\psi \in \mathbb{F}^R$ to the value $P(\tilde{\psi})$ of any realizable extension $\tilde{\psi}$ of ψ .

To prove this theorem, we must first show that $\min_{S \setminus R} P$ is well-defined as a function.

Lemma 6.23. *Let P be a Dirichlet form over \mathbb{F} on S , let $R \subseteq S$ be an inclusion of finite sets, and let $\psi \in \mathbb{F}^R$. Then for all realizable extensions $\tilde{\psi}, \tilde{\psi}' \in \mathbb{F}^S$ of ψ we have $P(\tilde{\psi}) = P(\tilde{\psi}')$.*

Proof. This follows from the formal version of the multivariable Taylor theorem for polynomial rings over a field of characteristic zero. Let $\tilde{\psi}, \tilde{\psi}' \in \mathbb{F}^S$ be realizable extensions of ψ , and note that $dP_{\tilde{\psi}}(\tilde{\psi} - \tilde{\psi}') = 0$, since for all $s \in R$ we have $\tilde{\psi}(s) - \tilde{\psi}'(s) = 0$, and for all $s \in S \setminus R$ we have

$$\left. \frac{\partial P}{\partial \varphi(s)} \right|_{\varphi=\tilde{\psi}} = 0.$$

We may take the Taylor expansion of P around $\tilde{\psi}$ and evaluate at $\tilde{\psi}'$. As P is a quadratic form, this gives

$$\begin{aligned} P(\tilde{\psi}') &= P(\tilde{\psi}) + dP_{\tilde{\psi}}(\tilde{\psi}' - \tilde{\psi}) + P(\tilde{\psi}' - \tilde{\psi}) \\ &= P(\tilde{\psi}) + P(\tilde{\psi}' - \tilde{\psi}). \end{aligned}$$

Similarly, we arrive at

$$P(\tilde{\psi}) = P(\tilde{\psi}') + P(\tilde{\psi} - \tilde{\psi}').$$

But again as P is a quadratic form, we then see that

$$P(\tilde{\psi}') - P(\tilde{\psi}) = P(\tilde{\psi}' - \tilde{\psi}) = P(\tilde{\psi} - \tilde{\psi}') = P(\tilde{\psi}) - P(\tilde{\psi}').$$

This implies that $P(\tilde{\psi}') - P(\tilde{\psi}) = 0$, as required. \square

It remains to show that $\min P$ remains a Dirichlet form. We do this inductively.

Lemma 6.24. *Let P be a Dirichlet form over \mathbb{F} on S , and let $s \in S$ be an element of S . Then the map $\min_{\{s\}} P : \mathbb{F}^{S \setminus \{s\}} \rightarrow \mathbb{F}$ sending ψ to $P(\tilde{\psi})$ is a Dirichlet form on $S \setminus \{s\}$.*

Proof. Write $P(\phi) = \sum_{i,j} c_{ij}(\phi_i - \phi_j)^2$, assuming without loss of generality that $c_{sk} = 0$ for all k . We then have

$$\left. \frac{\partial P}{\partial \varphi(s)} \right|_{\varphi=\phi} = \sum_k 2c_{ks}(\phi_s - \phi_k),$$

and this is equal to zero when

$$\phi_s = \frac{\sum_k c_{ks}\phi_k}{\sum_k c_{ks}}.$$

Thus $\min_{\{s\}} P$ may be given explicitly by the expression

$$\min_{\{s\}} P(\psi) = \sum_{i,j \in S \setminus \{s\}} c_{ij}(\psi_i - \psi_j)^2 + \sum_{\ell \in S \setminus \{s\}} c_{\ell s} \left(\psi_\ell - \frac{\sum_k c_{ks}\psi_k}{\sum_k c_{ks}} \right)^2.$$

We must show this is a Dirichlet form on $S \setminus \{s\}$.

As the sum of Dirichlet forms is evidently Dirichlet, it suffices to check that the expression

$$\sum_{\ell} c_{\ell s} \left(\psi_{\ell} - \frac{\sum_k c_{ks} \psi_k}{\sum_k c_{ks}} \right)^2$$

is Dirichlet on $S \setminus \{s\}$. Multiplying through by the constant $(\sum_k c_{ks})^2 \in \mathbb{F}^+$, it further suffices to check

$$\begin{aligned} \sum_{\ell} c_{\ell s} \left(\sum_k c_{ks} \psi_{\ell} - \sum_k c_{ks} \psi_k \right)^2 &= \sum_{\ell} c_{\ell s} \left(\sum_k c_{ks} (\psi_{\ell} - \psi_k) \right)^2 \\ &= \sum_{\ell} c_{\ell s} \left(2 \sum_{\substack{k,m \\ k \neq m}} c_{ks} c_{ms} (\psi_{\ell} - \psi_k)(\psi_{\ell} - \psi_m) + \sum_k c_{ks}^2 (\psi_{\ell} - \psi_k)^2 \right) \\ &= 2 \sum_{\substack{k,\ell,m \\ k \neq m}} c_{\ell s} c_{ks} c_{ms} (\psi_{\ell} - \psi_k)(\psi_{\ell} - \psi_m) + \sum_{k,\ell} c_{\ell s} c_{ks}^2 (\psi_{\ell} - \psi_k)^2 \end{aligned}$$

is Dirichlet. But

$$\begin{aligned} &(\psi_k - \psi_{\ell})(\psi_k - \psi_m) + (\psi_{\ell} - \psi_k)(\psi_{\ell} - \psi_m) + (\psi_m - \psi_k)(\psi_m - \psi_{\ell}) \\ &= \psi_k^2 + \psi_{\ell}^2 + \psi_m^2 - \psi_k \psi_{\ell} - \psi_k \psi_m - \psi_{\ell} \psi_m \\ &= \frac{1}{2} ((\psi_k - \psi_{\ell})^2 + (\psi_k - \psi_m)^2 + (\psi_{\ell} - \psi_m)^2), \end{aligned}$$

so this expression is indeed Dirichlet. Indeed, pasting these computations together shows that

$$\min_{\{s\}} P(\psi) = \sum_{i,j} \left(c_{ij} + \frac{c_{is} c_{js}}{\sum_k c_{ks}} \right) (\psi_i - \psi_j)^2. \quad \square$$

With these two lemmas, the proof of Theorem 6.22 becomes straightforward.

Proof of Theorem 6.22. Lemma 6.23 shows that $\min_{S \setminus R} P$ is a well-defined function. As R is a finite set, we may write it $R = \{s_1, \dots, s_n\}$ for some natural number n . Then we may define a sequence of functions $P_i = \min_{\{s_1, \dots, s_i\}} P_{i-1}$, $1 \leq i \leq n$. Define also $P_0 = P$, and note that $P_n = \min_{S \setminus R} P$. Then, by Lemma 6.24, each P_i is Dirichlet as P_{i-1} is. This proves the proposition. \square

We can thus define the power functional of a circuit by analogy with circuits made of resistors:

Definition 6.25. The **power functional** $Q: \mathbb{R}^{\partial N} \rightarrow \mathbb{R}$ of a circuit with extended power functional P is given by

$$Q = \min_{N \setminus \partial N} P.$$

As before, we define two circuits to be **equivalent** if they have the same power functional, and define the **behavior** of a circuit to be its equivalence class.

6.3.5 Composition of Dirichlet forms

It would be nice to have a category in which circuits are morphisms, and a category in which Dirichlet forms are morphisms, such that the map sending a circuit to its behavior is a functor. Here we present a naïve attempt to construct the category with Dirichlet forms as morphisms, using the principle of minimum power to compose these morphisms. Unfortunately the proposed category does not include identity morphisms. However, it points in the right direction, and underlines the importance of the cospan formalism we then turn to develop.

We can define a composition rule for Dirichlet forms that reflects composition of circuits. Given finite sets S and T , let $S + T$ denote their disjoint union. Let $D(S, T)$ be the set of Dirichlet forms on $S + T$. There is a way to compose these Dirichlet forms

$$\circ: D(T, U) \times D(S, T) \rightarrow D(S, U)$$

defined as follows. Given $P \in D(T, U)$ and $Q \in D(S, T)$, let

$$(P \circ Q)(\alpha, \gamma) = \min_T Q(\alpha, \beta) + P(\beta, \gamma),$$

where $\alpha \in F^S, \gamma \in F^U$. This operation has a clear interpretation in terms of electrical circuits: the power used by the entire circuit is just the sum of the power used by its parts.

It is immediate from Theorem 6.22 that this composition rule is well-defined: the composite of two Dirichlet forms is again a Dirichlet form. Moreover, this composition is associative. However, it fails to provide the structure of a category, as there is typically no Dirichlet form $1_S \in D(S, S)$ playing the role of the identity for this composition. For an indication of why this is so, let $\{\bullet\}$ be a set with one element, and suppose that some Dirichlet form $I(\beta, \gamma) = k(\beta - \gamma)^2 \in D(\{\bullet\}, \{\bullet\})$ acts as an identity on the right for this composition. Then for all $Q(\alpha, \beta) = c(\alpha - \beta)^2 \in D(\{\bullet\}, \{\bullet\})$,

we must have

$$\begin{aligned}
 c\alpha^2 &= Q(\alpha, 0) \\
 &= (I \circ Q)(\alpha, 0) \\
 &= \min_{\beta \in \mathbb{F}} Q(\alpha, \beta) + I(\beta, 0) \\
 &= \min_{\beta \in \mathbb{F}} k(\alpha - \beta)^2 + c\beta^2 \\
 &= \frac{kc}{k+c} \alpha^2,
 \end{aligned}$$

where we have noted that $\frac{kc}{k+c}\alpha^2$ minimizes $k(\alpha - \beta)^2 + c\beta^2$ with respect to β . But for any choice of $k \in \mathbb{F}$ this equality only holds when $c = 0$, so no such Dirichlet form exists. Note, however, that for $k \gg c$ we have $c\alpha^2 \approx \frac{kc}{k+c}\alpha^2$, so Dirichlet forms with large values of k —corresponding to resistors with resistance close to zero—act as ‘approximate identities’.

In this way we might interpret the identities we wish to introduce into this category as the behaviors of idealized components with zero resistance: perfectly conductive wires. Unfortunately, the power functional of a purely conductive wire is undefined: the formula for it involves division by zero. In real life, coming close to this situation leads to the disaster that electricians call a ‘short circuit’: a huge amount of power dissipated for even a small voltage. This is why we have fuses and circuit breakers.

Nonetheless, we have most of the structure required for a category. A ‘category without identity morphisms’ is called a **semicategory**, so we see

Proposition 6.26. *There is a semicategory where:*

- *the objects are finite sets,*
- *a morphism from T to S is a Dirichlet form $Q \in D(S, T)$.*
- *composition of morphisms is given by*

$$(R \circ Q)(\gamma, \alpha) = \min_T Q(\gamma, \beta) + R(\beta, \alpha).$$

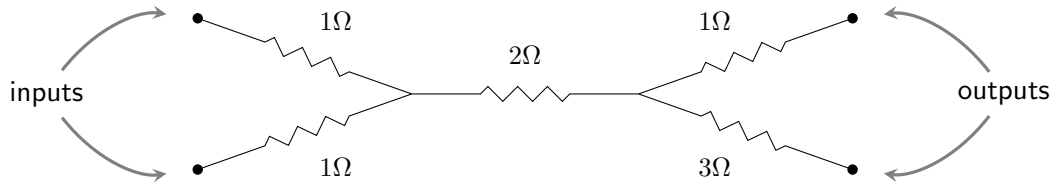
We would like to make this into a category. One easy way to do this is to formally adjoin identity morphisms; this trick works for any semicategory. However, we obtain a better category if we include *more* morphisms: more behaviors corresponding to circuits made of perfectly conductive wires. As the expression for the extended power functional includes the reciprocals of impedances, such circuits cannot be expressed within the framework we have developed thus far. Indeed, for these idealized circuits

there is no function taking boundary potentials to boundary currents: the vanishing impedance would imply that any difference in potentials at the boundary induces ‘infinite’ currents. To deal with this issue, we generalize Dirichlet forms to Lagrangian relations. First, however, we develop a category theoretic framework, based around decorated cospans, to define the category of circuits itself and understand its basic properties.

6.4 The category of passive linear circuits

In this part we move our focus from the semantics of circuit diagrams to the syntax, addressing the question “How do we interact with circuit diagrams?”. Informally, the answer to this is that we interact with them by connecting them to each other, perhaps after moving them into the right form by rotating or reflecting them, or by crossing or bending some of the wires. To formalize this, we adopt a category theoretic viewpoint, defining various dagger compact categories with circuits and their behaviors as morphisms. We claim a formal analysis of this structure, especially of the composition or connection of circuits, has been overlooked in analysis of circuits thus far. This part culminates in the definition of two important categories, the category Circ of circuit diagrams, and the category LagrRel containing all behaviors of circuits. We also develop the technical material required to appreciate the structure of these categories, and that aids understanding of the relationship between the two, to be addressed in Part .

In Part we defined a circuit of linear resistors to be a labelled graph with marked input and output terminals, as in the example:



We then defined general passive linear circuits by replacing resistances with impedances chosen from a set of positive elements \mathbb{F}^+ in any field \mathbb{F} . In fact, these circuits are examples of decorated cospans. This gives a dagger compact category Circ whose morphisms are circuits, with the dagger compact structure expressing standard operations on circuits.

We actually give two constructions of this category, first arriving at a category of cospans decorated by \mathbb{F}^+ -graphs, and then showing that this is a full subcategory of the decategorification of a bicategory of cospans of \mathbb{F}^+ -graphs.

6.4.1 A decorated cospan construction

We officially defined a ‘circuit’ in Definition 6.19, but now we can give an equivalent definition using cospans:

Lemma 6.27. *A circuit is a cospan of finite sets $X \xrightarrow{i} N \xleftarrow{o} Y$ together with an \mathbb{F}^+ -graph whose set of nodes is N .*

Proof. This is just a matter of remembering the terminology: recall that an \mathbb{F}^+ -graph is a graph whose edges are labelled with elements of \mathbb{F}^+ , our chosen set of positive elements in the field \mathbb{F} . \square

This suggests that circuits should be morphisms in a decorated cospan category. Indeed, we can show that the map taking a finite set N to the set of \mathbb{F}^+ -graphs with set N of nodes in fact forms a lax symmetric monoidal functor. This allows us to apply decorated cospans to construct a category of circuits.

To this end, define the functor

$$\text{Circuit}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

to take a finite set N , as an object of FinSet , to the set $\text{Circuit}(N)$ of \mathbb{F}^+ -graphs (N, E, s, t, r) with N as their set of nodes. On morphisms let it take a function $f: N \rightarrow M$ to the function that pushes labelled graph structures on a set N forward onto the set M :

$$\begin{aligned} \text{Circuit}(f): \text{Circuit}(N) &\longrightarrow \text{Circuit}(M); \\ (N, E, s, t, r) &\longmapsto (M, E, f \circ s, f \circ t, r). \end{aligned}$$

Note that as this map simply acts by post-composition, our map Circuit is indeed functorial.

We then arrive at a lax symmetric monoidal functor by equipping this functor with the natural transformation

$$\begin{aligned} \rho_{N,M}: \text{Circuit}(N) \times \text{Circuit}(M) &\longrightarrow \text{Circuit}(N + M); \\ ((N, E, s, t, r), (M, F, s', t', r')) &\longmapsto (N + M, E + F, s + s', t + t', [r, r']), \end{aligned}$$

together with the unit map

$$\begin{aligned} \rho_1: 1 &\longrightarrow \text{Circuit}(\emptyset); \\ \bullet &\longmapsto (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset), \end{aligned}$$

where we use \emptyset to denote both the empty set and the unique function of the appropriate codomain with domain the empty set. The naturality of this collection of morphisms, as well as the coherence laws for lax symmetric monoidal functors, follow from the universal property of the coproduct.

Definition 6.28. We define

$$\mathbf{Circ} = \mathbf{CircuitCospan}.$$

Corollary 6.29. *The category \mathbf{Circ} is a dagger compact category.*

The different structures of this category capture different operations that can be performed with circuits. The composition expresses the fact that we can connect the outputs of one circuit to the inputs of the next, while the monoidal composition models the placement of circuits side-by-side. The symmetric monoidal structure allows us reorder input and output wires, and the compactness captures the interchangeability between input and outputs of circuits—that is, the fact that we can choose any input to our circuit and consider it instead as an output, and vice versa. Finally, the dagger structure expresses the fact that we may reflect a whole circuit, switching all inputs with all outputs.

6.4.2 A bicategory of circuits

As shown by Bénabou [?], cospans are most naturally thought of as 1-morphisms in a bicategory. To obtain the category we are calling $\mathbf{Cospan}(\mathcal{C})$, we ‘deategorify’ this bicategory by discarding 2-morphisms and identifying isomorphic 1-morphisms. Since we are studying circuits using decorated cospans, this suggests that circuits, too, are most naturally thought of as 1-morphisms in a bicategory. Indeed this is the case.

One route to the bicategory whose 1-morphisms are circuits would be to take the theory of decorated cospans [?] and show that it can be enhanced to create not merely categories whose morphisms are isomorphism classes of decorated cospans, but bicategories whose 1-morphisms are exactly decorated cospans. A less powerful but easier approach is as follows.

Recall that we define a graph to be a pair of functions $s, t: E \rightarrow N$ where E and N are finite sets, and an L -graph to be a graph further equipped with a function $r: E \rightarrow L$. Thus, an L -graph looks like this:

$$L \xleftarrow{r} E \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} N$$

Given L -graphs $\Gamma = (E, N, s, t, r)$ and $\Gamma' = (E', N', s', t', r')$, a morphism of L -graphs $\Gamma \rightarrow \Gamma'$ is a pair of functions $\epsilon: E \rightarrow E'$, $v: N \rightarrow N'$ such that the following diagrams commute:

$$\begin{array}{ccc}
 & E & \\
 r \swarrow & & \downarrow \epsilon \\
 L & & E' \\
 & r' \searrow & \\
 & E' &
 \end{array}
 \qquad
 \begin{array}{ccc}
 E & \xrightarrow{s} & N \\
 \epsilon \downarrow & & \downarrow v \\
 E' & \xrightarrow{s'} & N'
 \end{array}
 \qquad
 \begin{array}{ccc}
 E & \xrightarrow{t} & N \\
 \epsilon \downarrow & & \downarrow v \\
 E' & \xrightarrow{t'} & N'
 \end{array}$$

These L -graphs and their morphisms form a category **L -Graph**. Using results about colimits in the category of sets, it is straightforward to check that this category has finite colimits.

Bénabou [?] gave a general construction of bicategories starting from any category with pushouts. Since L -Graph has pushouts, it follows that there is a bicategory **Cospan(L -Graph)** with

- L -labelled graphs as objects,
- cospans in L -Graphs as morphisms, and
- maps of cospans as 2-morphisms.

Since L -Graph also has coproducts, this bicategory is symmetric monoidal and in fact compact, thanks to the work of Stay [?]. We make the following definition:

Definition 6.30. The bicategory **2-Circ** is the full and 2-full sub-bicategory of **Cospan(\mathbb{F}^+ -Graph)** with objects those \mathbb{F}^+ -graphs with no edges.

It can be shown that every object in **Cospan(\mathbb{F}^+ -Graph)** is self-dual, and this implies that **2-Circ** is again a compact closed bicategory. Note that the category obtained by decategorifying this bicategory has finite sets as objects, with morphisms being isomorphism classes of cospans in \mathbb{F}^+ -Graph with feet such objects. Thus, decategorifying **2-Circ** gives a category equivalent to our previously defined category **Circ**.

6.5 Circuits as Lagrangian relations

In the first part of this paper, we explored the semantic content contained in circuit diagrams, leading to an understanding of circuit diagrams as expressing some relationship between the potentials and currents that can simultaneously be imposed on some subset, the so-called terminals, of the nodes of the circuit. We called this collection of possible relationships the behavior of the circuit. While in that setting

we used the concept of Dirichlet forms to describe this relationship, we saw in the end that describing circuits as Dirichlet forms does not allow for a straightforward notion of composition of circuits.

In this section, inspired by the principle of least action of classical mechanics in analogy with the principle of minimum power, we develop a setting for describing behaviors that allows for easy discussion of composite behaviors: Lagrangian subspaces of symplectic vector spaces. These Lagrangian subspaces provide a more direct, invariant perspective, comprising precisely the set of vectors describing the possible simultaneous potential and current readings at all terminals of a given circuit. As we shall see, one immediate and important advantage of this setting is that we may model wires of zero resistance.

Recall that we write \mathbb{F} for some field, which for our applications is usually the field \mathbb{R} of real numbers or the field $\mathbb{R}(s)$ of rational functions of one real variable.

6.5.1 Symplectic vector spaces

A circuit made up of wires of positive resistance defines a function from boundary potentials to boundary currents. A wire of zero resistance, however, does not define a function: the principle of minimum power is obeyed as long as the potentials at the two ends of the wire are equal. More generally, we may thus think of circuits as specifying a set of allowed voltage-current pairs, or as a relation between boundary potentials and boundary currents. This set forms what is called a Lagrangian subspace, and is given by the graph of the differential of the power functional. More generally, Lagrangian submanifolds graph derivatives of smooth functions: they describe the point evaluated and the tangent to that point within the same space.

The material in this section is all known, and follows without great difficulty from the definitions. To keep this section brief we omit proofs. See any introduction to symplectic vector spaces, such as Cimasoni and Turaev [?] or Piccione and Tausk [?], for details.

Definition 6.31. Given a finite-dimensional vector space V over a field \mathbb{F} , a **symplectic form** $\omega: V \times V \rightarrow \mathbb{F}$ on V is an alternating nondegenerate bilinear form. That is, a symplectic form ω is a function $V \times V \rightarrow \mathbb{F}$ that is

- (i) bilinear: for all $\lambda \in \mathbb{F}$ and all $u, v \in V$ we have $\omega(\lambda u, v) = \omega(u, \lambda v) = \lambda \omega(u, v)$;
- (ii) alternating: for all $v \in V$ we have $\omega(v, v) = 0$; and
- (iii) nondegenerate: given $v \in V$, $\omega(u, v) = 0$ for all $u \in V$ if and only if $v = 0$.

A **symplectic vector space** (V, ω) is a vector space V equipped with a symplectic form ω .

Given symplectic vector spaces $(V_1, \omega_1), (V_2, \omega_2)$, a **symplectic map** is a linear map

$$f: (V_1, \omega_1) \longrightarrow (V_2, \omega_2)$$

such that $\omega_2(f(u), f(v)) = \omega_1(u, v)$ for all $u, v \in V_1$. A **symplectomorphism** is a symplectic map that is also an isomorphism.

An alternating form is always **antisymmetric**, meaning that $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$. The converse is true except in characteristic 2. A **symplectic basis** for a symplectic vector space (V, ω) is a basis $\{p_1, \dots, p_n, q_1, \dots, q_n\}$ such that $\omega(p_i, p_j) = \omega(q_i, q_j) = 0$ for all $1 \leq i, j \leq n$, and $\omega(p_i, q_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$, where δ_{ij} is the Kronecker delta, equal to 1 when $i = j$, and 0 otherwise. A symplectomorphism maps symplectic bases to symplectic bases, and conversely, any map that takes a symplectic basis to another symplectic basis is a symplectomorphism.

Example 6.32 (The symplectic vector space generated by a finite set). Given a finite set N , we consider the vector space $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ a symplectic vector space $(\mathbb{F}^N \oplus (\mathbb{F}^N)^*, \omega)$, with symplectic form

$$\omega((\phi, i), (\phi', i')) = i'(\phi) - i(\phi').$$

Let $\{\phi_n\}_{n \in N}$ be the basis of \mathbb{F}^N consisting of the functions $N \rightarrow \mathbb{F}$ mapping n to 1 and all other elements of n to 0, and let $\{i_n\}_{n \in N} \subseteq (\mathbb{F}^N)^*$ be the dual basis. Then $\{(\phi_n, 0), (0, i_n)\}_{n \in N}$ forms a symplectic basis for $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$.

There are two common ways we will build symplectic spaces from other symplectic spaces: conjugation and summation. Given a symplectic form ω , we may define its **conjugate** symplectic form $\bar{\omega} = -\omega$, and write the conjugate symplectic space $(V, \bar{\omega})$ as \bar{V} . Given two symplectic vector spaces $(U, \nu), (V, \omega)$, we consider their direct sum $U \oplus V$ a symplectic vector space with the symplectic form $\nu + \omega$, and call this the **sum** of the two symplectic vector spaces. Note that this is not a product in the category of symplectic vector spaces and symplectic maps.

The symplectic form provides a notion of orthogonal complement. Given a subspace S of V , we define its **complement**

$$S^\circ = \{v \in V \mid \omega(v, s) = 0 \text{ for all } s \in S\}.$$

Note that this construction obeys the following identities, where S and T are subspaces of V :

$$\begin{aligned}\dim S + \dim S^\circ &= \dim V \\ (S^\circ)^\circ &= S \\ (S + T)^\circ &= S^\circ \cap T^\circ \\ (S \cap T)^\circ &= S^\circ + T^\circ.\end{aligned}$$

In the symplectic vector space $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$, the subspace \mathbb{F}^N has the property of being a maximal subspace such that the symplectic form restricts to the zero form on this subspace. Subspaces with this property are known as Lagrangian subspaces, and they may all be realized as the image of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ under symplectomorphisms from $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ to itself.

Definition 6.33. Let S be a linear subspace of a symplectic vector space (V, ω) . We say that S is **isotropic** if $\omega|_{S \times S} = 0$, and that S is **coisotropic** if S° is isotropic. A subspace is **Lagrangian** if it is both isotropic and coisotropic, or equivalently, if it is a maximal isotropic subspace.

Lagrangian subspaces are also known as Lagrangian correspondences and canonical relations. Note that a subspace S is isotropic if and only if $S \subseteq S^\circ$. This fact helps with the following characterizations of Lagrangian subspaces.

Proposition 6.34. *Given a subspace $L \subset V$ of a symplectic vector space (V, ω) , the following are equivalent:*

- (i) L is Lagrangian.
- (ii) L is maximally isotropic.
- (iii) L is minimally coisotropic.
- (iv) $L = L^\circ$.
- (v) L is isotropic and $\dim L = \frac{1}{2} \dim V$.

From this proposition it follows easily that the direct sum of two Lagrangian subspaces is Lagrangian in the sum of their ambient spaces. We also observe that an advantage of isotropy is that there is a good way to take a quotient of a symplectic vector space by an isotropic subspace—that is, there is a way to put a natural symplectic structure on the quotient space.

Proposition 6.35. *Let S be an isotropic subspace of a symplectic vector space (V, ω) . Then S°/S is a symplectic vector space with symplectic form $\omega'(v+S, u+S) = \omega(v, u)$.*

Proof. The function ω' is a well-defined due to the isotropy of S —by definition adding any pair (s, s') of elements of S to a pair (v, u) of elements of S° does not change the value of $\omega(v+s, u+s')$. As ω is a symplectic form, one can check that ω' is too. \square

6.5.2 Lagrangian subspaces from quadratic forms

Lagrangian subspaces are of relevance to us here as the behavior of any passive linear circuit forms a Lagrangian subspace of the symplectic vector space generated by the nodes of the circuit. We think of this vector space as comprising two parts: a space \mathbb{F}^N of potentials at each node, and a dual space $(\mathbb{F}^N)^*$ of currents. To make clear how circuits can be interpreted as Lagrangian subspaces, here we describe how Dirichlet forms on a finite set N give rise to Lagrangian subspaces of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. More generally, we show that there is a one-to-one correspondence between Lagrangian subspaces and quadratic forms.

Proposition 6.36. *Let N be a finite set. Given a quadratic form Q over \mathbb{F} on N , the subspace*

$$L_Q = \{(\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*,$$

where $dQ_\phi \in (\mathbb{F}^N)^$ is the formal differential of Q at $\phi \in \mathbb{F}^N$, is Lagrangian. Moreover, this construction gives a one-to-one correspondence*

$$\left\{ \text{Quadratic forms over } \mathbb{F} \text{ on } N \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Lagrangian subspaces of } \mathbb{F}^N \oplus (\mathbb{F}^N)^* \\ \text{with trivial intersection with } \{0\} \oplus (\mathbb{F}^N)^* \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^* \end{array} \right\}$$

Proof. The symplectic structure on $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ and our notation for it is given in Example 6.32.

Note that for all $n, m \in N$ the corresponding basis elements

$$\frac{\partial^2 Q}{\partial \phi_n \partial \phi_m} = dQ_{\phi_n}(\phi_m) = dQ_{\phi_m}(\phi_n),$$

so $dQ_\phi(\psi) = dQ_\psi(\phi)$ for all $\phi, \psi \in \mathbb{F}^N$. Thus L_Q is indeed Lagrangian: for all $\phi, \psi \in \mathbb{F}^N$

$$\omega((\phi, dQ_\phi), (\psi, dQ_\psi)) = dQ_\psi(\phi) - dQ_\phi(\psi) = 0.$$

Observe also that for all quadratic forms Q we have $dQ_0 = 0$, so the only element of L_Q of the form $(0, i)$, where $i \in (\mathbb{F}^N)^*$, is $(0, 0)$. Thus L_Q has trivial intersection

with the subspace $\{0\} \oplus (\mathbb{F}^N)^*$ of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. This L_Q construction forms the leftward direction of the above correspondence.

For the rightward direction, suppose that L is a Lagrangian subspace of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ such that $L \cap (\{0\} \oplus (\mathbb{F}^N)^*) = \{(0, 0)\}$. Then for each $\phi \in \mathbb{F}^N$, there exists a unique $i_\phi \in (\mathbb{F}^N)^*$ such that $(\phi, i_\phi) \in L$. Indeed, if i_ϕ and i'_ϕ were distinct elements of $(\mathbb{F}^N)^*$ with this property, then by linearity $(0, i_\phi - i'_\phi)$ would be a nonzero element of $L \cap (\{0\} \oplus (\mathbb{F}^N)^*)$, contradicting the hypothesis about trivial intersection. We thus can define a function, indeed a linear map, $\mathbb{F}^N \rightarrow (\mathbb{F}^N)^*$; $\phi \mapsto i_\phi$. This defines a bilinear form $Q(\phi, \psi) = i_\phi(\psi)$ on $\mathbb{F}^N \oplus \mathbb{F}^N$, and so $Q(\phi) = i_\phi(\phi)$ defines a quadratic form on \mathbb{F}^N .

Moreover, L is Lagrangian, so

$$\omega((\phi, i_\phi), (\psi, i_\psi)) = i_\psi(\phi) - i_\phi(\psi) = 0,$$

and so $Q(-, -)$ is a symmetric bilinear form. This gives a one-to-one correspondence between Lagrangian subspaces of specified type, symmetric bilinear forms, and quadratic forms, and so in particular gives the claimed one-to-one correspondence. \square

In particular, every Dirichlet form defines a Lagrangian subspace.

6.5.3 Lagrangian relations

Recall that a relation between sets X and Y is a subset R of their product $X \times Y$. Furthermore, given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, there is a composite relation $(S \circ R) \subseteq X \times Z$ given by pairs (x, z) such that there exists $y \in Y$ with $(x, y) \in R$ and $(y, z) \in S$ —a direct generalization of function composition. A Lagrangian relation between symplectic vector spaces V_1 and V_2 is a relation between V_1 and V_2 that forms a Lagrangian subspace of the symplectic vector space $\overline{V_1} \oplus V_2$. This gives us a way to think of certain Lagrangian subspaces, such as those arising from circuits, as morphisms, giving a way to compose them.

Recall discussion of names/conames in Chapter refONE.

Definition 6.37. A **Lagrangian relation** $L: V_1 \rightarrow V_2$ is a Lagrangian subspace L of $\overline{V_1} \oplus V_2$.

This is a generalization of the notion of symplectomorphism: any symplectomorphism $f: V_1 \rightarrow V_2$ forms a Lagrangian subspace when viewed as a relation $f \subseteq \overline{V_1} \oplus V_2$. More generally, any symplectic map $f: V_1 \rightarrow V_2$ forms an isotropic subspace when viewed as a relation in $\overline{V_1} \oplus V_2$.

Importantly for us, the composite of two Lagrangian relations is again a Lagrangian relation. This is well-known [?], but sufficiently easy and important to us that we provide a proof.

Proposition 6.38. *Let $L: V_1 \rightarrow V_2$ and $L': V_2 \rightarrow V_3$ be Lagrangian relations. Then their composite relation $L' \circ L$ is a Lagrangian relation $V_1 \rightarrow V_3$.*

We prove this proposition by way of two lemmas detailing how the Lagrangian property is preserved under various operations. The first lemma says that the intersection of a Lagrangian space with a coisotropic space is in some sense Lagrangian, once we account for the complement.

Lemma 6.39. *Let $L \subseteq V$ be a Lagrangian subspace of a symplectic vector space V , and $S \subseteq V$ be an isotropic subspace of V . Then $(L \cap S^\circ) + S \subseteq V$ is Lagrangian in V .*

Proof. Recall from Proposition 6.34 that a subspace is Lagrangian if and only if it is equal to its complement. The lemma is then immediate from the way taking the symplectic complement interacts with sums and intersections:

$$((L \cap S^\circ) + S)^\circ = (L \cap S^\circ)^\circ \cap S^\circ = (L^\circ + (S^\circ)^\circ) \cap S^\circ = (L + S) \cap S^\circ = (L \cap S^\circ) + (S \cap S^\circ) = (L \cap S^\circ) + S.$$

Since $(L \cap S^\circ) + S$ is equal to its complement, it is Lagrangian. \square

The second lemma says that if a subspace of a coisotropic space is Lagrangian, taking quotients by the complementary isotropic space does not affect this.

Lemma 6.40. *Let $L \subseteq V$ be a Lagrangian subspace of a symplectic vector space V , and $S \subseteq L$ an isotropic subspace of V contained in L . Then $L/S \subseteq S^\circ/S$ is Lagrangian in the quotient symplectic space S°/S .*

Proof. As L is isotropic and the symplectic form on S°/S is given by $\omega'(v+S, u+S) = \omega(v, u)$, the quotient L/S is immediately isotropic. Recall from Proposition 6.34 that an isotropic subspace S of a symplectic vector space V is Lagrangian if and only if $\dim S = \frac{1}{2} \dim V$. Also recall that for any subspace $\dim S + \dim S^\circ = \dim V$. Thus

$$\begin{aligned} \dim(L/S) &= \dim L - \dim S = \frac{1}{2} \dim V - \dim S \\ &= \frac{1}{2}(\dim S + \dim S^\circ) - \dim S = \frac{1}{2}(\dim S^\circ - \dim S) = \frac{1}{2} \dim(S^\circ/S). \end{aligned}$$

Thus L/S is Lagrangian in S°/S . \square

Combining these two lemmas gives a proof that the composite of two Lagrangian relations is again a Lagrangian relation.

Proof of Proposition 6.38. Let Δ be the diagonal subspace

$$\Delta = \{(0, v_2, v_2, 0) \mid v_2 \in V_2\} \subseteq \overline{V}_1 \oplus V_2 \oplus \overline{V}_2 \oplus V_3.$$

Observe that Δ is isotropic, and has coisotropic complement

$$\Delta^\circ = \{(v_1, v_2, v_2, v_3) \mid v_i \in V_i\} \subseteq \overline{V}_1 \oplus V_2 \oplus \overline{V}_2 \oplus V_3.$$

As Δ is the kernel of the restriction of the projection map $\overline{V}_1 \oplus V_2 \oplus \overline{V}_2 \oplus V_3 \rightarrow \overline{V}_1 \oplus V_3$ to Δ° , and after restriction this map is still surjective, the quotient space Δ°/Δ is isomorphic to $\overline{V}_1 \oplus V_3$.

Now, by definition of composition of relations,

$$L' \circ L = \{(v_1, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}.$$

But note also that

$$L \oplus L' = \{(v_1, v_2, v'_2, v_3) \mid (v_1, v_2) \in L, (v'_2, v_3) \in L'\},$$

so

$$(L \oplus L') \cap \Delta^\circ = \{(v_1, v_2, v_2, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}.$$

Quotienting by Δ then gives

$$L' \circ L = ((L \oplus L') \cap \Delta^\circ + \Delta) / \Delta.$$

As $L' \oplus L$ is Lagrangian in $\overline{V}_1 \oplus V_2 \oplus \overline{V}_2 \oplus V_3$, Lemma 6.39 says that $(L' \oplus L) \cap \Delta^\circ + \Delta$ is also Lagrangian in $\overline{V}_1 \oplus V_2 \oplus \overline{V}_2 \oplus V_3$. Lemma 6.40 thus shows that $L' \circ L$ is Lagrangian in $\Delta^\circ/\Delta = \overline{V}_1 \oplus V_3$, as required. \square

Note that this composition is associative. We shall see later that this composition agrees with composition of Dirichlet forms, and hence also composition of circuits.

6.5.4 The dagger compact category of Lagrangian relations

Lagrangian relations solve the identity problems we had with Dirichlet forms: given a symplectic vector space V , the Lagrangian relation $\text{id}: V \rightarrow V$ specified by the Lagrangian subspace

$$\text{id} = \{(v, v) \mid v \in V\} \subseteq \overline{V} \oplus V,$$

acts as an identity for composition of relations. We thus have a category.

Definition 6.41. We write **LagrRel** for the category with symplectic vector spaces as objects and Lagrangian relations as morphisms.

In fact the move to the setting of Lagrangian relations, rather than Dirichlet forms, adds far richer structure than just identity morphisms. The category **LagrRel** can be viewed as endowed with the structure of a dagger compact category. We lay this out in steps.

Symmetric monoidal structure

We define the tensor product of two objects of **LagrRel** to be their direct sum. Similarly, we define the tensor product of two morphisms $L: U \rightarrow V$, $L \subseteq \overline{U} \oplus V$ and $K: T \rightarrow W$, $K \subseteq \overline{T} \oplus W$ to be their direct sum

$$L \oplus K \subseteq \overline{U} \oplus V \oplus \overline{T} \oplus W,$$

but considered as a subspace of the naturally isomorphic space $\overline{U \oplus T} \oplus V \oplus W$. Despite this subtlety, we abuse our notation and write their tensor product $L \oplus K: U \oplus T \rightarrow V \oplus W$, and move on having sounded this note of caution.

Note that the direct sum of two Lagrangian subspaces is again Lagrangian in the direct sum of their ambient spaces, and the zero dimensional vector space $\{0\}$ acts as an identity for direct sum. Indeed, defining for all objects U, V, W in **LagrRel** unitors:

$$\begin{aligned} \lambda_V &= \{(0, v, v)\} \subseteq \overline{\{0\}} \oplus V \oplus V, \\ \rho_V &= \{(v, 0, v)\} \subseteq \overline{V \oplus \{0\}} \oplus V, \end{aligned}$$

associators:

$$\alpha_{U,V,W} = \{(u, v, w, u, v, w)\} \subseteq \overline{(U \oplus V) \oplus W} \oplus U \oplus (V \oplus W),$$

and braidings:

$$\sigma_{U,V} = \{(u, v, v, u) \mid u \in U, v \in V\} \subseteq \overline{U \oplus V} \oplus V \oplus U,$$

we have a symmetric monoidal category. Note that all these structure maps come from symplectomorphisms between the domain and codomain. From this viewpoint it is immediate that all the necessary diagrams commute, so we have a symmetric monoidal category.

Duals for objects

Each object V of LagrRel is dual to its conjugate space \overline{V} , with cup $\eta: \{0\} \rightarrow \overline{V} \oplus V$ given by

$$\eta = \{(0, v, v) \mid v \in V\} \subseteq \overline{\{0\}} \oplus \overline{V} \oplus V$$

and cap $\epsilon: V \oplus \overline{V} \rightarrow \{0\}$ given by

$$\epsilon = \{(v, v, 0) \mid v \in V\} \subseteq \overline{V} \oplus \overline{V} \oplus \{0\}.$$

It is straightforward to check these satisfy the zigzag identities.

Dagger structure

Given symplectic vector spaces U, V , observe that the map

$$\begin{aligned} (-)^\dagger: \overline{U} \oplus V &\longrightarrow \overline{V} \oplus U; \\ (u, v) &\longmapsto (v, u) \end{aligned}$$

takes Lagrangian subspaces of the domain to Lagrangian subspaces of the codomain. Thus we can view it as a map $(-)^\dagger$ taking morphisms $L: U \rightarrow V$ of LagrRel to morphisms $L^\dagger: V \rightarrow U$. This defines a dagger structure on LagrRel , which makes this category into a symmetric monoidal dagger category.

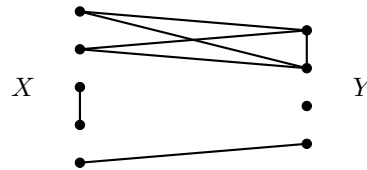
Moreover, every object in LagrRel has a dagger dual: it is clear that $\eta^\dagger = \epsilon \circ \sigma$. This category thus becomes a dagger compact category.

6.6 Ideal wires and corelations

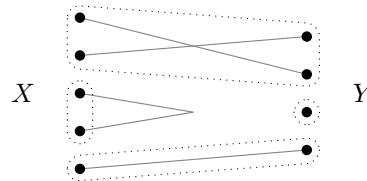
In the previous section our exploration of the meaning of circuit diagrams culminated with our understanding of behaviors as Lagrangian subspaces. We now turn our attention to how circuit components fit together, and the category of operations. In this section we shall see that the algebra of connections is described by the concept of corelations, a generalization of the notion of function that forgets the directionality from the domain to the codomain. We then observe that Kirchhoff's laws follow directly from interpreting these structures in the category of linear relations.

6.6.1 Ideal wires

To motivate the definition of this category, let us start with a set of input terminals X , and a set of output terminals Y . We may connect these terminals with ideal wires of zero impedance, whichever way we like—input to input, output to output, input to output—producing something like:

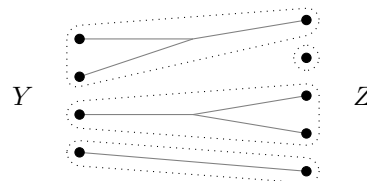


In doing so, we introduce a notion of equivalence on our terminals, where two terminals are equivalent if we, or if electrons, can traverse from one to another via some sequence of wires. Because of this, we consider our perfectly-conducting components to be equivalence relations on $X + Y$, transforming the above picture into

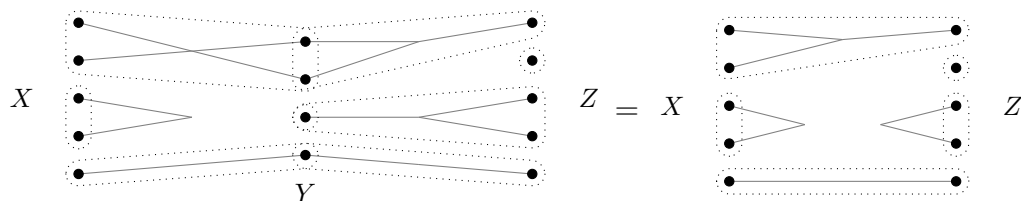


The dotted lines indicate equivalence classes of points, while for reference the grey lines indicate ideal wires connecting these points, running through a central hub.

Given another circuit of this sort, say from sets Y to Z ,



we may combine these circuits in to a circuit X to Z



by taking the transitive closure of the two equivalence relations, and then restricting this to an equivalence relation on $X + Z$. This in fact defines a dagger compact category of fundamental importance: the category of *corelations*.

Ellerman gives a detailed treatment of corelations from a logic viewpoint in [?], while basic category theoretic aspects can be found in Lawvere and Rosebrugh [?]. However, to make this paper self-contained, we explain them here.

6.6.2 The category of corelations

In the category of sets we hold the fundamental relationship between sets to be that of functions. These encode the idea of a deterministic process that takes each element of one set to a unique element of the other. For the study of networks this is less appropriate, as the relationship between terminals is not an input-output one, but rather one of interconnection.

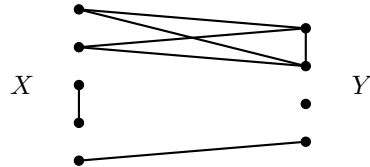
In particular, the direction of a function becomes irrelevant, and to describe these interconnections via the category of sets we must develop an understanding of how to compose functions head to head and tail to tail. We have so far used cospans and pushouts to address this. Cospans, however, come with an apex, which represents extraneous structure beyond the two sets we wish to specify a relationship between. Corelations arise from omitting this information.

Definition 6.42. A **corelation** $\alpha: X \rightarrow Y$ between finite sets X and Y is a partition α of the disjoint union $X + Y$.

That is, for finite sets X and Y , a corelation is a collection of nonempty subsets $\alpha = \{A_1, A_2, \dots, A_n\}$ of $X + Y$ such that

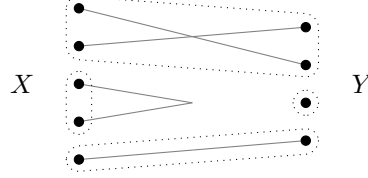
- (i) α does not contain the empty set.
- (ii) $\bigcup_{i=1}^n A_i = X + Y$.
- (iii) $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

For example, we can take a circuit of ideal wires with X as the set of inputs and Y as the set of outputs:



and define a corelation $\alpha: X \rightarrow Y$ for which terminals lie in the same set of the partition $\alpha = \{A_1, A_2, \dots, A_n\}$ when we can travel from one terminal to another

following a path of wires:



The discussion in the previous section then motivates a rule for composing correlations. We compose a correlation $\alpha: X \rightarrow Y$ and a correlation $\beta: Y \rightarrow Z$ by finding the finest partition on $X + Y + Z$ that is coarser than both α and β when restricted to $X + Y$ and $Y + Z$ respectively, and then restricting this to a partition on $X + Z$. More explicitly, the correlation $\beta \circ \alpha = \{C_1, C_2, \dots, C_m\}$ is the unique set of pairwise disjoint C_i of the form

$$C_i = \bigcup_j A_j \cap X \cup \bigcup_k B_k \cap Z$$

for j, k varying over indices such that

$$\bigcup_j A_j \cap Y = \bigcup_k B_k \cap Y.$$

This rule for composition is associative, as both pairwise methods of composing relations $\alpha: X \rightarrow Y$, $\beta: Y \rightarrow Z$, and $\gamma: Z \rightarrow W$ amount to finding the finest partition on $X + Y + Z + W$ that is coarser than each of α , β , and γ when restricted to the relevant subset, and then restricting this partition to a partition on $X + W$. The pictures in the previous section make this clear.

The identity correlation $1_X: X \rightarrow X$ on a set X is the map $[1_X, 1_X]: X + X \rightarrow X$. Equivalently, it is the partition of $X + X$ such that each partition comprises exactly two elements: an element $x \in X$ considered as an element of both the first and then the second summand of $X + X$. We thus define a category:

Definition 6.43. Let **Corel** be the category with finite sets as objects and correlations between finite sets as morphisms.

This category becomes monoidal under disjoint union of finite sets, using the fact that the union of partition of $X + Y$ and a partition of $X' + Y'$ is a partition of $(X + X') + (Y + Y')$. Moreover, the commutativity and associativity of the coproduct of finite sets also allows implies the category is symmetric monoidal and indeed dagger compact: the braiding and cups and caps are respectively given by the partitions of $(X + Y) + (Y + X)$ and $X + X$ where two elements are in the same part of the

partition if and only if they are equal as elements of X or Y , while a dagger is given by simply considering a partition of $X + Y$ as a partition of $Y + X$.⁴

We can get a corelation $\alpha: X \rightarrow Y$ from a surjection $f: X + Y \rightarrow S$ for some set S , by taking the sets in the partition $\alpha = (A_1, \dots, A_n)$ of $X + Y$ to be the inverse images of the points of S . Two surjections $f: X + Y \rightarrow S$, $f': X + Y \rightarrow S'$ give the same corelation if and only if there is an isomorphism $g: S \rightarrow S'$ making the obvious triangle commute: $f' = g \circ f$.

By the universal property of the coproduct, surjections $X + Y \rightarrow A$ are in one-to-one correspondence with jointly epic cospans $X \rightarrow A \leftarrow Y$. Thus, corelations can also be seen as isomorphism classes of jointly epic cospans. This lets us compose corelations by composing cospans and applying a correction. That is: given corelations $\alpha: X \rightarrow Y$, $\beta: Y \rightarrow Z$, we may choose cospans representing them and compose these cospans. The composite cospan may not be jointly epic. However, we can then replace the apex of the composite cospan by the joint image of the feet. The resulting jointly epic cospan gives the composite corelation $\beta \circ \alpha: X \rightarrow Z$. More precisely, we have the following proposition.

Proposition 6.44. *There is a strict symmetric monoidal dagger functor*

$$\text{Cospan}(\text{FinSet}) \longrightarrow \text{Corel}$$

mapping any cospan to the corelation it defines.

Proof. This functor maps each finite sets to itself. We thus need only discuss how it acts on morphisms. A cospan in FinSet comprises a pair of functions $X \xrightarrow{f} N \xleftarrow{g} Y$. Restricting the apex N down to the joint image $f(X) \cup g(Y)$ gives a jointly epic cospan $X \xrightarrow{f} f(X) \cup g(Y) \xleftarrow{g} Y$, and so a corelation $X \rightarrow Y$. As the elements of the apex not in the image of maps from the feet play only a trivial role in the pushout, and hence in composition of cospans, this map is functorial. It is now readily observed that the functor is symmetric monoidal. \square

All this is dual to the more familiar connection between spans and relations, where a relation is seen as an isomorphism class of jointly monic spans. This explains the name ‘corelation’. Note that neither relations nor corelations are a generalization of the other. The key property of corelations here is that it forms a *compact* category

⁴In fact, if we take a skeleton of Corel we obtain the PROP for ‘extraspecial commutative Frobenius monoids’, as defined by Baez and Erbele [?]. We do not need this here, but it helps tie our current work to other work on categories in electrical engineering. A proof can be assembled from Baez–Erbele together with the work of Lack [?].

with disjoint union of finite sets as the monoidal product. This is not true of the category of relations between finite sets.

Via the above proposition, the symmetric monoidal dagger functor embedding \mathbf{FinSet} into $\mathbf{Cospan}(\mathbf{FinSet})$ gives rise to a symmetric monoidal dagger functor embedding \mathbf{FinSet} into \mathbf{Corel} . Composing this with the dagger structure on \mathbf{Corel} , we also obtain a symmetric monoidal dagger functor embedding $\mathbf{FinSet}^{\text{op}}$ into \mathbf{Corel} . Corelations thus give a method of composing functions regardless of the direction of those functions. Given some not necessarily directed path of functions, for example

$$A \longrightarrow B \longleftarrow C \longleftarrow D \longrightarrow E \longrightarrow F,$$

considering these functions as corelations gives a way to compose them.

In particular, while this mode of composition is simply composition of functions for two functions head to tail, and turning a cospan into a corelation for two functions head to head, for two tail to tail functions $C \leftarrow D \rightarrow E$ we compute the composite of cospans

$$\begin{array}{ccccc} & & C & & E \\ & \swarrow & \parallel & \nwarrow & \parallel \\ C & & 1_C & & 1_E \\ & \searrow & & \swarrow & \\ & & D & & E \end{array}$$

before restricting the apex to arrive at a surjective function from $C + E$. This implies the following lemma.

Lemma 6.45. *Let*

$$\begin{array}{ccc} & P & \\ A & \nearrow & \nwarrow B \\ & N & \end{array}$$

be a pushout square in \mathbf{FinSet} . Then the composites of corelations $A \rightarrow P \leftarrow B$ and $A \leftarrow N \rightarrow B$ are equal.

6.6.3 Potentials on corelations

Chasing our interpretation of corelations as ideal wires, our aim for the remainder of this section is to build a functor

$$S: \mathbf{Corel} \longrightarrow \mathbf{LagrRel}$$

that expresses this interpretation. We break this functor down into the sum of two parts, according to the behaviors of potentials and currents respectively.

The consideration of potentials gives a functor $\Phi: \text{Corel} \rightarrow \text{LinRel}$, where LinRel is the symmetric monoidal dagger category of finite-dimensional \mathbb{F} -vector spaces, linear relations, direct sum, and transpose. In particular, this functor expresses Kirchhoff's voltage law: it requires that if two elements are in the same part of the corelation partition—that is, if two nodes are connected by ideal wires—then the potential at those two points must be the same.

This functor is a generalization of the contravariant functor $\text{FinSet} \rightarrow \text{Vect}$ that maps a set to the vector space of \mathbb{F} -valued functions on that set.

Proposition 6.46. *Define the functor*

$$\Phi: \text{Corel} \longrightarrow \text{LinRel},$$

on objects by sending a finite set X to the vector space \mathbb{F}^X , and on morphisms by sending a corelation $\alpha: X \rightarrow Y$ to the linear subspace $\Phi(\alpha)$ of $\mathbb{F}^X \oplus \mathbb{F}^Y$ comprising functions $\phi = [\phi_X, \phi_Y]: X + Y \rightarrow \mathbb{F}$ that are constant on each element of α . This is a symmetric monoidal dagger functor.

Proof. For coherence maps we take the usual natural isomorphisms $\mathbb{F}^X \oplus \mathbb{F}^Y \cong \mathbb{F}^{X \times Y}$ and $\{0\} \cong \mathbb{F}^\emptyset$. We detail only the proof that Φ preserves composition; the other properties are straightforward to check.

Let $\alpha: X \rightarrow Y$ and $\beta: Y \rightarrow Z$ be corelations. As Φ maps corelations to relations, it is enough to check both inclusions $\Phi(\beta) \circ \Phi(\alpha) \subseteq \Phi(\beta \circ \alpha)$ and $\Phi(\beta \circ \alpha) \subseteq \Phi(\beta) \circ \Phi(\alpha)$.

$\Phi(\beta) \circ \Phi(\alpha) \subseteq \Phi(\beta \circ \alpha)$: Let $\phi = [\phi_X, \phi_Z] \in \Phi(\beta) \circ \Phi(\alpha)$. We wish to show that for all $C_i \in \beta \circ \alpha$, for all $c, c' \in C_i$, we have $\phi(c) = \phi(c')$. To this end, note that there exists some $\phi_Y: Y \rightarrow \mathbb{F}$ such that $\phi_{XY} := [\phi_X, \phi_Y] \in \Phi(\alpha)$ and $\phi_{YZ} := [\phi_Y, \phi_Z] \in \Phi(\beta)$. Furthermore, by definition this ϕ_Y has the property that for all $A_j \in \alpha$, for all $a, a' \in A_j$, we have $\phi_{XY}(a) = \phi_{XY}(a')$, and for all $B_k \in \beta$, for all $b, b' \in B_k$, we have $\phi_{YZ}(b) = \phi_{YZ}(b')$. Write $\phi_{XYZ} = [\phi_X, \phi_Y, \phi_Z]: X + Y + Z \rightarrow \mathbb{F}$.

Our desired fact is thus true: for all $c, c' \in C_i$, there exists a sequence $c = c_0, c_1, \dots, c_n = c'$ in $X + Y + Z$ such that for all $\ell = 0, 1, \dots, n-1$ we have either $c_\ell, c_{\ell+1} \in A_j$ for some j or $c_\ell, c_{\ell+1} \in B_k$ for some k , and hence that

$$\phi(c) = \phi_{X+Y+Z}(c_0) = \phi_{X+Y+Z}(c_1) = \dots = \phi_{X+Y+Z}(c_n) = \phi(c')$$

as required.

$\Phi(\beta \circ \alpha) \subseteq \Phi(\beta) \circ \Phi(\alpha)$: Let $\phi = [\phi_X, \phi_Z] \in \Phi(\beta \circ \alpha)$. We must show that there exists $\phi_Y: Y \rightarrow \mathbb{F}$ such that $[\phi_X, \phi_Y] \in \Phi(\alpha)$ and $[\phi_Y, \phi_Z] \in \Phi(\beta)$. We claim

$$\phi_Y(y) := \begin{cases} \phi(x) & \text{if } x \in X, x, y \in A_j \text{ for some } j; \\ \phi(z) & \text{if } z \in Z, y, z \in B_k \text{ for some } k; \\ 0 & \text{if there exist no such } x \in X \text{ or } z \in Z \end{cases}$$

satisfies this. This is well-defined as for all A_j , $A_j \cap X \subseteq C_i$ for some i , so ϕ is constant on A_j , and similarly for all B_j . Thus it does not matter if there are many such x or z with $x, y \in A_j$ or $y, z \in B_k$. Furthermore, if there exists y such that both $x, y \in A_j$ for some x, A_j and $y, z \in B_k$ for some z, B_k , then the C_i with $A_j \cap X \subseteq C_i$ and $B_k \cap Z \subseteq C_i$ is unique, so the definitions of $\phi_Y(y)$ do not conflict.

Moreover, by construction $[\phi_X, \phi_Y]$ is constant on all A_j , and $[\phi_Y, \phi_Z]$ is constant on all B_k , so $[\phi_X, \phi_Y] \in \Phi(\alpha)$ and $[\phi_Y, \phi_Z] \in \Phi(\beta)$ as required. \square

To recap, we have now constructed a functor $\Phi: \text{Corel} \rightarrow \text{LinRel}$ expressing the behavior of potentials on corelations interpreted as ideal wires. We now do the same for currents.

6.6.4 Currents on corelations

Next we consider the case of currents, described by a functor $I: \text{Corel} \rightarrow \text{LinRel}$. This functor now expresses Kirchhoff's current law: it requires that the sum of currents flowing into each part of the corelation partition must equal to the sum of currents flowing out. It may also be understood as a generalization of the covariant functor $\text{Set} \rightarrow \text{Vect}$ that maps a set to the vector space of \mathbb{F} -linear combinations of elements of that set.

Proposition 6.47. *Define the functor*

$$I: \text{Corel} \longrightarrow \text{LinRel}.$$

as follows. On objects send a finite set X to the vector space $(\mathbb{F}^X)^$. On morphisms send a corelation $\alpha: X \rightarrow Y$ to the linear relation $I(\alpha)$ comprising precisely those*

$$(i_X, i_Y) = \left(\sum_{x \in X} \lambda_x dx, \sum_{y \in Y} \lambda_y dy \right) \in (\mathbb{F}^X)^* \oplus (\mathbb{F}^Y)^*$$

such that for all $A_i \in \alpha$ the sum of the coefficients of the elements of $A_i \cap X$ is equal to that for $A_i \cap Y$:

$$\sum_{x \in A_i \cap X} \lambda_x = \sum_{y \in A_i \cap Y} \lambda_y.$$

This is a symmetric monoidal dagger functor.

Proof. The coherence maps are the natural isomorphisms $(\mathbb{F}^X)^* \oplus (\mathbb{F}^Y)^* \rightarrow (\mathbb{F}^{X+Y})^*$ and $\{0\} \rightarrow (\mathbb{F}^\emptyset)^*$. Again the only nontrivial task is to check this map I preserves composition. Again we do this by checking inclusions $I(\beta) \circ I(\alpha) \subseteq I(\beta \circ \alpha)$ and $I(\beta \circ \alpha) \subseteq I(\beta) \circ I(\alpha)$.

$I(\beta) \circ I(\alpha) \subseteq I(\beta \circ \alpha)$: Let $(i_X, i_Z) \in I(\beta) \circ I(\alpha)$, with $i_X = \sum_{x \in X} \lambda_x dx$ and $i_Z = \sum_{z \in Z} \lambda_z dz$. Note that this implies that there is some $i_Y = \sum_{y \in Y} \lambda_y dy$ such that $(i_X, i_Y) \in I(\alpha)$, $(i_Y, i_Z) \in I(\beta)$. Then for each $C_i \in \beta \circ \alpha$ we have

$$\begin{aligned}
\sum_{x \in C_i \cap X} \lambda_x &= \sum_{\substack{x \in A_j \cap X \\ A_j \cap X \subseteq C_i}} \lambda_x \\
&= \sum_{\substack{y \in A_j \cap Y \\ A_j \cap X \subseteq C_i}} \lambda_y && \text{(By definition of } I(\alpha)) \\
&= \sum_{\substack{y \in B_k \cap Y \\ B_k \cap Z \subseteq C_i}} \lambda_y && \text{(See composition of correlations)} \\
&= \sum_{\substack{z \in B_k \cap Z \\ B_k \cap Z \subseteq C_i}} \lambda_z && \text{(By definition of } I(\beta)) \\
&= \sum_{z \in C_i \cap Z} \lambda_z.
\end{aligned}$$

Thus $(i_X, i_Z) \in I(\beta \circ \alpha)$.

$I(\beta \circ \alpha) \subseteq I(\beta) \circ I(\alpha)$: The reverse inclusion requires a bit more effort. Let $(i_X, i_Z) \in I(\beta \circ \alpha)$. We wish to construct some $i_Y = \sum_{y \in Y} \lambda_y dy \in (\mathbb{F}^Y)^*$ such that $(i_X, i_Y) \in I(\alpha)$ and $(i_Y, i_Z) \in I(\beta)$. This means that we must find a vector $i_Y \in (\mathbb{F}^Y)^*$ satisfying the $\#\alpha$ linear constraints

$$\sum_{y \in A_j \cap Y} \lambda_y = \sum_{x \in A_j \cap X} \lambda_x$$

and the $\#\beta$ linear constraints

$$\sum_{y \in B_j \cap Y} \lambda_y = \sum_{z \in B_j \cap Z} \lambda_z.$$

Note, however, that for each $C_i \in \beta \circ \alpha$, summing over the A_j and B_k that intersect C_i shows a linear dependence between these constraints themselves, as

$$\sum_{\substack{y \in A_j \cap Y \\ A_j \cap X \subseteq C_i}} \lambda_y = \sum_{x \in C_i \cap X} \lambda_x = \sum_{z \in C_i \cap Z} \lambda_z = \sum_{\substack{y \in B_k \cap Y \\ B_k \cap Z \subseteq C_i}} \lambda_y.$$

Moreover, as each element of Y lies in exactly one element of each α and β , we may view them as edges in a graph on $\alpha + \beta$, with exactly $\#(\beta \circ \alpha)$ connected components. This implies that

$$\#Y > \#\alpha + \#\beta - \#(\beta \circ \alpha).$$

Thus these constraints define an affine subspace of $(\mathbb{F}^Y)^*$ of positive dimension, and hence we can always find a vector i_Y with the desired property. This proves the claim. \square

Using elementary methods, an algorithm can also be given to construct an explicit solution.

6.6.5 The functor from Corel to LagrRel

We have now defined functors that, when interpreting corelations as connections of ideal wires, describe the behaviors of the currents and potentials at the terminals of these wires. In this section, we combine these to define a single functor $S: \text{Corel} \rightarrow \text{LagrRel}$ describing the behavior of both currents and potentials as a Lagrangian subspace.

Proposition 6.48. *We define the symplectification functor*

$$S: \text{Corel} \longrightarrow \text{LagrRel}$$

sending a finite set X to the symplectic vector space

$$S(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*,$$

and a corelation $\alpha: X \rightarrow Y$ to the Lagrangian relation

$$S(\alpha) = \Phi(\alpha) \oplus I(\alpha) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Then S is a symmetric monoidal dagger functor, with coherence maps inherited from Φ and I .

Proof. As S is the tensor product in LinRel of the symmetric monoidal dagger functors $\Phi, I: \text{Corel} \rightarrow \text{LinRel}$, it is itself a symmetric monoidal dagger functor $\text{Corel} \rightarrow \text{LinRel}$. Thus it only remains to be checked that, with respect to the symplectic structure we put on the objects $S(X)$, the image of each corelation $S(\alpha)$ is Lagrangian.

This follows from condition (v) of Proposition 6.34: $S(\alpha)$ is (i) isotropic as, for all $(\phi_X, i_X, \phi_Y, i_Y), (\phi'_X, i'_X, \phi'_Y, i'_Y) \in S(\alpha)$ we have

$$\begin{aligned}
& \omega((\phi_X, i_X, \phi_Y, i_Y), (\phi'_X, i'_X, \phi'_Y, i'_Y)) \\
&= - (i'_X(\phi_X) - i_X(\phi'_X)) + i'_Y(\phi_Y) - i_Y(\phi'_Y) \\
&= i_X(\phi'_X) - i_Y(\phi'_Y) + i'_X(\phi_X) - i'_Y(\phi_Y) \\
&= \sum_{x \in X} \lambda_x dx(\phi'_X) - \sum_{y \in Y} \lambda_y dy(\phi'_Y) + \sum_{y \in Y} \lambda'_y dy(\phi_Y) - \sum_{x \in X} \lambda'_x dx(\phi_X) \\
&= \sum_{A_j \in \alpha} \left(\sum_{x \in A_j \cap X} \lambda_x dx(\phi'_X) - \sum_{y \in A_j \cap Y} \lambda_y dy(\phi'_Y) \right) + \sum_{A_j \in \alpha} \left(\sum_{x \in A_j \cap X} \lambda'_x dx(\phi_X) - \sum_{y \in A_j \cap Y} \lambda'_y dy(\phi_Y) \right) \\
&= \sum_{A_j \in \alpha} \left(\sum_{x \in A_j \cap X} \lambda_x - \sum_{y \in A_j \cap Y} \lambda_y \right) k'_{A_j} + \sum_{A_j \in \alpha} \left(\sum_{x \in A_j \cap X} \lambda'_x - \sum_{y \in A_j \cap Y} \lambda'_y \right) k_{A_j} \\
&= 0,
\end{aligned}$$

and (ii) has dimension equal to

$$\dim(\Phi(\alpha)) + \dim(I(\alpha)) = \#\alpha + \#(X + Y) - \#\alpha = \#(X + Y).$$

This proves the proposition. \square

We have thus shown we do indeed have a functor $S: \text{Corel} \rightarrow \text{LagrRel}$. In the next section we shall see that this functor provides the engine of our black box functor, playing the key role in showing that we may indeed treat circuit components as black boxes: that is, that circuits that behave the same compose the same. Before we get there, we quickly make two relevant observations.

Example 6.49 (Symplectification of functions). Let $f: X \rightarrow Y$ be a function. In this example we show that Sf has the form

$$Sf = \{(\phi_X, i_X, \phi_Y, i_Y) \mid \phi_X = f^* \phi_Y, i_Y = f_* i_X\} \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*,$$

where f^* is the pullback map

$$\begin{aligned}
f^*: \mathbb{F}^Y &\longrightarrow \mathbb{F}^X; \\
\phi &\longmapsto \phi \circ f,
\end{aligned}$$

and f_* is the pushforward map

$$\begin{aligned}
f_*: (\mathbb{F}^X)^* &\longrightarrow (\mathbb{F}^Y)^*; \\
i(-) &\longmapsto i(- \circ f).
\end{aligned}$$

(We shall also write f_* for the more general map from functions on \mathbb{F}^X to functions on \mathbb{F}^Y that takes a function $i(-)$ on \mathbb{F}^X to the function $i(- \circ f)$.) The claim is then that these pullback and pushforward constructions express Kirchhoff's laws.

Recall that the corelation corresponding to f partitions $X + Y$ into $\#Y$ parts, each of the form $f^{-1}(y) \cup \{y\}$. The linear relation $\Phi(f)$ requires that if $x \in X$ and $y \in Y$ lie in the same part of the partition, then they have the same potential: that is, $\phi_X(x) = \phi_Y(y)$. This is precisely the arrangement imposed by $f^*\phi_Y = \phi_X$:

$$\phi_X(x) = \phi_Y(f(x)) = \phi_Y(y).$$

On the other hand, the linear relation $I(f)$ requires that if $i_X = \sum_{x \in X} \lambda_x dx$, $i_Y = \sum_{y \in Y} \lambda_y dy$, then for each $y \in Y$ we have

$$\sum_{x \in f^{-1}(y)} \lambda_x = \lambda_y.$$

This is precisely what is required by f_* : given any $\phi \in \mathbb{F}^Y$, we have

$$f_*i_X(\phi) = f_* \sum_{x \in X} \lambda_x dx(\phi) = \sum_{x \in X} \lambda_x dx(\phi \circ f) = \sum_{y \in Y} \left(\sum_{x \in f^{-1}(y)} \lambda_x \right) dy.$$

This gives us the above representation of Sf when f is a function.

Remark 6.50. We make a remark on our conventions for string diagrams representing the cups and caps of the dualities in Corel and LagrRel .

Note that by the cap duality diagram

$$\begin{array}{c} X \quad X \\ \frown \end{array}$$

in Corel we mean the corelation $\cup_X: (X + X \xrightarrow{[1,1]} X \xleftarrow{!} \emptyset)$, whereas by the cap

$$\begin{array}{c} V \quad \bar{V} \\ \smile \end{array}$$

in LagrRel we mean the Lagrangian relation $\cup_V: V \oplus \bar{V} \rightarrow 0$ given by the Lagrangian subspace $\{(v, v) \in \overline{V \oplus \bar{V}} \mid v \in V\}$. Although we represent them similarly, the functor S does not map these directly onto each other: the image of \cup_X under S is the Lagrangian relation $S(\cup_X): \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \rightarrow 0$ given by the Lagrangian subspace

$$\{(\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^*\} \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^*},$$

and in particular a Lagrangian relation $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \rightarrow 0$ and not $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \rightarrow 0$. As the cup diagrams in these categories simply denote the dagger image of these caps, the analogous statements apply to them too.

This is an expression of the fact that in cospan categories and Corel there is a canonical self-duality, while for Lagrangian relations a basis must be picked before there is an isomorphism between a symplectic vector space and its conjugate, the conjugate being a canonical dual object. Nonetheless as the functor S uses the set X to generate a symplectic vector space, each object in the image of S has a canonical symplectomorphism with its dual $S^tX: \mathbb{F}^X \oplus (\mathbb{F}^X)^* \rightarrow \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*}$, with name the Lagrangian subspace $\{(\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^*\}$.⁵ The image of the canonical self-duality Corel is thus given by the composite of this symplectomorphism with canonical duality in LagrRel; that is

$$S \left(\begin{array}{c} X \quad X \\ \cup \end{array} \right) = \begin{array}{c} S(X) \quad S(X) \\ \cup \quad \boxed{S^tX} \end{array}$$

6.7 Defining the black box functor

We have now developed enough machinery to prove Theorem 6.1: there is a symmetric monoidal dagger functor, the black box functor

$$\blacksquare: \text{Circ} \rightarrow \text{LagrRel}$$

taking passive linear circuits to their behaviors. To recap, we have so far developed two categories: Circ, in which morphisms are passive linear circuits, and LagrRel, which captures the external behavior of such circuits. We now define a functor that maps each circuit to its behavior, before proving it is indeed a symmetric monoidal dagger functor.

The role of the functor we construct here is to identify all circuits with the same external behavior, making the internal structure of the circuit inaccessible. Circuits treated this way are frequently referred to as ‘black boxes’, so we call this functor the **black box functor**,

$$\blacksquare: \text{Circ} \rightarrow \text{LagrRel}.$$

In this section we first provide the definition of this functor, and then check that our definition really does map a circuit to its behavior.

⁵While we could treat S^tX as an atomic notation, we write it here to evoke the idea of the concept of ‘twisted’ symplectification introduced in the introduction.

6.7.1 Definition

It should be no surprise that the black box functor maps a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$ of potentials and currents that may be placed on that set. The challenge is to provide a succinct statement of its action on the circuits themselves. To do this, we take advantage of four processes we developed in Parts and .

Let $\Gamma: X \rightarrow Y$ be a circuit, represented by the decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y, \Gamma).$$

Recall that this means that X and Y are finite sets, Γ is a \mathbb{F} -graph (N, E, s, t, r) , and we have a cospan of finite sets

$$\begin{array}{ccc} & \Gamma & \\ i \nearrow & & \nwarrow o \\ X & & Y. \end{array}$$

To define the image of Γ under our functor \blacksquare , by definition a Lagrangian relation $\blacksquare(\Gamma): \blacksquare(X) \rightarrow \blacksquare(Y)$, we must specify a Lagrangian subspace

$$\blacksquare(\Gamma) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Recall that to each \mathbb{F}^+ -graph Γ we can associate a Dirichlet form, the extended power functional

$$\begin{aligned} P_\Gamma: \mathbb{F}^N &\longrightarrow \mathbb{F}; \\ \phi &\longmapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\phi(t(e)) - \phi(s(e)))^2, \end{aligned}$$

and to this Dirichlet form we associate a Lagrangian subspace

$$\text{Graph}(dP_\Gamma) = \{(\phi, d(P_\Gamma)_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*.$$

We consider this Lagrangian subspace as a Lagrangian relation $\{0\} \rightarrow \mathbb{F}^N \oplus (\mathbb{F}^N)^*$.

From the legs of the cospan Γ , the symplectification functor S gives the Lagrangian relation

$$S[i, o]^\dagger: \mathbb{F}^N \oplus (\mathbb{F}^N)^* \longrightarrow \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Writing $Y: Y \rightarrow Y$ for the identity morphism on the finite set Y , S also provides a way of writing the identity morphism $SY: \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \rightarrow \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$.

Lastly, we have the symplectomorphism

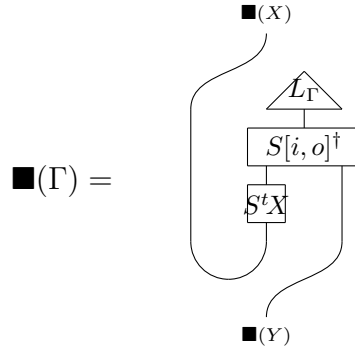
$$\begin{aligned} S^tX: \mathbb{F}^X \oplus (\mathbb{F}^X)^* &\longrightarrow \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*}, \\ (\phi, i) &\longmapsto (\phi, -i). \end{aligned}$$

The black box functor maps a circuit Γ to the Lagrangian relation

$$(S^tX \oplus SY) \circ S[i, o]^\dagger \circ \text{Graph}(P_\Gamma).$$

As isomorphisms of cospans of \mathbb{F}^+ -graphs amount to no more than a relabelling of nodes and edges, this construction is independent of the cospan chosen as representative of the isomorphism class of cospans forming the circuit.

We picture this as



To summarize:

Definition 6.51. We define the black box functor

$$\blacksquare: \text{Circ} \rightarrow \text{LagrRel}$$

on objects by mapping a finite set X to the symplectic vector space

$$\blacksquare(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*.$$

and on morphisms by mapping a circuit $\Gamma: X \rightarrow Y$, represented by the decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y, \Gamma)$$

to the Lagrangian relation

$$\blacksquare(\Gamma) = (S^tX \oplus SY) \circ S[i, o]^\dagger \circ \text{Graph}(dP_\Gamma).$$

The coherence maps are given by the natural isomorphisms

$$\blacksquare(X) \oplus \blacksquare(Y) = \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \cong \mathbb{F}^{X+Y} \oplus (\mathbb{F}^{X+Y})^* = \blacksquare(X + Y)$$

and

$$\{0\} \cong \mathbb{F}^\emptyset \oplus (\mathbb{F}^\emptyset)^* = \blacksquare(\emptyset).$$

Theorem 6.52. *The black box functor is a well-defined symmetric monoidal dagger functor.*

The next, and final, section is devoted to the proof of this theorem. Before we get there, we first assure ourselves that we have indeed arrived at the theorem we set out to prove.

6.7.2 Minimization via composition of relations

At this point the reader might voice two concerns: firstly, why does the *black box* functor refer to the *extended* power functional P and, secondly, since it fails to talk about power minimization, how is it the same functor as that defined in Theorem 6.1? These fears are allayed by the remarkable trinity of minimization, the symplectification of functions, and Kirchhoff's laws.

We have seen that symplectification of functions views the cograph of the function as a picture of ideal wires, governed by Kirchhoff's laws (Example 6.49). We have also seen that Kirchhoff's laws are closely related to the principle of minimum power (Theorems 6.20 and 6.22). The final aspect of this relationship is that we may use symplectification of functions to enact minimization.

Theorem 6.53. *Let $\iota : \partial N \rightarrow N$ be an injection, and let P be a Dirichlet form on N . Write $Q = \min_{N \setminus \partial N} P$ for the Dirichlet form on ∂N given by minimization over $N \setminus \partial N$. Then we have an equality of Lagrangian subspaces*

$$S\iota^\dagger(\text{Graph}(dP)) = \text{Graph}(dQ).$$

Proof. Recall from Example 6.49 that $S\iota^\dagger$ is the Lagrangian relation

$$S\iota^\dagger = \{(\phi, \iota_*i, \phi \circ \iota, i) \mid \phi \in \mathbb{F}^N, i \in (\mathbb{F}^{\partial N})^*\} \subseteq \overline{\mathbb{F}^N \oplus (\mathbb{F}^N)^*} \oplus \mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*,$$

where $\iota_*i(\phi) = i(\phi \circ \iota)$, and note that

$$\text{Graph}(dP) = \{(\phi, dP_\phi) \mid \phi \in \mathbb{F}^N\}.$$

This implies that their composite is given by the set

$$S\iota^\dagger \circ \text{Graph}(dP) = \{(\phi \circ \iota, i) \mid \phi \in \mathbb{F}^N, i \in (\mathbb{F}^{\partial N})^*, dP_\phi = \iota_*i\}.$$

We must show this Lagrangian subspace is equal to $\text{Graph}(dQ)$.

Consider the constraint $dP_\phi = \iota_* i$. This states that for all $\varphi \in \mathbb{F}^N$ we have $dP_\phi(\varphi) = i(\varphi \circ \iota)$. Letting $\chi_n : N \rightarrow \mathbb{F}$ be the function sending $n \in N$ to 1 and all other elements of N to 0, we see that when $n \in N \setminus \partial N$ we must have

$$\left. \frac{dP}{d\varphi(n)} \right|_{\varphi=\phi} = dP_\phi(\chi_n) = i(\chi_n \circ \iota) = i(0) = 0.$$

So ϕ must be a realizable extension of $\psi = \phi \circ \iota$. We henceforth write $\tilde{\psi} = \phi$. As ι is injective, $\psi = \phi \circ \iota$ gives no constraint on $\psi \in \mathbb{F}^{\partial N}$.

We next observe that we can write $S\iota^\dagger \circ \text{Graph}(dP) = \text{Graph}(dO)$ for some quadratic form O . Recall that Proposition 6.36 states that a Lagrangian subspace L of $\mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*$ is of the form $\text{Graph}(dO)$ if and only if L has trivial intersection with $\{0\} \oplus (\mathbb{F}^{\partial N})^*$. But indeed, if $\psi = 0$ then 0 is a realizable extension of ψ , so $\iota_* i = dP_0 = 0$, and hence $i = 0$.

It remains to check that $O = Q$. This is a simple computation:

$$O(\psi) = dO_\psi(\psi) = dO_\psi(\tilde{\psi} \circ \iota) = \iota_* dQ_\psi(\tilde{\psi}) = dP_{\tilde{\psi}}(\tilde{\psi}) = P(\tilde{\psi}) = Q(\psi),$$

where $\tilde{\psi}$ is any realizable extension of $\psi \in \mathbb{F}^{\partial N}$. □

Write $\iota : \partial N \rightarrow N$ for the inclusion of the terminals into the set of nodes of the circuit, and $i|_{\partial N} : X \rightarrow \partial N$, $o|_{\partial N} : Y \rightarrow \partial N$ for the respective corestrictions of the input and output map to ∂N . Note that $[i, o] = \iota \circ [i|_{\partial N}, o|_{\partial N}]$. Then we have the equalities of sets, and thus Lagrangian relations:

$$\begin{aligned} (S^t X \oplus 1_Y) \circ S[i, o]^\dagger \circ \text{Graph}(dP_\Gamma) &= (S^t X \oplus SY) \circ S[i|_{\partial N}, o|_{\partial N}]^\dagger \circ S\iota^\dagger \circ \text{Graph}(dP_\Gamma) \\ &= (S^t X \oplus SY) \circ S[i|_{\partial N}, o|_{\partial N}]^\dagger \circ \text{Graph}(dQ_\Gamma) \\ &= \bigcup_{v \in \text{Graph}(dQ)} S^t i|_{\partial N}(v) \times S o|_{\partial N}(v) \end{aligned}$$

We see now that Theorem 6.52 is a restatement of Theorem 6.1 in the introduction.

6.8 Proof of functoriality

To prove that the black box construction is indeed functorial, we factor it into three functors. These functors are each symmetric monoidal dagger functors, so the black box functor is too.

We first make use, twice, of our results on decorated cospans, showing the existence of categories of cospans decorated by Dirichlet forms and then Lagrangian

subspaces, and the existence of functors from the category of circuits to each of these. This proceeds by defining functors Dirich and Lagr to construction decorated cospan categories, and then by defining the relevant natural transformations to construct the desired decorated cospan functors.

The third functor takes cospans decorated by Lagrangian subspaces, and black boxes them to give Lagrangian relations between the feet of such cospans. The functoriality of this process relies on interpreting corelations as Lagrangian relations.

This gives a factorization

$$\blacksquare: \text{Circ} \longrightarrow \text{DirichCospan} \longrightarrow \text{LagrCospan} \longrightarrow \text{LagrRel}$$

The following subsections deal with these functors in sequence, defining and proving the existence of each one using the techniques of Part .

6.8.1 From circuits to Dirichlet cospans

To begin, we define a category of cospans of finite sets decorated by Dirichlet forms. This might be seen as a solution to our sought after composition rule for Dirichlet forms—but not quite, as it also requires the additional data of a cospan. Nonetheless, it has the property that there is a functor from the category of circuits to this so-called category of Dirichlet cospans.

The category of Dirichlet cospans

Proposition 6.54. *Let*

$$\text{Dirich}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

map a finite set X to the set $\text{Dirich}(X)$ of Dirichlet forms on X , and map a function $f: X \rightarrow Y$ between finite sets to the pushforward function

$$\begin{aligned} \text{Dirich}(f): \text{Dirich}(X) &\longrightarrow \text{Dirich}(Y); \\ Q &\longmapsto f_*Q, \end{aligned}$$

*where f_*Q maps $\phi \in \mathbb{F}^Y$ to $Q(\phi \circ f)$. This defines a functor.*

Moreover, equipping this functor with the family of maps

$$\begin{aligned} \delta_{N,M}: \text{Dirich}(N) \times \text{Dirich}(M) &\longrightarrow \text{Dirich}(N + M); \\ (Q_N, Q_M) &\longmapsto Q_N(- \circ \iota_N) + Q_M(- \circ \iota_M), \end{aligned}$$

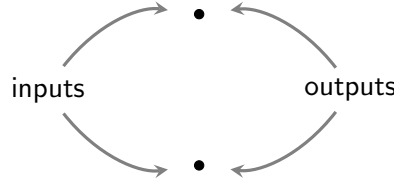
where $\iota_N: N \rightarrow N + M$, $\iota_M: M \rightarrow N + M$ are the injections for the coproduct, and with unit

$$\begin{aligned} \delta_1: 1 &\longrightarrow \text{Dirich}(\emptyset); \\ \bullet &\longmapsto (\mathbb{F}^\emptyset \rightarrow \mathbb{F}; \emptyset \mapsto 0) \end{aligned}$$

defines a lax symmetric monoidal functor.

Proof. Functoriality is just the associativity of composition of functions; lax symmetric monoidality is the associativity, unitality, and commutativity of addition of Dirichlet forms. \square

By decorated cospans, we thus obtain a category DirichCospan where a morphism is a cospan of finite sets whose apex is equipped with a Dirichlet form. In particular, note that we have overcome our inability to define a category whose morphisms are Dirichlet forms. An identity morphism in DirichCospan is a circuit whose set of inputs is equal to its set of outputs, with all of its nodes being terminals, and no edges:



The functor $\text{Circ} \rightarrow \text{DirichCospan}$

We have now constructed two symmetric monoidal functors

$$(\text{Circuit}, \rho), (\text{Dirich}, \delta): (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

that describe the circuit structures and Dirichlet forms we may put on the set respectively. We have also seen, motivating our discussion of Dirichlet forms, that from any circuit we can obtain a Dirichlet form describing the power usage of that circuit. Now we shall see that this process respects the tensor product. More precisely, it specifies a monoidal natural transformation between Circuit and Dirich . By the decorated cospan construction, this gives us a strict symmetric monoidal dagger functor $\text{Circ} \rightarrow \text{DirichCospan}$.

Proposition 6.55. *Let*

$$\alpha: (\text{Circuit}, \rho) \Longrightarrow (\text{Dirich}, \delta)$$

be the collection of functions

$$\alpha_N: \text{Circuit}(N) \longrightarrow \text{Dirich}(N);$$

$$(N, E, s, t, r) \longmapsto \left(\phi \in \mathbb{F}^N \mapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\phi(s(e)) - \phi(t(e)))^2 \right).$$

Then α is a monoidal natural transformation.

Proof. Naturality requires that the square

$$\begin{array}{ccc} \text{Circuit}(N) & \xrightarrow{\alpha_N} & \text{Dirich}(N) \\ \text{Circuit}(f) \downarrow & & \downarrow \text{Dirich}(f) \\ \text{Circuit}(M) & \xrightarrow{\alpha_M} & \text{Dirich}(M) \end{array}$$

commutes. Let (N, E, s, t, r) be an \mathbb{F}^+ -graph on N and $f: N \rightarrow M$ be a function N to M . Then both $\text{Dirich}(f) \circ \alpha_N$ and $\alpha_M \circ \text{Circuit}(f)$ map (N, E, s, t, r) to the Dirichlet form

$$\mathbb{F}^M \longrightarrow \mathbb{F};$$

$$\psi \longmapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} (\psi(f(s(e))) - \psi(f(t(e))))^2.$$

Thus both methods of constructing a power functional on a set of nodes M from a circuit on N and a function $N \rightarrow M$ produce the same power functional.

To show that α is a monoidal natural transformation, we must check that the square

$$\begin{array}{ccc} \text{Circuit}(N) \times \text{Circuit}(M) & \xrightarrow{\alpha_N \times \alpha_M} & \text{Dirich}(N) \times \text{Dirich}(M) \\ \rho_{N,M} \downarrow & & \downarrow \delta_{N,M} \\ \text{Circuit}(N + M) & \xrightarrow{\alpha_{N+M}} & \text{Dirich}(N + M) \end{array}$$

and the triangle

$$\begin{array}{ccc} & 1 & \\ \rho_{\emptyset} \swarrow & & \searrow \delta_{\emptyset} \\ \text{Circuit}(\emptyset) & \xrightarrow{\alpha_{\emptyset}} & \text{Dirich}(\emptyset) \end{array}$$

commute. It is readily observed that both paths around the square lead to taking two graphs and summing their corresponding Dirichlet forms, and that the triangle commutes immediately as all objects in it are the one element set. \square

From decorated cospans, we thus obtain a strict symmetric monoidal dagger functor

$$Q: \text{Circ} = \text{CircuitCospan} \longrightarrow \text{DirichCospan}.$$

Roughly, this says that the process of composition for circuit diagrams is the same as that of composition for Dirichlet cospans. Note that although this functor preserves much of the information in circuit diagrams, it is not a faithful functor. For example, applying Q to a circuit with edges e, e' from the node m to the node n of resistance r_e and $r_{e'}$ respectively, we obtain the same result as for the circuit with just one edge e'' from m to n whose resistance $r_{e''}$ is given by

$$\frac{1}{r_{e''}} = \frac{1}{r_e} + \frac{1}{r_{e'}}.$$

6.8.2 From Dirichlet cospans to Lagrangian cospans

The next step is to show that our process of turning a Dirichlet form into a Lagrangian subspace—by taking the graph of its differential—is functorial.

The category of Lagrangian cospans

We begin by describing a category where morphisms are cospans decorated by Lagrangian subspaces.

Proposition 6.56. *Define*

$$\text{Lagr}: (\text{FinSet}, +) \longrightarrow (\text{Set}, \times)$$

as follows. For objects let Lagr map a finite set X to the set $\text{Lagr}(X)$ of Lagrangian subspaces of the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^$. For morphisms, recall that a function $f: X \rightarrow Y$ between finite sets may be considered as a corelation, and the symplectification functor S thus maps this corelation to some Lagrangian relation $Sf: \mathbb{F}^X \oplus (\mathbb{F}^X)^* \rightarrow \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$. As Lagrangian relations map Lagrangian subspaces to Lagrangian subspaces (Proposition 6.38), this gives a map:*

$$\begin{aligned} \text{Lagr}(f): \text{Lagr}(X) &\longrightarrow \text{Lagr}(Y); \\ L &\longmapsto Sf(L). \end{aligned}$$

Moreover, equipping this functor with the family of maps

$$\begin{aligned} \lambda_{N,M}: \text{Lagr}(N) \times \text{Lagr}(M) &\longrightarrow \text{Lagr}(N + M); \\ (L_N, L_M) &\longmapsto L_N \oplus L_M, \end{aligned}$$

and unit

$$\begin{aligned}\lambda_1 &: 1 \longrightarrow \text{Lagr}(\emptyset); \\ \bullet &\longmapsto \{0\}\end{aligned}$$

defines a lax symmetric monoidal functor.

Proof. The functoriality of this construction follows from the functoriality of S ; the lax symmetric monoidality from the relevant properties of the direct sum of vector spaces. \square

We thus obtain a dagger compact category LagrCospan .

The functor $\text{DirichCospan} \rightarrow \text{LagrCospan}$

We now wish to construct a strict symmetric monoidal dagger functor $\text{DirichCospan} \rightarrow \text{LagrCospan}$. For this we need a monoidal natural transformation

$$(\text{Dirich}, \delta), (\text{Lagr}, \lambda): (\text{FinSet}, +) \longrightarrow (\text{Set}, \times).$$

Proposition 6.57. *Let*

$$\beta: (\text{Dirich}, \delta) \Longrightarrow (\text{Lagr}, \lambda)$$

be the collection of functions

$$\begin{aligned}\beta_N &: \text{Dirich}(N) \longrightarrow \text{Lagr}(N); \\ Q &\longmapsto \{(\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*.\end{aligned}$$

Then β is a monoidal natural transformation.

Proof. Naturality requires that the square

$$\begin{array}{ccc}\text{Dirich}(N) & \xrightarrow{\beta_N} & \text{Lagr}(N) \\ \text{Dirich}(f) \downarrow & & \downarrow \text{Lagr}(f) \\ \text{Dirich}(M) & \xrightarrow{\beta_M} & \text{Lagr}(M)\end{array}$$

commutes for every function $f: N \rightarrow M$. This is primarily a consequence of the fact that the differential commutes with pullbacks. As we did in Example 6.49, write f^* for the pullback map and f_* for the pushforward map. Then $\text{Dirich}(f)$ maps a Dirichlet form Q on N to the form f_*Q , and β_M in turn maps this to the Lagrangian subspace

$$\{(\psi, d(f_*Q)_\psi) \mid \psi \in \mathbb{F}^M\} \subseteq \mathbb{F}^M \oplus (\mathbb{F}^M)^*.$$

On the other hand, β_N maps a Dirichlet form Q on N to the Lagrangian subspace

$$\{(\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*,$$

before $\text{Lagr}(f)$ maps this to the Lagrangian subspace

$$\{(\psi, f_* dQ_\phi) \mid \psi \in \mathbb{F}^M, \phi = f^*(\psi)\} \subseteq \mathbb{F}^M \oplus (\mathbb{F}^M)^*.$$

But

$$f_* dQ_{f^*\psi} = d(f_* Q)_\psi,$$

so these two processes commute.

Monoidality requires that the diagrams

$$\begin{array}{ccc} \text{Dirich}(N) \times \text{Dirich}(M) & \xrightarrow{\beta_N \times \beta_M} & \text{Lagr}(N) \times \text{Lagr}(M) \\ \delta_{N,M} \downarrow & & \downarrow \lambda_{N,M} \\ \text{Dirich}(N+M) & \xrightarrow{\beta_{N+M}} & \text{Lagr}(N+M) \end{array} \quad \text{and} \quad \begin{array}{ccc} & 1 & \\ \delta_\emptyset \swarrow & & \searrow \lambda_\emptyset \\ \text{Dirich}(\emptyset) & \xrightarrow{\beta_\emptyset} & \text{Lagr}(\emptyset) \end{array}$$

commute. These do: the Lagrangian subspace corresponding to the sum of Dirichlet forms is equal to the sum of the Lagrangian subspaces that correspond to the summand Dirichlet forms, while there is only a unique map $1 \rightarrow \text{Lagr}(\emptyset)$. \square

We thus obtain a strict symmetric monoidal dagger functor

$$\text{DirichCospan} \rightarrow \text{LagrCospan},$$

which simply replaces the decoration on each cospan in DirichCospan with the corresponding Lagrangian subspace.

6.8.3 From cospans to relations

At this point we have checked that the process of reinterpreting a circuit as a Lagrangian subspace of behaviors is functorial. Our task is now to integrate this information as more than just a ‘decoration’ on our morphisms. This process constitutes a monoidal dagger functor

$$\text{LagrCospan} \longrightarrow \text{LagrRel}.$$

This factor of the black box functor is the one that gives it its name; through this functor we finally seal off the inner structure of our circuits, leaving us access only to the behavior at the terminals. Its purpose is to take a Lagrangian cospan, which captures information about the behaviors of a circuit measured at every node, and restrict it down to a relation detailing the behaviors simply on the terminals.

Proposition 6.58. *We may define a symmetric monoidal dagger functor*

$$\text{LagrCospan} \longrightarrow \text{LagrRel}$$

as follows. On objects let this functor take a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^$. On morphisms let it take a Lagrangian cospan*

$$(X \xrightarrow{i} N \xleftarrow{o} Y; L \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*)$$

to the Lagrangian relation

$$(S^t X \oplus S Y) \circ S[i, o]^\dagger \circ L \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Proof. The coherence maps here are the usual natural isomorphisms

$$\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \xrightarrow{\sim} \mathbb{F}^{X+Y} \oplus (\mathbb{F}^{X+Y})^*$$

and

$$\{0\} \xrightarrow{\sim} \mathbb{F}^\emptyset \oplus (\mathbb{F}^\emptyset)^*.$$

It is now routine to observe the symmetric monoidality and dagger-preserving nature of this construction, as well as that it preserves identities. Finally, we must check that composition is preserved.

Using the concept of names introduced in Chapter refONE, this comes down to checking this equality of Lagrangian subspaces:

where the left hand side is the composite in LagrRel of the images of the Lagrangian cospans $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y; L)$ and $(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z; K)$, and the right hand side is the image of their composite in LagrCospan . (Recall that we write $j_N : N \rightarrow N +_Y M$ and $j_M : M \rightarrow N +_Y M$ for the maps given by the pushout.)

Recalling Remark 6.50 pertaining to the functor S and duals for objects, this implies that it is enough to check the equality of corelations

Writing this instead as a commutative diagram, we wish to prove the equality of the two corelations $N + M \rightarrow X + Y$:

$$\begin{array}{ccccc}
 & & N +_Y M & & \\
 & \swarrow [j_N, j_M] & & \nwarrow [j_N \circ i_X, j_M \circ o_Z] & \\
 N + M & & & & X + Z \\
 & \searrow [i_X, o_Y] + [i_Y, o_Z] & & \swarrow \text{incl}_{X+Z} & \\
 & X + Y + Y + Z & \xrightarrow{\text{id}_X + [\text{id}_Y, \text{id}_Y] + \text{id}_Z} & X + Y + Z &
 \end{array}$$

As the right-hand part admits the factorization

$$\begin{array}{ccc}
 N +_Y M & & \\
 & \nwarrow [j_N \circ i_X, j_M \circ o_Z] & \\
 & X + Z & \\
 & \swarrow \text{incl}_{X+Z} & \\
 & X + Y + Z &
 \end{array}$$

it is enough to check the following diagram commutes when interpreted as a pair of morphisms from $N + M$ to $X + Y + Z$ in the category of corelations:

$$\begin{array}{ccc}
 & N +_Y M & \\
 \swarrow [j_N, j_M] & & \nwarrow [j_N \circ i_X, j_M \circ o_Y, j_M \circ o_Z] \\
 N + M & & X + Y + Z \\
 \searrow [i_X, o_Y] + [i_Y, o_Z] & & \swarrow \text{id}_X + [\text{id}_Y, \text{id}_Y] + \text{id}_Z \\
 & X + Y + Y + Z &
 \end{array}$$

By Lemma 6.45, this is equivalent to checking that it is a pushout square in FinSet . This is so: the square commutes in FinSet as it is the sum along the lower right edges of the three commutative squares

$$\begin{array}{ccc}
 & N +_Y M & \\
 \swarrow [j_N, j_M] & & \nwarrow j_N \circ i_X \\
 N + M & & X \\
 \searrow i_X & & \swarrow \text{id}_X
 \end{array}
 \quad
 \begin{array}{ccc}
 & N +_Y M & \\
 \swarrow [j_N, j_M] & & \nwarrow j_N \circ o_Y = j_M \circ i_Y \\
 N + M & & Y \\
 \searrow o_Y + i_Y & & \swarrow [\text{id}_Y, \text{id}_Y]
 \end{array}
 \quad
 \begin{array}{ccc}
 & N +_Y M & \\
 \swarrow [j_N, j_M] & & \nwarrow j_M \circ o_Z \\
 N + M & & Z \\
 \searrow o_Z & & \swarrow \text{id}_Z
 \end{array}$$

and given any other object T and maps f, g such that

$$\begin{array}{ccc}
 & T & \\
 \swarrow f & & \nwarrow g \\
 N + M & & X + Y + Z \\
 \searrow [i_X, o_Y] + [i_Y, o_Z] & & \swarrow \text{id}_X + [\text{id}_Y, \text{id}_Y] + \text{id}_Z \\
 & X + Y + Y + Z &
 \end{array}$$

commutes, there is a unique map $N +_Y M \rightarrow T$ defined by sending a in $N +_Y M$ to $f(\hat{a})$, where \hat{a} is a preimage of a in $N + M$ under the coproduct of pushout maps $[j_N, j_M]$. This is well-defined as the preimage of a is either unique or equal to $\{o_Y(y), i_Y(y)\}$ for some element $y \in Y$, and the commutativity of the above square containing T implies that $f(o_Y(y)) = f(i_Y(y))$. This proves the functoriality of the map $\text{LagrCospan} \rightarrow \text{LagrRel}$ defined above. \square

The three functors of this section compose to give the black box functor

$$\blacksquare: \text{Circ} \rightarrow \text{LagrRel}.$$

Since they are each separately symmetric monoidal dagger functors, the black box functor is too.

Chapter 7

Further directions

Open systems.

7.1 Open Markov processes

Bring in notions of entropy, summary statistics.

7.2 Flow networks

cf linear algebra. diagrams+rewrite rules, algebraic semantics, summary arrays.

7.3 Chemical reaction networks

7.4 Bounded colimits

Grothendieck construction

7.5 Other categorical structures

Not all diagrammatic structures have the same hypergraph network style. For example, some have substantive notions of input and output. Spivak's work on operads of wiring diagrams

Bibliography

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