

The Algebra of Open and Interconnected Systems



Brendan Fong
Hertford College
University of Oxford

A thesis submitted for the degree of
Doctor of Philosophy in Computer Science

Trinity 2016

Contents

Lay introduction	v
Introduction	viii
0.1 Outline	xi
0.2 Related work	xii
0.3 Contributions of collaborators	xii
 I Mathematical Foundations	 1
1 Hypergraph categories: the algebra of interconnection	2
1.1 The algebra of interconnection	2
1.2 Symmetric monoidal categories	5
1.2.1 Monoidal categories	5
1.2.2 String diagrams	7
1.2.3 Symmetry	8
1.3 Hypergraph categories	9
1.3.1 Frobenius monoids	10
1.3.2 Hypergraph categories	11
1.3.3 Hypergraph categories are self-dual compact closed	13
1.3.4 Coherence	14
1.4 Example: cospan categories	16
 2 Decorated cospans: from closed systems to open	 20
2.1 From closed systems to open	20
2.2 Decorated cospan categories	23
2.2.1 Composing decorated cospans	23
2.2.2 The hypergraph structure	28
2.2.3 A bicategory of decorated cospans	33

2.3	Functors between decorated cospan categories	33
2.4	Decorations in \mathbf{Set} are general	38
2.5	Examples	40
2.5.1	Labelled graphs	40
2.5.2	Linear relations	41
2.5.3	Electrical circuits	42
3	Corelations: a tool for black boxing	45
3.1	The idea of black boxing	45
3.2	Corelations	46
3.2.1	Factorisation systems.	46
3.2.2	Corelations	49
3.2.3	Categories of corelations	50
3.3	Functors between corelation categories	53
3.4	Examples	57
3.4.1	Equivalence relations as corelations in \mathbf{Set}	57
3.4.2	Linear relations as corelations in \mathbf{Vect}	59
4	Decorated corelations	61
4.1	Black-boxed open systems	61
4.2	Decorated corelations	63
4.2.1	Adjoining right adjoints	64
4.2.2	Decorated corelations	65
4.2.3	Categories of decorated corelations	66
4.3	Functors between decorated corelation categories	67
4.4	All hypergraph categories are decorated corelation categories	71
4.4.1	The global sections construction	71
4.4.2	A categorical equivalence	75
4.4.3	Factorisations as decorations	76
4.5	Examples	79
4.5.1	Matrices	79
4.5.2	Two constructions for linear relations	81

II	Applications	82
5	Signal flow diagrams	79
5.1	Introduction	79
5.1.1	Behavioural control	79
5.1.2	Signal flow	82
5.2	Preliminaries	86
5.3	Linear time-invariant dynamical systems	87
5.4	Presentation of LTI	91
5.4.1	Syntax	91
5.4.2	Presentation of $\mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$	93
5.4.3	Presentation of $\mathbf{Cospan} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$	94
5.4.4	Corelations	95
5.4.5	Presentation of $\mathbf{Corel} \, \mathbf{Mat} \, \mathbf{k}[s, s^{-1}]$	97
5.5	Controllability	100
5.5.1	A categorical characterisation	100
5.5.2	Control and interconnection	102
5.5.3	Comparison to matrix methods	104
6	Passive linear networks	106
6.1	Introduction	106
6.1.1	Finding your way through this paper	115
6.2	Circuits of linear resistors	116
6.2.1	Circuits as labelled graphs	116
6.2.2	Ohm's law, Kirchhoff's laws, and the principle of minimum power	118
6.2.3	A Dirichlet problem	121
6.2.4	Equivalent circuits	124
6.2.5	Dirichlet forms	127
6.3	Inductors and capacitors	129
6.3.1	The frequency domain and Ohm's law revisited	130
6.3.2	The mechanical analogy	133
6.3.3	Generalized Dirichlet forms	134
6.3.4	A generalized minimizability result	136
6.3.5	Composition of Dirichlet forms	139
6.4	The category of passive linear circuits	141
6.4.1	A decorated cospan construction	142
6.4.2	A bicategory of circuits	143

6.5	Circuits as Lagrangian relations	144
6.5.1	Symplectic vector spaces	145
6.5.2	Lagrangian subspaces from quadratic forms	148
6.5.3	Lagrangian relations	149
6.5.4	The dagger compact category of Lagrangian relations	152
6.6	Ideal wires and corelations	153
6.6.1	Ideal wires	154
6.6.2	The category of corelations	155
6.6.3	Potentials on corelations	158
6.6.4	Currents on corelations	160
6.6.5	The functor from Corel to LagrRel	162
6.7	Defining the black box functor	165
6.7.1	Definition	166
6.7.2	Minimization via composition of relations	168
6.8	Proof of functoriality	169
6.8.1	From circuits to Dirichlet cospans	170
6.8.2	From Dirichlet cospans to Lagrangian cospans	173
6.8.3	From cospans to relations	175
7	Further directions	179
7.1	Theoretical	179
7.1.1	Bounded colimits	179
7.1.2	Classifying hypergraph categories	179
7.1.3	Other categorical structures	179
7.2	Applied	179
7.2.1	Passive linear networks	179
7.2.2	Open Markov processes	179
7.2.3	Flow networks	179
7.2.4	Chemical reaction networks	180
	Notation and conventions	181
	Bibliography	182

Part I

Mathematical Foundations

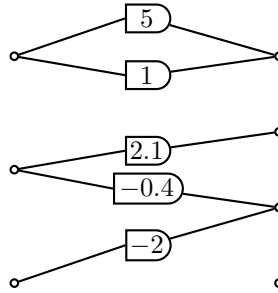
Chapter 4

Decorated corelations

When enough structure is available to us, we may decorate corelations too. Furthermore, and key to the idea of ‘black-boxing’, we get a hypergraph functor from decorated cospans to decorated corelations. The decorated corelations framework adds compositional operations to network-diagram representations, and handles composition of semantics too.

4.1 Black-boxed open systems

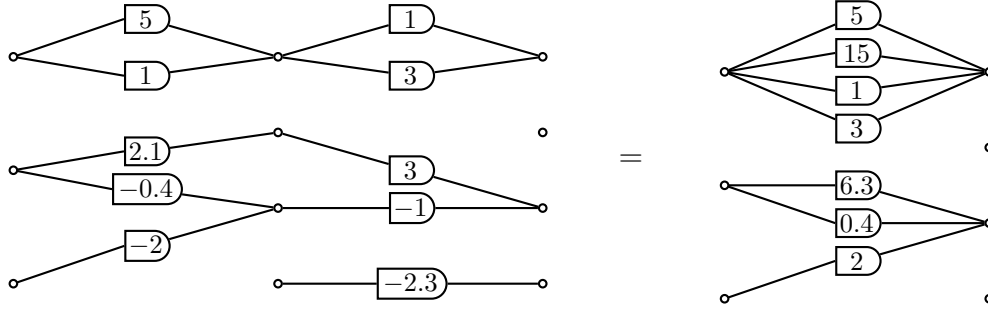
Suppose we have devices built from paths that take the signal at some input, amplify it, and deliver it to some output. For simplicity let these signals be real numbers, and amplification be linear: we just multiply by some fixed scalar. We depict an example device like so:



Here there are three inputs, four outputs, and five paths. Formally, we might model these devices as finite sets of inputs X , outputs Y , and paths N , together with functions $i: N \rightarrow X$ and $o: N \rightarrow Y$ describing the start and end of each path, and a function $s: N \rightarrow \mathbb{R}$ describing the amplification along it. In other words, these are cospans $X \rightarrow N \leftarrow Y$ in $\mathbf{FinSet}^{\text{opp}}$ decorated by ‘scalar assignment’ functions $N \rightarrow \mathbb{R}$. This suggests a decorated cospans construction.

Such a construction can be obtained from the lax symmetric monoidal functor $\mathbb{R}^{(-)}: \text{FinSet}^{\text{opp}} \rightarrow \text{Set}$ taking a finite set N to the set \mathbb{R}^N of functions $s: N \rightarrow \mathbb{R}$, and opposite functions $f^{\text{opp}}: N \rightarrow M$ to the map taking $s: N \rightarrow \mathbb{R}$ to $s \circ f: M \rightarrow \mathbb{R}$. Note that the coproduct in $\text{FinSet}^{\text{opp}}$ is the cartesian product of sets. The coherence maps of the functor which, recall, are critical for the composition of the decorations, are given by $\varphi_{N,M}: \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^{N \times M}$, taking $(s, t) \in \mathbb{R}^N \times \mathbb{R}^M$ to the function $s \cdot t: N \times M \rightarrow \mathbb{R}$ defined by pointwise multiplication in \mathbb{R} .

Composition in this decorated cospans category is thus given by the multiplication in \mathbb{R} . In detail, given decorated cospans $(X \xrightarrow{i_X^{\text{opp}}} N \xleftarrow{o_Y^{\text{opp}}} Y, N \xrightarrow{s} \mathbb{R})$ and $(Y \xrightarrow{i_Y^{\text{opp}}} M \xleftarrow{o_Z^{\text{opp}}} Z, M \xrightarrow{t} \mathbb{R})$, the composite has a path from $x \in X$ to $z \in Z$ for every triple (y, n, m) where $y \in Y$, $n \in N$, and $m \in M$, such that n is a path from x to y and m is a path from y to z . The scalar assigned to this path is the product of those assigned to n and m . For example, we have the following composite

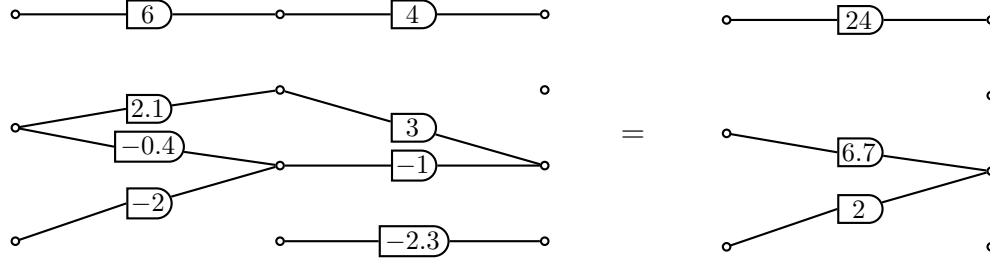


There are four paths between the top-most element x_1 of the domain and the top-most element z_1 of the codomain: we may first take the path that amplifies by $5\times$ and then the path that amplifies by $1\times$ for a total amplification of $5\times$, or $5\times$ and $3\times$ for $15\times$, and so on. This means we end up with four elements relating x_1 and z_1 in the composite. The apex of the composite is in fact given by the pullback $N \times_Y M$ of the cospan $N \xrightarrow{o_Y} Y \xleftarrow{i_Y} M$ in FinSet .

Here we again see the problem of decorated cospans: the composite of the above puts decorations on $N \times_Y M$, which can be of much larger cardinality than N and M . We wish to avoid the size of our decorated cospan from growing so fast. Moreover, from our open systems perspective, we care not about the path but by the total amplification of the signal from some chosen input to some chosen output. The intuition is that if we black box the system, then we cannot tell what paths the signal took through the system, only the total amplification from input to output.

We thus want to restrict our apex to contain at most one point for each input-output pair (x, y) , pushing the decoration along the quotient map e in the epi-mono

factorisation of the function $N \times_Y M \xrightarrow{e} \overline{N \times_Y M} \xrightarrow{m} X \times Z$. Put another way, we want the category of decorated corelations, not decorated cospans. Represented as decorated corelations, the above composite becomes



Note that composite is not simply the composite as decorated cospans, but the composite decorated cospan reduced to a decorated corelation. We will show that this decorated corelations category is equivalent to the category of real vector spaces and linear maps, with monoidal product the tensor product.

We prove two main theorems in this section. The first is that these decorated cospans do form a hypergraph category. That is, given a category \mathcal{C} with finite colimits, factorisation system $(\mathcal{E}, \mathcal{M})$ such that \mathcal{M} is stable under pushouts, and a lax symmetric monoidal functor

$$F: \mathcal{C}; \mathcal{M}^{\text{opp}} \longrightarrow \text{Set},$$

we may define a hypergraph category $F\text{Corel}$ with objects those of \mathcal{C} and morphisms decorated corelations. As usual, functors between these so-called decorated corelations categories can be defined from natural transformations between these functors.

The second major theorem is that every hypergraph category and hypergraph functor can be constructed in this way. This shows our decorated corelations construction is as general as we need for the study of hypergraph categories, and hence network-style diagrammatic languages.

4.2 Decorated corelations

The key difference is to decorate cospans we need to know how to push decorations up. To decorate corelations we all need to know how to pull decorations back down. This is related to the existence of an extraspecial commutative Frobenius monoid in our main applications.

4.2.1 Adjoining right adjoints

Suppose we have a cospan $X + Y \rightarrow N$ with a decoration on N . Reducing this to a corelation requires us to factor this to $X + Y \xrightarrow{c} \overline{N} \xrightarrow{m} N$. To define a category of decorated corelations, then, we must specify how to take decoration on N and ‘pull it back’ along m to a decoration on \overline{N} .

For decorated cospans, it is enough to have a functor F from a category \mathcal{C} with finite colimits; the image Ff of morphisms f in \mathcal{C} describes how to move decorations forward along f . In this subsection we explain how to expand \mathcal{C} to include morphisms m^{opp} for each $m \in \mathcal{C}$, so that the image of m^{opp} describes how to move morphisms backwards along m .

Proposition 4.1. *Let \mathcal{C} be a category with finite colimits, and let \mathcal{M} be a subcategory of \mathcal{C} stable under pushouts. Then we define the category $\mathcal{C}; \mathcal{M}^{\text{opp}}$ as follows*

The symmetric monoidal category $(\mathcal{C}; \mathcal{M}^{\text{opp}}, +)$	
objects	the objects of \mathcal{C}
morphisms	isomorphism classes of cospans of the form $\xrightarrow{c} \xleftarrow{m}$, where c lies in \mathcal{C} and m in \mathcal{M}
composition	given by pushout
monoidal product	the coproduct in \mathcal{C}
coherence maps	the coherence maps in \mathcal{C}

Proof. This is a symmetric monoidal subcategory of $\text{Cospan}(\mathcal{C})$. Our data is well-defined: composition because \mathcal{M} is stable under pushouts, and monoidal composition by Lemma 3.9. \square

This category can be viewed as a bicategory, with 2-morphisms given by maps of cospans. In this bicategory every morphism of \mathcal{M} has a right adjoint.

Examples 4.2. Our familiar examples:

- $\mathcal{C}; \mathcal{C}^{\text{opp}}$ is by definition equal to $\text{Cospan}(\mathcal{C})$.
- $\mathcal{C}; \mathcal{I}_{\mathcal{C}}^{\text{opp}}$ is naturally isomorphic to \mathcal{C} .
- $\text{Set}; \text{Inj}^{\text{opp}}$ is the category of partial functions.

Lemma 4.3. *Let $\mathcal{C}, \mathcal{C}'$ be categories with finite colimits, and let $\mathcal{M}, \mathcal{M}'$ be subcategories each stable under pushouts. Let $A: \mathcal{C} \rightarrow \mathcal{C}'$ be functor that preserves colimits*

and such that the image of \mathcal{M} lies in \mathcal{M}' . Then A extends to a symmetric strong monoidal functor

$$A: \mathcal{C}; \mathcal{M}^{\text{opp}} \longrightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}.$$

mapping X to AX and $\xrightarrow{c} \xleftarrow{m}$ to $\xrightarrow{Ac} \xleftarrow{Am}$.

Proof. Note $A(\mathcal{M}) \subseteq \mathcal{M}'$, so $\xrightarrow{Ac} \xleftarrow{Am}$ is indeed a morphism in $\mathcal{C}'; \mathcal{M}'^{\text{opp}}$. This is then a restriction and corestriction of the usual functor $\text{Cospan}(\mathcal{C}) \rightarrow \text{Cospan}(\mathcal{C}')$ to the above domain and codomain. \square

That this ‘subcospan category’ construction could be defined more generally using any two isomorphism-containing wide subcategories stable under pushout, but the above suffices for decorated corelations.

4.2.2 Decorated corelations

Decorated corelations are constructed from a lax monoidal functor from $\mathcal{C}; \mathcal{M}^{\text{opp}}$ to Set .

Definition 4.4. Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . Suppose we also have a lax monoidal functor

$$F: (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

We define an F -**decorated corelation** to a pair

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right)$$

where the cospan is jointly \mathcal{E} -like. A morphism of decorated corelations is a morphism of decorated cospans between two decorated corelations.

Suppose we have decorated corelations

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}, \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right) \quad \text{and} \quad \left(\begin{array}{ccc} & M & \\ i \nearrow & & \nwarrow o \\ Y & & Z \end{array}, \quad \begin{array}{c} FM \\ \uparrow s \\ 1 \end{array} \right).$$

Then their composite is given by the composite corelation

$$\begin{array}{ccc} & \overline{N +_Y M} & \\ i \nearrow & & \nwarrow o \\ X & & Z \end{array}$$

paired with the decoration

$$1 \longrightarrow F(N + M) \longrightarrow F(N +_Y M) \xrightarrow{F(m^{\text{opp}})} F(\overline{N +_Y M})$$

As composition of corelations and decorated cospans are both well-defined up to isomorphism, this too is well-defined up to isomorphism. Again, we will be lazy about the distinction between a decorated corelation and its isomorphism class.

4.2.3 Categories of decorated corelations

In this subsection we give a definition of the hypergraph category $F\text{Corel}$ of decorated corelations. In analogy with how we defined the hypergraph category of corelations, we leverage the fact that decorated cospans form a hypergraph category, this time using a structure preserving map

$$\square: F\text{Cospan} \longrightarrow F\text{Corel}.$$

Here $F\text{Cospan}$ denotes the decorated cospan category constructed from the restriction of the functor $F: \mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \text{Set}$ to the domain \mathcal{C} .

Given a cospan $X \rightarrow N \leftarrow Y$, write $m: \overline{N} \rightarrow N$ for the \mathcal{M} factor of the copairing $X + Y \rightarrow N$. The functor \square takes a decorated cospan

$$(X \xrightarrow{i} N \xleftarrow{o} Y, 1 \xrightarrow{s} FN)$$

to the decorated corelation

$$(X \xrightarrow{\bar{i}} \overline{N} \xleftarrow{\bar{o}} Y, 1 \xrightarrow{Fm^{\text{opp}} \circ s} F\overline{N}),$$

where the corelation is given by the jointly \mathcal{E} -part of the cospan, and the decoration is given by composing s with the F -image $Fm^{\text{opp}}: FN \rightarrow F\overline{N}$ of the map $N \xrightarrow{1_N} N \xleftarrow{m} \overline{N}$ in $\mathcal{C}; \mathcal{M}^{\text{opp}}$. We call $Fm^{\text{opp}} \circ s$ the **restricted decoration** of the decorated cospan $(X \rightarrow N \leftarrow Y, 1 \xrightarrow{s} FN)$.

The monoidal product of two decorated corelations is their monoidal product as decorated cospans.

Theorem 4.5. *Let \mathcal{C} be a category with finite colimits and factorisation system $(\mathcal{E}, \mathcal{M})$ with \mathcal{M} stable under pushout, and let*

$$F: (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times)$$

be a lax symmetric monoidal functor. Then we may define

<i>The hypergraph category $(F\text{Corel}, +)$</i>	
objects	<i>the objects of \mathcal{C}</i>
morphisms	<i>isomorphism classes of F-decorated corelations in \mathcal{C}</i>
composition	<i>given by \mathcal{E}-part of pushout with restricted decoration</i>
monoidal product	<i>the coproduct in \mathcal{C}</i>
coherence maps	<i>maps from $\text{Cospan}(\mathcal{C})$ with restricted empty decoration</i>
hypergraph maps	<i>maps from $\text{Cospan}(\mathcal{C})$ with restricted empty decoration</i>

Similar to the corelations theorem (Theorem 3.10), we have specified well-defined data and now need to check a collection of coherence axioms. As before, we prove this alongside the theorem about decorated corelation functors in the next section.

Example 4.6. Note that decorated cospans are a special case of decorated corelations: simply use an morphism–isomorphism factorisation system.

Example 4.7. Note that ‘undecorated’ corelations are a special case of decorated corelations: they are corelations decorated by the functor $1: \mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \text{Set}$ that maps each object to the one element set 1, and each morphism to the identity function on 1. This is a symmetric monoidal functor with the coherence maps all also the identity function on 1.

4.3 Functors between decorated corelation categories

Functors between decorated corelation categories hold no surprises: their requirements combine the requirements of corelations and decorated cospans. Recall that Lemma 4.3 says that we can extend a colimit-preserving functor $\mathcal{C} \rightarrow \mathcal{C}'$ to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}$.

Proposition 4.8. *Let $\mathcal{C}, \mathcal{C}'$ have finite colimits and respective factorisation systems $(\mathcal{E}, \mathcal{M}), (\mathcal{E}', \mathcal{M}')$, such that \mathcal{M} and \mathcal{M}' are stable under pushout, and suppose that we have lax symmetric monoidal functors*

$$F : (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times)$$

and

$$G : (\mathcal{C}'; \mathcal{M}'^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

Further let $A: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor that preserves finite colimits and such that the image of \mathcal{M} lies in \mathcal{M}' . This functor A extends to a symmetric monoidal functor $\mathcal{C}; \mathcal{M}^{\text{opp}} \rightarrow \mathcal{C}'; \mathcal{M}'^{\text{opp}}$.

Suppose we have a monoidal natural transformation θ :

$$\begin{array}{ccc} \mathcal{C}; \mathcal{M}^{\text{opp}} & \xrightarrow{F} & \text{Set} \\ A \downarrow & \Downarrow \theta & \uparrow G \\ \mathcal{C}'; \mathcal{M}'^{\text{opp}} & \xrightarrow{G} & \text{Set} \end{array}$$

Then we may define a hypergraph functor $T: F\text{Corel} \rightarrow G\text{Corel}$ sending each object $X \in F\text{Corel}$ to $AX \in G\text{Corel}$ and each decorated corelation

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, \quad 1 \xrightarrow{s} FN)$$

to

$$(AX \xrightarrow{Ai_X} \overline{AN} \xleftarrow{Ao_Y} AY, \quad 1 \xrightarrow{s} FN \xrightarrow{\theta_N} GAN \xrightarrow{Gm_{AN}^{\text{opp}}} \overline{GAN}).$$

The coherence maps $\overline{\kappa_{X,Y}}$ are given by the coherence maps of A with the restricted empty decoration.

Proof of Theorem 4.5 and Proposition 4.8. In the proof of Theorem 3.10 and Proposition 3.11 we proved that the map

$$\square: \text{Corel}(\mathcal{C}) \longrightarrow \text{Corel}(\mathcal{C}')$$

preserved composition and had natural coherence maps. Specialising to the case when $\text{Corel}(\mathcal{C}) = \text{Cospan}(\mathcal{C}')$, we saw that this bijective-on-objects, surjective-on-morphisms, composition and monoidal product preserving map proved $\text{Corel}(\mathcal{C}')$ is a hypergraph category, and it immediately followed that \square is a hypergraph functor.

The analogous argument holds here: we simply need to prove

$$\square: F\text{Corel} \longrightarrow G\text{Corel}$$

preserves composition and has natural coherence maps. Theorem 4.5 then follows from examining the map $F\text{Cospan} \rightarrow F\text{Corel}$ obtained by choosing $\mathcal{C} = \mathcal{C}'$, $(\mathcal{E}, \mathcal{M}) = (\mathcal{C}', \mathcal{I}_{\mathcal{C}'})$, F the restriction of G to \mathcal{C}' , A the identity functor on \mathcal{C}' , and θ the identity natural transformation. Subsequently Proposition 4.8 follows from noting that all the axioms hold for the corresponding maps in $G\text{Cospan}$.

□ **preserves composition.** Suppose we have decorated corelations

$$f = (X \xrightarrow{i_X} N \xleftarrow{o_Y} Y, 1 \xrightarrow{s} FN) \quad \text{and} \quad g = (Y \xrightarrow{i_Y} M \xleftarrow{o_Y} Z, 1 \xrightarrow{t} FM)$$

We know the functor □ preserves composition on the cospan part; this is precisely the content of Proposition 3.11. It remains to check that □($g \circ f$) and □ $g \circ$ □ f have isomorphic decorations. This is expressed by the commutativity of the following diagram:

$$\begin{array}{ccccc}
 \overline{GA(N+Y\ M)} & \xrightarrow{Gn} & \overline{G(\overline{AN}+_{AY}\ \overline{AM})} & & \\
 \uparrow Gm_{A(N+Y\ M)}^{\text{opp}} & & \uparrow Gm_{\overline{AN}+_{AY}\ \overline{AM}}^{\text{opp}} & & \\
 \overline{GA(N+Y\ M)} & & \overline{G(\overline{AN}+_{AY}\ \overline{AM})} & & \\
 \uparrow \theta_{N+Y\ M} & \swarrow GAm_{N+Y\ M}^{\text{opp}} & & \searrow G(m_{AN}^{\text{opp}}+_{AY}m_{AM}^{\text{opp}}) & \\
 F(\overline{N+Y\ M}) & \xleftarrow{G\sim} & G(\overline{AN}+_{AY}\ \overline{AM}) & \xleftarrow{(c)} & G(\overline{AN}+_{AY}\ \overline{AM}) \\
 \uparrow Fm_{N+Y\ M}^{\text{opp}} & & \uparrow GA[j_N, j_M] & & \uparrow G[j_{AN}, j_{AM}] \\
 F(N+Y\ M) & \xleftarrow{(TN)} & GA(N+Y\ M) & \xleftarrow{(A)} & G(AN+_{AY}\ AM) \\
 \uparrow F[j_N, j_M] & & \uparrow G\alpha_{N,M} & & \uparrow G(m_{AN}^{\text{opp}}+m_{AM}^{\text{opp}}) \\
 F(N+Y\ M) & \xleftarrow{\theta_{N+M}} & GA(N+M) & \xleftarrow{(GM)} & G\overline{AN} \times G\overline{AM} \\
 \uparrow & & \uparrow & & \uparrow \gamma_{\overline{AN}, \overline{AM}} \\
 F(N+M) & \xleftarrow{\varphi_{N,M}} & FN \times FM & \xleftarrow{\theta_N \times \theta_M} & GAN \times GAM \\
 \uparrow \rho_1 \circ (s \times t) & & \uparrow & & \uparrow Gm_{AN}^{\text{opp}} \times Gm_{AM}^{\text{opp}} \\
 1 & & 1 & & 1
 \end{array}$$

This diagram does indeed commute. To check this, first observe that (TM) commutes by the monoidality of θ , (GM) commutes by the monoidality of G , and (TN) commutes by the naturality of θ . The remaining three diagrams commute as they are G -images of diagrams that commute in \mathcal{C}' ; $\mathcal{M}'^{\text{opp}}$. Indeed, (A) commutes since A preserves colimits and G is functorial, (C) commutes as it is the G -image of a pushout square in \mathcal{C}' , so

$$m_{AN}^{\text{opp}} + m_{AM}^{\text{opp}} [j_{AN}, j_{AM}] \quad \text{and} \quad [j_{AN}, j_{AM}] m_{AN}^{\text{opp}} + m_{AM}^{\text{opp}}$$

are equal as morphisms of \mathcal{C}' ; $\mathcal{M}'^{\text{opp}}$, and (**) commutes as it is the G -image of the right-hand subdiagram of (*) used to define n in the proof of Lemma 3.12.

Coherence maps are natural. Let $f = (X \longrightarrow N \longleftarrow Y, 1 \rightarrow FN)$, $g = (Z \longrightarrow M \longleftarrow W, 1 \rightarrow FM)$ be F -decorated corelations in \mathcal{C} . We wish to show that

$$\begin{array}{ccc} AX + AY & \xrightarrow{\square f + \square g} & AZ + AW \\ \overline{\kappa_{X,Y}} \downarrow & & \downarrow \overline{\kappa_{Z,W}} \\ A(X + Y) & \xrightarrow{\square(f+g)} & A(Z + W) \end{array}$$

commutes in $G\text{Corel}$, where the coherence maps are given by

$$\overline{\kappa_{X,Y}} = \left(\begin{array}{ccc} & \overline{A(X + Y)} & \\ \nearrow & & \nwarrow \\ AX + AY & & A(X + Y) \end{array} \right), \quad \begin{array}{c} G(\overline{A(X + Y)}) \\ \uparrow Gm_{AX+AY}^{\text{opp}} \\ GA(X + Y) \\ \uparrow G! \\ G\emptyset \\ \uparrow \gamma_1 \\ 1 \end{array} \right).$$

Lemma 3.14 shows that the composites of corelations agree. It remains to check that the decorations also agree.

Here Lemma 2.5 is helpful. Since \square is composition preserving, we can replace the $\overline{\kappa}$ with the empty decorated coherence maps κ of $G\text{Cospan}$, and compute these composites in $G\text{Cospan}$, before restricting to the \mathcal{E}' -parts. Lemma 2.5 then implies that the restricted empty decorations on the isomorphisms $\overline{\kappa}$ play no role in determining the composite decorations. It is thus enough to prove that the decorations of $\square f + \square g$ and $\square(f + g)$ are the same up to the isomorphism $p: G(\overline{AN} + \overline{AM}) \rightarrow G\overline{A(N + M)}$ between their apices, as defined in the diagram (#) in the proof of Lemma 3.14.

This comes down to proving the following diagram commutes:

$$\begin{array}{ccccc} & GAN \times GAM & \xrightarrow{Gm \times Gm} & G\overline{AN} \times G\overline{AM} & \xrightarrow{\gamma} & G(\overline{AN} + \overline{AM}) \\ & \uparrow \theta & & \downarrow \gamma & \nearrow G(m+m) & \downarrow \sim Gp \\ 1 \xrightarrow{\langle s, t \rangle} & FN \times FM & & G(AN + AM) & & \\ & \downarrow \varphi & & \downarrow G\kappa & \nearrow (\#\#) & \\ & F(N + M) & \xrightarrow{\theta} & GA(N + M) & \xrightarrow{Gm} & G\overline{A(N + M)} \end{array}$$

It is straightforward to check this commutes: (T) by the monoidality of θ , (G) by the monoidality of G , and ($\#\#$) as it is the G -image of the rightmost square in (#). \square

Corollary 4.9. Let \mathcal{C} be a category with finite colimits, and let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on \mathcal{C} . Suppose that we also have a lax monoidal functor

$$F : (\mathcal{C}; \mathcal{M}^{\text{opp}}, +) \longrightarrow (\text{Set}, \times).$$

Then we may define a category $F\text{Corel}$ with objects the objects of \mathcal{C} and morphisms isomorphism classes of F -decorated corelations.

Write also F for the restriction of F to the wide subcategory \mathcal{C} of $\mathcal{C}; \mathcal{M}^{\text{opp}}$. We can thus also obtain the category $FCospan$ of F -decorated cospan. We moreover have a functor

$$FCospan \rightarrow F\text{Corel}$$

which takes each object of $FCospan$ to itself as an object of $F\text{Corel}$, and each decorated cospan

$$\left(\begin{array}{ccc} & N & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array} , \quad \begin{array}{c} FN \\ \uparrow s \\ 1 \end{array} \right)$$

to its jointly- \mathcal{E} -part

$$\begin{array}{ccc} & \overline{N} & \\ i \nearrow & & \nwarrow o \\ X & & Y \end{array}$$

decorated by the composite

$$1 \xrightarrow{s} FN \xrightarrow{Fm_N^{\text{opp}}} F\overline{N}.$$

4.4 All hypergraph categories are decorated correlation categories

Not all hypergraph categories are decorated *cospan* categories. To see this, we can count morphisms. In a decorated cospan category the possible apices and decorations are the same for all morphisms. So unless the category \mathcal{C} and thus $\text{Cospan}(\mathcal{C})$ is trivial, there must be a number of morphisms $0 \rightarrow 0$. On the other hand, the hypergraph category of finite sets and equivalence relations, discussed in Subsection 3.4.1, has a unique morphism $0 \rightarrow 0$.

Decorated correlation categories, however, are more powerful. In this section we show that all hypergraph categories are decorated correlation categories, and provide some examples for intuition.

4.4.1 The global sections construction

Suppose we have a hypergraph category and wish to construct a hypergraph equivalent decorated correlation category. The main idea is to take advantage of the compact closed structure: recall that morphisms $X \rightarrow Y$ in a hypergraph category

are in one-to-one correspondence with morphisms $I \rightarrow X \otimes Y$. Using an isomorphism–morphism factorisation system on the category comprising just the objects and the coherence and hypergraph maps, we can use the monoidal global sections functor to decorate each corelation $X \rightarrow X \otimes Y \leftarrow Y$ with the monoidal elements of $X \otimes Y$. This recovers the original hypergraph category.

Theorem 4.10. *Every hypergraph category is hypergraph equivalent to a decorated corelation category.*

Proof. Let (\mathcal{H}, \otimes) be a hypergraph category. By Theorem 1.5, we have a hypergraph equivalent strict hypergraph category \mathcal{H}_{str} , with objects finite lists of objects in \mathcal{H} . Write \mathcal{O} for the collection of objects in \mathcal{H} , and $\text{FinSet}_{\mathcal{O}}$ for \mathcal{O} -labelled finite sets (i.e. an object is a finite set S together with each element in S labelled by some object in \mathcal{O} , and a morphism is a function that preserves labels). The category $\text{FinSet}_{\mathcal{O}}$ has finite colimits: the colimit is just the colimit set in FinSet with elements labelled by the label of any element that maps to it in the colimit diagram. This labelling exists and is unique by the universal property of the colimit. Write $+$ for the coproduct; we consider $\text{FinSet}_{\mathcal{O}}$ a symmetric monoidal category. Replacing FinSet with an equivalent skeleton, we think of the objects of $\text{FinSet}_{\mathcal{O}}$ as finite lists of objects of \mathcal{H} , and hence as the same as the objects of \mathcal{H}_{str} .

We define a wide hypergraph embedding

$$H: (\text{Cospan}(\text{FinSet}_{\mathcal{O}}), +) \longrightarrow (\mathcal{H}_{\text{str}}, \otimes)$$

whose ‘image’ is the hypergraph structure of \mathcal{H} . More precisely, H sends each object of $\text{Cospan}(\text{FinSet}_{\mathcal{O}})$ to the same as an object of \mathcal{H}_{str} . As all functions can be generated under composition and coproduct from the unique functions $\mu: 2 \rightarrow 1$ and $\eta: 0 \rightarrow 1$, the morphisms of $\text{FinSet}_{\mathcal{O}}$ are generated by the unique morphisms $\mu_X: [X, X] = X + X \rightarrow X$ and $\eta_X: \emptyset \rightarrow X$, and so the morphisms of $\text{Cospan}(\text{FinSet}_{\mathcal{O}})$ are generated by the cospans $\mu_X, \eta_X, \delta_X = \mu_X^{\text{opp}}$, and $\epsilon_X = \eta_X^{\text{opp}}$. The functor H maps these generators to the corresponding Frobenius maps on X in \mathcal{H}_{str} . Since $\text{Cospan}(\text{FinSet})$ is the ‘generic special commutative monoid’ (see Proposition 1.6), it is straightforward to check this defines a hypergraph functor.

Next, recall that the monoidal global sections functor $\mathcal{H}(I, -): \mathcal{H} \rightarrow \text{Set}$ is a lax symmetric monoidal functor taking each object X of \mathcal{H} to the homset $\mathcal{H}(I, X)$, and recall we have a hypergraph equivalence $\mathcal{H}_{\text{str}} \rightarrow \mathcal{H}$. Write F for the composite of these three functors:

$$F: \text{Cospan}(\text{FinSet}_{\mathcal{O}}) \xrightarrow{H} \mathcal{H}_{\text{str}} \xrightarrow{\sim} \mathcal{H} \xrightarrow{\mathcal{H}(I, -)} \text{Set}$$

This is lax symmetric monoidal functor mapping X to the homset $\mathcal{H}(I, X)$, and a cospan between lists of objects in \mathcal{H} to the corresponding Frobenius map.

Now consider $\text{FinSet}_{\mathcal{O}}$ with an (isomorphism, morphism)-factorisation system. This implies F defines a decorated correlation category $F\text{Corel}$ with objects those of \mathcal{H}_{str} , and morphisms $X \rightarrow Y$ trivial corelations $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$ decorated by some morphism $s \in \mathcal{H}(I, X \otimes Y)$. We will show $(F\text{Corel}, +)$ and $(\mathcal{H}_{\text{str}}, \otimes)$ are isomorphic as hypergraph categories.

First, let us examine composition in $F\text{Corel}$. As morphisms $X \rightarrow Y$ are completely specified by their decoration $s \in \mathcal{H}(I, X \otimes Y)$, we will abuse terminology and refer to the decorations themselves as the morphisms. Given morphisms $s \in \mathcal{H}(I, X \otimes Y)$ and $t \in \mathcal{H}(I, Y \otimes Z)$ in $F\text{Corel}$, composition is given by the map

$$H(I, X \otimes Y \otimes Y \otimes Z) \rightarrow H(I, X \otimes Z)$$

arising as the F -image of the cospan $X + Y + Y + Z \xrightarrow{[j_{X+Y}, j_{Y+Z}]} X + Y + Z \xleftarrow{[\iota_X, \iota_Z]} X + Z$, where these maps come from the pushout square

$$\begin{array}{ccccc}
 & & X + Y + Z & & \\
 & \nearrow^{j_{X+Y}} & & \nwarrow_{j_{Y+Z}} & \\
 X + Y & & & & Y + Z \\
 \nwarrow_{\iota_X} & & & & \nearrow_{\iota_Z} \\
 X & & Y & & Z
 \end{array}$$

and the trivial isomorphism-morphism factorisation $X + Z = X + Z \xrightarrow{[\iota_X, \iota_Z]} X + Y + Z$ in $\text{FinSet}_{\mathcal{O}}$.

In terms of string diagrams in \mathcal{H} , this means composing the maps

$$\begin{array}{c} \boxed{s} \end{array} \begin{array}{l} \text{---} X \\ \text{---} Y \end{array} \quad \begin{array}{c} \boxed{t} \end{array} \begin{array}{l} \text{---} Y \\ \text{---} Z \end{array}$$

with the Frobenius map

$$\begin{array}{c} X \text{---} X \\ Y \text{---} \\ Y \text{---} \\ Z \text{---} Z \end{array} \quad = \quad \begin{array}{c} X \text{---} X \\ Y \text{---} \bullet \\ Y \text{---} \\ Z \text{---} Z \end{array}$$

to get

$$\begin{array}{c} \boxed{t \circ s} \end{array} \begin{array}{l} \text{---} X \\ \text{---} Z \end{array} \quad = \quad \begin{array}{c} \boxed{s} \\ \boxed{t} \end{array} \begin{array}{l} \text{---} X \\ \text{---} Z \end{array}$$

in $\mathcal{H}(I, X \otimes Z)$.

The monoidal product is given by

$$\begin{array}{|c|} \hline s \\ \hline \end{array} \begin{array}{c} X \\ Y \end{array} + \begin{array}{|c|} \hline t \\ \hline \end{array} \begin{array}{c} Z \\ W \end{array} = \begin{array}{|c|} \hline s \\ \hline \end{array} \begin{array}{c} X \\ Z \\ Y \\ W \end{array}$$

Taking a hint from the compact closed structure, the isomorphism between $F\text{Corel}$ and \mathcal{H}_{str} is the clear: the functors act as the identity on objects, and on morphisms take $f : A \rightarrow B$ in \mathcal{H} to its name $\hat{f} : I \rightarrow A + B$ as a morphism of $F\text{Corel}$, and vice versa.

It is straightforward to check that these are strict hypergraph functors. To demonstrate the most subtle aspect, the Frobenius structure, we consider the multiplication on an object X of $F\text{Corel}$. To obtain this, we start with the multiplication in $F\text{Cospan}$, the cospan $X + X \xrightarrow{[1,1]} X \xleftarrow{1} X$ decorated with the empty decoration, which in this case is $\eta_X \in \mathcal{H}(I, X)$, and restrict this empty decoration along the map $X \xleftarrow{[1,1,1]} X + X + X$. Thus the multiplication on X in $F\text{Corel}$ is

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \\ X \\ X \\ X \end{array} \in \mathcal{H}(I, X \otimes X \otimes X),$$

which corresponds under our isomorphism to the map $\mu_X : X \otimes X \rightarrow X \in \mathcal{H}_{\text{str}}$, as required. \square

We can also recover hypergraph functors using decorated corelations.

Theorem 4.11. *All hypergraph functors are decorated corelation functors.*

Proof. Let $(T, \tau) : \mathcal{H} \rightarrow \mathcal{H}'$ be a hypergraph functor between hypergraph categories. The previous theorem implies there exist lax symmetric monoidal functors

$$F : \text{Cospan}(\text{FinSet}_{\mathcal{O}}) \rightarrow \text{Set} \quad \text{and} \quad F' : \text{Cospan}(\text{FinSet}_{\mathcal{O}'}) \rightarrow \text{Set}$$

such that we have isomorphisms $\mathcal{H}_{\text{str}} \cong F\text{Corel}$ and $\mathcal{H}'_{\text{str}} \cong F'\text{Corel}$. We will realise T as the functor induced by some colimit-preserving functor $A : \text{FinSet}_{\mathcal{O}} \rightarrow \text{FinSet}_{\mathcal{O}'}$ and monoidal natural transformation:

$$\begin{array}{ccc} \text{Cospan}(\text{FinSet}_{\mathcal{O}}) & \xrightarrow{F} & \text{Set} \\ \downarrow A & \searrow \Downarrow \theta & \\ \text{Cospan}(\text{FinSet}_{\mathcal{O}'}) & \xrightarrow{F'} & \end{array}$$

Define the functor A to take each \mathcal{O} -labelled set $[X_1, \dots, X_n]$ to the same set now \mathcal{O}' -labelled $[TX_1, \dots, TX_n]$, and to take a function between \mathcal{O} -labelled sets to the same function, noting that the function now preserves the \mathcal{O}' -labels. This functor A clearly preserves colimits: again, colimits in $\text{FinSet}_{\mathcal{O}}$ are just colimits in FinSet with the labels inherited as described above.

Let $S = [X_1, \dots, X_n]$ be an \mathcal{O} -labelled set. Then we define

$$\begin{aligned} \theta_S: \mathcal{H}(I, X_1 \otimes \dots \otimes X_n) &\longrightarrow \mathcal{H}'(I, TX_1 \otimes \dots \otimes TX_n); \\ \left(I \xrightarrow{s} X_1 \otimes \dots \otimes X_n \right) &\longmapsto \left(I \xrightarrow{\tau} TI \xrightarrow{T_s} T(X_1 \otimes \dots \otimes X_n) \xrightarrow{\tau} TX_1 \otimes \dots \otimes TX_n \right), \end{aligned}$$

where the τ are the appropriate coherence isomorphisms of T . This collection of maps θ is natural as functions in $\text{FinSet}_{\mathcal{O}}$ act as the Frobenius maps on the homsets $\mathcal{H}(I, X)$ and $\mathcal{H}'(I, TX)$, and the functor T is hypergraph and so maps, loosely speaking, Frobenius maps to Frobenius maps. Moreover θ is monoidal as T is monoidal. Thus θ defines a monoidal natural transformation.

The hypergraph functor $F\text{Corel} \rightarrow F'\text{Corel}$ induced by θ is by definition the map taking $s \in \mathcal{H}(I, X \otimes Y)$ to $\tau^{-1} \circ Ts \circ \tau_1 \in \mathcal{H}'(I, TX \otimes TY)$. Composing this functor with the appropriate equivalences we recover $T: \mathcal{H} \rightarrow \mathcal{H}'$. \square

4.4.2 A categorical equivalence

Choose Grothendieck universes so that we may talk about categories Set now of all ‘small’ sets, and Cat of all ‘small’ categories. If we restrict our attention to small hypergraph categories, we may summarise these results as a categorical equivalence.

Indeed, given some fixed object set \mathcal{O} , we have a category of lax symmetric monoidal functors and monoidal natural transformations

$$\text{LaxSymMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}}), \text{Set}).$$

This is equivalent to some subcategory of the category HypCat of hypergraph categories. To get a category equivalent to all hypergraph categories, we must patch together these functor categories by varying the object set and specifying morphisms from objects in one category to objects in another. For this we use the Grothendieck construction.

Given a (contravariant) functor $S: \mathcal{B} \rightarrow \text{Cat}$ from some category \mathcal{B} to Cat , we define the **(contravariant) Grothendieck construction** $\mathcal{B} \int S$ to be the category with objects pairs (\mathcal{O}, F) where \mathcal{O} is an object of \mathcal{B} and F is an object of $S\mathcal{O}$, and morphisms pairs

$$(f, \theta): (\mathcal{O}, F) \longrightarrow (\mathcal{O}', F')$$

where $f: \mathcal{O} \rightarrow \mathcal{O}'$ is a morphism in \mathcal{B} and $\theta: F \rightarrow Sr(F')$ is a morphism in $S\mathcal{O}$.

Now, define the functor

$$\text{LaxSymMon}(\text{Cospan}(\text{FinSet}_-), \text{Set}): \text{Set} \longrightarrow \text{Cat}$$

as follows. On objects let it map a set \mathcal{O} to the lax symmetric monoidal functor category $\text{LaxSymMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}}), \text{Set})$. For morphisms, recall from the above proof that a function $r: \mathcal{O} \rightarrow \mathcal{O}'$ induces a functor $R: \text{Cospan}(\text{FinSet}_{\mathcal{O}}) \rightarrow \text{Cospan}(\text{FinSet}_{\mathcal{O}'})$ taking each \mathcal{O} -labelled set $N \xrightarrow{l} \mathcal{O}$ to the \mathcal{O}' -labelled set $N \xrightarrow{r \circ l} \mathcal{O}'$. This in turn defines a functor

$$\begin{aligned} \rho: \text{LaxSymMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}'}) , \text{Set}) &\longrightarrow \text{LaxSymMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}}), \text{Set}); \\ (F': \text{Cospan}(\text{FinSet}_{\mathcal{O}'}) \rightarrow \text{Set}) &\longmapsto (R \circ F': \text{Cospan}(\text{FinSet}_{\mathcal{O}}) \rightarrow \text{Set}). \end{aligned}$$

The Grothendieck construction

$$\text{Set} \int \text{LaxSymMon}(\text{Cospan}(\text{FinSet}_-), \text{Set})$$

thus gives a category where the objects are some label set \mathcal{O} together with an object in $\text{LaxSymMon}(\text{Cospan}(\text{FinSet}_{\mathcal{O}}), \text{Set})$ —that is, with a lax symmetric monoidal functor $F: (\text{Cospan}(\text{FinSet}_{\mathcal{O}}), +) \rightarrow (\text{Set}, \times)$. The morphisms $(\mathcal{O}, F) \rightarrow (\mathcal{O}', F')$ are functions of label sets $r: \mathcal{O} \rightarrow \mathcal{O}'$ together with a natural transformation $\theta: R \circ F' \Rightarrow F$.

In the case of small categories, it is not difficult to show Theorems 4.10 and 4.11 imply:

Theorem 4.12. *There is an equivalence of categories*

$$\text{HypCat} \cong \text{Set} \int \text{LaxSymMon}(\text{Cospan}(\text{FinSet}_-), \text{Set}).$$

The 2-categorical version of the above equivalence is the also the subject of a forthcoming paper by Vagner, Spivak, and Schultz, from their operadic perspective [?].

4.4.3 Factorisations as decorations

We have seen that every hypergraph category can be constructed as a decorated correlations category. More precisely, we have seen that every hypergraph category can be constructed as a decorated correlation category with the factorisation system the trivial isomorphism-morphism factorisation system. But decorated correlation

categories may be constructed using other factorisation systems, and these are also hypergraph categories. How do the two constructions relate to each other?

For each hypergraph category there is a poset of lax symmetric monoidal functors that give decorated corelation constructions of an equivalent category. Although the constructed categories are all equivalent, decorated corelations only allows functors decrease the number of corelations available; corelations are ultimately a ‘quotienting’ construction. The utility of decorated corelations is that we can maintain equivalence by putting this extra, quotiented information into the decoration.

For illustration, we first example this interaction for the simplest hypergraph category: $\text{Cospan}(\text{FinSet})$, the free hypergraph category on a single object.

Example 4.13. As per Example 4.7, $\text{Cospan}(\text{FinSet})$ is the hypergraph category of undecorated (morphism-isomorphism)-corelations in FinSet . Theorem 4.10 shows it is also the partition-decorated (isomorphism-morphism)-corelations in FinSet .

First, the global sections functor $G: \text{Cospan}(\text{FinSet}) \rightarrow \text{Set}$ takes each finite set X to the set of (equivalence classes of) cospans $0 \rightarrow D \leftarrow X$; that is, to the set GX of functions $s: X \rightarrow D$, where a unique codomain D is chosen for each finite cardinality. Given a cospan $X \xrightarrow{f} N \xleftarrow{g} Y$, its image under the global sections functor is the functor $GX \rightarrow GY$ mapping $s: X \rightarrow D$ in GX to the function $Y \rightarrow N +_Y D$ in

$$\begin{array}{ccccc} X & \xrightarrow{s} & D \\ f \downarrow & & \downarrow \\ Y & \xrightarrow{g} & N \longrightarrow N +_Y D \end{array}$$

where the square is a pushout square. The coherence maps $\gamma_1: 1 \rightarrow G\emptyset$ map the unique element of 1 to the unique function $!: \emptyset \rightarrow \emptyset$, and $\gamma_{X,Y}$ maps a pair of functions $a: X \rightarrow D$, $b: Y \rightarrow E$ to $a + b: X + Y \rightarrow D + E$. This defines a lax symmetric monoidal functor (G, γ) .

A decorated *cospan* in FinSet with respect to (G, γ) is a cospan of finite sets $X \rightarrow N \leftarrow Y$ together with a function of finite sets $N \rightarrow D$. Using the isomorphism-morphism factorisation, a decorated *corelation* is thus a cospan $X \xrightarrow{\iota_X} X + Y \xleftarrow{\iota_Y} Y$ together with a function $X + Y \rightarrow D$. A decorated corelation is thus specified by the decoration $X + Y \rightarrow D$ alone. Note that these are in one-to-one correspondence with cospans $X \rightarrow D \leftarrow Y$ in FinSet via the coproduct maps.

As observed in the proof of Theorem 4.10, the hypergraph structure on $G\text{Corel}$ agrees with that on $\text{Cospan}(\text{FinSet})$ via this correspondence; the multiplication $\mu_X: X + X \rightarrow X$ in $G\text{Corel}$ is simply given by the decoration $(X + X) + X \rightarrow X$, and so on.

The intuition is that in the global sections construction on a cospan category, the shift from cospans to corelations takes the ‘factored part’ and puts it into the decoration.

This suggests an isomorphism; indeed, the isomorphism given by Theorem 4.10 does precisely this. We can construct one direction, the one from the smaller to larger \mathcal{M} —that is, from $\mathcal{M} = \mathcal{I}_{\text{FinSet}}$ to $\mathcal{M} = \text{FinSet}$ —as a decorated corelations functor. Note that the identity on FinSet maps the subcategory $\mathcal{I}_{\text{FinSet}}$ into FinSet , and so extends to a morphism $\text{FinSet} \rightarrow \text{Cospan}(\text{FinSet})$. Also recall that the ‘undecorated’ cospan category $\text{Cospan}(\text{FinSet})$ is equal to the decorated cospan category given by the functor $1: \text{FinSet} \rightarrow \text{Set}$ mapping each finite set to some chosen one element set 1 , and each morphism to the identity morphism on 1 . Define monoidal natural transformation

$$\begin{array}{ccc}
 \text{FinSet} = \text{FinSet}; \mathcal{I}_{\text{FinSet}}^{\text{opp}} & \xrightarrow{1} & \text{Set} \\
 \downarrow \iota & & \Downarrow \theta \\
 \text{Cospan}(\text{FinSet}) = \text{FinSet}; \text{FinSet}^{\text{opp}} & \xrightarrow{\quad} & \text{Set}
 \end{array}$$

with each $\theta_X: 1 \rightarrow GX$ mapping the unique element to the identity function $1_X: X \rightarrow X$. This gives the hypergraph functor we expect, mapping the undecorated cospan $X \rightarrow N \leftarrow Y$ to the trivial cospan $X \rightarrow X + Y \leftarrow Y$ decorated by $X + Y \rightarrow N$. It is now routine to verify this is an isomorphism.

The previous example extends to any finitely cocomplete category \mathcal{C} : the hypergraph category $\text{Cospan}(\mathcal{C})$ can always be constructed as (i) trivially decorated $(\mathcal{C}, \mathcal{I}_{\mathcal{C}})$ -corelations, or (ii) $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by equivalence classes of morphisms with domain the apex of the corelation, and moreover the isomorphism of these hypergraph categories is given by the analogous monoidal natural transformation between the decorating functors.

More general still, a category of trivially decorated $(\mathcal{E}, \mathcal{M})$ -corelations in \mathcal{C} can always be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by equivalence classes of morphisms in \mathcal{E} with domain the apex of the corelation, and the isomorphism of these hypergraph categories a decorated corelations functor.

Most generally, we can still perform this construction on decorated corelation categories: Theorem 4.10 implies any category of $(\mathcal{E}, \mathcal{M})$ -decorated corelations can be constructed also as $(\mathcal{I}_{\mathcal{C}}, \mathcal{C})$ -corelations decorated by codomain decorated morphisms in \mathcal{E} . Given some lax symmetric monoidal functor

$$F: \mathcal{C}; \mathcal{M}^{\text{opp}} \longrightarrow \text{Set},$$

the decorated $(\mathcal{I}_C, \mathcal{C})$ -corelations are specified by the (global sections) functor

$$F': \text{Cospan}(\mathcal{C}) \longrightarrow \text{Set}$$

taking any object $Z \in \text{Cospan}(\mathcal{C})$ to the pair $(Z \xrightarrow{e} N, 1 \xrightarrow{s} FN)$, where e is a morphism in \mathcal{E} .

What is the utility of this variety of constructions? The first is that some categories are very naturally constructed as corelation (or, dually, relation) categories, like the category of equivalence relations, relations, or linear relations. Here the factorisation system has an intuitive interpretation, like epimorphisms retaining only the structure in the apex that is ‘accessible’ or ‘mapped onto’ by the feet/boundary. Decorating this corelations retains the same sort of intuition.

On the other hand, the ‘fully decorated’ latter form, where all interesting structure is carried by the apex of the decorated corelation, is useful for constructing functors into the given decorated corelation category. The ability to construct functors from one hypergraph category to another is essential for the understanding of hypergraph categories as network-type diagrammatic languages, with functors giving rise to notions of semantics, equivalence of diagrams, and reasoning.

We will illustrate these principles in greater depth in the next path, which delves into applications of this philosophy and framework. We conclude this section and Part I with a collection of examples of a more abstract nature.

4.5 Examples

We give two extended examples. Our first example is to revisit the matrix example from the introduction, having now developed the necessary material. Our second example is to revisit the category of linear relations once more, showing that we can also construct it as a decorated corelation category.

4.5.1 Matrices

Let R be a commutative ring. We will construct matrices over R as decorated corelations over $\text{FinSet}^{\text{opp}}$. Here the coproduct is the Cartesian product \times of sets, the initial object is the one element set 1 , and cospans are spans in FinSet . The notation will thus be less confusing if we talk of decorated spans on (FinSet, \times) given by the

contravariant lax monoidal functor

$$\begin{aligned} R^{(-)} : (\mathbf{FinSet}, \times) &\longrightarrow (\mathbf{Set}, \times); \\ N &\longmapsto R^N \\ (f : N \rightarrow M) &\longmapsto (R^f : R^M \rightarrow R^N; v \mapsto v \circ f). \end{aligned}$$

The coherence maps $\varphi_{N,M} : R^N \times R^M \rightarrow R^{N \times M}$ take a pair (s, t) of maps $s : N \rightarrow R$, $t : M \rightarrow R$ to the pointwise product $s \cdot t : N \times M \rightarrow R$; $(n, m) \mapsto s(n) \cdot t(m)$. The unit coherence map $\varphi_1 : 1 \rightarrow R^1$ sounds almost tautological: it takes the unique element of the one element set 1 to the function $1 \rightarrow R$ that maps the unique element of the one element set to the multiplicative identity 1_R of the ring R . As in the introduction, we show $R^{(-)}\mathbf{Cospan}$ is the category of ‘multivalued matrices’ over R , and $R^{(-)}\mathbf{Corel}$ is the category of matrices over R .

It is the coherence maps $\varphi_{N,M}$ that enact multiplication of scalars, with the composite of $(X \xleftarrow{i_X} N \xrightarrow{o_Y} Y, N \xrightarrow{s} R)$ and $(Y \xleftarrow{i_Y} M \xrightarrow{o_Z} Z, M \xrightarrow{t} R)$ the span $X \leftarrow N \times_Y M \rightarrow Z$ decorated by the map

$$N \times_Y M \hookrightarrow N \times M \xrightarrow{s \cdot t} R,$$

where the inclusion from $N \times_Y M$ into $N \times M$ is that given by the categorical product. We discussed the intuition for this composition rule in the introduction.

These morphisms are multivalued matrices in the following sense: the cardinality of the domain and the codomain give the dimension of the matrix, and the apex N indexes the entries of the matrix.

As φ_1 essentially selects the multiplicative identity 1_R of R , the empty decoration on any set N is the function that sends every element of N to 1_R .

This becomes a decorated relations example if we extend $R^{(-)}$ to the contravariant functor

$$R^{(-)} : (\mathbf{Span}(\mathbf{FinSet}), \times) \longrightarrow (\mathbf{Set}, \times)$$

mapping now a span $N \xleftarrow{f} A \xrightarrow{g} M$ to the function

$$\begin{aligned} R^{f^{\text{opp}};g} : R^M &\longrightarrow R^N; \\ v &\longmapsto \left(n \mapsto \sum_{a \in f^{-1}(n)} v \circ g(a) \right). \end{aligned}$$

empty decoration

Many aspects of this example are ‘atypical’ for the intuition we have been working towards. Note that the monoidal product here is the tensor product of matrices, not

the biproduct. Indeed, there is no special commutative Frobenius algebra in \mathbf{Vect} if we use the biproduct, but if we use the tensor product then these correspond to orthonormal bases (Vicary). The comultiplication is the diagonal map, multiplication is codiagonal. unit produces basis.

We note that you could take decorations here in the category $R\mathbf{Mod}$ of R -modules. While Proposition 2.9 shows that the resulting decorated cospans category would be isomorphic, this may hint at an enriched version of the theory.

4.5.2 Two constructions for linear relations

We saw earlier that linear relations are epi-mono corelations in \mathbf{Vect} . The hypergraph structure is given by addition. We show how to recover this in another construction. We also get a hypergraph functor between them. This is very useful for compositional linear relations semantics of diagrams.

We can also construct linear relations in $\mathbf{Vect}^{\text{opp}}$.

$$\begin{aligned}
 & : \mathbf{Cospan}(\mathbf{FinSet}) \longrightarrow \mathbf{Set} \\
 & X \longmapsto \{\text{subspaces of } k^X\} \\
 & f : X \rightarrow Y \longmapsto L \mapsto \{v \mid v \circ f \in L\} \\
 & f^{\text{opp}} : X \rightarrow Y \longmapsto L \mapsto \{v = u \circ f \mid u \in L\}
 \end{aligned}$$

Then \mathbf{Cospan} is cospans decorated by subspaces, and \mathbf{Corel} is linear relations. This is important for circuits work [?, ?].

Part II

Applications

Bibliography

- [Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page [20](#).)