Hypergraph categories and their applications

Brendan Fong
Hertford College
University of Oxford

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Hypergraph categories

1.0.1 Hypergraph categories.

A **Frobenius monoid** $(X, \mu, \delta, \eta, \epsilon)$ in a monoidal category $(\mathcal{C}, \otimes)$ is an object X together with monoid (X, μ, η) and comonoid (X, δ, ϵ) structures such that

$$(1 \otimes \mu) \circ (\delta \otimes 1) = \delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \delta) \colon X \otimes X \longrightarrow X \otimes X.$$

A Frobenius monoid is further called **special** if

$$\mu \circ \delta = 1 \colon X \longrightarrow X,$$

and further called **commutative** if the ambient monoidal category is symmetric and the monoid and comonoid structures that comprise the Frobenius monoid are commutative and cocommutative respectively. Note that for Frobenius monoids commutativity of the monoid structure implies cocommutativity of the comoniod structure, and vice versa, so the use of the term 'commutativity' for both the Frobenius monoid and the constituent monoid is not ambiguous.

A **hypergraph category** is a symmetric monoidal category in which each object is equipped with a special commutative Frobenius structure $(X, \mu_X, \delta_X, \eta_X, \epsilon_X)$ such that

$$\begin{aligned} \mu_{X\otimes Y} &= (\mu_X \otimes \mu_Y) \circ (1_X \otimes \sigma_{YX} \otimes 1_Y) & \eta_{X\otimes Y} &= \eta_X \otimes \eta_Y \\ \delta_{X\otimes Y} &= (1_X \otimes \sigma_{XY} \otimes 1_Y) \circ (\delta_X \otimes \delta_Y) & \epsilon_{X\otimes Y} &= \epsilon_X \otimes \epsilon_Y. \end{aligned}$$

A functor (F, φ) of hypergraph categories, or **hypergraph functor**, is a strong symmetric monoidal functor (F, φ) that preserves the hypergraph structure. More precisely, the latter condition means that given an object X, the special commutative Frobenius structure on FX must be

$$(FX,\ F\mu_X \circ \varphi_{X,X},\ \varphi^{-1} \circ F\delta_X,\ F\eta_X \circ \varphi_1,\ \varphi_1 \circ \epsilon_X).$$

This terminology was introduced recently [?], in reference to the fact that these special commutative Frobenius monoids provide precisely the structure required to draw graphs with 'hyperedges': wires connecting any number of inputs to any number of outputs. Commutative special Frobenius monoids are also known as commutative separable algebras [?], and hypergraph categories as well-supported compact closed categories [?].

Note that if an object X is equipped with a Frobenius monoid structure then the maps $\epsilon \circ \mu \colon X \otimes X \longrightarrow 1$ and $\delta \circ \eta \colon 1 \longrightarrow X \otimes X$ obey

$$\big(1 \otimes (\epsilon \circ \mu)\big) \circ \big((\delta \circ \eta) \otimes 1\big) = 1_X = \big((\epsilon \circ \mu) \otimes 1\big) \circ \big(1 \otimes (\delta \circ \eta)\big) \colon X \longrightarrow X.$$

Thus if an object carries a Frobenius monoid it is also self-dual, and any hypergraph category is a fortiori self-dual compact closed. Mapping each morphism $f \colon X \to Y$ to its dual morphism

$$\big((\epsilon_Y \circ \mu_Y) \otimes 1_X\big) \circ \big(1_Y \otimes f \otimes 1_X\big) \circ \big(1_Y \otimes (\delta_X \circ \eta_X)\big) \colon Y \longrightarrow X$$

further equips each hypergraph category with a so-called dagger functor—an involutive contravariant endofunctor that is the identity on objects—such that the category is a dagger compact category. Dagger compact categories were first introduced in the context of categorical quantum mechanics [?], under the name strongly compact closed category, and have been demonstrated to be a key structure in diagrammatic reasoning and the logic of quantum mechanics.

We shall see that every decorated cospan category is a hypergraph category, and hence also a dagger compact category.

Example 1.1. A central example of a hypergraph category is the category $\mathrm{Cospan}(\mathcal{C})$ of cospans in any category $\mathcal{C}$ with finite colimits. We will later see that decorated cospan categories are a generalisation of such categories, and each inherits a hypergraph structure from such.

First, $\mathrm{Cospan}(\mathcal{C})$ inherits a symmetric monoidal structure from $\mathcal{C}$. We call a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ **wide** if $\mathcal{C}$ contains all objects of $\mathcal{D}$, and call a functor that is faithful and bijective-on-objects a **wide embedding**. Note then that we have a wide embedding

$$\mathcal{C} \hookrightarrow \mathrm{Cospan}(\mathcal{C})$$

that takes each object of $\mathcal{C}$ to itself as an object of $\mathrm{Cospan}(\mathcal{C})$, and each morphism $f\colon X \to Y$ in $\mathcal{C}$ to the cospan

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] & & Y, \arrow[ul, equal] \end{tikzcd}$$

where the extended 'equals' sign denotes an identity morphism. This allows us to view $\mathcal{C}$ as a wide subcategory of $\mathrm{Cospan}(\mathcal{C})$.

Now as $\mathcal{C}$ has finite colimits, it can be given a symmetric monoidal structure with the coproduct the monoidal product; we write this monoidal category $(\mathcal{C}, +)$, and write $\varnothing$ for the initial object, the monoidal unit of this category. Then $\mathrm{Cospan}(\mathcal{C})$ inherits the same symmetric monoidal structure: since the monoidal product $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is left adjoint to the diagram functor, it preserves colimits, and so extends to a functor $+\colon \mathrm{Cospan}(\mathcal{C}) \times \mathrm{Cospan}(\mathcal{C}) \to \mathrm{Cospan}(\mathcal{C})$. The remainder of the monoidal structure is inherited because $\mathcal{C}$ is a wide subcategory of $\mathrm{Cospan}(\mathcal{C})$.

Next, the Frobenius structure comes from copairings of identity morphisms. We call cospans

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & N & \\ Y \arrow[ur, "o"] & & X \arrow[ul, "i"'] \end{tikzcd}$$

that are reflections of each other **opposite** cospans. Given any object X in $\mathcal{C}$, the copairing $[1_X, 1_X]\colon X + X \to X$ of two identity maps on X, together with the unique map $!\colon \varnothing \to X$ from the initial object to X, define a monoid structure on X. Considering these maps as morphisms in $\mathrm{Cospan}(\mathcal{C})$, we may take them together with their opposites to give a special commutative Frobenius structure on X. In this way we consider each category $\mathrm{Cospan}(\mathcal{C})$ a hypergraph category.

It is a simple computation to check that the resulting dagger functor simply takes a cospan $X \xrightarrow{i} N \xleftarrow{o} Y$ to its opposite cospan $Y \xrightarrow{o} N \xleftarrow{i} X$.

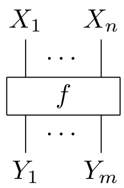
1.0.2 Dagger compact categories

In a dagger category, the distinction between the source and target of a morphism is arbitrary, and can be reversed. This is true of electrical circuits: if we like we may treat the set of inputs as the set of outputs instead, and the set of outputs as the set of inputs.

Recall that a **dagger category** is a category equipped with an involutive, contravariant endofunctor that is the identity on objects [**?**, **?**]. In other words, a **dagger structure** on a category $\mathcal{C}$ is a contravariant functor $(-)^\dagger \colon \mathcal{C} \to \mathcal{C}$ such that $A^\dagger = A$ for all objects $A \in \mathrm{Ob}\mathcal{C}$ and $(f^\dagger)^\dagger = f$ for all morphisms f in $\mathcal{C}$. A dagger category is then a category equipped with a dagger structure.

When other structure is present, we prefer this dagger to be compatible with it. We say that a morphism f is **unitary** if its dagger provides it with an inverse morphism. A **symmetric monoidal dagger category** is a symmetric monoidal category equipped with a **symmetric monoidal dagger structure**—that is, a dagger structure that coherently preserves the symmetric monoidal structure. Concretely, this requires that the functor $(-)^\dagger \colon \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ be symmetric monoidal, and that the associator, unitors, and braiding of $\mathcal{C}$ be unitary. Furthermore, letting L and R be dual objects of a symmetric monoidal dagger category, with monoidal unit I, braiding $\sigma_{L,R} \colon L \otimes R \to R \otimes L$, and unit $\eta \colon I \to R \otimes L$ and counit $\epsilon \colon L \otimes R \to I$, we say that L and R are **dagger dual** if $\eta = \sigma \circ \epsilon^\dagger$. A **dagger compact category** is a symmetric monoidal dagger category in which every object has a dagger dual, while a **dagger functor** is a functor F between dagger compact categories that preserves the dagger structures: $F((-)^\dagger) = (F(-))^\dagger$.

Importantly for our applications, dagger compact categories come with a graphical calculus, where each morphism is represented by a diagram such that two diagrams are considered equal by the rules of this calculus if and only if they are equal according to the defining laws of dagger compact categories [**?**]. In brief, to set up our conventions, we represent a morphism $f \colon X_1 \otimes \cdots \otimes X_n \to Y_1 \otimes \cdots \otimes Y_m$ as a 'downwards flow chart':



Composition is then represented by connecting the lines (or wires) representing the codomain of one morphism with the domain of the another placed below it, the monoidal product of two morphisms is represented by their side-by-side juxtaposition, the swap map by crossing two wires, the compact structure by bending a wire 180 degrees, and the dagger functor by flipping a diagram in the horizontal axis. We believe these operations on diagrams—placing diagrams on the same page, rearranging their wires/flipping them, and then connecting their wires to form a larger diagram—represents the collection of operations used for reasoning with circuit diagrams, and hence that dagger compact categories are an appropriate structure for the formalization of such.

Chapter 2

Decorated cospans

This chapter is based on [Fon15]

2.1 Introduction

There is a well-known way to compose cospans in a category with finite colimits: given cospans

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & M & \\ Y \arrow[ur, "i_Y"] & & Z, \arrow[ul, "o_Z"'] \end{tikzcd}$$

we take the pushout over their shared foot Y

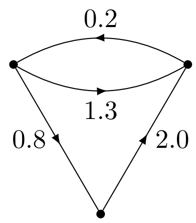
$$\begin{tikzcd} & & P & & \\ & N \arrow[ur, "j"] & & M \arrow[ul, "j'"'] & \\ X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z \arrow[ul, "o_Z"'] \end{tikzcd}$$

to get a cospan from X to Z. In many situations, however, we wish to compose 'decorated' cospans, where the apex of each cospan is equipped with some extra structure. In this article we detail a method for composing such decorated cospans.

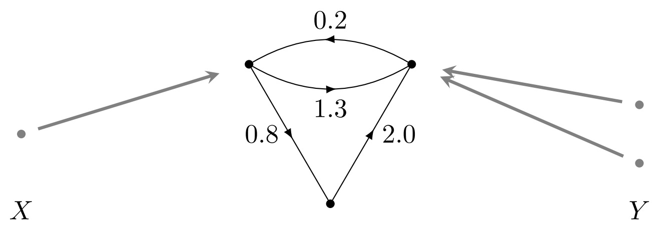
Beyond category theoretic interest, the motivation for such a method lies in developing compositional accounts of semantics associated to topological diagrams. While this has long been a technique associated with topological quantum field theory, dating back to [?], it has most recently had significant influence in the nascent field of categorical network theory, with application to automata and computation [?, ?], electrical circuits [?], signal flow diagrams [?, ?], Markov processes [?, ?], and dynamical systems [?], among others.

It has been recognised for some time that spans and cospans provide an intuitive framework for composing network diagrams [**?**], and the material we develop here is a variant on this theme. In the case of finite graphs, the intuition reflected is this: given two graphs, we may construct a third by gluing chosen vertices of the first with chosen vertices of the second. It is our goal in this article to view this process as composition of morphisms in a category, in a way that also facilitates the construction of a composition rule for any semantics associated to the diagrams, and a functor between these two resulting categories.

To see how this works, let us start with the following graph:

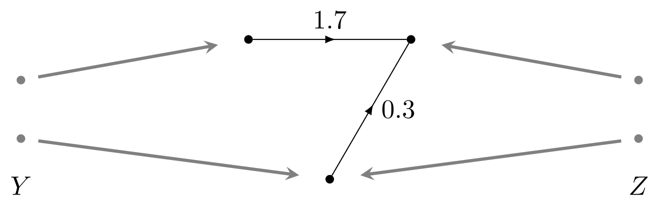


We shall work with labelled, directed graphs, as the additional data help highlight the relationships between diagrams. Now, for this graph to be a morphism, we must equip it with some notion of 'input' and 'output'. We do this by marking vertices using functions from finite sets:

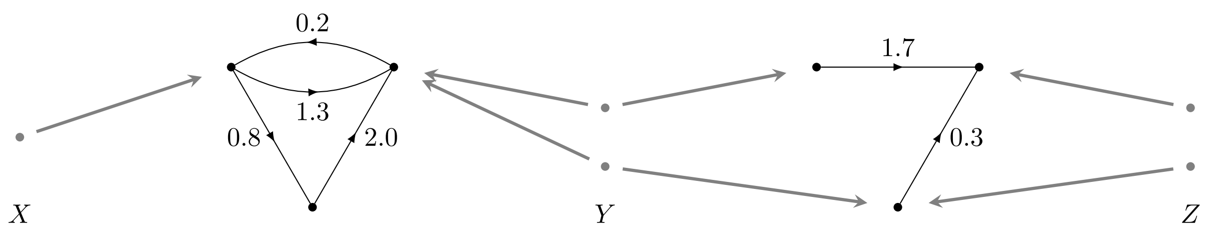


Let N be the set of vertices of the graph. Here the finite sets X, Y, and N comprise one, two, and three elements respectively, drawn as points, and the values of the functions $X \to N$ and $Y \to N$ are indicated by the grey arrows. This forms a cospan in the category of finite sets, one with the set at the apex decorated by our given graph.

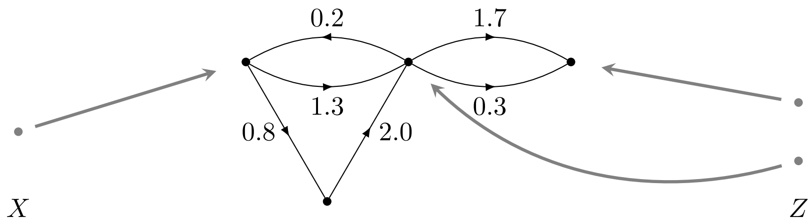
Given another such decorated cospan with input set equal to the output of the above cospan



composition involves gluing the graphs along the identifications



specified by the shared foot of the two cospans. This results in the decorated cospan



The decorated cospan framework generalises this intuitive construction.

More precisely: fix a set L. Then given a finite set N, we may talk of the collection of finite L-labelled directed multigraphs, to us just L-graphs or simply graphs, that have N as their set of vertices. Write such a graph (N, E, s, t, r), where E is a finite set of edges, $s\colon E \to N$ and $t\colon E \to N$ are functions giving the source and target of each edge respectively, and $r\colon E \to L$ equips each edge with a label from the set L. Next, given a function $f\colon N \to M$, we may define a function from graphs on N to graphs on M mapping (N, E, s, t, r) to $(M, E, f \circ s, f \circ t, r)$. After dealing appropriately with size issues, this gives a lax monoidal functor from $(\mathrm{FinSet}, +)$ to $(\mathrm{Set}, \times)$.[1]

Now, taking any lax monoidal functor $(F, \varphi)\colon (\mathcal{C}, +) \to (\mathcal{D}, \otimes)$ with $\mathcal{C}$ having finite colimits and coproduct written $+$, the decorated cospan category associated to F has as objects the objects of $\mathcal{C}$, and as morphisms pairs comprising a cospan in $\mathcal{C}$ together with some morphism $1 \to FN$, where 1 is the unit in $(\mathcal{D}, \otimes)$ and N is the apex of the cospan. In the case of our graph functor, this additional data is equivalent to equipping the apex N of the cospan with a graph. We thus think of our morphisms as having two distinct parts: an instance of our chosen structure on the apex, and a

[1] Here $(\mathrm{FinSet}, +)$ is the monoidal category of finite sets and functions with disjoint union as monoidal product, and $(\mathrm{Set}, \times)$ is the category of sets and functions with cartesian product as monoidal product. One might ensure the collection of graphs forms a set in a number of ways. One such method is as follows: the categories of finite sets and finite graphs are essentially small; replace them with equivalent small categories. We then constrain the graphs (N, E, s, t, r) to be drawn only from the objects of our small category of finite graphs.

cospan describing interfaces to this structure. Our first theorem says that when $(\mathcal{D}, \otimes)$ is braided monoidal and (F, φ) lax braided monoidal, we may further give this data a composition rule and monoidal product such that the resulting 'decorated cospan category' is symmetric monoidal with a special commutative Frobenius monoid on each object.

Suppose now we have two such lax monoidal functors; we then have two such decorated cospan categories. Our second theorem is that, given also a monoidal natural transformation between these functors, we may construct a strict monoidal functor between their corresponding decorated cospan categories. These natural transformations can often be specified by some semantics associated to some type of topological diagram. A trivial case of such is assigning to a finite graph its number of vertices, but richer examples abound, including assigning to a directed graph with edges labelled by rates its depicted Markov process, or assigning to an electrical circuit diagram the current–voltage relationship such a circuit would impose.

An advantage of the decorated cospan framework is that the resulting categories are hypergraph categories, and the resulting functors respect this structure. As dagger compact categories, hypergraph categories themselves have a rich diagrammatic nature [**?**], and in cases when our decorated cospan categories are inspired by diagrammatic applications, the hypergraph structure provides language to describe natural operations on our diagrams, such as juxtaposing, rotating, and reflecting them.

2.1.1 Outline.

The structure of this paper is straightforward: in the following section we review some basic background material, which then allows us to give the constructions of decorated cospan categories and their functors in Sections 2.3 and 2.4 respectively. We then explicate these definitions through some examples in Section 2.5.

2.1.2 Notation.

We shall assume the following standard names for certain distinguished objects and morphisms, only disambiguating the symbols with subscripts when we judge that the extra clarity is worth the clutter. We write:

- 1 for both identity morphisms and monoidal units, leaving context to determine which one we mean.
- λ, ρ, a, and σ for respectively the left unitor, right unitor, associator, and, if present, braiding, in a monoidal category.

- $\varnothing$ for the initial object in a category.
- ! for the unique map from the initial object to a given object.

2.2 Background

2.2.1 Cospan categories.

Recall that a **cospan** from X to Y in a category $\mathcal{C}$ is an object N in $\mathcal{C}$ with a pair of morphisms $(i\colon X \to N,\, o\colon Y \to N)$:

$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y. \arrow[ul, "o"'] \end{tikzcd}$$

We shall refer to X and Y as the **feet**, and N as the **apex** of the cospan. Cospans may be composed using the pushout from the common foot, when such a pushout exists: given cospans $X \xrightarrow{i_X} N \xleftarrow{o_Y} Y$ from X to Y and $Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z$ from Y to Z, their composite cospan is $X \xrightarrow{j \circ i_X} P \xleftarrow{j' \circ i_Z} Z$, where P, $(j\colon N \to P)$, and $(j'\colon M \to P)$ form the top half of the pushout square

$$\begin{tikzcd} & & P & & \\ & N \arrow[ur, "j"] & & M \arrow[ul, "j'"'] & \\ X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z. \arrow[ul, "o_Z"'] \end{tikzcd}$$

A **map of cospans** is a morphism $n\colon N \to N'$ in $\mathcal{C}$ between the apices of two cospans $X \xrightarrow{i} N \xleftarrow{o} Y$ and $X \xrightarrow{i'} N' \xleftarrow{o'} Y$ with the same feet, such that both triangles

$$\begin{tikzcd} & N \arrow[dd, "n"] & \\ X \arrow[ur, "i"] \arrow[dr, "i'"'] & & Y \arrow[ul, "o"'] \arrow[dl, "o'"] \\ & N' & \end{tikzcd}$$

commute. Given a category $\mathcal{C}$ with pushouts, we may define a category $\mathrm{Cospan}(\mathcal{C})$ with objects the objects of $\mathcal{C}$ and morphisms isomorphism classes of cospans [?]. We will often abuse our terminology and refer to cospans themselves as morphisms in some cospan category $\mathrm{Cospan}(\mathcal{C})$; we of course refer instead to the isomorphism class of the said cospan.

2.3 Decorated cospan categories

We now detail our central construction and state the main theorem.

Definition 2.1. Let $\mathcal{C}$ be a category with finite colimits, and

$$(F,\varphi)\colon (\mathcal{C},+) \longrightarrow (\mathcal{D},\otimes)$$

be a lax monoidal functor. We define a **decorated cospan**, or more precisely an F-decorated cospan, to be a pair

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd}, \qquad \begin{tikzcd} FN \\ 1 \arrow[u, "s"'] \end{tikzcd}\right)$$

comprising a cospan $X \stackrel{i}{\rightarrow} N \stackrel{o}{\leftarrow} Y$ in $\mathcal{C}$ together with an element $1 \stackrel{s}{\rightarrow} FN$ of the F-image FN of the apex of the cospan. We shall call the element $1 \stackrel{s}{\rightarrow} FN$ the **decoration** of the decorated cospan. A morphism of decorated cospans

$$n\colon \big(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN\big) \longrightarrow \big(X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y,\ 1 \xrightarrow{s'} FN'\big)$$

is a morphism $n\colon N \to N'$ of cospans such that $Fn \circ s = s'$.

Proposition 2.2. *There is a category $F\mathrm{Cospan}$ of F-decorated cospans, with objects the objects of $\mathcal{C}$, and morphisms isomorphism classes of F-decorated cospans. On representatives of the isomorphism classes, composition in this category is given by pushout of cospans in $\mathcal{C}$*

$$\begin{tikzcd} & & N+_Y M & & \\ & N \arrow[ur, "j_N"] & & M \arrow[ul, "j_M"'] & \\ X \arrow[ur, "i_X"] & & Y \arrow[ul, "o_Y"'] \arrow[ur, "i_Y"] & & Z \arrow[ul, "o_Z"'] \end{tikzcd}$$

paired with the composite

$$1 \xrightarrow{\lambda^{-1}} 1\otimes 1 \xrightarrow{s\otimes t} FN\otimes FM \xrightarrow{\varphi_{N,M}} F(N+M) \xrightarrow{F[j_N,j_M]} F(N+_Y M)$$

of the tensor product of the decorations with the F-image of the copairing of the pushout maps.

Proof. The identity morphism on an object X in a decorated cospan category is simply the identity cospan decorated as follows:

$$\left(\begin{tikzcd} & X & \\ X \arrow[ur, equal] & & X \arrow[ul, equal] \end{tikzcd}, \quad \begin{tikzcd} FX \\ F\varnothing \arrow[u, "F!"'] \\ 1 \arrow[u, "\varphi_1"'] \end{tikzcd}\right).$$

We must check that the composition defined is well-defined on isomorphism classes, is associative, and, with the above identity maps, obeys the unitality axiom. These are straightforward, but lengthy, exercises in using the available colimits and monoidal structure to show that the relevant diagrams of decorations commute.

Representation independence for composition of isomorphism classes of decorated cospans. Let

$$n\colon \big(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN\big) \longrightarrow \big(X \xrightarrow{i'_X} N' \xleftarrow{o'_Y} Y,\ 1 \xrightarrow{s'} FN'\big)$$

and

$$m\colon \big(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{t} FM\big) \longrightarrow \big(Y \xrightarrow{i'_Y} M' \xleftarrow{o'_Z} Z,\ 1 \xrightarrow{t'} FM'\big)$$

be isomorphisms of decorated cospans. We wish to show that the composite of the decorated cospans on the left is isomorphic to the composite of the decorated cospans on the right. As discussed, it is well-known that the composite cospans are isomorphic, and it remains to us to check the decorations agree too. Let $p\colon N+_Y M \to N'+_Y M'$ be the isomorphism given by the universal property of the pushout and the isomorphisms $n\colon N \to N'$ and $m\colon M \to M'$. Then the two decorations in question are given by the top and bottom rows of the following diagram.

$$\begin{tikzcd} & & FN \otimes FM \arrow[r, "\varphi_{N,M}"] \arrow[d, "\sim"', "Fn\otimes Fm"] \arrow[dr, phantom, "\text{(F)}"] & F(N+M) \arrow[r, "F{[j_N,j_M]}"] \arrow[d, "\sim"', "F(n+m)"] \arrow[dr, phantom, "\text{(C)}"] & F(N+_Y M) \arrow[d, "\sim"', "Fp"] \\ 1 \arrow[r, "\lambda^{-1}"] & 1\otimes 1 \arrow[ur, "s\otimes t"] \arrow[dr, "s'\otimes t'"'] \arrow[r, phantom, "\text{(I)}"] & {} & & \\ & & FN' \otimes FM' \arrow[r, "\varphi_{N',M'}"'] & F(N'+M') \arrow[r, "F{[j_{N'},j_{M'}]}"'] & F(N' +_Y M') \end{tikzcd}$$

The triangle (I) commutes as n and m are morphisms of decorated cospans and $-\otimes-$ is functorial, (F) commutes by the monoidality of F, and (C) commutes by properties of colimits in $\mathcal{C}$ and the functoriality of F. This proves the claim.

Associativity. Suppose we have morphisms

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN),$$

$$(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{t} FM),$$

$$(Z \xrightarrow{i_Z} P \xleftarrow{o_W} W,\ 1 \xrightarrow{u} FP).$$

It is well-known that composition of isomorphism classes of cospans via pushout of representatives is associative; this follows from the universal properties of the relevant colimit. We must check that the pushforward of the decorations is also an associative process. Write

$$\tilde{a}\colon (N +_Y M) +_Z P \longrightarrow N +_Y (M +_Z P)$$

for the unique isomorphism between the two pairwise pushouts constructions from the above three cospans. Consider then the following diagram, with leftmost column the decoration obtained by taking the composite of the first two morphisms first, and the rightmost column the decoration obtained by taking the composite of the last two morphisms first.

$$
\begin{tikzcd}
F((N+_Y M)+_Z P) \arrow[rrrrrr, "F\tilde{a}"] & & & & & & F(N+_Y(M+_Z P)) \\
F((N+_Y M)+P) \arrow[u, "{F[j_{N+_Y M},j_P]}"] \arrow[rrrrrr, phantom, "\text{(C)}"] & & & & & & F(N+(M+_Z P)) \arrow[u, "{F[j_N,j_{M+_Z P}]}"'] \\
& F((N+M)+P) \arrow[ul, "{F([j_N,J_M]+1_P)}"'] \arrow[rrrr, "Fa"] & & & & F(N+(M+P)) \arrow[ur, "{F(1_N+[j_M,j_P])}"] & \\
F(N+_Y M)\otimes FP \arrow[uu, "\varphi_{N+_Y M,P}"] \arrow[ur, phantom, "\text{(F2)}"] \arrow[rrrrrr, phantom, "\text{(F1)}"] & & & & & & FN\otimes F(M+_Z P) \arrow[uu, "\varphi_{N,M+_Z P}"'] \arrow[ul, phantom, "\text{(F3)}"] \\
& F(N+M)\otimes FP \arrow[ul, "{F[j_N,j_M]\otimes 1_{FP}}"] \arrow[uu, "\varphi_{N+M,P}"'] & & & & FN\otimes F(M+P) \arrow[ur, "{1_{FN}\otimes F[j_M,j_P]}"'] \arrow[uu, "\varphi_{N,M+P}"] & \\
& & (FN\otimes FM)\otimes FP \arrow[ul, "\varphi_{N,M}\otimes 1_{FP}"] \arrow[rr, "a"] & & FN\otimes(FM\otimes FP) \arrow[ur, "1_{FN}\otimes\varphi_{M,P}"'] & & \\
& & (1\otimes 1)\otimes 1 \arrow[u, "(s\otimes t)\otimes u"] \arrow[rr, "a"] \arrow[urr, phantom, "\text{(D2)}"] & & 1\otimes(1\otimes 1) \arrow[u, "s\otimes(t\otimes u)"'] & & \\
& & & 1\otimes 1 \arrow[ul, "\rho^{-1}\otimes 1"] \arrow[ur, "1\otimes\lambda^{-1}"'] \arrow[u, phantom, "\text{(D1)}"] & & & \\
& & & 1 \arrow[u, "\lambda^{-1}"] & & &
\end{tikzcd}
$$

This diagram commutes as (D1) is the triangle coherence equation for the monoidal category $(\mathcal{D}, \otimes)$, (D2) is naturality for the associator a, (F1) is the associativity condition for the monoidal functor F, (F2) and (F3) commute by the naturality of φ, and (C) commutes as it is the F-image of a hexagon describing the associativity of the pushout. This shows that the two decorations obtained by the two different orders of composition of our three morphisms are equal up to the unique isomorphism $\tilde{a}$ between the two different pushouts that may be obtained. Our composition rule is hence associative.

Identity morphisms. We shall show that the claimed identity morphism on Y, the decorated cospan

$$(Y \xrightarrow{1_Y} Y \xleftarrow{1_Y} Y,\ 1 \xrightarrow{F!\circ\varphi_1} FY),$$

is an identity for composition on the right; the case for composition on the left is similar. The cospan in this pair is known to be the identity cospan in $\mathrm{Cospan}(\mathcal{C})$. We thus need to check that, given a morphism

$$(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN),$$

the composite of the product $s\otimes(F!\circ\varphi_1)$ with the F-image of the copairing $[1_N, i_Y]\colon N+Y \to N$ of the pushout maps is again the same element s; this composite being, by definition, the decoration of the composite of the given morphism and the claimed identity map. This is shown by the commutativity of the diagram below, with the path along the lower edge equal to the aforementioned pushforward.

$$\begin{tikzcd}
1 \arrow[rrrrr, "s"] \arrow[dr, "\rho_1^{-1}"'] & & \arrow[d, phantom, "\text{(D1)}"] & & \arrow[d, phantom, "\text{(F1)}"] & FN \\
& 1\otimes 1 \arrow[r, "s\otimes 1"] \arrow[drr, "s\otimes(F!\circ\varphi_1)"'] & FN\otimes 1 \arrow[r, "1_{FN}\otimes\varphi_1"] \arrow[urrr, "\rho_{FN}"] \arrow[d, phantom, "\text{(D2)}"] & FN\otimes F\varnothing \arrow[r, "\varphi_{N,\varnothing}"] \arrow[d, "1_{FN}\otimes F!"'] \arrow[dr, phantom, "\text{(F2)}"] & F(N+\varnothing) \arrow[ur, "F\rho_N=F{[1_N,!]}"] \arrow[d, "F(1_N+!)"'] \arrow[r, phantom, "\text{(C)}"] & {} \\
& & {} & FN\otimes FY \arrow[r, "\varphi_{N,M}"'] & F(N+Y) \arrow[uur, "F{[1_N,i_Y]}"'] &
\end{tikzcd}$$

This diagram commutes as each subdiagram commutes: (D1) commutes by the naturality of ρ, (D2) by the functoriality of the monoidal product in $\mathcal{D}$, (F1) by the unit axiom for the monoidal functor F, (F2) by the naturality of φ, and (C) due to the properties of colimits in $\mathcal{C}$ and the functoriality of F.

We thus have a category. $\square$

Remark 2.3. While at first glance it might seem surprising that we can construct a composition rule for decorations $s\colon 1 \to FN$ and $t\colon 1 \to FM$ just from monoidal structure, the copair $[j_N, j_M]\colon N+M \to N+_Y M$ of the pushout maps contains the data necessary to compose them. Indeed, this is the key insight of the decorated cospan construction. To wit, the coherence maps for the lax monoidal functor allow us to construct an element of $F(N+M)$ from the monoidal product $s\otimes t$ of the decorations, and we may then post-compose with $F[j_N, j_M]$ to arrive at an element of $F(N +_Y M)$. The map $[j_N, j_M]$ encodes the identification of the image of Y in N with the image of the same in M, and so describes merging the 'overlap' of the two decorations.

Our main theorem is that when *braided* monoidal structure is present, the category of decorated cospans is a hypergraph category, and moreover one into which the category of 'undecorated' cospans widely embeds. This embedding motivates the monoidal and hypergraph structures we put on $F\mathrm{Cospan}$.

Theorem 2.4. *Let $\mathcal{C}$ be a category with finite colimits, $(\mathcal{D}, \otimes)$ a braided monoidal category, and $(F, \varphi)\colon (\mathcal{C}, +) \to (\mathcal{D}, \otimes)$ be a lax braided monoidal functor. Then we may give $F\mathrm{Cospan}$ a symmetric monoidal and hypergraph structure such that there is a wide embedding of hypergraph categories*

$$\mathrm{Cospan}(\mathcal{C}) \hookrightarrow F\mathrm{Cospan}.$$

Proof. Recall that the identity decorated cospan has apex decorated by $1 \xrightarrow{\varphi_1} F\varnothing \xrightarrow{F!} FX$. Given any cospan $X \to N \leftarrow Y$, we call the decoration $1 \xrightarrow{\varphi_1} F\varnothing \xrightarrow{F!} FN$ the **empty decoration** on N.

Embedding. We define a functor

$$\mathrm{Cospan}(\mathcal{C}) \hookrightarrow F\mathrm{Cospan}.$$

mapping each object of $\mathrm{Cospan}(\mathcal{C})$ to itself as an object of $F\mathrm{Cospan}$, and each cospan in $\mathcal{C}$ to the same cospan decorated with the empty decoration on its apex. Should this functor be well-defined, it is evidently a wide embedding.

To see that the embedding is well-defined, we must check that the composite of two empty-decorated cospans is again empty-decorated. To do this, generalise the previous observation for identity morphisms.

Let

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN),$$

be a decorated cospan, and suppose we have an empty-decorated cospan

$$(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{\varphi \circ F!} FM).$$

Here we show that the composite of these decorated cospans is

$$\big(X \xrightarrow{j_N \circ i_X} N +_Y M \xleftarrow{j_M \circ o_Z} Z,\ 1 \xrightarrow{Fj_N \circ s} F(N +_Y M)\big).$$

In particular, the decoration on the composite is the decoration s pushed forward along the F-image of the map $j_N\colon N \to N +_Y M$ to become an F-decoration on $N +_Y M$. We say that the empty decoration acts trivially on other decorations. The analogous statement holds for composition with an empty-decorated cospan on the left.

As is now familiar, a statement of this sort is proved by a large commutative diagram:

$$\begin{tikzcd}
1 \arrow[rrr, "s"] \arrow[dr, "\rho_1^{-1}"'] \arrow[drr, phantom, "\text{(D1)}"] & & & FN \arrow[rr, "Fj_N"] \arrow[d, phantom, "\text{(F1)}"] & & F(N +_Y M) \\
& 1 \otimes 1 \arrow[r, "s\otimes 1"] \arrow[drr, "s\otimes(F!\circ\varphi_1)"'] & FN \otimes 1 \arrow[ur, "\rho_{FN}"] \arrow[r, "1_{FN}\otimes\varphi_1"] \arrow[dr, phantom, "\text{(D2)}"] & FN \otimes F\varnothing \arrow[r, "\varphi_{N,\varnothing}"] \arrow[d, "1_{FN}\otimes F!"] \arrow[dr, phantom, "\text{(F2)}"] & F(N + \varnothing) \arrow[ul, "F\rho_N = F[1_N,!]"'] \arrow[ur, "F{[j_N,!]}"] \arrow[d, "F(1_N+!)"'] \arrow[u, phantom, "\text{(C1)}"] \arrow[r, phantom, "\text{(C2)}"] & {} \\
& & & FN \otimes FM \arrow[r, "\varphi_{N,M}"'] & F(N + M) \arrow[uur, "F{[j_N,j_M]}"'] &
\end{tikzcd}$$

The subdiagrams in this diagram commute for the same reasons as their corresponding regions in the previous diagram for identity morphisms.

A consequence of the trivial action of the empty decoration is that the composite of two empty-decorated cospans is again empty-decorated. This implies the functoriality of the embedding $\mathrm{Cospan}(\mathcal{C}) \hookrightarrow F\mathrm{Cospan}$.

Monoidal structure. We define the monoidal product of objects X and Y of $F\mathrm{Cospan}$ to be their coproduct $X + Y$ in $\mathcal{C}$, and define the monoidal product of decorated cospans $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN)$ and $(X' \xrightarrow{i_{X'}} N' \xleftarrow{o_{Y'}} Y',\ 1 \xrightarrow{t} FN')$ to be

$$\left(
\begin{tikzcd}
& N + N' & \\
X + X' \arrow[ur, "i_X + i_{X'}"] & & Y + Y' \arrow[ul, "o_Y + o_{Y'}"']
\end{tikzcd}
,\quad
\begin{tikzcd}
F(N + N') \\
FN \otimes FN' \arrow[u, "\varphi_{N,N'}"'] \\
1 \otimes 1 \arrow[u, "s\otimes t"'] \\
1 \arrow[u, "\lambda^{-1}"']
\end{tikzcd}
\right).$$

Using the braiding in $\mathcal{D}$, we can show that this proposed monoidal product is functorial.

Indeed, suppose we have decorated cospans

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN), \qquad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{t} FM),$$

$$(U \xrightarrow{i_U} P \xleftarrow{o_V} V,\ 1 \xrightarrow{u} FP), \qquad (V \xrightarrow{i_V} Q \xleftarrow{o_W} W,\ 1 \xrightarrow{v} FQ).$$

We must check the so-called interchange law: that the composite of the column-wise monoidal products is equal to the monoidal product of the row-wise composites.

Again, for the cospans we take this equality as familiar fact. Write

$$b\colon (N+P)+_{(Y+V)}(M+Q) \xrightarrow{\sim} (N+_Y M)+(P+_V Q).$$

for the isomorphism between the two resulting representatives of the isomorphism class of cospans. The two resulting decorations are then given by the leftmost and rightmost columns respectively of the diagram below.

$$\begin{tikzcd}
F((N+P)+_{(Y+V)}(M+Q)) \arrow[rr, "Fb", "\sim"'] & & F((N+_Y M)+(P+_V Q)) \\
F((N+P)+(M+Q)) \arrow[u, "{F[j_{N+P},j_{M+Q}]}"] \arrow[urr, dotted] \arrow[ur, phantom, "\text{(C)}"] & \text{(F2)} & F(N+_Y M)\otimes F(P+_V Q) \arrow[u, "\varphi_{N+_Y M,P+_V Q}"'] \\
F(N+P)\otimes F(M+Q) \arrow[u, "\varphi_{N+P,M+Q}"] & & F(N+M)\otimes F(P+Q) \arrow[u, "{F[j_N,j_M]\otimes F[j_P,j_Q]}"'] \arrow[ull, dotted] \\
(FN\otimes FP)\otimes(FM\otimes FQ) \arrow[u, "\varphi_{N,P}\otimes\varphi_{M,Q}"] \arrow[rr, dotted] \arrow[urr, phantom, "\text{(F1)}"] & & (FN\otimes FM)\otimes(FP\otimes FQ) \arrow[u, "\varphi_{N,M}\otimes\varphi_{P,Q}"'] \\
& (1\otimes 1)\otimes(1\otimes 1) \arrow[ul, "(s\otimes u)\otimes(t\otimes v)"] \arrow[ur, "(s\otimes t)\otimes(u\otimes v)"'] \arrow[u, phantom, "\text{(D)}"] & \\
& 1 \arrow[u, "(\lambda^{-1}\otimes\lambda^{-1})\circ\lambda^{-1}"'] &
\end{tikzcd}$$

These two decorations are related by the isomorphism b as the diagram commutes. We argue this more briefly than before, as the basic structure of these arguments is now familiar to us. Briefly then, there exist dotted arrows of the above types such that the subdiagram (D) commutes by the naturality of the associators and braiding in $\mathcal{D}$, (F1) commutes by the coherence diagrams for the braided monoidal functor F, (F2) commutes by the naturality of the coherence map φ for F, and (C) commutes by the properties of colimits in $\mathcal{C}$ and the functoriality of F.

Using now routine methods, it also is straightforward to show that the monoidal product of identity decorated cospans on objects X and Y is the identity decorated cospan on $X+Y$; for the decorations this amounts to the observation that the monoidal product of empty decorations is again an empty decoration.

Choosing associator, unitors, and braiding in FCospan to be the images of those in Cospan$(\mathcal{C})$, we have a symmetric monoidal category. These transformations remain natural transformations when viewed in the category of F-decorated cospans as they have empty decorations.

We consider the case of the left unitor in detail; the naturality of the right unitor, associator, and braiding follows similarly, using the relevant axiom where here we use the left unitality axiom.

Given a decorated cospan $(X \xrightarrow{i} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN)$, we must show that the diagram of decorated cospans

$$\begin{tikzcd} X+\varnothing \arrow[r, "i+1"] \arrow[d, "\lambda_{\mathcal{C}}"'] & N+\varnothing & Y+\varnothing \arrow[l, "i+1"'] \arrow[d, "\lambda_{\mathcal{C}}"] \\ X \arrow[r, "i"'] & N & Y \arrow[l, "o"] \end{tikzcd}$$

commutes, where the $\lambda_{\mathcal{C}}$ are the maps of the left unitor in $\mathcal{C}$ considered as empty-decorated cospans, and where the top cospan has decoration

$$1 \xrightarrow{\lambda_{\mathcal{D}}^{-1}} 1 \otimes 1 \xrightarrow{s\otimes\varphi_1} FN \otimes F\varnothing \xrightarrow{\varphi_{N,\varnothing}} F(N+\varnothing),$$

and the lower cospan simply has decoration $1 \xrightarrow{s} FN$.

Now as the λ are isomorphisms in $\mathcal{C}$ and have empty decorations, the composite through the upper right corner is isomorphic to the decorated cospan

$$\big(X+\varnothing \xrightarrow{i+1} N+\varnothing \xleftarrow{(o+1)\circ\lambda_{\mathcal{C}}^{-1}} Y,\ 1 \xrightarrow{\varphi_{N,\varnothing}\circ(s\otimes\varphi_1)\circ\lambda_{\mathcal{D}}^{-1}} F(N+\varnothing)\big),$$

while the composite through the lower left corner is isomorphic to the decorated cospan

$$(X+\varnothing \xrightarrow{[i,!]} N \xleftarrow{o} Y,\ 1 \xrightarrow{s} FN).$$

Furthermore, $\lambda_{\mathcal{C}}\colon N + \varnothing \to N$ gives an isomorphism between these two cospans, and the naturality of the left unitor and the left unitality axiom in $\mathcal{D}$ imply that this is in fact an isomorphism of decorated cospans:

$$\begin{tikzcd}
1 \arrow[rrrr, "s"] \arrow[d, "\lambda_{\mathcal{D}}^{-1}"'] & & & & FN \\
1 \otimes 1 \arrow[r, "s\otimes 1"'] & FN \otimes 1 \arrow[rrru, "\lambda_{\mathcal{D}}"] \arrow[r, "1\otimes\varphi_1"'] & FN \otimes F\varnothing \arrow[rr, "\varphi_{N,\varnothing}"'] & & F(N+\varnothing). \arrow[u, "F\lambda_{\mathcal{C}}"']
\end{tikzcd}$$

This the coherence maps are indeed natural transformations. Moreover, they obey the required coherence laws as they are images of maps that obey these laws in $\mathrm{Cospan}(\mathcal{C})$.

Similarly, to arrive at the hypergraph structure on $F\mathrm{Cospan}$, we simply equip each object X with the image of the special commutative Frobenius monoid specified by the hypergraph structure of $\mathrm{Cospan}(\mathcal{C})$. It is evident that this choice of structures implies the above wide embedding is a hypergraph functor. □

Note that if the monoidal unit in $(\mathcal{D}, \otimes)$ is the initial object, then each object only has one possible decoration: the empty decoration. This immediately implies the following corollary.

Corollary 2.5. *Let $1_{\mathcal{C}}\colon (\mathcal{C}, +) \to (\mathcal{C}, +)$ be the identity functor on a category $\mathcal{C}$ with finite colimits. Then* $\mathrm{Cospan}(\mathcal{C})$ *and* $1_{\mathcal{C}}\mathrm{Cospan}$ *are isomorphic as hypergraph categories.*

Thus we see that there is always a hypergraph functor between decorated cospan categories $1_{\mathcal{C}}\mathrm{Cospan} \to F\mathrm{Cospan}$. This provides an example of a more general way to construct hypergraph functors between decorated cospan categories. We detail this in the next section.

2.4 Functors between decorated cospan categories

Decorated cospans provide a setting for formulating various operations that we might wish to enact on the decorations, including the composition of these decorations, both sequential and monoidal, as well as dagger, dualising, and other operations afforded by the Frobenius structure. We now observe that these operations are formulated in a systematic way, so that transformations of the decorating structure—that is, monoidal transformations between the lax monoidal functors defining decorated cospan categories—respect these operations.

Theorem 2.6. *Let $\mathcal{C}$, $\mathcal{C}'$ be categories with finite colimits, abusing notation to write the coproduct in each category $+$, and $(\mathcal{D}, \otimes)$, $(\mathcal{D}', \boxtimes)$ be braided monoidal categories. Further let*

$$(F, \varphi) \colon (\mathcal{C}, +) \longrightarrow (\mathcal{D}, \otimes)$$

and

$$(G, \gamma) \colon (\mathcal{C}', +) \longrightarrow (\mathcal{D}', \boxtimes)$$

be lax braided monoidal functors. This gives rise to decorated cospan categories $F\mathrm{Cospan}$ *and* $G\mathrm{Cospan}$.

Suppose then that we have a finite colimit-preserving functor $A \colon \mathcal{C} \to \mathcal{C}'$ with accompanying natural isomorphism $\alpha \colon A(-) + A(-) \Rightarrow A(- + -)$, a lax monoidal functor $(B, \beta) \colon (\mathcal{D}, \otimes) \to (\mathcal{D}', \boxtimes)$, and a monoidal natural transformation $\theta \colon (B \circ F, B\varphi \circ \beta) \Rightarrow (G \circ A, G\alpha \circ \gamma)$. This may be depicted by the diagram:

$$\begin{tikzcd}
(\mathcal{C}, +) \arrow[r, "{(F,\varphi)}"] \arrow[d, "{(A,\alpha)}"'] \arrow[dr, phantom, "\not\Downarrow_\theta"] & (\mathcal{D}, \otimes) \arrow[d, "{(B,\beta)}"] \\
(\mathcal{C}', +) \arrow[r, "{(G,\gamma)}"'] & (\mathcal{D}', \boxtimes).
\end{tikzcd}$$

Then we may construct a hypergraph functor

$$(T, \tau) \colon F\mathrm{Cospan} \longrightarrow G\mathrm{Cospan}$$

mapping objects $X \in F\mathrm{Cospan}$ to $AX \in G\mathrm{Cospan}$, and morphisms

$$\left(\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y \arrow[ul, "o"'] \end{tikzcd} , \quad \begin{tikzcd} FN \\ 1_{\mathcal{D}} \arrow[u, "s"'] \end{tikzcd} \right) \quad \text{to} \quad \left(\begin{tikzcd} & AN & \\ AX \arrow[ur, "Ai"] & & AY \arrow[ul, "Ao"'] \end{tikzcd} , \quad \begin{tikzcd} GAN \\ BFN \arrow[u, "\theta_N"'] \\ B1_{\mathcal{D}} \arrow[u, "Bs"'] \\ 1_{\mathcal{D}'} \arrow[u, "\beta_1"'] \end{tikzcd} \right).$$

Moreover, (T, τ) is a strict monoidal functor if and only if (A, α) is.

Proof. We must prove that (T, τ) is a functor, is strong symmetric monoidal, and that it preserves the special commutative Frobenius structure on each object.

Checking the functoriality of T is again an exercise in applying the properties of structure available—in this case the colimit-preserving nature of A and the monoidality of $(\mathcal{D}, \boxtimes)$, (B, β), and θ—to show that the relevant diagrams of decorations commute.

In detail, let

$$(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y,\ 1 \xrightarrow{s} FN) \qquad \text{and} \qquad (Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z,\ 1 \xrightarrow{t} FM),$$

be morphisms in $F\mathrm{Cospan}$. As the composition of the cospan part is by pushout in $\mathcal{C}$ in both cases, and as T acts as the colimit preserving functor A on these cospans, it is clear that T preserves composition of isomorphism classes of cospans. Write

$$c\colon AX +_{AY} AZ \xrightarrow{\sim} A(X +_Y Z)$$

for the isomorphism from the cospan obtained by composing the A-images of the above two decorated cospans to the cospan obtained by taking the A-image of their composite. To see that this extends to an isomorphism of decorated cospans, observe that the decorations of these two cospans are given by the rightmost and leftmost columns respectively in the following diagram:

$$\begin{tikzcd}
GA(N +_Y M) & & G(AN +_{AY} AM) \arrow[ll, "Gc", "\sim"'] \arrow[dl, phantom, "\text{(A)}"] \\
BF(N +_Y M) \arrow[u, "\theta_{N+_Y M}"] \arrow[r, phantom, "\text{(T2)}"] & GA(N+M) \arrow[ul, "{GA[j_N,j_M]}"'] & G(AN+AM) \arrow[l, "G\alpha", "\sim"'] \arrow[u, "{G[j_{AN},j_{AM}]}"'] \\
BF(N+M) \arrow[u, "{BF[j_N,j_M]}"] \arrow[ur, "\theta_{N,M}"'] \arrow[rr, phantom, "\text{(T1)}"] & & GAN \boxtimes GAM \arrow[u, "\gamma_{AN,AM}"'] \\
B(FN \otimes FM) \arrow[u, "B\varphi_{N,M}"] & & BFN \boxtimes BFM \arrow[ll, "\beta_{FN,FM}"'] \arrow[u, "\theta_N \boxtimes \theta_M"'] \\
B(1_{\mathcal{D}} \otimes 1_{\mathcal{D}}) \arrow[u, "B(s\otimes t)"] & & B1_{\mathcal{D}} \boxtimes B1_{\mathcal{D}} \arrow[ll, "\beta_{1,1}"'] \arrow[u, "Bs\boxtimes Bt"'] \\
B1_{\mathcal{D}} \arrow[u, "\beta\lambda_1^{-1}"] \arrow[r, "\lambda_{B1}^{-1}"] & 1_{\mathcal{D}'} \boxtimes B1_{\mathcal{D}} \arrow[ur, "\beta_1 \boxtimes 1"] & 1_{\mathcal{D}'} \boxtimes 1_{\mathcal{D}'} \arrow[l, "1\boxtimes\beta_1"'] \arrow[u, "\beta_1\boxtimes\beta_1"'] \\
& 1_{\mathcal{D}'} \arrow[ul, "\beta_1"] \arrow[ur, "\lambda_1^{-1}"'] &
\arrow[from=4-1, to=5-3, phantom, "\text{(B2)}"]
\arrow[from=5-1, to=6-2, phantom, "\text{(B1)}"]
\arrow[from=6-2, to=5-3, phantom, "\text{(D2)}"']
\arrow[from=6-2, to=7-2, phantom, "\text{(D1)}"]
\end{tikzcd}$$

From bottom to top, (D1) commutes by the naturality of λ, (D2) by the functoriality of the monoidal product $-\boxtimes-$, (B1) by the unit law for (B,β), (B2) by the naturality of β, (T1) by the monoidality of the natural transformation θ, (T2) by the naturality of θ, and (A) by the colimit preserving property of A and the functoriality of G.

We must also show that identity morphisms are mapped to identity morphisms. Let

$$(X \xrightarrow{1_X} X \xleftarrow{1_X} X,\ 1 \xrightarrow{F!\circ\varphi_1} FX)$$

be the identity morphism on some object X in the category of F-decorated cospans. Now this morphism has T-image

$$(AX \xrightarrow{1_{AX}} AX \xleftarrow{1_{AX}} AX,\ 1 \xrightarrow{\theta_X\circ B(F!\circ\varphi_1)\circ\beta_1} GAX).$$

But we have the following diagram

$$\begin{tikzcd}
& & BF\varnothing_{\mathcal{C}} \arrow[d, "\theta_\varnothing"] \arrow[dr, "BF!"] & & \\
& B1_{\mathcal{D}} \arrow[ur, "B\varphi_1"] \arrow[dr, phantom, "\text{(T1)}"] & & BFX \arrow[dr, "\theta_X"] \arrow[dl, phantom, "\text{(T2)}"] & \\
1_{\mathcal{D}'} \arrow[ur, "\beta_1"] \arrow[r, "\gamma_1"'] \arrow[drr, "\gamma_1"'] & G\varnothing_{\mathcal{C}'} \arrow[r, "G\alpha_1"'] \arrow[dr, phantom, "\text{(A1)}"] & GA\varnothing_{\mathcal{C}} \arrow[rr, "GA!"'] \arrow[dr, phantom, "\text{(A2)}"] & & GAX \\
& & G\varnothing_{\mathcal{C}'} \arrow[u, "\sim", "G!"'] \arrow[urr, "G!"'] & &
\end{tikzcd}$$

Here (A1) and (A2) commute by the fact A preserves colimits, (T1) commutes by the unit law for the monoidal natural transformation θ, and (T2) commutes by the naturality of θ.

Thus we have the equality of decorations $\theta_X \circ B(F!\circ\varphi_1)\circ\beta_1 = G!\circ\gamma_1 \colon 1 \to GAX$, and so T sends identity morphisms to identity morphisms.

The coherence maps of the functor are given by the coherence maps for the monoidal functor A, viewed now as cospans with the empty decoration. That is, we define the coherence maps τ to be the collection of isomorphisms

$$\tau_1 = \left(\begin{tikzcd}
& A\varnothing_{\mathcal{C}} \arrow[dr, equal] & \\
\varnothing_{\mathcal{C}'} \arrow[ur, "\alpha_1"] & & A\varnothing_{\mathcal{C}}
\end{tikzcd} , \quad \begin{tikzcd}
GA\varnothing_{\mathcal{C}} \\
G\varnothing_{\mathcal{C}'} \arrow[u, "G!"'] \\
1_{\mathcal{D}'} \arrow[u, "\gamma_1"']
\end{tikzcd} \right),$$

$$\tau_{X,Y} = \left(\begin{tikzcd}
& A(X+Y) \arrow[dr, equal] & \\
AX+AY \arrow[ur, "\alpha_{X,Y}"] & & A(X+Y)
\end{tikzcd} , \quad \begin{tikzcd}
GA(X+Y) \\
G\varnothing_{\mathcal{C}'} \arrow[u, "G!"'] \\
1_{\mathcal{D}} \arrow[u, "\gamma_1"']
\end{tikzcd} \right),$$

where X, Y are objects of GCospan. As (A, α) is already strong symmetric monoidal and τ merely views these maps in $\mathcal{C}$ as empty-decorated cospans in GCospan, τ is natural in X and Y, and obeys the required coherence axioms for (T, τ) to also be strong symmetric monoidal.

Indeed, the monoidality of a functor (T, τ) has two aspects: the naturality of the transformation τ, and the coherence axioms. We discuss the former; since τ is just an empty-decorated version of α, the latter then immediately follow from the coherence of α.

The naturality of τ may be proved via the same method as that employed for the naturality of the coherence maps of decorated cospan categories: we first use the composition of empty decorations to compute the two paths around the naturality square, and then use the naturality of the coherence map α to show that these two decorated cospans are isomorphic.

In slightly more detail, suppose we have decorated cospans

$$(X \xrightarrow{i_X} N \xleftarrow{o_Z} Z,\ 1 \xrightarrow{s} FN) \quad \text{and} \quad (Y \xrightarrow{i_Y} M \xleftarrow{o_W} W,\ 1 \xrightarrow{t} FM).$$

Then naturality demands that the cospans

$$AX + AY \xrightarrow{Ai_X + Ai_Y} AN + AM \xleftarrow{(o_Z + o_W)\circ\alpha^{-1}} A(Z+W)$$

and

$$AX + AY \xrightarrow{A(i_X + i_Y)\circ\alpha} A(N+M) \xleftarrow{A(o_Z + o_W)} A(Z+W)$$

are isomorphic as decorated cospans, with decorations the top and bottom rows of the diagram below respectively.

$$\begin{tikzcd}
 & 1\otimes 1 \arrow[r, "\beta_1\otimes\beta_1"] & B1\otimes B1 \arrow[r, "Bs\otimes Bt"] & BFN\otimes BFM \arrow[r, "\theta_N\otimes\theta_M"] & GAN\otimes GAM \arrow[r, "\gamma_{AN,AM}"] & G(AN+AM) \arrow[d, "G\alpha_{N,M}"] \\
1 \arrow[ur, "\lambda^{-1}"] \arrow[r, "\beta_1"'] & B1 \arrow[r, "B\lambda^{-1}"'] & B(1\otimes 1) \arrow[r, "B(s\otimes t)"'] & B(FN\otimes FM) \arrow[r, "B\varphi_{N,M}"'] & B(F(N+M)) \arrow[r, "\theta_{N,M}"'] & G(A(N+M)).
\end{tikzcd}$$

As it is a subdiagram of the large functoriality commutative diagram, this diagram commutes. The diagrams required for $\alpha_{N,M}$ to be a morphism of cospans also commute, so our decorated cospans are indeed isomorphic. This proves τ is a natural tranfomation.

Moreover, as A is coproduct-preserving and the Frobenius structures on FCospan and GCospan are built using various copairings of the identity map, (T, τ) preserves the hypergraph structure.

Finally, it is straightforward to observe that the maps τ are identity maps if and only if the maps α are, so (T, τ) is a strict monoidal functor if and only (A, α) is. $\square$

When the decorating structure comprises some notion of topological diagram, such as a graph, these natural transformations θ might describe some semantic interpretation of the decorating structure. In this setting the above theorem constructs functorial semantics for the decorated cospan category of diagrams. We conclude this paper with an example of this application of decorated cospans.

2.5 Examples

In this final section we outline two constructions of decorated cospan categories, based on labelled graphs and linear subspaces respectively, and a functor between these two categories interpreting each graph as an electrical circuit. We shall see that the decorated cospan framework allows us to take a notion of closed system and construct a corresponding notion of open or composable system, together with functorial semantics for these systems.

This electrical circuits example outlines the motivating application for the decorated cospan construction; further details can be found in [**?**].

2.5.1 Labelled graphs.

To begin we return to the example of this paper's introduction.

Recall that a $\mathbf{(0,\infty)}$**-graph** (N,E,s,t,r) comprises a finite set N of vertices (or nodes), a finite set E of edges, functions $s,t\colon E \to N$ describing the source and target of each edge, and a function $r\colon E \to (0,\infty)$ labelling each edge. The decorated cospan framework allows us to construct a category with, roughly speaking, these graphs as morphisms. More precisely, our morphisms will consist of these graphs, together with subsets of the nodes marked, with multiplicity, as 'input' and 'output' connection points.

As suggested in the introduction, pick small categories equivalent to the categories of finite sets and $(0,\infty)$-graphs such that we may talk about the set of all $(0,\infty)$-graphs on each finite set N. Then we may consider the functor

$$\mathrm{Graph}\colon (\mathrm{FinSet},+) \longrightarrow (\mathrm{Set},\times)$$

taking a finite set N to the set $\mathrm{Graph}(N)$ of $(0,\infty)$-graphs (N,E,s,t,r) with set of nodes N. On morphisms let it take a function $f\colon N \to M$ to the function that pushes labelled graph structures on a set N forward onto the set M:

$$\begin{aligned} \mathrm{Graph}(f)\colon\ \mathrm{Graph}(N) &\longrightarrow \mathrm{Graph}(M); \\ (N,E,s,t,r) &\longmapsto (M,E,f\circ s,f\circ t,r). \end{aligned}$$

As this map simply acts by post-composition, our map Graph is indeed functorial.

We then arrive at a lax braided monoidal functor (Graph, ζ) by equipping this functor with the natural transformation

$$\begin{aligned}\zeta_{N,M} \colon \mathrm{Graph}(N) \times \mathrm{Graph}(M) &\longrightarrow \mathrm{Graph}(N+M);\\ \big((N,E,s,t,r),(M,F,s',t',r')\big) &\longmapsto \big(N+M, E+F, s+s', t+t', [r,r']\big),\end{aligned}$$

together with the unit map

$$\begin{aligned}\zeta_1 \colon 1 = \{\bullet\} &\longrightarrow \mathrm{Graph}(\varnothing);\\ \bullet &\longmapsto (\varnothing, \varnothing, !, !, !),\end{aligned}$$

where we remind ourselves that we write $[r, r']$ for the copairing of the functions r and r'. The naturality of this collection of morphisms, as well as the coherence laws for lax braided monoidal functors, follow from the universal property of the coproduct.

Theorem 2.4 thus allows us to construct a hypergraph category GraphCospan. For an intuitive visual understanding of the morphisms of this category and its composition rule, see this paper's introduction.

Chapter 3

Decorated corelations

Part II

Applications

Chapter 4

Passive linear circuits

4.1 Introduction

In late 1940s, just as Feynman was developing his diagrams for processes in particle physics, Eilenberg and Mac Lane initiated their work on category theory. Over the subsequent decades, and especially in the work of Joyal and Street in the 1980s [?, ?], it became clear that these developments were profoundly linked: monoidal categories have a precise graphical representation in terms of string diagrams, and conversely monoidal categories provide an algebraic foundation for the intuitions behind Feynman diagrams. The key insight is the use of categories where morphisms describe physical processes, rather than structure-preserving maps between mathematical objects [?, ?].

In work on fundamental physics, the cutting edge has moved from categories to higher categories [?]. But the same techniques have filtered into more immediate applications, particularly in computation and quantum computation [?, ?, ?]. This paper is part of a still nascent program of applying string diagrams to engineering, with the aim of giving diverse diagram languages a unified foundation based on category theory [?, ?, ?, ?, ?].

Indeed, even before physicists began using Feynman diagrams, various branches of engineering were using diagrams that in retrospect are closely related. Foremost among these are the ubiquitous electrical circuit diagrams. Although less well-known, similar diagrams are used to describe networks consisting of mechanical, hydraulic, thermodynamic and chemical systems. Further work, pioneered in particular by Forrester [?] and Odum [?], applies similar diagrammatic methods to biology, ecology, and economics.

As discussed in detail by Olsen [?], Paynter [?] and others, there are mathematically precise analogies between these different systems. In each case, the system's state is described by variables that come in pairs, with one variable in each pair playing the role of 'displacement' and the other playing the role of 'momentum'. In engineering, the time derivatives of these variables are sometimes called 'flow' and 'effort'. In classical mechanics, this pairing of variables is well understood using symplectic geometry. Thus, any mathematical formulation of the diagrams used to describe networks in engineering needs to take symplectic geometry as well as category theory into account.

	displacement q	flow $\dot{q}$	momentum p	effort $\dot{p}$
Electronics	charge	current	flux linkage	voltage
Mechanics (translation)	position	velocity	momentum	force
Mechanics (rotation)	angle	angular velocity	angular momentum	torque
Hydraulics	volume	flow	pressure momentum	pressure
Thermodynamics	entropy	entropy flow	temperature momentum	temperature
Chemistry	moles	molar flow	chemical momentum	chemical potent

While diagrams of networks have been independently introduced in many disciplines, we do not expect formalizing these diagrams to immediately help the practitioners of these disciplines. At first the flow of information will mainly go in the other direction: by translating ideas from these disciplines into the language of modern mathematics, we can provide mathematicians with food for thought and interesting new problems to solve. We hope that in the long run mathematicians can return the favor by bringing new insights to the table.

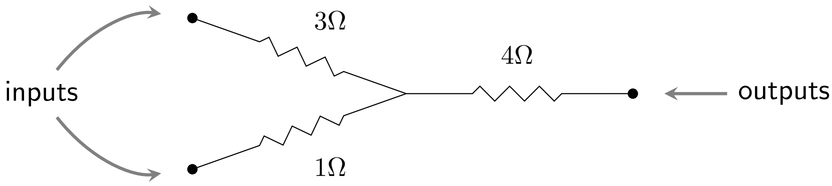
Although we shall keep the broad applicability of network diagrams in the back of our minds, we couch our discussion in terms of electrical circuits, for the sake of familiarity. In this paper our goal is somewhat limited. We only study circuits built from 'passive' components: that is, those that do not produce energy. Thus, we exclude batteries and current sources. We only consider components that respond linearly to an applied voltage. Thus, we exclude components such as nonlinear resistors or diodes. Finally, we only consider components with one input and one output, so that a circuit can be described as a graph with edges labeled by components. Thus, we also exclude transformers. The most familiar components our framework covers are linear resistors, capacitors and inductors.

While we hope to expand our scope in future work, the class of circuits made from these components has appealing mathematical properties, and is worthy of deep study. Indeed, this class has been studied intensively for many decades by electrical engineers [?, ?, ?]. Even circuits made exclusively of resistors have inspired work by mathematicians of the caliber of Weyl [?] and Smale [?].

Our work relies on this research. All we are adding is an emphasis on symplectic geometry and an explicitly 'compositional' framework, which clarifies the way a larger circuit can be built from smaller pieces. This is where monoidal categories become important: the main operations for building circuits from pieces are composition and tensoring.

Our strategy is most easily illustrated for circuits made of linear resistors. Such a resistor dissipates power, turning useful energy into heat at a rate determined by the voltage across the resistor. However, a remarkable fact is that a circuit made of these resistors always acts to *minimize* the power dissipated this way. This 'principle of minimum power' can be seen as the reason symplectic geometry becomes important in understanding circuits made of resistors, just as the principle of least action leads to the role of symplectic geometry in classical mechanics.

Here is a circuit made of linear resistors:



The wiggly lines are resistors, and their resistances are written beside them: for example, 3Ω means 3 ohms, an 'ohm' being a unit of resistance. To formalize this, define a circuit of linear resistors to consist of:

- a set N of nodes,
- a set E of edges,
- maps $s, t \colon E \to N$ sending each edge to its source and target node,
- a map $r \colon E \to (0, \infty)$ specifying the resistance of the resistor labelling each edge,
- maps $i \colon X \to N$, $o \colon Y \to N$ specifying the inputs and outputs of the circuit.

When we run electric current through such a circuit, each node $n \in N$ gets a 'potential' $\phi(n)$. The 'voltage' across an edge $e \in E$ is defined as the change in potential as we move from to the source of e to its target, $\phi(t(e)) - \phi(s(e))$, and the power dissipated by the resistor on this edge equals

$$\frac{1}{r(e)}\big(\phi(t(e)) - \phi(s(e))\big)^2.$$

The total power dissipated by the circuit is therefore twice

$$P(\phi) = \frac{1}{2}\sum_{e\in E}\frac{1}{r(e)}\big(\phi(t(e)) - \phi(s(e))\big)^2.$$

The factor of $\frac{1}{2}$ is convenient in some later calculations. Note that P is a nonnegative quadratic form on the vector space $\mathbb{R}^N$. However, not every nonnegative definite quadratic form on $\mathbb{R}^N$ arises in this way from some circuit of linear resistors with N as its set of nodes. The quadratic forms that do arise are called 'Dirichlet forms'. They have been extensively investigated [**?**, **?**, **?**, **?**], and they play a major role in our work.

We write $\partial N = i(X) \cup o(Y)$ for the set of 'terminals': that is, nodes corresponding to inputs and outputs. The principle of minimum power says that if we fix the potential at the terminals, the circuit will choose the potential at other nodes to minimize the total power dissipated. An element ψ of the vector space $\mathbb{R}^{\partial N}$ assigns a potential to each terminal. Thus, if we fix ψ, the total power dissipated will be twice

$$Q(\psi) = \min_{\substack{\phi\in\mathbb{R}^N \\ \phi|_{\partial N}=\psi}} P(\phi)$$

The function $Q\colon \mathbb{R}^{\partial N} \to \mathbb{R}$ is again a Dirichlet form. We call it the 'power functional' of the circuit.

Now, suppose we are unable to see the internal workings of a circuit, and can only observe its 'external behavior': that is, the potentials at its terminals and the currents flowing into or out of these terminals. Remarkably, this behavior is completely determined by the power functional Q. The reason is that the current at any terminal can be obtained by differentiating Q with respect to the potential at this terminal, and relations of this form are *all* the relations that hold between potentials and currents at the terminals.

The Laplace transform allows us to generalize this immediately to circuits that can also contain linear inductors and capacitors, simply by changing the field we work over, replacing $\mathbb{R}$ by the field $\mathbb{R}(s)$ of rational functions of a single real variable, and talking of 'impedance' where we previously talked of resistance. We obtain a category Circ where an object is a finite set, a morphism $f\colon X \to Y$ is a circuit with input set

X and output set Y, and composition is given by identifying the outputs of one circuit with the inputs of the next, and taking the resulting union of labelled graphs. Each such circuit gives rise to a Dirichlet form, now defined over $\mathbb{R}(s)$, and this Dirichlet form completely describes the externally observable behavior of the circuit.

We can take equivalence classes of circuits, where two circuits count as the same if they have the same Dirichlet form. We wish for these equivalence classes of circuits to form a category. Although there is a notion of composition for Dirichlet forms, we find that it lacks identity morphisms or, equivalently, it lacks morphisms representing ideal wires of zero impedance. To address this we turn to Lagrangian subspaces of symplectic vector spaces. These generalize quadratic forms via the map

$$\Big(Q\colon \mathbb{F}^{\partial N} \to \mathbb{F}\Big) \longmapsto \Big(\mathrm{Graph}(dQ) = \{(\psi, dQ_\psi) \mid \psi \in \mathbb{F}^{\partial N}\} \subseteq \mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*\Big)$$

taking a quadratic form Q on the vector space $\mathbb{F}^{\partial N}$ over the field $\mathbb{F}$ to the graph of its differential dQ. Here we think of the symplectic vector space $\mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*$ as the state space of the circuit, and the subspace $\mathrm{Graph}(dQ)$ as the subspace of attainable states, with $\psi \in \mathbb{F}^{\partial N}$ describing the potentials at the terminals, and $dQ_\psi \in (\mathbb{F}^{\partial N})^*$ the currents.

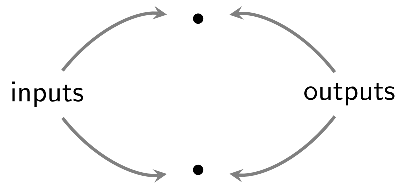
This construction is well-known in classical mechanics [**?**], where the principle of least action plays a role analogous to that of the principle of minimum power here. The set of Lagrangian subspaces is actually an algebraic variety, the 'Lagrangian Grassmannian', which serves as a compactification of the space of quadratic forms. The Lagrangian Grassmannian has already played a role in Sabot's work on circuits made of resistors [**?**, **?**]. For us, its importance it that we can find identity morphisms for the composition of Dirichlet forms by taking circuits made of parallel resistors and letting their resistances tend to zero: the limit is not a Dirichlet form, but it exists in the Lagrangian Grassmannian. Indeed, there exists a category LagrRel with finite dimensional symplectic vector spaces as objects and 'Lagrangian relations' as morphisms: that is, linear relations from V to W that are given by Lagrangian subspaces of $\overline{V} \oplus W$, where $\overline{V}$ is the symplectic vector space conjugate to V.

To move from the Lagrangian subspace defined by the graph of the differential of the power functional to a morphism in the category LagrRel—that is, to a Lagrangian relation—we must treat seriously the input and output functions of the circuit. These express the circuit as built upon a cospan

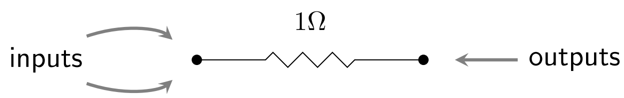
$$\begin{tikzcd} & N & \\ X \arrow[ur, "i"] & & Y. \arrow[ul, "o"'] \end{tikzcd}$$

Applicable far more broadly than this present formalization of circuits, cospans model systems with two 'ends', an input and output end, albeit without any connotation of directionality: we might just as well exchange the role of the inputs and outputs by taking the mirror image of the above diagram. The role of the input and output functions, as we have discussed, is to mark the terminals we may glue onto the terminals of another circuit, and the pushout of cospans gives formal precision to this gluing construction.

One upshot of this cospan framework is that we may consider circuits with elements of N that are both inputs and outputs, such as this one:



This corresponds to the identity morphism on the finite set with two elements. Another is that some points may be considered an input or output multiple times; we draw this:



This allows us to connect two distinct outputs to the above double input.

Given a set X of inputs or outputs, we understand the electrical behavior on this set by considering the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$, the direct sum of the space $\mathbb{F}^X$ of potentials and the space $(\mathbb{F}^X)^*$ of currents at these points. A Lagrangian relation specifies which states of the output space $\mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$ are allowed for each state of the input space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$. Turning the Lagrangian subspace Graph(dQ) of a circuit into this information requires that we understand the 'symplectification'

$$Sf\colon \mathbb{F}^B \oplus (\mathbb{F}^B)^* \to \mathbb{F}^A \oplus (\mathbb{F}^A)^*$$

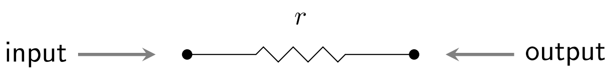
and 'twisted symplectification'

$$S^tf\colon \mathbb{F}^B \oplus (\mathbb{F}^B)^* \to \overline{\mathbb{F}^A \oplus (\mathbb{F}^A)^*}$$

of a function $f\colon A \to B$ between finite sets. In particular we need to understand how these apply to the input and output functions with codomain restricted to ∂N; abusing notation, we also write these $i\colon X \to \partial N$ and $o\colon Y \to \partial N$.

The symplectification is a Lagrangian relation, and the catch phrase is that it 'copies voltages' and 'splits currents'. More precisely, for any given potential-current pair (ψ, ι) in $\mathbb{F}^B \oplus (\mathbb{F}^B)^*$, its image under Sf comprises all elements of $(\psi', \iota') \in \mathbb{F}^A \oplus (\mathbb{F}^A)^*$ such that the potential at $a \in A$ is equal to the potential at $f(a) \in B$, and such that, for each fixed $b \in B$, collectively the currents at the $a \in f^{-1}(b)$ sum to the current at b. We use the symplectification So of the output function to relate the state on ∂N to that on the outputs Y. As our current framework is set up to report the current *out* of each node, to describe input currents we define the twisted symplectification $S^t f \colon \mathbb{F}^B \oplus (\mathbb{F}^B)^* \to \overline{\mathbb{F}^A \oplus (\mathbb{F}^A)^*}$ almost identically to the above, except that we flip the sign of the currents $\iota' \in (\mathbb{F}^A)^*$. We use the twisted symplectification $S^t i$ of the input function to relate the state on ∂N to that on the inputs.

The Lagrangian relation corresponding to a circuit is then the set of all potential–current pairs that are possible at the inputs and outputs of that circuit. For instance, consider a resistor of resistance r, with one end considered as an input and the other as an output:



To obtain the corresponding Lagrangian relation, we must first specify domain and codomain symplectic vector spaces. In this case, as the input and output sets each consist of a single point, these vector spaces are both $\mathbb{F} \oplus \mathbb{F}^*$, where the first summand is understood as the space of potentials, and the second the space of currents.

Now, the resistor has power functional $Q \colon \mathbb{F}^2 \to \mathbb{F}$ given by

$$Q(\psi_1, \psi_2) = \frac{1}{2r}(\psi_2 - \psi_1)^2,$$

and the graph of the differential of Q is

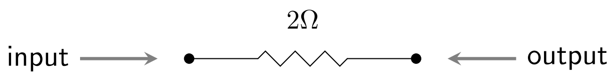
$$\mathrm{Graph}(dQ) = \left\{ \left(\psi_1, \psi_2, \tfrac{1}{r}(\psi_1 - \psi_2), \tfrac{1}{r}(\psi_2 - \psi_1)\right) \,\middle|\, \psi_1, \psi_2 \in \mathbb{F} \right\} \subseteq \mathbb{F}^2 \oplus (\mathbb{F}^2)^*.$$

In this example the input and output functions i, o are simply the identity functions on a one element set, so the symplectification of the output function is simply the identity linear transformation, and the twisted symplectification of the input function is the isomorphism between conjugate symplectic vector spaces $\mathbb{F} \oplus \mathbb{F}^* \to \overline{\mathbb{F} \oplus \mathbb{F}^*}$ mapping (ϕ, i) to $(\phi, -i)$ This implies that the behavior associated to this circuit is the Lagrangian relation

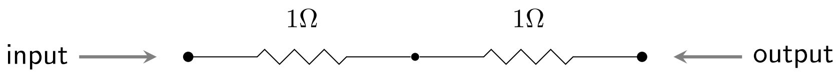
$$\left\{ (\psi_1, i, \psi_2, i) \,\middle|\, \psi_1, \psi_2 \in \mathbb{F}, i = \tfrac{1}{r}(\psi_2 - \psi_1) \right\} \subseteq \overline{\mathbb{F} \oplus \mathbb{F}^*} \oplus \mathbb{F} \oplus \mathbb{F}^*.$$

This is precisely the set of potential-current pairs that are allowed at the input and output of a resistor of resistance r. In particular, the relation $i = \frac{1}{r}(\psi_2 - \psi_1)$ is well-known in electrical engineering: it is ‘Ohm's law’.

A crucial fact is that the process of mapping a circuit to its corresponding Lagrangian relation identifies distinct circuits. For example, a single 2-ohm resistor:



has the same Lagrangian relation as two 1-ohm resistors in series:



The Lagrangian relation does not shed any light on the internal workings of a circuit. Thus, we call the process of computing this relation 'black boxing': it is like encasing the circuit in an opaque box, leaving only its terminals accessible. Fortunately, the Lagrangian relation of a circuit is enough to completely characterize its external behavior, including how it interacts when connected with other circuits.

Put more precisely, the black boxing process is *functorial*: we can compute the black boxed version of a circuit made of parts by computing the black boxed versions of the parts and then composing them. In fact we shall prove that Circ and LagrRel are dagger compact categories, and the black box functor preserves all this extra structure:

Theorem 4.1. *There exists a symmetric monoidal dagger functor, the* **black box functor**

$$\blacksquare \colon \mathrm{Circ} \to \mathrm{LagrRel},$$

mapping a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$ it generates, and a circuit $\big((N, E, s, t, r), i, o\big)$ to the Lagrangian relation

$$\bigcup_{v \in \mathrm{Graph}(dQ)} S^t i(v) \times So(v) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*,$$

where Q is the circuit's power functional.

The goal of this paper is to prove and explain this result. The proof itself is more tricky than one might first expect, but our approach introduces various concepts that should be useful throughout the study of networks, such as 'decorated cospans' and 'corelations'. These provide a general framework for discussing open networked systems—and not only the passive linear systems discussed here, but also others, such as Markov processes [?].

Cospans are already familiar as a formalism for making entities with an arbitrarily designated 'input end' and 'output end' into the morphisms of a category. For example, in topological quantum field theory we use special cospans called 'cobordisms' to describe pieces of spacetime [?, ?]. Earlier we introduced 'decorated cospans' to describe circuits with specified inputs and outputs. Later, with the help of machinery developed in a companion paper [?], we use decorated cospans to set up several functors that appear as factors of the black box functor, as steps in proving its functoriality.

Just as a relation is an isomorphism class of spans of a special sort, namely jointly monic spans, a 'corelation' is an isomorphism class of jointly epic cospans. While corelations are less widely used than relations, they turn out to be perfectly suited for describing circuits of ideal perfectly conductive wires—precisely the class of circuits that cannot be modeled simply as Dirichlet forms, since the power functional becomes infinite when it includes terms where the resistance is zero. We introduce corelations in Section 4.8, and they too play a key role in constructing the black box functor.

With these tools in hand, the black box functor turns out to rely on a tight relationship between Kirchhoff's laws, the minimization of Dirichlet forms, and the 'symplectification' of corelations. It is well-known that away from the terminals, a circuit must obey two rules known as Kirchhoff's laws. We have already noted that the principle of minimum power states that a circuit will 'choose' potentials on its interior that minimize the power functional. We clarify the relation between these points in Theorems 4.20 and 4.22, which together show that minimizing a Dirichlet form over some subset amounts to assuming that the corresponding circuit obeys Kirchhoff's laws on that subset.

We have also mentioned the symplectification of functions above. Extending this to allow symplectification of corelations, this process gives a map sending corelations to Lagrangian relations that describe the behavior of ideal perfectly conductive wires. We prove that these symplectified corelations simultaneously impose Kirchhoff's laws (Proposition 4.48 and Example 4.49) and accomplish the minimization of Dirichlet forms (Theorem 4.53).

Together, our results show that these three concepts—Kirchhoff's laws from circuit theory, the analytic idea of minimizing power dissipation, and the algebraic idea of symplectification of corelations—are merely different faces of one law: the law of composition of circuits.

4.1.1 Finding your way through this paper

This paper is split into three parts, addressing in turn the questions:

I. What do circuit diagrams mean?

II. How do we interact with circuit diagrams?

III. How is meaning preserved under these interactions?

We begin Part 4.2, on the semantics of circuit diagrams, with a discussion of circuits of linear resistors, developing the intuition for the governing laws of passive linear circuits—Ohm's law, Kirchhoff's voltage law, and Kirchhoff's current law—in a time-independent setting (Section 4.3). This allows us to develop the concept of Dirichlet form as a representation of power consumption, and understand their composition as minimizing power, an expression of the current law. In Section 4.4, the Laplace transform then allows us to recapitulate these ideas after introducing inductors and capacitors, speaking of impedance where we formerly spoke of resistance, and generalizing Dirichlet forms from the field $\mathbb{R}$ to the field $\mathbb{R}(s)$ of real rational functions. While in this setting the principle of minimum power is replaced by a variational principle, the intuitions gained from circuits of resistors still remain useful.

At the end of this part, we show that Dirichlet forms alone do not provide the flexibility to construct a category representing the semantics of circuit diagrams. The fundamental reason is that they do not allow us to describe circuits involving ideal perfectly conductive wires. This motivates the development of more powerful machinery.

Part 4.5, on the syntax of circuit diagrams, contains the main technical contributions of the paper. It begins with Section, which develops machinery to construct what we call categories of decorated cospans. These are categories where the objects are finite sets and the morphisms are cospans in the category of finite sets together with some extra structure on the apex. Circuits, as defined above, are naturally an example of such a construction, and Section 4.6 lays out the details of this. In Section 4.7 we then review the basic theory of linear Lagrangian relations, giving details to the correspondence we have defined between Dirichlet forms, and hence passive linear circuits, and Lagrangian relations. Section 4.8 then takes immediate advantage of the added flexibility of Lagrangian relations, discussing the 'trivial' circuits comprising only perfectly conductive wires, which mediate the notion of composition of circuits.

Having introduced these prerequisites, we get to the point in Part 4.9. In Section 4.10 we introduce the black box functor, and in Section 4.11 we prove our main result.

4.2 Passive Linear Circuits

In this part we review the meaning of circuit diagrams comprising resistors, inductors, and capacitors, giving an answer to the question "What do circuit diagrams mean?".

To elaborate, while circuit diagrams model electric circuits according to their physical form, another, often more relevant, way to understand a circuit is by its external behavior. This means the following. To an electric circuit we associate two quantities to each edge: voltage and current. We are not free, however, to choose these quantities as we like; circuits are subject to governing laws that imply voltages and currents must obey certain relationships. From the perspective of control theory we are particularly interested in the values these quantities take at the so-called terminals, and how altering one value will affect the other values. We call two circuits equivalent when they determine the same relationship. Our main task in this first part is to explore when two circuits are equivalent.

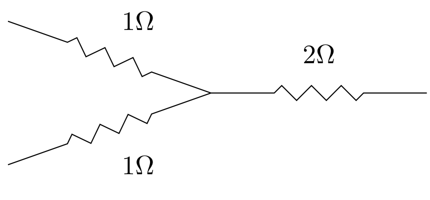
4.3 Circuits of linear resistors

In order to let physical intuition lead the way, we begin by specialising to the case of linear resistors. In this section we describe how to find the function of a circuit from its form, advocating in particular the perspective of the principle of minimum power. This allows us to identify the external behavior of a circuit with a so-called Dirichlet form representing the dependence of its power consumption on potentials at its terminals.

4.3.1 Circuits as labelled graphs

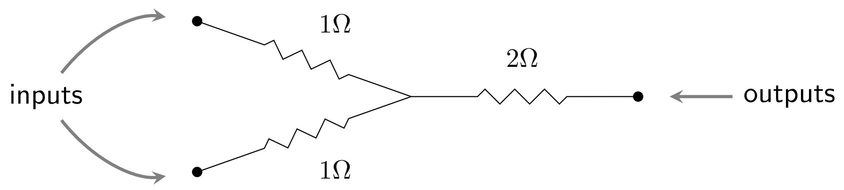
The concept of an abstract open electrical circuit made of linear resistors is well-known in electrical engineering, but we shall need to formalize it with more precision than usual. The basic idea is that a circuit of linear resistors is a graph whose edges are labelled by positive real numbers called 'resistances', and whose sets of vertices is equipped with two subsets: the 'inputs' and 'outputs'. This unfolds as follows.

A (closed) circuit of resistors looks like this:

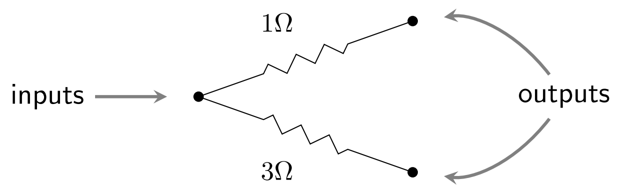


We can consider this a labelled graph, with each resistor an edge of the graph, its resistance its label, and the vertices of the graph the points at which resistors are connected.

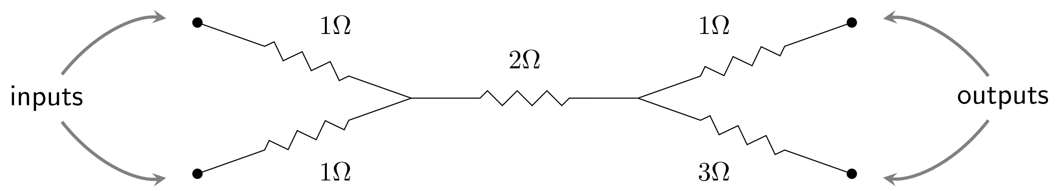
A circuit is 'open' if it can be connected to other circuits. To do this we first mark points at which connections can be made by denoting some vertices as input and output terminals:



Then, given a second circuit, we may choose a relation between the output set of the first and the input set of this second circuit, such as the simple relation of the single output vertex of the circuit above with the single input vertex of the circuit below.



We connect the two circuits by identifying output and input vertices according to this relation, giving in this case the composite circuit:



More formally, we define a **graph**[1] to be a pair of functions $s, t \colon E \to N$ where E and N are finite sets. We call elements of E **edges** and elements of N **vertices** or **nodes**. We say that the edge $e \in E$ has **source** $s(e)$ and **target** $t(e)$, and also say that e is an edge **from** $s(e)$ **to** $t(e)$.

To study circuits we need graphs with labelled edges:

Definition 4.2. Given a set L of **labels**, an ***L*-graph** is a graph equipped with a function $r \colon E \to L$:

$$L \xleftarrow{r} E \overset{s}{\underset{t}{\rightrightarrows}} N.$$

[1]In this paper we refer to directed multigraphs simply as graphs.

For circuits made of resistors we take $L = (0, \infty)$, but later we shall take L to be a set of positive elements in some more general field. In either case, a circuit will be an L-graph with some extra structure:

Definition 4.3. Given a set L, a **circuit over $\boldsymbol{L}$** is an L-graph $L \xleftarrow{r} E \overset{s}{\underset{t}{\rightrightarrows}} N$ together with finite sets X, Y, and functions $i\colon X \to N$ and $o\colon Y \to N$. We call the sets $i(X)$, $o(Y)$, and $\partial N = i(X) \cup o(Y)$ the **inputs**, **outputs**, and **terminals** or **boundary** of the circuit, respectively.

We will later make use of the notion of connectedness in graphs. Recall that given two vertices $v, w \in N$ of a graph, a **path from $\boldsymbol{v}$ to $\boldsymbol{w}$** is a finite sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ and edges $e_1, \ldots, e_n$ such that for each $1 \le i \le n$, either e_i is an edge from v_i to v_{i+1}, or an edge from v_{i+1} to v_i. A subset S of the vertices of a graph is **connected** if, for each pair of vertices in S, there is a path from one to the other. A **connected component** of a graph is a maximal connected subset of its vertices.[2]

In the rest of this section we take $L = (0, \infty) \subseteq \mathbb{R}$ and fix a circuit over $(0, \infty)$. The edges of this circuit should be thought of as 'wires'. The label $r_e \in (0, \infty)$ stands for the **resistance** of the resistor on the wire e. There will also be a voltage and current on each wire. In this section, these will be specified by functions $V \in \mathbb{R}^E$ and $I \in \mathbb{R}^E$. Here, as customary in engineering, we use I for 'intensity of current', following Ampère.

4.3.2 Ohm's law, Kirchhoff's laws, and the principle of minimum power

In 1827, Georg Ohm published a book which included a relation between the voltage and current for circuits made of resistors [**?**]. At the time, the critical reception was harsh: one contemporary called Ohm's work "a web of naked fancies, which can never find the semblance of support from even the most superficial of observations", and the German Minister of Education said that a professor who preached such heresies was unworthy to teach science [**?**, **?**]. However, a simplified version of his relation is now widely used under the name of 'Ohm's law'. We say that **Ohm's law** holds if for all edges $e \in E$ the voltage and current functions of a circuit obey:

$$V(e) = r(e)I(e).$$

[2]In the theory of directed graphs the qualifier 'weakly' is commonly used before the word 'connected' in these two definitions, in distinction from a stronger notion of connectedness requiring paths to respect edge directions. As we never consider any other sort of connectedness, we omit this qualifier.

Kirchhoff's laws date to Gustav Kirchhoff in 1845, generalising Ohm's work. They were in turn generalized into Maxwell's equations a few decades later. We say **Kirchhoff's voltage law** holds if there exists $\phi \in \mathbb{R}^N$ such that

$$V(e) = \phi(t(e)) - \phi(s(e)).$$

We call the function ϕ a **potential**, and think of it as assigning an electrical potential to each node in the circuit. The voltage then arises as the differences in potentials between adjacent nodes. If Kirchhoff's voltage law holds for some voltage V, the potential ϕ is unique only in the trivial case of the empty circuit: when the set of nodes N is empty. Indeed, two potentials define the same voltage function if and only if their difference is constant on each connected component of the graph Γ.

We say **Kirchhoff's current law** holds if for all nonterminal nodes $n \in N \setminus \partial N$ we have

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e).$$

This is an expression of conservation of charge within the circuit; it says that the total current flowing in or out of any nonterminal node is zero. Even when Kirchhoff's current law is obeyed, terminals need not be sites of zero net current; we call the function $\iota \in \mathbb{R}^{\partial N}$ that takes a terminal to the difference between the outward and inward flowing currents,

$$\begin{aligned} \iota : \partial N &\longrightarrow \mathbb{R} \\ n &\longmapsto \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e), \end{aligned}$$

the **boundary current** for I.

A **boundary potential** is also a function in $\mathbb{R}^{\partial N}$, but instead thought of as specifying potentials on the terminals of a circuit. As we think of our circuits as open circuits, with the terminals points of interaction with the external world, we shall think of these potentials as variables that are free for us to choose. Using the above three principles—Ohm's law, Kirchhoff's voltage law, and Kirchhoff's current law—it is possible to show that choosing a boundary potential determines unique voltage and current functions on that circuit.

The so-called 'principle of minimum power' gives some insight into how this occurs, by describing a way potentials on the terminals might determine potentials at all nodes. From this, Kirchhoff's voltage law then gives rise to a voltage function on the edges, and Ohm's law gives us a current function too. We shall show, in fact, that a potential satisfies the principle of minimum power for a given boundary potential if and only if this current obeys Kirchhoff's current law.

A circuit with current I and voltage V dissipates energy at a rate equal to

$$\sum_{e \in E} I(e)V(e).$$

Ohm's law allows us to rewrite I as V/r, while Kirchhoff's voltage law gives us a potential ϕ such that $V(e)$ can be written as $\phi(t(e)) - \phi(s(e))$, so for a circuit obeying these two laws the power can also be expressed in terms of this potential. We thus arrive at a functional mapping potentials ϕ to the power dissipated by the circuit when Ohm's law and Kirchhoff's voltage law are obeyed for ϕ.

Definition 4.4. The **extended power functional** $P \colon \mathbb{R}^N \to \mathbb{R}$ of a circuit is defined by

$$P(\phi) = \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} \big(\phi(t(e)) - \phi(s(e))\big)^2.$$

The factor of $\frac{1}{2}$ is inserted to cancel the factor of 2 that appears when we differentiate this expression. We call P the *extended* power functional as we shall see that it is defined even on potentials that are not compatible with the three governing laws of electric circuits. We shall later restrict the domain of this functional so that it is defined precisely on those potentials that *are* compatible with the governing laws. Note that this functional does not depend on the directions chosen for the edges of the circuit.

This expression lets us formulate the 'principle of minimum power', which gives us information about the potential ϕ given its restriction to the boundary of Γ. Call a potential $\phi \in \mathbb{R}^N$ an **extension** of a boundary potential $\psi \in \mathbb{R}^{\partial N}$ if ϕ is equal to ψ when restricted to $\mathbb{R}^{\partial N}$—that is, if $\phi|_{\partial N} = \psi$.

Definition 4.5. We say a potential $\phi \in \mathbb{R}^N$ **obeys the principle of minimum power** for a boundary potential $\psi \in \mathbb{R}^{\partial N}$ if ϕ minimizes the extended power functional P subject to the constraint that ϕ is an extension of ψ.

As promised, in the presence of Ohm's law and Kirchhoff's voltage law, the principle of minimum power is equivalent to Kirchhoff's current law.

Proposition 4.6. *Let ϕ be a potential extending some boundary potential ψ. Then ϕ obeys the principle of minimum power for ψ if and only if the current*

$$I(e) = \frac{1}{r(e)}(\phi(t(e)) - \phi(s(e)))$$

obeys Kirchhoff's current law.

Proof. Fixing the potentials at the terminals to be those given by the boundary potential ψ, the power is a nonnegative quadratic function of the potentials at the nonterminals. This implies that an extension ϕ of ψ minimizes P precisely when

$$\left.\frac{\partial P(\varphi)}{\partial \varphi(n)}\right|_{\varphi=\phi} = 0$$

for all nonterminals $n \in N \setminus \partial N$. Note that the partial derivative of the power with respect to the potential at n is given by

$$\begin{aligned}\left.\frac{\partial P}{\partial \varphi(n)}\right|_{\varphi=\phi} &= \sum_{t(e)=n} \frac{1}{r(e)}\big(\phi(t(e)) - \phi(s(e))\big) - \sum_{s(e)=n} \frac{1}{r(e)}\big(\phi(t(e)) - \phi(s(e))\big) \\ &= \sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e).\end{aligned}$$

Thus ϕ obeys the principle of minimum power for ψ if and only if

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e)$$

for all $n \in N \setminus \partial N$, and so if and only if Kirchhoff's current law holds. □

4.3.3 A Dirichlet problem

We remind ourselves that we are in the midst of understanding circuits as objects that define relationships between boundary potentials and boundary currents. This relationship is defined by the stipulation that voltage–current pairs on a circuit must obey Ohm's law and Kirchhoff's laws—or equivalently, Ohm's law, Kirchhoff's voltage law, and the principle of minimum power. In this subsection we show these conditions imply that for each boundary potential ψ on the circuit there exists a potential ϕ on the circuit extending ψ, unique up to what may be interpreted as a choice of reference potential on each connected component of the circuit. From this potential ϕ we can then compute the unique voltage, current, and boundary current functions compatible with the given boundary potential.

Fix again a circuit with extended power functional $P\colon \mathbb{R}^N \to \mathbb{R}$. Let $\nabla\colon \mathbb{R}^N \to \mathbb{R}^N$ be the operator that maps a potential $\phi \in \mathbb{R}^N$ to the function from N to $\mathbb{R}$ given by

$$n \longmapsto \left.\frac{\partial P}{\partial \varphi(n)}\right|_{\varphi=\phi}.$$

As we have seen, this function takes potentials to twice the pointwise currents that they induce. We have also seen that a potential ϕ is compatible with the governing laws of circuits if and only if

$$\nabla\phi\big|_{\mathbb{R}^{\partial N}} = 0. \tag{4.1}$$

The operator ∇ acts as a discrete analogue of the Laplacian for the graph Γ, so we call this operator the **Laplacian** of Γ, and say that equation (4.1) is a version of Laplace's equation. We then say that the problem of finding an extension ϕ of some fixed boundary potential ψ that solves this Laplace's equation—or, equivalently, the problem of finding a ϕ that obeys the principle of minimum power for ψ—is a discrete version of the **Dirichlet problem**.

As we shall see, this version of the Dirichlet problem always has a solution. However, the solution is not necessarily unique. If we take a solution ϕ and some $\alpha \in \mathbb{R}^N$ that is constant on each connected component and vanishes on the boundary of Γ, it is clear that $\phi + \alpha$ is still an extension of ψ and that

$$\left.\frac{\partial P(\varphi)}{\partial \varphi(n)}\right|_{\varphi=\phi} = \left.\frac{\partial P(\varphi)}{\partial \varphi(n)}\right|_{\varphi=\phi+\alpha},$$

so $\phi + \alpha$ is another solution. We say that a connected component of a circuit **touches the boundary** if it contains a vertex in ∂N. Note that such an α must vanish on all connected components touching the boundary.

With these preliminaries in hand, we can solve the Dirichlet problem:

Proposition 4.7. *For any boundary potential $\psi \in \mathbb{R}^{\partial N}$ there exists a potential ϕ obeying the principle of minimum power for ψ. If we also demand that ϕ vanish on every connected component of Γ not touching the boundary, then ϕ is unique.*

Proof. For existence, observe that the power is a nonnegative quadratic form, the extensions of ψ form an affine subspace of $\mathbb{R}^N$, and a nonnegative quadratic form restricted to an affine subspace of a real vector space must reach a minimum somewhere on this subspace.

For uniqueness, suppose that both ϕ and ϕ' obey the principle of minimum power for ψ. Let

$$\alpha = \phi' - \phi.$$

Then

$$\alpha\big|_{\partial N} = \phi'\big|_{\partial N} - \phi\big|_{\partial N} = \psi - \psi = 0,$$

so $\phi + \lambda\alpha$ is an extension of ψ for all $\lambda \in \mathbb{R}$. This implies that

$$f(\lambda) := P(\phi + \lambda\alpha)$$

is a smooth function attaining its minimum value at both $t = 0$ and $t = 1$. In particular, this implies that $f'(0) = 0$. But this means that when writing f as a quadratic, the coefficient of λ must be 0, so we can write

$$\begin{aligned}
2f(\lambda) &= \sum_{e\in E} \frac{1}{r(e)} \big((\phi+\lambda\alpha)(t(e)) - (\phi+\lambda\alpha)(s(e))\big)^2 \\
&= \sum_{e\in E} \frac{1}{r(e)} \Big(\big(\phi(t(e)) - \phi(s(e))\big) + \lambda\big(\alpha(t(e)) - \alpha(s(e))\big)\Big)^2 \\
&= \sum_{e\in E} \frac{1}{r(e)} \big(\phi(t(e)) - \phi(s(e))\big)^2 + \lambda\text{-term} + \lambda^2 \sum_{e\in E} \frac{1}{r(e)} \big(\alpha(t(e)) - \alpha(s(e))\big)^2 \\
&= \sum_{e\in E} \frac{1}{r(e)} \big(\phi(t(e)) - \phi(s(e))\big)^2 + \lambda^2 \sum_{e\in E} \frac{1}{r(e)} \big(\alpha(t(e)) - \alpha(s(e))\big)^2.
\end{aligned}$$

Then

$$f(1) - f(0) = \frac{1}{2} \sum_{e\in E} \frac{1}{r(e)} \big(\alpha(t(e)) - \alpha(s(e))\big)^2 = 0,$$

so $\alpha(t(e)) = \alpha(s(e))$ for every edge $e \in E$. This implies that α is constant on each connected component of the graph Γ of our circuit.

Note that as $\alpha|_{\partial N} = 0$, α vanishes on every connected component of Γ touching the boundary. Thus, if we also require that ϕ and ϕ' vanish on every connected component of Γ not touching the boundary, then $\alpha = \phi' - \phi$ vanishes on all connected components of Γ, and hence is identically zero. Thus $\phi' = \phi$, and this extra condition ensures a unique solution to the Dirichlet problem. □

We have also shown the following:

Proposition 4.8. *Suppose $\psi \in \mathbb{R}^{\partial N}$ and ϕ is a potential obeying the principle of minimum power for ψ. Then ϕ' obeys the principle of minimum power for ψ if and only if the difference $\phi' - \phi$ is constant on every connected component of Γ and vanishes on every connected component touching the boundary of Γ.*

Furthermore, ϕ depends linearly on ψ:

Proposition 4.9. *Fix $\psi \in \mathbb{R}^{\partial N}$, and suppose $\phi \in \mathbb{R}^N$ is the unique potential obeying the principle of minimum power for ψ that vanishes on all connected components of Γ not touching the boundary. Then ϕ depends linearly on ψ.*

Proof. Fix $\psi, \psi' \in \mathbb{R}^{\partial N}$, and suppose $\phi, \phi' \in \mathbb{R}^N$ obey the principle of minimum power for ψ, ψ' respectively, and that both ϕ and ϕ' vanish on all connected components of Γ not touching the boundary.

Then, for all $\lambda \in \mathbb{R}$,

$$(\phi + \lambda\phi')\big|_{\mathbb{R}^{\partial N}} = \phi\big|_{\mathbb{R}^{\partial N}} + \lambda\phi'\big|_{\mathbb{R}^{\partial N}} = \psi + \lambda\psi'$$

and

$$(\nabla(\phi + \lambda\phi'))\big|_{\mathbb{R}^{\partial N}} = (\nabla\phi)\big|_{\mathbb{R}^{\partial N}} + \lambda(\nabla\phi')\big|_{\mathbb{R}^{\partial N}} = 0.$$

Thus $\phi + \lambda\phi'$ solves the Dirichlet problem for $\psi + \lambda\psi'$, and thus ϕ depends linearly on ψ. □

Bamberg and Sternberg [?] describe another way to solve the Dirichlet problem, going back to Weyl [?].

4.3.4 Equivalent circuits

We have seen that boundary potentials determine, essentially uniquely, the value of all the electric properties across the entire circuit. But from the perspective of control theory, this internal structure is irrelevant: we can only access the circuit at its terminals, and hence only need concern ourselves with the relationship between boundary potentials and boundary currents. In this section we streamline our investigations above to state the precise way in which boundary currents depend on boundary potentials. In particular, we shall see that the relationship is completely captured by the functional taking boundary potentials to the minimum power used by any extension of that boundary potential. Furthermore, each such power functional determines a different boundary potential–boundary current relationship, and so we can conclude that two circuits are equivalent if and only if they have the same power functional.

An 'external behavior', or **behavior** for short, is an equivalence class of circuits, where two are considered equivalent when the boundary current is the same function of the boundary potential. The idea is that the boundary current and boundary potential are all that can be observed 'from outside', i.e. by making measurements at the terminals. Restricting our attention to what can be observed by making measurements at the terminals amounts to treating a circuit as a 'black box': that is, treating its interior as hidden from view. So, two circuits give the same behavior when they behave the same as 'black boxes'.

First let us check that the boundary current is a function of the boundary potential. For this we introduce an important quadratic form on the space of boundary potentials:

Definition 4.10. The **power functional** $Q\colon \mathbb{R}^{\partial N} \to \mathbb{R}$ of a circuit with extended power functional P is given by

$$Q(\psi) = \min_{\phi|_{\mathbb{R}^{\partial N}} = \psi} P(\phi).$$

Proposition 4.7 shows the minimum above exists, so the power functional is well-defined. Thanks to the principle of minimum power, $Q(\psi)$ equals $\frac{1}{2}$ times the power dissipated by the circuit when the boundary voltage is ψ. We will later see that in fact $Q(\psi)$ is a nonnegative quadratic form on $\mathbb{R}^{\partial N}$.

Since Q is a smooth real-valued function on $\mathbb{R}^{\partial N}$, its differential dQ at any given point $\psi \in \mathbb{R}^{\partial N}$ defines an element of the dual space $(\mathbb{R}^{\partial N})^*$, which we denote by dQ_ψ. In fact, this element is equal to the boundary current ι corresponding to the boundary voltage ψ:

Proposition 4.11. *Suppose $\psi \in \mathbb{R}^{\partial N}$. Suppose ϕ is any extension of ψ minimizing the power. Then $dQ_\psi \in (\mathbb{R}^{\partial N})^* \cong \mathbb{R}^{\partial N}$ gives the boundary current of the current induced by the potential ϕ.*

Proof. Note first that while there may be several choices of ϕ minimizing the power subject to the constraint that $\phi|_{\mathbb{R}^{\partial N}} = \psi$, Proposition 4.8 says that the difference between any two choices vanishes on all components touching the boundary of Γ. Thus, these two choices give the same value for the boundary current $\iota\colon \partial N \to \mathbb{R}$. So, with no loss of generality we may assume ϕ is the unique choice that vanishes on all components not touching the boundary. Write $\bar{\iota}\colon N \to \mathbb{R}$ for the extension of $\iota\colon \partial N \to \mathbb{R}$ to N taking value 0 on $N \setminus \partial N$.

By Proposition 4.9, there is a linear operator

$$f\colon \mathbb{R}^{\partial N} \longrightarrow \mathbb{R}^N$$

sending $\psi \in \mathbb{R}^{\partial N}$ to this choice of ϕ, and then

$$Q(\psi) = P(f\psi).$$

Given any $\psi' \in \mathbb{R}^{\partial N}$, we thus have

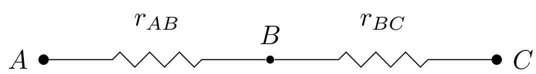
$$\begin{aligned}
dQ_\psi(\psi') &= \frac{d}{d\lambda} Q(\phi + \lambda\psi')\bigg|_{\lambda=0} \\
&= \frac{d}{d\lambda} P(f(\psi + \lambda\psi'))\bigg|_{\lambda=0} \\
&= \frac{1}{2}\frac{d}{d\lambda}\sum_{e\in E}\frac{1}{r(e)}\Big(f(\psi+\lambda\psi'))(t(e)) - (f(\psi+\lambda\psi'))(s(e))\Big)^2\bigg|_{\lambda=0} \\
&= \frac{1}{2}\frac{d}{d\lambda}\sum_{e\in E}\frac{1}{r(e)}\Big((f\psi(t(e)) - f\psi(s(e))) \ + \ \lambda(f\psi'(t(e)) - f\psi'(s(e)))\Big)^2\bigg|_{\lambda=0} \\
&= \sum_{e\in E}\frac{1}{r(e)}(f\psi(t(e)) - f\psi(s(e)))(f\psi'(t(e)) - f\psi'(s(e))) \\
&= \sum_{e\in E} I(e)(f\psi'(t(e)) - f\psi'(s(e))) \\
&= \sum_{n\in N}\left(\sum_{t(e)=n} I(e) - \sum_{s(e)=n} I(e)\right) f\psi'(n) \\
&= \sum_{n\in N}\bar{\iota}(n) f\psi'(n) \\
&= \sum_{n\in\partial N}\iota(n)\psi'(n).
\end{aligned}$$

This shows that $dQ^*_\psi = \iota$, as claimed. Note that this calculation explains why we inserted a factor of $\frac{1}{2}$ in the definition of P: it cancels the factor of 2 obtained from differentating a square. $\square$

Note this only depends on Q, which makes no mention of the potentials at nonterminals. This is amazing: the way power depends on boundary potentials completely characterizes the way boundary currents depend on boundary potentials. In particular, in Part 4.9 we shall see that this allows us to define a composition rule for behaviors of circuits.

To demonstrate these notions, we give a basic example of equivalent circuits.

Example 4.12 (Resistors in series)**.** Resistors are said to be placed in **series** if they are placed end to end or, more precisely, if they form a path with no self-intersections. It is well known that resistors in series are equivalent to a single resistor with resistance equal to the sum of their resistances. To prove this, consider the following circuit comprising two resistors in series, with input A and output C:



Now, the extended power functional $P\colon \mathbb{R}^{\{A,B,C\}} \to \mathbb{R}$ for this circuit is

$$P(\phi) = \frac{1}{2}\left(\frac{1}{r_{AB}}\big(\phi(A) - \phi(B)\big)^2 + \frac{1}{r_{BC}}\big(\phi(B) - \phi(C)\big)^2\right),$$

while the power functional $Q\colon \mathbb{R}^{\{A,C\}} \to \mathbb{R}$ is given by minimization over values of $\phi(B) = x$:

$$Q(\psi) = \min_{x\in\mathbb{R}} \frac{1}{2}\left(\frac{1}{r_{AB}}\big(\psi(A) - x\big)^2 + \frac{1}{r_{BC}}\big(x - \psi(C)\big)^2\right).$$

Differentiating with respect to x, we see that this minimum occurs when

$$\frac{1}{r_{AB}}\big(x - \psi(A)\big) + \frac{1}{r_{BC}}\big(x - \psi(C)\big) = 0,$$

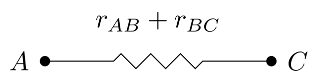
and hence when x is the r-weighted average of $\psi(A)$ and $\psi(C)$:

$$x = \frac{r_{BC}\psi(A) + r_{AB}\psi(C)}{r_{BC} + r_{AB}}.$$

Substituting this value for x into the expression for Q above and simplifying gives

$$Q(\psi) = \frac{1}{2}\cdot\frac{1}{r_{AB} + r_{BC}}\big(\psi(A) - \psi(C)\big)^2.$$

This is also the power functional of the circuit



and so the circuits are equivalent.

4.3.5 Dirichlet forms

In the previous subsection we claimed that power functionals are quadratic forms on the boundary of the circuit whose behavior they represent. They comprise, in fact, precisely those quadratic forms known as Dirichlet forms.

Definition 4.13. Given a finite set S, a **Dirichlet form** on S is a quadratic form $Q : \mathbb{R}^S \to \mathbb{R}$ given by the formula

$$Q(\psi) = \sum_{i,j} c_{ij}(\psi_i - \psi_j)^2$$

for some nonnegative real numbers c_{ij}.

Note that we may assume without loss of generality that $c_{ii} = 0$ and $c_{ij} = c_{ji}$; we do this henceforth. Any Dirichlet form is nonnegative: $Q(\psi) \geq 0$ for all $\psi \in \mathbb{R}^S$. However, not all nonnegative quadratic forms are Dirichlet forms. For example, if $S = \{1, 2\}$, the nonnegative quadratic form $Q(\psi) = (\psi_1 + \psi_2)^2$ is not a Dirichlet form. That said, the concept of Dirichlet form is vastly more general than the above definition: such quadratic forms are studied not just on finite-dimensional vector spaces $\mathbb{R}^S$ but on L^2 of any measure space. When this measure space is just a finite set, the concept of Dirichlet form reduces to the definition above. For a thorough introduction to Dirichlet forms, see the text by Fukushima [**?**]. For a fun tour of the underlying ideas, see the paper by Doyle and Snell [**?**].

The following characterizations of Dirichlet forms help illuminate the concept:

Proposition 4.14. *Given a finite set S and a quadratic form $Q\colon \mathbb{R}^S \to \mathbb{R}$, the following are equivalent:*

(i) Q is a Dirichlet form.

(ii) $Q(\phi) \leq Q(\psi)$ whenever $|\phi_i - \phi_j| \leq |\psi_i - \psi_j|$ for all i, j.

(iii) $Q(\phi) = 0$ whenever ϕ_i is independent of i, and Q obeys the **Markov property***: $Q(\phi) \leq Q(\psi)$ when $\phi_i = \min(\psi_i, 1)$.*

Proof. See Fukushima [**?**]. □

While the extended power functionals of circuits are evidently Dirichlet forms, it is not immediate that all power functionals are. For this it is crucial that the property of being a Dirichlet form is preserved under minimising over linear subspaces of the domain that are generated by subsets of the given finite set.

Proposition 4.15. *If $Q\colon \mathbb{R}^{S+T} \to \mathbb{R}$ is a Dirichlet form, then*

$$\min_{\nu \in \mathbb{R}^T} Q(-, \nu)\colon \mathbb{R}^S \to \mathbb{R}$$

is Dirichlet.

Proof. We first note that $\min_{\nu \in \mathbb{R}^S} Q(-, \nu)$ is a quadratic form. Again, $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is well-defined as a nonnegative quadratic form also attains its minimum on an affine subspace of its domain. Furthermore $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$ is itself a quadratic form, as the partial derivatives of Q are linear, and hence the points at with these minima are attained depend linearly on the argument of $\min_{\nu \in \mathbb{R}^T} Q(-, \nu)$.

Now by Proposition 4.14, $Q(\phi) \leq Q(\phi')$ whenever $|\phi_i - \phi_j| \leq |\phi'_i - \phi'_j|$ for all $i, j \in S+T$. In particular, this implies $\min_{\nu\in\mathbb{R}^T} Q(\psi,\nu) \leq \min_{\nu\in\mathbb{R}^T} Q(\psi',\nu)$ whenever $|\psi_i - \psi_j| \leq |\psi'_i - \psi'_j|$ for all $i, j \in S$. Using Proposition 4.14 again then implies that $\min_{\nu\in\mathbb{R}^T} Q(-,\nu)$ is a Dirichlet form. □

Corollary 4.16. *Let $Q\colon \mathbb{R}^{\partial N} \to \mathbb{R}$ be the power functional for some circuit. Then Q is a Dirichlet form.*

Proof. The extended power functional P is a Dirichlet form, and writing $\mathbb{R}^N = \mathbb{R}^{\partial N} \oplus \mathbb{R}^{N\setminus\partial N}$ allows us to write

$$Q(-) = \min_{\phi\in\mathbb{R}^{N\setminus\partial N}} P(-,\phi).$$ □

The converse is also true: simply construct the circuit with set of vertices ∂N and an edge of resistance $\frac{1}{2c_{ij}}$ between any $i, j \in \partial N$ such that the term $c_{ij}(\psi_i - \psi_j)$ appears in the Dirichlet form. This gives:

Proposition 4.17. *A function Q is the power functional for some circuit if and only if Q is a Dirichlet form.*

This is an expression of the 'star-mesh transform', a well-known fact of electrical engineering stating that every circuit of linear resistors is equivalent to some complete graph of resistors between its terminals. For more details see [**?**]. We may interpret the proof of Proposition 4.15 as showing that intermediate potentials at minima depend linearly on boundary potentials, in fact a weighted average, and that substituting these into a quadratic form still gives quadratic form.

In summary, in this section we have shown the existence of a surjective function

$$\left\{ \begin{array}{c} \text{circuits of linear resistors} \\ \text{with boundary } \partial N \end{array} \right\} \longrightarrow \left\{ \text{Dirichlet forms on } \partial N \right\}$$

mapping two circuits to the same Dirichlet form if and only if they have the same external behavior. In the next section we extend this result to encompass inductors and capacitors too.

4.4 Inductors and capacitors

The intuition gleaned from the study of resistors carries over to inductors and capacitors too, to provide a framework for studying what are known as passive linear networks. To understand inductors and capacitors in this way, however, we must introduce a notion of time dependency and subsequently the Laplace transform, which allows us to work in the so-called frequency domain. Here, like resistors, inductors and capacitors simply impose a relationship of proportionality between the voltages and currents that run across them. The constant of proportionality is known as the impedance of the component.

As for resistors, the interconnection of such components may be understood, at least formally, as a minimization of some quantity, and we may represent the behaviors of this class of circuits with a more general idea of Dirichlet form. We conclude this section by noting an obstruction to building a composition rule for Dirichlet forms, motivating our work in Part 4.5.

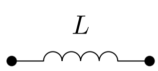
4.4.1 The frequency domain and Ohm's law revisited

In broadening the class of electrical circuit components under examination, we find ourselves dealing with components whose behaviors depend on the rates of change of current and voltage with respect to time. We thus now consider time-varying voltages $v\colon [0,\infty) \to \mathbb{R}$ and currents $i\colon [0,\infty) \to \mathbb{R}$, where $t \in [0,\infty)$ is a real variable representing time. For mathematical reasons, we restrict these voltages and currents to only those with (i) zero initial conditions (that is, $f(0) = 0$) and (ii) Laplace transform lying in the field

$$\mathbb{R}(s) = \left\{ Z(s) = \tfrac{P(s)}{Q(s)} \;\middle|\; P, Q \text{ polynomials over } \mathbb{R} \text{ in } s,\ Q \neq 0 \right\}$$

of real rational functions of one variable. While it is possible that physical voltages and currents might vary with time in a more general way, we restrict to these cases as the rational functions are, crucially, well-behaved enough to form a field, and yet still general enough to provide arbitrarily close approximations to currents and voltages found in standard applications.

An **inductor** is a two-terminal circuit component across which the voltage is proportional to the rate of change of the current. By convention we draw this as follows, with the inductance L the constant of proportionality:[3]

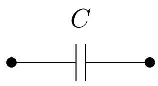


Writing $v_L(t)$ and $i_L(t)$ for the voltage and current over time t across this component respectively, and using a dot to denote the derivative with respect to time t, we thus have the relationship

$$v_L(t) = L\,\dot{i}_L(t).$$

[3]We follow the standard convention of denoting inductance by the letter L, after the work of Heinrich Lenz and to avoid confusion with the I used for current.

Permuting the roles of current and voltage, a **capacitor** is a two-terminal circuit component across which the current is proportional to the rate of change of the voltage. We draw this as follows, with the capacitance C the constant of proportionality:



Writing $v_C(t)$, $i_C(t)$ for the voltage and current across the capacitor, this gives the equation

$$i_C(t) = C\,\dot{v}_C(t).$$

We assume here that inductances L and capacitances C are positive real numbers.

Although inductors and capacitors impose a linear relationship if we involve the derivatives of current and voltage, to mimic the above work on resistors we wish to have a constant of proportionality between functions representing the current and voltage themselves. Various integral transforms perform just this role; electrical engineers typically use the Laplace transform. This lets us write a function of time t instead as a function of frequencies s, and in doing so turns differentiation with respect to t into multiplication by s, and integration with respect to t into division by s.

In detail, given a function $f(t)\colon [0,\infty) \to \mathbb{R}$, we define the **Laplace transform** of f

$$\mathfrak{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt.$$

We also use the notation $\mathfrak{L}\{f\}(s) = F(s)$, denoting the Laplace transform of a function in upper case, and refer to the Laplace transforms as lying in the **frequency domain** or ***s*-domain**. For us, the three crucial properties of the Laplace transform are then:

(i) linearity: $\mathfrak{L}\{af + bg\}(s) = aF(s) + bG(s)$ for $a, b \in \mathbb{R}$;

(ii) differentiation: $\mathfrak{L}\{\dot{f}\}(s) = sF(s) - f(0)$;

(iii) integration: if $g(t) = \int_0^t f(\tau)d\tau$ then $G(s) = \frac{1}{s}F(s)$.

Writing $V(s)$ and $I(s)$ for the Laplace transform of the voltage $v(t)$ and current $i(t)$ across a component respectively, and recalling that by assumption $v(t) = i(t) = 0$ for $t \leq 0$, the s-domain behaviors of components become, for a resistor of resistance R:

$$V(s) = RI(s),$$

for an inductor of inductance L:

$$V(s) = sLI(s),$$

and for a capacitor of capacitance C:

$$V(s) = \frac{1}{sC} I(s).$$

Note that for each component the voltage equals the current times a rational function of the real variable s, called the **impedance** and in general denoted by Z. Note also that the impedance is a **positive real function**, meaning that it lies in the set

$$\mathbb{R}(s)^+ = \{Z \in \mathbb{R}(s) : \forall s \in \mathbb{C} \;\; \mathrm{Re}(s) > 0 \implies \mathrm{Re}(Z(s)) > 0\}.$$

While Z is a quotient of polynomials with real cofficients, in this definition we are applying it to complex values of s, and demanding that its real part be positive in the open left half-plane. Positive real functions were introduced by Otto Brune in 1931, and they play a basic role in circuit theory [**?**].

Indeed, Brune convincingly argued that for any conceivable passive linear component we have this generalization of Ohm's law:

$$V(s) = Z(s)I(s)$$

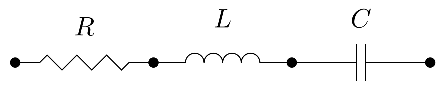
where $I \in \mathbb{R}(s)$ is the **current**, $V \in \mathbb{R}(s)$ is the **voltage** and $Z \in \mathbb{R}(s)^+$ is the **impedance** of the component. As we shall see, generalizing from circuits of linear resistors to arbitrary passive linear circuits is just a matter of formally replacing resistances by impedances. This amounts to replacing the field $\mathbb{R}$ by the larger field $\mathbb{R}(s)$, and replacing the set of positive reals, $\mathbb{R}^+ = (0, \infty)$, by the set of positive real functions, $\mathbb{R}(s)^+$. From a mathematical perspective we might as well work with any field with a mildly well-behaved notion of 'positive element', and we do this in the next section.

4.4.2 The mechanical analogy

Now that we have introduced inductors and capacitors, it is worth taking another glance at the analogy chart in Section 4.1. What are the analogues of resistance, inductance and capacitance in mechanics? If we restrict attention to systems with translational degrees of freedom, the answer is given in the following chart.

Electronics	Mechanics (translation)
charge Q	position q
current $i = \dot{Q}$	velocity $v = \dot{q}$
flux linkage λ	momentum p
voltage $v = \dot{\lambda}$	force $F = \dot{p}$
resistance R	damping coefficient c
inductance L	mass m
inverse capacitance C^{-1}	spring constant k

A famous example concerns an electric circuit with a resistor of resistance R, an inductor of inductance L, and a capacitor of capacitance C, all in series:



We saw in Example 4.12 that for resistors in series, the resistances add. The same fact holds more generally for passive linear circuits, so the impedance of this circuit is the sum

$$Z = sL + R + (sC)^{-1}.$$

Thus, the voltage across this circuit is related to the current through the circuit by

$$V(s) = (sL + R + (sC)^{-1})I(s)$$

If $v(t)$ and $i(t)$ are the voltage and current as functions of time, we conclude that

$$v(t) = L\frac{d}{dt}i(t) + Ri(t) + C^{-1}\int_0^t i(s)\, ds$$

It follows that

$$L\ddot{Q} + R\dot{Q} + C^{-1}Q = v$$

where $Q(t) = \int_0^t i(t)ds$ has units of charge. As the chart above suggests, this equation is analogous to that of a damped harmonic oscillator:

$$m\ddot{q} + c\dot{q} + kq = F$$

where m is the mass of the oscillator, c is the damping coefficient, k is the spring constant and F is a time-dependent external force.

For details, and many more analogies of this sort, see the book by Karnopp, Margolis and Rosenberg [?] or Brown's enormous text [?]. While it would be a distraction to discuss them further here, these analogies mean that our work applies to a wide class of networked systems, not just electrical circuits.

4.4.3 Generalized Dirichlet forms

To understand the behavior of passive linear circuits we need to understand how the behaviors of individual components, governed by Ohm's law, fit together give the behavior of an entire network. Kirchhoff's laws still hold, and so does a version of the principle of minimum power. To see this, we generalize our remarks so far to any field with a set of 'positive elements'.

Definition 4.18. Given a field $\mathbb{F}$, we define a **set of positive elements** for $\mathbb{F}$ to be any subset $\mathbb{F}^+ \subset \mathbb{F}$ containing 1 but not 0, and closed under addition, multiplication and division.

Our first motivating example arises from circuits made of resistors. Here $\mathbb{F} = \mathbb{R}$ is the field of real numbers and we take $\mathbb{F}^+ = (0, \infty)$. Our second motivating example arises from general passive linear circuits. Here $\mathbb{F} = \mathbb{R}(s)$ is the field of rational functions in one real variable, and we take $\mathbb{F}^+ = \mathbb{R}(s)^+$ to be the positive real functions, as defined in the last section.

In all that follows, we fix a field $\mathbb{F}$ equipped with a set of positive elements $\mathbb{F}^+$. By a 'circuit', we shall henceforth mean a circuit over $\mathbb{F}^+$, as explained in Definition 4.3. To fix the notation:

Definition 4.19. A **(passive linear) circuit** is a graph $s, t \colon E \to N$ with E as its set of **edges** and N as its set of **nodes**, equipped with function $Z \colon E \to \mathbb{F}^+$ assigning each edge an **impedance**, together with finite sets X, Y, and functions $i \colon X \to N$ and $o \colon Y \to N$. We call the sets $i(X)$, $o(Y)$, and $\partial N = i(X) \cup o(Y)$ the **inputs**, **outputs**, and **terminals** or **boundary** of the circuit, respectively.

Generalizing from circuits of resistors, we define the **extended power functional** $P \colon \mathbb{F}^N \to \mathbb{F}$ of any circuit by

$$P(\varphi) = \frac{1}{2} \sum_{e \in E} \frac{1}{Z(e)} \big(\varphi(t(e)) - \varphi(s(e))\big)^2.$$

and we call $\varphi \in \mathbb{F}^N$ a **potential**. Note that a field of characteristic 2 cannot be given a set of positive elements, so dividing by 2 is allowed. Note also that the extended power functional is a Dirichlet form on N.

Although it is not clear what it means to minimize over the field $\mathbb{F}$, we can use formal derivatives to formulate an analogue of the principle of minimum power. This will actually be a 'variational principle', saying the derivative of the power functional vanishes with respect to certain variations in the potential. As before we shall see that given Ohm's law, this principle is equivalent to Kirchhoff's current law.

Indeed, the extended power functional $P(\varphi)$ can be considered an element of the polynomial ring $\mathbb{F}[\{\varphi(n)\}_{n\in N}]$ generated by formal variables $\varphi(n)$ corresponding to potentials at the nodes $n \in N$. We may thus take formal derivatives of the extended power functional with respect to the $\varphi(n)$. We then call $\phi \in \mathbb{F}^N$ a **realizable potential** for the given circuit if for each nonterminal node, $n \in N \setminus \partial N$, the formal partial derivative of the extended power functional with respect to $\varphi(n)$ equals zero when evaluated at ϕ:

$$\left.\frac{\partial P}{\partial \varphi(n)}\right|_{\varphi=\phi} = 0$$

This terminology arises from the following fact, a generalization of Proposition 4.6:

Theorem 4.20. *The potential $\phi \in \mathbb{F}^N$ is a realizable potential for a given circuit if and only if the induced current*

$$I(e) = \frac{1}{Z(e)}(\phi(t(e)) - \phi(s(e)))$$

obeys Kirchhoff's current law:

$$\sum_{s(e)=n} I(e) = \sum_{t(e)=n} I(e)$$

for all $n \in N \setminus \partial N$.

Proof. The proof of this statement is exactly that for Proposition 4.6. □

A corollary of Theorem 4.20 is that the set of states—that is, potential–current pairs—that are compatible with the governing laws of a circuit is given by the set of realizable potentials together with their induced currents.

We now generalize the theory of Dirichlet forms:

Definition 4.21. Given a field $\mathbb{F}$ with a set of positive elements $\mathbb{F}^+$ and a finite set S, a **Dirichlet form over** $\mathbb{F}$ on S is a quadratic form $Q\colon \mathbb{F}^S \to \mathbb{F}$ given by the formula

$$Q(\psi) = \sum_{i,j\in S} c_{ij}(\psi_i - \psi_j)^2$$

for some choice of $c_{ij} \in \mathbb{F}^+ \cup \{0\}$.

Note that the extended power functional of any circuit is a Dirichlet form, since $Z(e) \in \mathbb{F}^+$ implies $\frac{1}{2}\frac{1}{Z(e)} \in \mathbb{F}^+$.

4.4.4 A generalized minimizability result

We begin to move from a discussion of the intrinsic behaviors of circuits to a discussion of their behaviors under composition. The key fact for composition of generalized Dirichlet forms is that, in analogy with Proposition 4.15, we may speak of a formal version of minimization of Dirichlet forms. We detail this here. In what follows, fix a field $\mathbb{F}$ with a set of positive elements $\mathbb{F}^+$, and let P be a Dirichlet form over $\mathbb{F}$ on some finite set S.

Recall that given $R \subseteq S$, we call $\tilde{\psi} \in \mathbb{F}^S$ an **extension** of $\psi \in \mathbb{F}^R$ if $\tilde{\psi}$ restricted to R equals ψ. We call such an extension **realizable** if

$$\left.\frac{\partial P}{\partial \varphi(s)}\right|_{\varphi=\tilde{\psi}} = 0$$

for all $s \in S \setminus R$. Note that over the real numbers $\mathbb{R}$ this means that among all the extensions of ψ, $\tilde{\psi}$ minimizes the function P.

Theorem 4.22. *Let P be a Dirichlet form over $\mathbb{F}$ on S, and let $R \subseteq S$ be an inclusion of finite sets. Then we may uniquely define a Dirichlet form*

$$\min_{S \setminus R} P : \mathbb{F}^R \to \mathbb{F}$$

on R by sending each $\psi \in \mathbb{F}^R$ to the value $P(\tilde{\psi})$ of any realizable extension $\tilde{\psi}$ of ψ.

To prove this theorem, we must first show that $\min_{S \setminus R} P$ is well-defined as a function.

Lemma 4.23. *Let P be a Dirichlet form over $\mathbb{F}$ on S, let $R \subseteq S$ be an inclusion of finite sets, and let $\psi \in \mathbb{F}^R$. Then for all realizable extensions $\tilde{\psi}, \tilde{\psi}' \in \mathbb{F}^S$ of ψ we have $P(\tilde{\psi}) = P(\tilde{\psi}')$.*

Proof. This follows from the formal version of the multivariable Taylor theorem for polynomial rings over a field of characteristic zero. Let $\tilde{\psi}, \tilde{\psi}' \in \mathbb{F}^S$ be realizable extensions of ψ, and note that $dP_{\tilde{\psi}}(\tilde{\psi} - \tilde{\psi}') = 0$, since for all $s \in R$ we have $\tilde{\psi}(s) - \tilde{\psi}'(s) = 0$, and for all $s \in S \setminus R$ we have

$$\left.\frac{\partial P}{\partial \varphi(s)}\right|_{\varphi=\tilde{\psi}} = 0.$$

We may take the Taylor expansion of P around $\tilde{\psi}$ and evaluate at $\tilde{\psi}'$. As P is a quadratic form, this gives

$$\begin{aligned} P(\tilde{\psi}') &= P(\tilde{\psi}) + dP_{\tilde{\psi}}(\tilde{\psi}' - \tilde{\psi}) + P(\tilde{\psi}' - \tilde{\psi}) \\ &= P(\tilde{\psi}) + P(\tilde{\psi}' - \tilde{\psi}). \end{aligned}$$

Similarly, we arrive at

$$P(\tilde{\psi}) = P(\tilde{\psi}') + P(\tilde{\psi} - \tilde{\psi}').$$

But again as P is a quadratic form, we then see that

$$P(\tilde{\psi}') - P(\tilde{\psi}) = P(\tilde{\psi}' - \tilde{\psi}) = P(\tilde{\psi} - \tilde{\psi}') = P(\tilde{\psi}) - P(\tilde{\psi}').$$

This implies that $P(\tilde{\psi}') - P(\tilde{\psi}) = 0$, as required. $\square$

It remains to show that $\min P$ remains a Dirichlet form. We do this inductively.

Lemma 4.24. *Let P be a Dirichlet form over $\mathbb{F}$ on S, and let $s \in S$ be an element of S. Then the map $\min_{\{s\}} P : \mathbb{F}^{S\setminus\{s\}} \to \mathbb{F}$ sending ψ to $P(\tilde{\psi})$ is a Dirichlet form on $S \setminus \{s\}$.*

Proof. Write $P(\phi) = \sum_{i,j} c_{ij}(\phi_i - \phi_j)^2$, assuming without loss of generality that $c_{sk} = 0$ for all k. We then have

$$\left.\frac{\partial P}{\partial \varphi(s)}\right|_{\varphi=\phi} = \sum_k 2c_{ks}(\phi_s - \phi_k),$$

and this is equal to zero when

$$\phi_s = \frac{\sum_k c_{ks}\phi_k}{\sum_k c_{ks}}.$$

Thus $\min_{\{s\}} P$ may be given explicitly by the expression

$$\min_{\{s\}} P(\psi) = \sum_{i,j\in S\setminus\{s\}} c_{ij}(\psi_i - \psi_j)^2 + \sum_{\ell\in S\setminus\{s\}} c_{\ell s}\left(\psi_\ell - \tfrac{\sum_k c_{ks}\psi_k}{\sum_k c_{ks}}\right)^2.$$

We must show this is a Dirichlet form on $S \setminus \{s\}$.

As the sum of Dirichlet forms is evidently Dirichlet, it suffices to check that the expression

$$\sum_\ell c_{\ell s}\left(\psi_\ell - \tfrac{\sum_k c_{ks}\psi_k}{\sum_k c_{ks}}\right)^2$$

is Dirichlet on $S\setminus\{s\}$. Multiplying through by the constant $(\sum_k c_{ks})^2 \in \mathbb{F}^+$, it further suffices to check

$$\begin{aligned}
\sum_\ell c_{\ell s}\left(\sum_k c_{ks}\psi_\ell - \sum_k c_{ks}\psi_k\right)^2 &= \sum_\ell c_{\ell s}\left(\sum_k c_{ks}(\psi_\ell - \psi_k)\right)^2 \\
&= \sum_\ell c_{\ell s}\left(2\sum_{\substack{k,m\\k\neq m}} c_{ks}c_{ms}(\psi_\ell - \psi_k)(\psi_\ell - \psi_m) + \sum_k c_{ks}^2(\psi_\ell - \psi_k)^2\right) \\
&= 2\sum_{\substack{k,\ell,m\\k\neq m}} c_{\ell s}c_{ks}c_{ms}(\psi_\ell - \psi_k)(\psi_\ell - \psi_m) + \sum_{k,\ell} c_{\ell s}c_{ks}^2(\psi_\ell - \psi_k)^2
\end{aligned}$$

is Dirichlet. But

$$\begin{aligned}&(\psi_k-\psi_\ell)(\psi_k-\psi_m)+(\psi_\ell-\psi_k)(\psi_\ell-\psi_m)+(\psi_m-\psi_k)(\psi_m-\psi_\ell)\\&=\psi_k^2+\psi_\ell^2+\psi_m^2-\psi_k\psi_\ell-\psi_k\psi_m-\psi_\ell\psi_m\\&=\tfrac{1}{2}\big((\psi_k-\psi_\ell)^2+(\psi_k-\psi_m)^2+(\psi_\ell-\psi_m)^2\big),\end{aligned}$$

so this expression is indeed Dirichlet. Indeed, pasting these computations together shows that

$$\min_{\{s\}} P(\psi)=\sum_{i,j}\left(c_{ij}+\frac{c_{is}c_{js}}{\sum_k c_{ks}}\right)(\psi_i-\psi_j)^2. \qquad \square$$

With these two lemmas, the proof of Theorem 4.22 becomes straightforward.

Proof of Theorem 4.22. Lemma 4.23 shows that $\min_{S\setminus R} P$ is a well-defined function. As R is a finite set, we may write it $R=\{s_1,\dots,s_n\}$ for some natural number n. Then we may define a sequence of functions $P_i=\min_{\{s_1,\dots,s_i\}} P_{i-1}$, $1\le i\le n$. Define also $P_0=P$, and note that $P_n=\min_{S\setminus R} P$. Then, by Lemma 4.24, each P_i is Dirichlet as P_{i-1} is. This proves the proposition. $\square$

We can thus define the power functional of a circuit by analogy with circuits made of resistors:

Definition 4.25. The **power functional** $Q\colon \mathbb{R}^{\partial N}\to\mathbb{R}$ of a circuit with extended power functional P is given by

$$Q=\min_{N\setminus\partial N} P.$$

As before, we define two circuits to be **equivalent** if they have the same power functional, and define the **behavior** of a circuit to be its equivalence class.

4.4.5 Composition of Dirichlet forms

It would be nice to have a category in which circuits are morphisms, and a category in which Dirichlet forms are morphisms, such that the map sending a circuit to its behavior is a functor. Here we present a naïve attempt to constructed the category with Dirichlet forms as morphisms, using the principle of minimum power to compose these morphisms. Unfortunately the proposed category does not include identity morphisms. However, it points in the right direction, and underlines the importance of the cospan formalism we then turn to develop.

We can define a composition rule for Dirichlet forms that reflects composition of circuits. Given finite sets S and T, let $S+T$ denote their disjoint union. Let $D(S,T)$

be the set of Dirichlet forms on $S+T$. There is a way to compose these Dirichlet forms

$$\circ\colon D(T,U) \times D(S,T) \to D(S,U)$$

defined as follows. Given $P \in D(T,U)$ and $Q \in D(S,T)$, let

$$(P \circ Q)(\alpha,\gamma) = \min_T Q(\alpha,\beta) + P(\beta,\gamma),$$

where $\alpha \in F^S, \gamma \in F^U$. This operation has a clear interpretation in terms of electrical circuits: the power used by the entire circuit is just the sum of the power used by its parts.

It is immediate from Theorem 4.22 that this composition rule is well-defined: the composite of two Dirichlet forms is again a Dirichlet form. Moreover, this composition is associative. However, it fails to provide the structure of a category, as there is typically no Dirichlet form $1_S \in D(S,S)$ playing the role of the identity for this composition. For an indication of why this is so, let $\{\bullet\}$ be a set with one element, and suppose that some Dirichlet form $I(\beta,\gamma) = k(\beta-\gamma)^2 \in D(\{\bullet\},\{\bullet\})$ acts as an identity on the right for this composition. Then for all $Q(\alpha,\beta) = c(\alpha-\beta)^2 \in D(\{\bullet\},\{\bullet\})$, we must have

$$\begin{aligned} c\alpha^2 &= Q(\alpha,0) \\ &= (I \circ Q)(\alpha,0) \\ &= \min_{\beta\in\mathbb{F}} Q(\alpha,\beta) + I(\beta,0) \\ &= \min_{\beta\in\mathbb{F}} k(\alpha-\beta)^2 + c\beta^2 \\ &= \frac{kc}{k+c}\alpha^2, \end{aligned}$$

where we have noted that $\frac{kc}{k+c}\alpha^2$ minimizes $k(\alpha-\beta)^2 + c\beta^2$ with respect to β. But for any choice of $k \in \mathbb{F}$ this equality only holds when $c = 0$, so no such Dirichlet form exists. Note, however, that for $k >> c$ we have $c\alpha^2 \approx \frac{kc}{k+c}\alpha^2$, so Dirichlet forms with large values of k—corresponding to resistors with resistance close to zero—act as 'approximate identities'.

In this way we might interpret the identities we wish to introduce into this category as the behaviors of idealized components with zero resistance: perfectly conductive wires. Unfortunately, the power functional of a purely conductive wire is undefined: the formula for it involves division by zero. In real life, coming close to this situation leads to the disaster that electricians call a 'short circuit': a huge amount of power dissipated for even a small voltage. This is why we have fuses and circuit breakers.

Nonetheless, we have most of the structure required for a category. A 'category without identity morphisms' is called a **semicategory**, so we see

Proposition 4.26. *There is a semicategory where:*

- *the objects are finite sets,*
- *a morphism from T to S is a Dirichlet form $Q \in D(S,T)$.*
- *composition of morphisms is given by*

$$(R \circ Q)(\gamma, \alpha) = \min_{T} Q(\gamma, \beta) + R(\beta, \alpha).$$

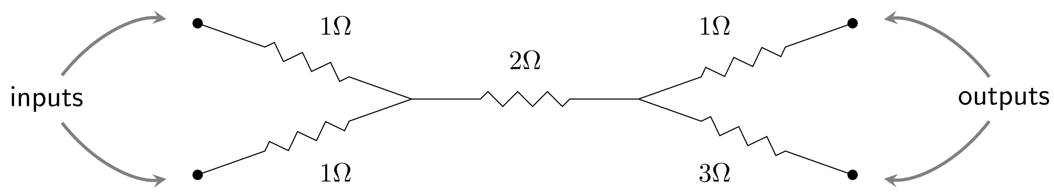
We would like to make this into a category. One easy way to do this is to formally adjoin identity morphisms; this trick works for any semicategory. However, we obtain a better category if we include *more* morphisms: more behaviors corresponding to circuits made of perfectly conductive wires. As the expression for the extended power functional includes the reciprocals of impedances, such circuits cannot be expressed within the framework we have developed thus far. Indeed, for these idealized circuits there is no function taking boundary potentials to boundary currents: the vanishing impedance would imply that any difference in potentials at the boundary induces 'infinite' currents. To deal with this issue, we generalize Dirichlet forms to Lagrangian relations. First, however, we develop a category theoretic framework, based around decorated cospans, to define the category of circuits itself and understand its basic properties.

4.5 Categories of Circuits

In this part we move our focus from the semantics of circuit diagrams to the syntax, addressing the question "How do we interact with circuit diagrams?". Informally, the answer to this is that we interact with them by connecting them to each other, perhaps after moving them into the right form by rotating or reflecting them, or by crossing or bending some of the wires. To formalize this, we adopt a category theoretic viewpoint, defining various dagger compact categories with circuits and their behaviors as morphisms. We claim a formal analysis of this structure, especially of the composition or connection of circuits, has been overlooked in analysis of circuits thus far. This part culminates in the definition of two important categories, the category Circ of circuit diagrams, and the category LagrRel containing all behaviors of circuits. We also develop the technical material required to appreciate the structure of these categories, and that aids understanding of the relationship between the two, to be addressed in Part 4.9.

4.6 The category of passive linear circuits

In Part 4.2 we defined a circuit of linear resistors to be a labelled graph with marked input and output terminals, as in the example:



We then defined general passive linear circuits by replacing resistances with impedances chosen from a set of positive elements $\mathbb{F}^+$ in any field $\mathbb{F}$. In fact, these circuits are examples of decorated cospans. This gives a dagger compact category Circ whose morphisms are circuits, with the dagger compact structure expressing standard operations on circuits.

We actually give two constructions of this category, first arriving at a category of cospans decorated by $\mathbb{F}^+$-graphs, and then showing that this is a full subcategory of the decategorification of a bicategory of cospans of $\mathbb{F}^+$-graphs.

4.6.1 A decorated cospan construction

We officially defined a 'circuit' in Definition 4.19, but now we can give an equivalent definition using cospans:

Lemma 4.27. *A circuit is a cospan of finite sets $X \xleftarrow{i} N \xrightarrow{o} Y$ together with an $\mathbb{F}^+$-graph whose set of nodes is N.*

Proof. This is just a matter of remembering the terminology: recall that an $\mathbb{F}^+$-graph is a graph whose edges are labelled with elements of $\mathbb{F}^+$, our chosen set of positive elements in the field $\mathbb{F}$. □

This suggests that circuits should be morphisms in a decorated cospan category. Indeed, we can show that the map taking a finite set N to the set of $\mathbb{F}^+$-graphs with set N of nodes in fact forms a lax symmetric monoidal functor. This allows us to apply decorated cospans to construct a category of circuits.

To this end, define the functor

$$\mathrm{Circuit}\colon (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

to take a finite set N, as an object of FinSet, to the set $\mathrm{Circuit}(N)$ of $\mathbb{F}^+$-graphs (N, E, s, t, r) with N as their set of nodes. On morphisms let it take a function $f\colon N \to M$ to the function that pushes labelled graph structures on a set N forward onto the set M:

$$\begin{aligned} \mathrm{Circuit}(f)\colon \mathrm{Circuit}(N) &\longrightarrow \mathrm{Circuit}(M); \\ (N, E, s, t, r) &\longmapsto (M, E, f \circ s, f \circ t, r). \end{aligned}$$

Note that as this map simply acts by post-composition, our map Circuit is indeed functorial.

We then arrive at a lax symmetric monoidal functor by equipping this functor with the natural transformation

$$\begin{aligned} \rho_{N,M}\colon \mathrm{Circuit}(N) \times \mathrm{Circuit}(M) &\longrightarrow \mathrm{Circuit}(N+M); \\ \big((N, E, s, t, r), (M, F, s', t', r')\big) &\longmapsto \big(N + M, E + F, s + s', t + t', [r, r']\big), \end{aligned}$$

together with the unit map

$$\begin{aligned} \rho_1\colon 1 &\longrightarrow \mathrm{Circuit}(\varnothing); \\ \bullet &\longmapsto (\varnothing, \varnothing, \varnothing, \varnothing, \varnothing), \end{aligned}$$

where we use $\varnothing$ to denote both the empty set and the unique function of the appropriate codomain with domain the empty set. The naturality of this collection of morphisms, as well as the coherence laws for lax symmetric monoidal functors, follow from the universal property of the coproduct.

Definition 4.28. We define

$$\mathbf{Circ} = \mathrm{Circuit}\mathrm{Cospan}.$$

Corollary 4.29. *The category* Circ *is a dagger compact category.*

The different structures of this category capture different operations that can be performed with circuits. The composition expresses the fact that we can connect the outputs of one circuit to the inputs of the next, while the monoidal composition models the placement of circuits side-by-side. The symmetric monoidal structure allows us reorder input and output wires, and the compactness captures the interchangeability between input and outputs of circuits—that is, the fact that we can choose any input to our circuit and consider it instead as an outputl, and vice versa. Finally, the dagger structure expresses the fact that we may reflect a whole circuit, switching all inputs with all outputs.

4.6.2 A bicategory of circuits

As shown by Bénabou [?], cospans are most naturally thought of as 1-morphisms in a bicategory. To obtain the category we are calling $\mathrm{Cospan}(\mathcal{C})$, we 'decategorify' this bicategory by discarding 2-morphisms and identifying isomorphic 1-morphisms. Since we are studying circuits using decorated cospans, this suggests that circuits, too, are most naturally thought of as 1-morphisms in a bicategory. Indeed this is the case.

One route to the bicategory whose 1-morphisms are circuits would be to take the theory of decorated cospans [?] and show that it can be enhanced to create not merely categories whose morphisms are isomorphism classes of decorated cospans, but bicategories whose 1-morphisms are exactly decorated cospans. A less powerful but easier approach is as follows.

Recall that we define a graph to be a pair of functions $s, t\colon E \to N$ where E and N are finite sets, and an L-graph to be a graph further equipped with a function $r\colon E \to L$. Thus, an L-graph looks like this:

$$\begin{tikzcd} L & E \arrow[l, "r"'] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & N \end{tikzcd}$$

Given L-graphs $\Gamma = (E, N, s, t, r)$ and $\Gamma' = (E', N', s', t', r')$, a morphism of L-graphs $\Gamma \to \Gamma'$ is a pair of functions $\epsilon\colon E \to E'$, $v\colon N \to N'$ such that the following diagrams commute:

$$\begin{tikzcd} & E \arrow[dl, "r"'] \arrow[dd, "\epsilon"] \\ L & \\ & E' \arrow[ul, "r'"] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "s"] \arrow[d, "\epsilon"'] & N \arrow[d, "v"] \\ E' \arrow[r, "s'"'] & N' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "t"] \arrow[d, "\epsilon"'] & N \arrow[d, "v"] \\ E' \arrow[r, "t'"'] & N'. \end{tikzcd}$$

These L-graphs and their morphisms form a category $\boldsymbol{L}$**-Graph**. Using results about colimits in the category of sets, it is straightforward to check that this category has finite colimits.

Bénabou [?] gave a general construction of bicategories starting from any category with pushouts. Since L-Graph has pushouts, it follows that there is a bicategory **Cospan(**$\boldsymbol{L}$**-Graph)** with

- L-labelled graphs as objects,
- cospans in L-Graphas morphisms, and
- maps of cospans as 2-morphisms.

Since L-Graph also has coproducts, this bicategory is symmetric monoidal and in fact compact, thanks to the work of Stay [**?**]. We make the following definition:

Definition 4.30. The bicategory **2-Circ** is the full and 2-full sub-bicategory of Cospan($\mathbb{F}^+$-Graph) with objects those $\mathbb{F}^+$-graphs with no edges.

It can be shown that every object in Cospan($\mathbb{F}^+$-Graph) is self-dual, and this implies that 2-Circ is again a compact closed bicategory. Note that the category obtained by decategorifying this bicategory has finite sets as objects, with morphisms being isomorphism classes of cospans in $\mathbb{F}^+$-Graph with feet such objects. Thus, decategorifying 2-Circ gives a category equivalent to our previously defined category Circ.

4.7 Circuits as Lagrangian relations

In the first part of this paper, we explored the semantic content contained in circuit diagrams, leading to an understanding of circuit diagrams as expressing some relationship between the potentials and currents that can simultaneously be imposed on some subset, the so-called terminals, of the nodes of the circuit. We called this collection of possible relationships the behavior of the circuit. While in that setting we used the concept of Dirichlet forms to describe this relationship, we saw in the end that describing circuits as Dirichlet forms does not allow for a straightforward notion of composition of circuits.

In this section, inspired by the principle of least action of classical mechanics in analogy with the principle of minimum power, we develop a setting for describing behaviors that allows for easy discussion of composite behaviors: Lagrangian subspaces of symplectic vector spaces. These Lagrangian subspaces provide a more direct, invariant perspective, comprising precisely the set of vectors describing the possible simultaneous potential and current readings at all terminals of a given circuit. As we shall see, one immediate and important advantage of this setting is that we may model wires of zero resistance.

Recall that we write $\mathbb{F}$ for some field, which for our applications is usually the field $\mathbb{R}$ of real numbers or the field $\mathbb{R}(s)$ of rational functions of one real variable.

4.7.1 Symplectic vector spaces

A circuit made up of wires of positive resistance defines a function from boundary potentials to boundary currents. A wire of zero resistance, however, does not define a function: the principle of minimum power is obeyed as long as the potentials at the two ends of the wire are equal. More generally, we may thus think of circuits as specifying a set of allowed voltage-current pairs, or as a relation between boundary potentials and boundary currents. This set forms what is called a Lagrangian subspace, and is given by the graph of the differential of the power functional. More generally, Lagrangian submanifolds graph derivatives of smooth functions: they describe the point evaluated and the tangent to that point within the same space.

The material in this section is all known, and follows without great difficulty from the definitions. To keep this section brief we omit proofs. See any introduction to symplectic vector spaces, such as Cimasoni and Turaev [**?**] or Piccione and Tausk [**?**], for details.

Definition 4.31. Given a finite-dimensional vector space V over a field $\mathbb{F}$, a **symplectic form** $\omega\colon V \times V \to \mathbb{F}$ on V is an alternating nondegenerate bilinear form. That is, a symplectic form ω is a function $V \times V \to \mathbb{F}$ that is

(i) bilinear: for all $\lambda \in \mathbb{F}$ and all $u, v \in V$ we have $\omega(\lambda u, v) = \omega(u, \lambda v) = \lambda\omega(u, v)$;

(ii) alternating: for all $v \in V$ we have $\omega(v, v) = 0$; and

(iii) nondegenerate: given $v \in V$, $\omega(u, v) = 0$ for all $u \in V$ if and only if $u = 0$.

A **symplectic vector space** (V, ω) is a vector space V equipped with a symplectic form ω.

Given symplectic vector spaces $(V_1, \omega_1), (V_2, \omega_2)$, a **symplectic map** is a linear map

$$f\colon (V_1, \omega_1) \longrightarrow (V_2, \omega_2)$$

such that $\omega_2(f(u), f(v)) = \omega_1(u, v)$ for all $u, v \in V_1$. A **symplectomorphism** is a symplectic map that is also an isomorphism.

An alternating form is always **antisymmetric**, meaning that $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$. The converse is true except in characteristic 2. A **symplectic basis** for a symplectic vector space (V, ω) is a basis $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ such that $\omega(p_i, p_j) = \omega(q_i, q_j) = 0$ for all $1 \le i, j \le n$, and $\omega(p_i, q_j) = \delta_{ij}$ for all $1 \le i, j \le n$, where δ_{ij} is the Kronecker delta, equal to 1 when $i = j$, and 0 otherwise. A symplectomorphism maps symplectic bases to symplectic bases, and conversely, any map that takes a symplectic basis to another symplectic basis is a symplectomorphism.

Example 4.32 (The symplectic vector space generated by a finite set)**.** Given a finite set N, we consider the vector space $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ a symplectic vector space $(\mathbb{F}^N \oplus (\mathbb{F}^N)^*, \omega)$, with symplectic form

$$\omega\big((\phi, i), (\phi', i')\big) = i'(\phi) - i(\phi').$$

Let $\{\phi_n\}_{n\in N}$ be the basis of $\mathbb{F}^N$ consisting of the functions $N \to \mathbb{F}$ mapping n to 1 and all other elements of n to 0, and let $\{i_n\}_{n\in N} \subseteq (\mathbb{F}^N)^*$ be the dual basis. Then $\{(\phi_n, 0), (0, i_n)\}_{n\in N}$ forms a symplectic basis for $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$.

There are two common ways we will build symplectic spaces from other symplectic spaces: conjugation and summation. Given a symplectic form ω, we may define its **conjugate** symplectic form $\overline{\omega} = -\omega$, and write the conjugate symplectic space $(V, \overline{\omega})$ as $\overline{V}$. Given two symplectic vector spaces $(U, \nu), (V, \omega)$, we consider their direct sum $U \oplus V$ a symplectic vector space with the symplectic form $\nu + \omega$, and call this the **sum** of the two symplectic vector spaces. Note that this is not a product in the category of symplectic vector spaces and symplectic maps.

The symplectic form provides a notion of orthogonal complement. Given a subspace S of V, we define its **complement**

$$S^\circ = \{v \in V \mid \omega(v, s) = 0 \text{ for all } s \in S\}.$$

Note that this construction obeys the following identities, where S and T are subspaces of V:

$$\begin{aligned}
\dim S + \dim S^\circ &= \dim V \\
(S^\circ)^\circ &= S \\
(S + T)^\circ &= S^\circ \cap T^\circ \\
(S \cap T)^\circ &= S^\circ + T^\circ.
\end{aligned}$$

In the symplectic vector space $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$, the subspace $\mathbb{F}^N$ has the property of being a maximal subspace such that the symplectic form restricts to the zero form on this subspace. Subspaces with this property are known as Lagrangian subspaces, and they may all be realized as the image of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ under symplectomorphisms from $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ to itself.

Definition 4.33. Let S be a linear subspace of a symplectic vector space (V, ω). We say that S is **isotropic** if $\omega|_{S \times S} = 0$, and that S is **coisotropic** if S° is isotropic. A subspace is **Lagrangian** if it is both isotropic and coisotropic, or equivalently, if it is a maximal isotropic subspace.

Lagrangian subspaces are also known as Lagrangian correspondences and canonical relations. Note that a subspace S is isotropic if and only if $S \subseteq S^\circ$. This fact helps with the following characterizations of Lagrangian subspaces.

Proposition 4.34. *Given a subspace $L \subset V$ of a symplectic vector space (V, ω), the following are equivalent:*

(i) L *is Lagrangian.*

(ii) L *is maximally isotropic.*

(iii) L *is minimally coisotropic.*

(iv) $L = L^\circ$.

(v) L *is isotropic and* $\dim L = \frac{1}{2} \dim V$.

From this proposition it follows easily that the direct sum of two Lagrangian subspaces in Lagrangian in the sum of their ambient spaces. We also observe that an advantage of isotropy is that there is a good way to take a quotient of a symplectic vector space by an isotropic subspace—that is, there is a way to put a natural symplectic structure on the quotient space.

Proposition 4.35. *Let S be an isotropic subspace of a symplectic vector space (V, ω). Then S°/S is a symplectic vector space with symplectic form $\omega'(v{+}S, u{+}S) = \omega(v, u)$.*

Proof. The function ω' is a well-defined due to the isotropy of S—by definition adding any pair (s, s') of elements of S to a pair (v, u) of elements of S° does not change the value of $\omega(v + s, u + s')$. As ω is a symplectic form, one can check that ω' is too. □

4.7.2 Lagrangian subspaces from quadratic forms

Lagrangian subspaces are of relevance to us here as the behavior of any passive linear circuit forms a Lagrangian subspace of the symplectic vector space generated by the nodes of the circuit. We think of this vector space as comprising two parts: a space $\mathbb{F}^N$ of potentials at each node, and a dual space $(\mathbb{F}^N)^*$ of currents. To make clear how circuits can be interpreted as Lagrangian subspaces, here we describe how Dirichlet forms on a finite set N give rise to Lagrangian subspaces of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. More generally, we show that there is a one-to-one correspondence between Lagrangian subspaces and quadratic forms.

Proposition 4.36. *Let N be a finite set. Given a quadratic form Q over $\mathbb{F}$ on N, the subspace*

$$L_Q = \big\{(\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N\big\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*,$$

where $dQ_\phi \in (\mathbb{F}^N)^*$ is the formal differential of Q at $\phi \in \mathbb{F}^N$, is Lagrangian. Moreover, this construction gives a one-to-one correspondence

$$\left\{ \textit{Quadratic forms over } \mathbb{F} \textit{ on } N \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{Lagrangian subspaces of } \mathbb{F}^N \oplus (\mathbb{F}^N)^* \\ \textit{with trivial intersection with } \{0\} \oplus (\mathbb{F}^N)^* \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^* \end{array} \right.$$

Proof. The symplectic structure on $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ and our notation for it is given in Example 4.32.

Note that for all $n, m \in N$ the corresponding basis elements

$$\frac{\partial^2 Q}{\partial\phi_n \partial\phi_m} = dQ_{\phi_n}(\phi_m) = dQ_{\phi_m}(\phi_n),$$

so $dQ_\phi(\psi) = dQ_\psi(\phi)$ for all $\phi, \psi \in \mathbb{F}^N$. Thus L_Q is indeed Lagrangian: for all $\phi, \psi \in \mathbb{F}^N$

$$\omega\big((\phi, dQ_\phi), (\psi, dQ_\psi)\big) = dQ_\psi(\phi) - dQ_\phi(\psi) = 0.$$

Observe also that for all quadratic forms Q we have $dQ_0 = 0$, so the only element of L_Q of the form $(0, i)$, where $i \in (\mathbb{F}^N)^*$, is $(0,0)$. Thus L_Q has trivial intersection with the subspace $\{0\} \oplus (\mathbb{F}^N)^*$ of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$. This L_Q construction forms the leftward direction of the above correspondence.

For the rightward direction, suppose that L is a Lagrangian subspace of $\mathbb{F}^N \oplus (\mathbb{F}^N)^*$ such that $L \cap (\{0\} \oplus (\mathbb{F}^N)^*) = \{(0,0)\}$. Then for each $\phi \in \mathbb{F}^N$, there exists a unique $i_\phi \in (\mathbb{F}^N)^*$ such that $(\phi, i_\phi) \in L$. Indeed, if i_ϕ and i'_ϕ were distinct elements of $(\mathbb{F}^N)^*$ with this property, then by linearity $(0, i_\phi - i'_\phi)$ would be a nonzero element of $L \cap (\{0\} \oplus (\mathbb{F}^N)^*$, contradicting the hypothesis about trivial intersection. We thus can define a function, indeed a linear map, $\mathbb{F}^N \to (\mathbb{F}^N)^*; \phi \mapsto i_\phi$. This defines a bilinear form $Q(\phi, \psi) = i_\phi(\psi)$ on $\mathbb{F}^N \oplus \mathbb{F}^N$, and so $Q(\phi) = i_\phi(\phi)$ defines a quadratic form on $\mathbb{F}^N$.

Moreover, L is Lagrangian, so

$$\omega\big((\phi, i_\phi), (\psi, i_\psi)\big) = i_\psi(\phi) - i_\phi(\psi) = 0,$$

and so $Q(-,-)$ is a symmetric bilinear form. This gives a one-to-one correspondence between Lagrangian subspaces of specified type, symmetric bilinear forms, and quadratic forms, and so in particular gives the claimed one-to-one correspondence. $\square$

In particular, every Dirichlet form defines a Lagrangian subspace.

4.7.3 Lagrangian relations

Recall that a relation between sets X and Y is a subset R of their product $X \times Y$. Furthermore, given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, there is a composite relation $(S \circ R) \subseteq X \times Z$ given by pairs (x, z) such that there exists $y \in Y$ with $(x, y) \in R$ and $(y, z) \in S$—a direct generalization of function composition. A Lagrangian relation between symplectic vector spaces V_1 and V_2 is a relation between V_1 and V_2 that forms a Lagrangian subspace of the symplectic vector space $\overline{V_1} \oplus V_2$. This gives us a way to think of certain Lagrangian subspaces, such as those arising from circuits, as morphisms, giving a way to compose them.

Definition 4.37. A **Lagrangian relation** $L\colon V_1 \to V_2$ is a Lagrangian subspace L of $\overline{V_1} \oplus V_2$.

This is a generalization of the notion of symplectomorphism: any symplectomorphism $f\colon V_1 \to V_2$ forms a Lagrangian subspace when viewed as a relation $f \subseteq \overline{V_1} \oplus V_2$. More generally, any symplectic map $f\colon V_1 \to V_2$ forms an isotropic subspace when viewed as a relation in $\overline{V_1} \oplus V_2$.

Importantly for us, the composite of two Lagrangian relations is again a Lagrangian relation. This is well-known [**?**], but sufficiently easy and important to us that we provide a proof.

Proposition 4.38. *Let $L\colon V_1 \to V_2$ and $L'\colon V_2 \to V_3$ be Lagrangian relations. Then their composite relation $L' \circ L$ is a Lagrangian relation $V_1 \to V_3$.*

We prove this proposition by way of two lemmas detailing how the Lagrangian property is preserved under various operations. The first lemma says that the intersection of a Lagrangian space with a coisotropic space is in some sense Lagrangian, once we account for the complement.

Lemma 4.39. *Let $L \subseteq V$ be a Lagrangian subspace of a symplectic vector space V, and $S \subseteq V$ be an isotropic subspace of V. Then $(L \cap S^\circ) + S \subseteq V$ is Lagrangian in V.*

Proof. Recall from Proposition 4.34 that a subspace is Lagrangian if and only if it is equal to its complement. The lemma is then immediate from the way taking the symplectic complement interacts with sums and intersections:

$$((L\cap S^\circ)+S)^\circ = (L\cap S^\circ)^\circ\cap S^\circ = (L^\circ+(S^\circ)^\circ)\cap S^\circ = (L+S)\cap S^\circ = (L\cap S^\circ)+(S\cap S^\circ) = (L\cap S^\circ)+S.$$

Since $(L \cap S^\circ) + S$ is equal to its complement, it is Lagrangian. □

The second lemma says that if a subspace of a coisotropic space is Lagrangian, taking quotients by the complementary isotropic space does not affect this.

Lemma 4.40. *Let $L \subseteq V$ be a Lagrangian subspace of a symplectic vector space V, and $S \subseteq L$ an isotropic subspace of V contained in L. Then $L/S \subseteq S^\circ/S$ is Lagrangian in the quotient symplectic space S°/S.*

Proof. As L is isotropic and the symplectic form on S°/S is given by $\omega'(v+S, u+S) = \omega(v, u)$, the quotient L/S is immediately isotropic. Recall from Proposition 4.34 that an isotropic subspace S of a symplectic vector space V is Lagrangian if and only if $\dim S = \frac{1}{2}\dim V$. Also recall that for any subspace $\dim S + \dim S^\circ = \dim V$. Thus

$$\begin{aligned}\dim(L/S) &= \dim L - \dim S = \tfrac{1}{2}\dim V - \dim S\\ &= \tfrac{1}{2}(\dim S + \dim S^\circ) - \dim S = \tfrac{1}{2}(\dim S^\circ - \dim S) = \tfrac{1}{2}\dim(S^\circ/S).\end{aligned}$$

Thus L/S is Lagrangian in S°/S. □

Combining these two lemmas gives a proof that the composite of two Lagrangian relations is again a Lagrangian relation.

Proof of Proposition 4.38. Let Δ be the diagonal subspace

$$\Delta = \{(0, v_2, v_2, 0) \mid v_2 \in V_2\} \subseteq \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3.$$

Observe that Δ is isotropic, and has coisotropic complement

$$\Delta^\circ = \{(v_1, v_2, v_2, v_3) \mid v_i \in V_i\} \subseteq \overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3.$$

As Δ is the kernel of the restriction of the projection map $\overline{V_1}\oplus V_2\oplus\overline{V_2}\oplus V_3 \to \overline{V_1}\oplus V_3$ to Δ°, and after restriction this map is still surjective, the quotient space Δ°/Δ is isomorphic to $\overline{V_1} \oplus V_3$.

Now, by definition of composition of relations,

$$L' \circ L = \{(v_1, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}.$$

But note also that

$$L \oplus L' = \{(v_1, v_2, v_2', v_3) \mid (v_1, v_2) \in L, (v_2', v_3) \in L'\},$$

so

$$(L \oplus L') \cap \Delta^\circ = \{(v_1, v_2, v_2, v_3) \mid \text{there exists } v_2 \in V_2 \text{ such that } (v_1, v_2) \in L, (v_2, v_3) \in L'\}.$$

Quotienting by Δ then gives

$$L' \circ L = ((L \oplus L') \cap \Delta^\circ) + \Delta)/\Delta.$$

As $L' \oplus L$ is Lagrangian in $\overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3$, Lemma 4.39 says that $(L' \oplus L) \cap \Delta^\circ) + \Delta$ is also Lagrangian in $\overline{V_1} \oplus V_2 \oplus \overline{V_2} \oplus V_3$. Lemma 4.40 thus shows that $L' \circ L$ is Lagrangian in $\Delta^\circ/\Delta = \overline{V_1} \oplus V_3$, as required. □

Note that this composition is associative. We shall see later that this composition agrees with composition of Dirichlet forms, and hence also composition of circuits.

4.7.4 The dagger compact category of Lagrangian relations

Lagrangian relations solve the identity problems we had with Dirichlet forms: given a symplectic vector space V, the Lagrangian relation $\mathrm{id}\colon V \to V$ specified by the Lagrangian subspace

$$\mathrm{id} = \{(v, v) \mid v \in V\} \subseteq \overline{V} \oplus V,$$

acts as an identity for composition of relations. We thus have a category.

Definition 4.41. We write **LagrRel** for the category with symplectic vector spaces as objects and Lagrangian relations as morphisms.

In fact the move to the setting of Lagrangian relations, rather than Dirichlet forms, adds far richer structure than just identity morphisms. The category LagrRel can be viewed as endowed with the structure of a dagger compact category. We lay this out in steps.

Symmetric monoidal structure

We define the tensor product of two objects of LagrRel to be their direct sum. Similarly, we define the tensor product of two morphisms $L\colon U \to V$, $L \subseteq \overline{U} \oplus V$ and $K\colon T \to W$, $K \subseteq \overline{T} \oplus W$ to be their direct sum

$$L \oplus K \subseteq \overline{U} \oplus V \oplus \overline{T} \oplus W,$$

but considered as a subspace of the naturally isomorphic space $\overline{U \oplus T} \oplus V \oplus W$. Despite this subtlety, we abuse our notation and write their tensor product $L \oplus K \colon U \oplus T \to V \oplus W$, and move on having sounded this note of caution.

Note that the direct sum of two Lagrangian subspaces is again Lagrangian in the direct sum of their ambient spaces, and the zero dimensional vector space $\{0\}$ acts as an identity for direct sum. Indeed, defining for all objects U, V, W in LagrRel unitors:

$$\lambda_V = \{(0, v, v)\} \subseteq \overline{\{0\} \oplus V} \oplus V,$$
$$\rho_V = \{(v, 0, v)\} \subseteq \overline{V \oplus \{0\}} \oplus V,$$

associators:

$$\alpha_{U,V,W} = \{(u, v, w, u, v, w)\} \subseteq \overline{(U \oplus V) \oplus W} \oplus U \oplus (V \oplus W),$$

and braidings:

$$\sigma_{U,V} = \{(u, v, v, u) \mid u \in U, v \in V\} \subseteq \overline{U \oplus V} \oplus V \oplus U,$$

we have a symmetric monoidal category. Note that all these structure maps come from symplectomorphisms between the domain and codomain. From this viewpoint it is immediate that all the necessary diagrams commute, so we have a symmetric monoidal category.

Duals for objects

Each object V of LagrRel is dual to its conjugate space $\overline{V}$, with cup $\eta \colon \{0\} \to \overline{V} \oplus V$ given by

$$\eta = \{(0, v, v) \mid v \in V\} \subseteq \overline{\{0\}} \oplus \overline{V} \oplus V$$

and cap $\epsilon \colon V \oplus \overline{V} \to \{0\}$ given by

$$\epsilon = \{(v, v, 0) \mid v \in V\} \subseteq \overline{V \oplus \overline{V}} \oplus \{0\}.$$

It is straightforward to check these satisfy the zigzag identities.

Dagger structure

Given symplectic vector spaces U, V, observe that the map

$$(-)^\dagger \colon \overline{U} \oplus V \longrightarrow \overline{V} \oplus U;$$
$$(u, v) \longmapsto (v, u)$$

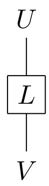
takes Lagrangian subspaces of the domain to Lagrangian subspaces of the codomain. Thus we can view it as a map $(-)^\dagger$ taking morphisms $L\colon U \to V$ of LagrRel to morphisms $L^\dagger\colon V \to U$. This defines a dagger structure on LagrRel, which makes this category into a symmetric monoidal dagger category.

Moreover, every object in LagrRel has a dagger dual: it is clear that $\eta^\dagger = \epsilon \circ \sigma$. This category thus becomes a dagger compact category.

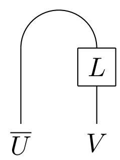
4.7.5 Names of Lagrangian relations

This brief subsection illustrates a guiding principle of this paper: *duals for objects allow us to blur the distinction between composition of morphisms and the tensor product of morphisms*. We will make use of this when we prove the functoriality of the black box functor.

Observe that a Lagrangian relation $L\colon \{0\} \to V$ is the same as a Lagrangian subspace of V. Moreover, given a Lagrangian relation $L\colon U \to V$,

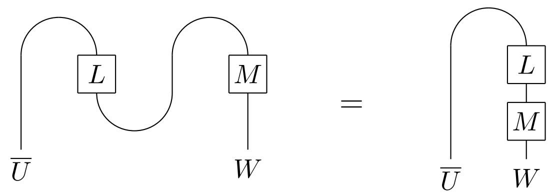


compactness allows us to view it as a Lagrangian relation $\{0\} \to \overline{U} \oplus V$:

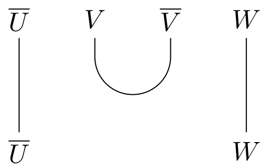


We call this subspace the **name** of the Lagrangian relation L; indeed, we have used this one-to-one correspondence between morphisms and their names to define Lagrangian relations.

By compactness, we have the equation



Here the right hand side is the name of the composite $M \circ L$ of Lagrangian relations, while the left hand side is the direct sum of the names of L and M post-composed with the Lagrangian relation



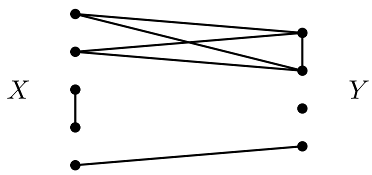
Thus this relation above, the product of a cap and two identity maps, enacts composition of Lagrangian relations. The proof of Proposition 4.38, that the composite of two Lagrangian relations is again a Lagrangian relation, makes use of this fact. We shall also return to it when discussing our functor Circ $\to$ LagrRel. Note the similarity in form between our diagrams of cospans and of names of Lagrangian relations.

4.8 Ideal wires and corelations

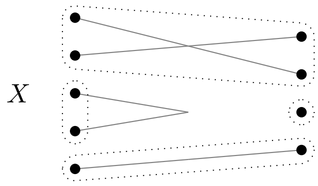
In the previous section our exploration of the meaning of circuit diagrams culminated with our understanding of behaviors as Lagrangian subspaces. We now turn our attention to how circuit components fit together, and the category of operations. In this section we shall see that the algebra of connections is described by the concept of corelations, a generalization of the notion of function that forgets the directionality from the domain to the codomain. We then observe that Kirchhoff's laws follow directly from interpreting these structures in the category of linear relations.

4.8.1 Ideal wires

To motivate the definition of this category, let us start with a set of input terminals X, and a set of output terminals Y. We may connect these terminals with ideal wires of zero impedance, whichever way we like—input to input, output to output, input to output—producing something like:

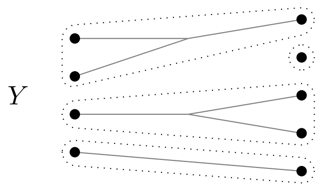


In doing so, we introduce a notion of equivalence on our terminals, where two terminals are equivalent if we, or if electrons, can traverse from one to another via some sequence of wires. Because of this, we consider our perfectly-conducting components to be equivalence relations on $X + Y$, transforming the above picture into

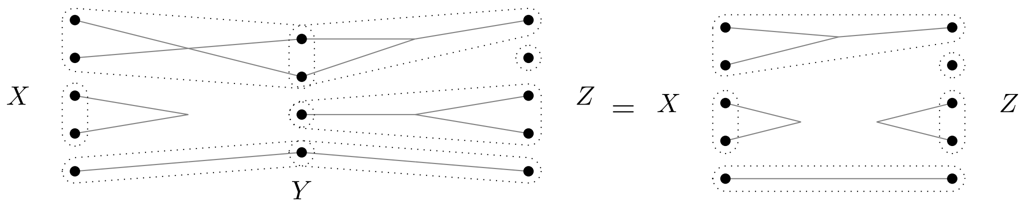


The dotted lines indicate equivalence classes of points, while for reference the grey lines indicate ideal wires connecting these points, running through a central hub.

Given another circuit of this sort, say from sets Y to Z,



we may combine these circuits in to a circuit X to Z



by taking the transitive closure of the two equivalence relations, and then restricting this to an equivalence relation on $X + Z$. This in fact defines a dagger compact category of fundamental importance: the category of *corelations*.

Ellerman gives a detailed treatment of corelations from a logic viewpoint in [?], while basic category theoretic aspects can be found in Lawvere and Rosebrugh [?]. However, to make this paper self-contained, we explain them here.

4.8.2 The category of corelations

In the category of sets we hold the fundamental relationship between sets to be that of functions. These encode the idea of a deterministic process that takes each element of one set to a unique element of the other. For the study of networks this is less appropriate, as the relationship between terminals is not an input-output one, but rather one of interconnection.

In particular, the direction of a function becomes irrelevant, and to describe these interconnections via the category of sets we must develop an understanding of how to compose functions head to head and tail to tail. We have so far used cospans and pushouts to address this. Cospans, however, come with an apex, which represents extraneous structure beyond the two sets we wish to specify a relationship between. Corelations arise from omitting this information.

Definition 4.42. A **corelation** $\alpha\colon X \to Y$ between finite sets X and Y is a partition α of the disjoint union $X + Y$.

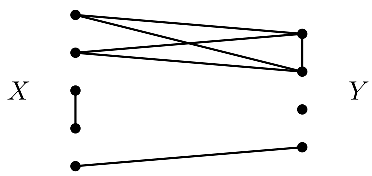
That is, for finite sets X and Y, a corelation is a collection of nonempty subsets $\alpha = \{A_1, A_2, \ldots, A_n\}$ of $X + Y$ such that

(i) α does not contain the empty set.

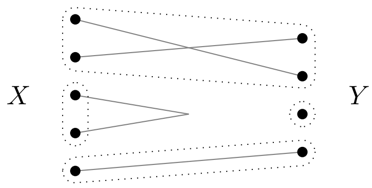
(ii) $\bigcup_{i=1}^{n} A_i = X + Y$.

(iii) $A_i \cap A_j = \varnothing$ whenever $i \neq j$.

For example, we can take a circuit of ideal wires with X as the set of inputs and Y as the set of outputs:



and define a corelation $\alpha\colon X \to Y$ for which terminals lie in the same set of the partition $\alpha = \{A_1, A_2, \ldots, A_n\}$ when we can travel from one terminal to another following a path of wires:



The discussion in the previous section then motivates a rule for composing corelations. We compose a corelation $\alpha\colon X \to Y$ and a corelation $\beta\colon Y \to Z$ by finding the finest partition on $X + Y + Z$ that is coarser than both α and β when restricted to $X + Y$ and $Y + Z$ respectively, and then restricting this to a partition on $X + Z$. More explicitly, the corelation $\beta \circ \alpha = \{C_1, C_2, \ldots, C_m\}$ is the unique set of pairwise disjoint C_i of the form

$$C_i = \bigcup_j A_j \cap X \cup \bigcup_k B_k \cap Z$$

for j, k varying over indices such that

$$\bigcup_j A_j \cap Y = \bigcup_k B_k \cap Y.$$

This rule for composition is associative, as both pairwise methods of composing relations $\alpha \colon X \to Y$, $\beta \colon Y \to Z$, and $\gamma \colon Z \to W$ amount to finding the finest partition on $X + Y + Z + W$ that is coarser than each of α, β, and γ when restricted to the relevant subset, and then restricting this partition to a partition on $X + W$. The pictures in the previous section make this clear.

The identity corelation $1_X \colon X \to X$ on a set X is the map $[1_X, 1_X] \colon X + X \to X$. Equivalently, it is the partition of $X + X$ such that each partition comprises exactly two elements: an element $x \in X$ considered as an element of both the first and then the second summand of $X + X$. We thus define a category:

Definition 4.43. Let **Corel** be the category with finite sets as objects and corelations between finite sets as morphisms.

This category becomes monoidal under disjoint union of finite sets, using the fact that the union of partition of $X + Y$ and a partition of $X' + Y'$ is a partition of $(X + X') + (Y + Y')$. Moreover, the commutativity and associativity of the coproduct of finite sets also allows implies the category is symmetric monoidal and indeed dagger compact: the braiding and cups and caps are respectively given by the partitions of $(X + Y) + (Y + X)$ and $X + X$ where two elements are in the same part of the partition if and only if they are equal as elements of X or Y, while a dagger is given by simply considering a partition of $X + Y$ as a partition of $Y + X$..[4]

We can get a corelation $\alpha \colon X \to Y$ from a surjection $f \colon X + Y \to S$ for some set S, by taking the sets in the partition $\alpha = (A_1, \ldots, A_n)$ of $X + Y$ to be the inverse images of the points of S. Two surjections $f \colon X + Y \to S$, $f \colon X + Y \to S'$ give the same corelation if and only if there is an isomorphism $g \colon S \to S'$ making the obvious triangle commute: $f' = g \circ f$.

By the universal property of the coproduct, surjections $X + Y \to A$ are in one-to-one correspondence with jointly epic cospans $X \to A \leftarrow Y$. Thus, corelations can also be seen as isomorphism classes of jointly epic cospans. This lets us compose corelations by composing cospans and applying a correction. That is: given corelations $\alpha \colon X \to Y$, $\beta \colon Y \to Z$, we may choose cospans representing them and compose

[4]In fact, if we take a skeleton of Corel we obtain the PROP for 'extraspecial commutative Frobenius monoids', as defined by Baez and Erbele [**?**]. We do not need this here, but it helps tie our current work to other work on categories in electrical engineering. A proof can be assembled from Baez–Erbele together with the work of Lack [**?**].

these cospans. The composite cospan may not be jointly epic. However, we can then replace the apex of the composite cospan by the joint image of the feet. The resulting jointly epic cospan gives the composite corelation $\beta \circ \alpha \colon X \to Z$. More precisely, we have the following proposition.

Proposition 4.44. *There is a strict symmetric monoidal dagger functor*

$$\mathrm{Cospan}(\mathrm{FinSet}) \longrightarrow \mathrm{Corel}$$

mapping any cospan to the corelation it defines.

Proof. This functor maps each finite sets to itself. We thus need only discuss how it acts on morphisms. A cospan in FinSet comprises a pair of functions $X \xrightarrow{f} N \xleftarrow{g} Y$. Restricting the apex N down to the joint image $f(X) \cup g(Y)$ gives a jointly epic cospan $X \xrightarrow{f} f(X) \cup g(Y) \xleftarrow{g} Y$, and so a corelation $X \to Y$. As the elements of the apex not in the image of maps from the feet play only a trivial role in the pushout, and hence in composition of cospans, this map is functorial. It is now readily observed that the functor is symmetric monoidal. □

All this is dual to the more familiar connection between spans and relations, where a relation is seen as an isomorphism class of jointly monic spans. This explains the name 'corelation'. Note that neither relations nor corelations are a generalization of the other. The key property of corelations here is that it forms a *compact* category with disjoint union of finite sets as the monoidal product. This is not true of the category of relations between finite sets.

Via the above proposition, the symmetric monoidal dagger functor embedding FinSet into Cospan(FinSet) gives rise to a symmetric monoidal dagger functor embedding FinSet into Corel. Composing this with the dagger structure on Corel, we also obtain a symmetric monoidal dagger functor embedding $\mathrm{FinSet}^{\mathrm{op}}$ into Corel. Corelations thus give a method of composing functions regardless of the direction of those functions. Given some not necessarily directed path of functions, for example

$$A \longrightarrow B \longleftarrow C \longleftarrow D \longrightarrow E \longrightarrow F,$$

considering these functions as corelations gives a way to compose them.

In particular, while this mode of composition is simply composition of functions for two functions head to tail, and turning a cospan into a corelation for two functions head to head, for two tail to tail functions $C \leftarrow D \rightarrow E$ we compute the composite of cospans

$$\begin{tikzcd} & C & & E & \\ C \arrow[ur, equal, "1_C"] & & D \arrow[ul] \arrow[ur] & & E \arrow[ul, equal, "1_E"'] \end{tikzcd}$$

before restricting the apex to arrive at a surjective function from $C + E$. This implies the following lemma.

Lemma 4.45. *Let*

$$\begin{tikzcd} & P & \\ A \arrow[ur] & & B \arrow[ul] \\ & N \arrow[ul] \arrow[ur] & \end{tikzcd}$$

be a pushout square in FinSet*. Then the composites of corelations $A \to P \leftarrow B$ and $A \leftarrow N \to B$ are equal.*

4.8.3 Potentials on corelations

Chasing our interpretation of corelations as ideal wires, our aim for the remainder of this section is to build a functor

$$S\colon \mathrm{Corel} \longrightarrow \mathrm{LagrRel}$$

that expresses this interpretation. We break this functor down into the sum of two parts, according to the behaviors of potentials and currents respectively.

The consideration of potentials gives a functor $\Phi\colon \mathrm{Corel} \to \mathrm{LinRel}$, where LinRel is the symmetric monoidal dagger category of finite-dimensional $\mathbb{F}$-vector spaces, linear relations, direct sum, and transpose. In particular, this functor expresses Kirchhoff's voltage law: it requires that if two elements are in the same part of the corelation partition—that is, if two nodes are connected by ideal wires—then the potential at those two points must be the same.

This functor is a generalization of the contravariant functor FinSet $\to$ Vect that maps a set to the vector space of $\mathbb{F}$-valued functions on that set.

Proposition 4.46. *Define the functor*

$$\Phi\colon \mathrm{Corel} \longrightarrow \mathrm{LinRel},$$

on objects by sending a finite set X to the vector space $\mathbb{F}^X$, and on morphisms by sending a corelation $\alpha\colon X \to Y$ to the linear subspace $\Phi(\alpha)$ of $\mathbb{F}^X \oplus \mathbb{F}^Y$ comprising functions $\phi = [\phi_X, \phi_Y]\colon X + Y \to \mathbb{F}$ that are constant on each element of α. This is a symmetric monoidal dagger functor.

Proof. For coherence maps we take the usual natural isomorphisms $\mathbb{F}^X \oplus \mathbb{F}^Y \cong \mathbb{F}^{X \times Y}$ and $\{0\} \cong \mathbb{F}^{\varnothing}$. We detail only the proof that Φ preserves composition; the other properties are straightforward to check.

Let $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ be corelations. As Φ maps corelations to relations, it is enough to check both inclusions $\Phi(\beta)\circ\Phi(\alpha) \subseteq \Phi(\beta\circ\alpha)$ and $\Phi(\beta\circ\alpha) \subseteq \Phi(\beta)\circ\Phi(\alpha)$.

$\Phi(\beta)\circ\Phi(\alpha) \subseteq \Phi(\beta\circ\alpha)$: Let $\phi = [\phi_X, \phi_Z] \in \Phi(\beta)\circ\Phi(\alpha)$. We wish to show that for all $C_i \in \beta \circ \alpha$, for all $c, c' \in C_i$, we have $\phi(c) = \phi(c')$. To this end, note that there exists some $\phi_Y\colon Y \to \mathbb{F}$ such that $\phi_{XY} := [\phi_X, \phi_Y] \in \Phi(\alpha)$ and $\phi_{YZ} := [\phi_Y, \phi_Z] \in \Phi(\beta)$. Furthermore, by definition this ϕ_Y has the property that for all $A_j \in \alpha$, for all $a, a' \in A_j$, we have $\phi_{XY}(a) = \phi_{XY}(a')$, and for all $B_k \in \beta$, for all $b, b' \in B_k$, we have $\phi_{YZ}(b) = \phi_{YZ}(b')$. Write $\phi_{XYZ} = [\phi_X, \phi_Y, \phi_Z]\colon X + Y + Z \to \mathbb{F}$.

Our desired fact is thus true: for all $c, c' \in C_i$, there exists a sequence $c = c_0, c_1, \dots, c_n = c'$ in $X + Y + Z$ such that for all $\ell = 0, 1, \dots n-1$ we have either $c_\ell, c_{\ell+1} \in A_j$ for some j or $c_\ell, c_{\ell+1} \in B_k$ for some k, and hence that

$$\phi(c) = \phi_{X+Y+Z}(c_0) = \phi_{X+Y+Z}(c_1) = \cdots = \phi_{X+Y+Z}(c_n) = \phi(c')$$

as required.

$\Phi(\beta \circ \alpha) \subseteq \Phi(\beta) \circ \Phi(\alpha)$: Let $\phi = [\phi_X, \phi_Z] \in \Phi(\beta \circ \alpha)$. We must show that there exists $\phi_Y\colon Y \to \mathbb{F}$ such that $[\phi_X, \phi_Y] \in \Phi(\alpha)$ and $[\phi_Y, \phi_Z] \in \Phi(\beta)$. We claim

$$\phi_Y(y) := \begin{cases} \phi(x) & \text{if } x \in X,\ x, y \in A_j \text{ for some } j; \\ \phi(z) & \text{if } z \in Z,\ y, z \in B_k \text{ for some } k; \\ 0 & \text{if there exist no such } x \in X \text{ or } z \in Z \end{cases}$$

satisfies this. This is well-defined as for all A_j, $A_j \cap X \subseteq C_i$ for some i, so ϕ is constant on A_j, and similarly for all B_j. Thus it does not matter if there are many such x or z with $x, y \in A_j$ or $y, z \in B_k$. Furthermore, if there exists y such that both $x, y \in A_j$ for some x, A_j and $y, z \in B_k$ for some z, B_k, then the C_i with $A_j \cap X \subseteq C_i$ and $B_k \cap Z \subseteq C_i$ is unique, so the definitions of $\phi_Y(y)$ do not conflict.

Moreover, by construction $[\phi_X, \phi_Y]$ is constant on all A_j, and $[\phi_Y, \phi_Z]$ is constant on all B_k, so $[\phi_X, \phi_Y] \in \Phi(\alpha)$ and $[\phi_Y, \phi_Z] \in \Phi(\beta)$ as required. □

To recap, we have now constructed a functor $\Phi\colon \mathrm{Corel} \to \mathrm{LinRel}$ expressing the behavior of potentials on corelations interpreted as ideal wires. We now do the same for currents.

4.8.4 Currents on corelations

Next we consider the case of currents, described by a functor $I\colon \mathrm{Corel} \to \mathrm{LinRel}$. This functor now expresses Kirchhoff's current law: it requires that the sum of currents flowing into each part of the corelation partition must equal to the sum of currents flowing out. It may also be understood as a generalization of the covariant functor $\mathrm{Set} \to \mathrm{Vect}$ that maps a set to the vector space of $\mathbb{F}$-linear combinations of elements of that set.

Proposition 4.47. *Define the functor*

$$I\colon \mathrm{Corel} \longrightarrow \mathrm{LinRel}.$$

as follows. On objects send a finite set X to the vector space $(\mathbb{F}^X)^*$. On morphisms send a corelation $\alpha\colon X \to Y$ to the linear relation $I(\alpha)$ comprising precisely those

$$(i_X, i_Y) = \left(\sum_{x\in X} \lambda_x dx, \sum_{y \in Y} \lambda_y dy\right) \in (\mathbb{F}^X)^* \oplus (\mathbb{F}^Y)^*$$

such that for all $A_i \in \alpha$ the sum of the coefficients of the elements of $A_i \cap X$ is equal to that for $A_i \cap Y$:

$$\sum_{x \in A_i \cap X} \lambda_x = \sum_{y \in A_i \cap Y} \lambda_y.$$

This is a symmetric monoidal dagger functor.

Proof. The coherence maps are the natural isomorphisms $(\mathbb{F}^X)^* \oplus (\mathbb{F}^Y)^* \to (\mathbb{F}^{X+Y})^*$ and $\{0\} \to (\mathbb{F}^{\varnothing})^*$. Again the only nontrivial task is to check this map I preserves composition. Again we do this by checking inclusions $I(\beta)\circ I(\alpha) \subseteq I(\beta\circ\alpha)$ and $I(\beta\circ\alpha) \subseteq I(\beta)\circ I(\alpha)$.

$I(\beta)\circ I(\alpha) \subseteq I(\beta\circ\alpha)$: Let $(i_X, i_Z) \in I(\beta)\circ I(\alpha)$, with $i_X = \sum_{x\in X} \lambda_x dx$ and $i_Z = \sum_{z \in Z} \lambda_z dz$. Note that this implies that there is some $i_Y = \sum_{y\in Y} \lambda_y dy$ such that

$(i_X, i_Y) \in I(\alpha)$, $(i_Y, i_Z) \in I(\beta)$. Then for each $C_i \in \beta \circ \alpha$ we have

$$\begin{aligned}
\sum_{x \in C_i \cap X} \lambda_x &= \sum_{\substack{x \in A_j \cap X \\ A_j \cap X \subseteq C_i}} \lambda_x \\
&= \sum_{\substack{y \in A_j \cap Y \\ A_j \cap X \subseteq C_i}} \lambda_y && \text{(By definition of } I(\alpha)\text{)} \\
&= \sum_{\substack{y \in B_k \cap Y \\ B_k \cap Z \subseteq C_i}} \lambda_y && \text{(See composition of corelations)} \\
&= \sum_{\substack{z \in B_k \cap Z \\ B_k \cap Z \subseteq C_i}} \lambda_z && \text{(By definition of } I(\beta)\text{)} \\
&= \sum_{z \in C_i \cap Z} \lambda_z.
\end{aligned}$$

Thus $(i_X, i_Z) \in I(\beta \circ \alpha)$.

$I(\beta \circ \alpha) \subseteq I(\beta) \circ I(\alpha)$: The reverse inclusion requires a bit more effort. Let $(i_X, i_Z) \in I(\beta \circ \alpha)$. We wish to construct some $i_Y = \sum_{y \in Y} \lambda_y dy \in (\mathbb{F}^Y)^*$ such that $(i_X, i_Y) \in I(\alpha)$ and $(i_Y, i_Z) \in I(\beta)$. This means that we must find a vector $i_Y \in (\mathbb{F}^Y)^*$ satisfying the $\#\alpha$ linear constraints

$$\sum_{y \in A_j \cap Y} \lambda_y = \sum_{x \in A_j \cap X} \lambda_x$$

and the $\#\beta$ linear constraints

$$\sum_{y \in B_j \cap Y} \lambda_y = \sum_{z \in B_j \cap Z} \lambda_z.$$

Note, however, that for each $C_i \in \beta \circ \alpha$, summing over the A_j and B_k that intersect C_i shows a linear dependence between these constraints themselves, as

$$\sum_{\substack{y \in A_j \cap Y \\ A_j \cap X \subseteq C_i}} \lambda_y = \sum_{x \in C_i \cap X} \lambda_x = \sum_{z \in C_i \cap Z} \lambda_z = \sum_{\substack{y \in B_k \cap Y \\ B_k \cap Z \subseteq C_i}} \lambda_y.$$

Moreover, as each element of Y lies in exactly one element of each α and β, we may view them as edges in a graph on $\alpha + \beta$, with exactly $\#(\beta \circ \alpha)$ connected components. This implies that

$$\#Y > \#\alpha + \#\beta - \#(\beta \circ \alpha).$$

Thus these constraints define an affine subspace of $(\mathbb{F}^Y)^*$ of positive dimension, and hence we can always find a vector i_Y with the desired property. This proves the claim. $\square$

Using elementary methods, an algorithm can also be given to construct an explicit solution.

4.8.5 The functor from Corel to LagrRel

We have now defined functors that, when interpreting corelations as connections of ideal wires, describe the behaviors of the currents and potentials at the terminals of these wires. In this section, we combine these to define a single functor $S\colon \mathrm{Corel} \to \mathrm{LagrRel}$ describing the behavior of both currents and potentials as a Lagrangian subspace.

Proposition 4.48. *We define the symplectification functor*

$$S\colon \mathrm{Corel} \longrightarrow \mathrm{LagrRel}$$

sending a finite set X to the symplectic vector space

$$S(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*,$$

and a corelation $\alpha\colon X \to Y$ to the Lagrangian relation

$$S(\alpha) = \Phi(\alpha) \oplus I(\alpha) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Then S is a symmetric monoidal dagger functor, with coherence maps inherited from Φ and I.

Proof. As S is the tensor product in LinRel of the symmetric monoidal dagger functors $\Phi, I\colon \mathrm{Corel} \to \mathrm{LinRel}$, it is itself a symmetric monoidal dagger functor $\mathrm{Corel} \to \mathrm{LinRel}$. Thus it only remains to be checked that, with respect to the symplectic structure we put on the objects $S(X)$, the image of each corelation $S(\alpha)$ is Lagrangian.

This follows from condition (v) of Proposition 4.34: $S(\alpha)$ is (i) isotropic as, for all $(\phi_X, i_X, \phi_Y, i_Y)$, $(\phi'_X, i'_X, \phi'_Y, i'_Y) \in S(\alpha)$ we have

$$
\begin{aligned}
&\omega\big((\phi_X, i_X, \phi_Y, i_Y), (\phi'_X, i'_X, \phi'_Y, i'_Y)\big) \\
&= -\big(i'_X(\phi_X) - i_X(\phi'_X)\big) + i'_Y(\phi_Y) - i_Y(\phi'_Y) \\
&= i_X(\phi'_X) - i_Y(\phi'_Y) + i'_X(\phi_X) - i'_Y(\phi_Y) \\
&= \sum_{x\in X} \lambda_x dx(\phi'_X) - \sum_{y\in Y} \lambda_y dy(\phi'_Y) + \sum_{y\in Y} \lambda'_y dy(\phi_Y) - \sum_{x\in X} \lambda'_x dx(\phi_X) \\
&= \sum_{A_j\in\alpha} \left(\sum_{x\in A_j\cap X} \lambda_x dx(\phi'_X) - \sum_{y\in A_j\cap Y} \lambda_y dy(\phi'_Y) \right) + \sum_{A_j\in\alpha} \left(\sum_{x\in A_j\cap X} \lambda'_x dx(\phi_X) - \sum_{y\in A_j\cap Y} \lambda'_y dy(\phi_Y) \right) \\
&= \sum_{A_j\in\alpha} \left(\sum_{x\in A_j\cap X} \lambda_x - \sum_{y\in A_j\cap Y} \lambda_y \right) k'_{A_j} + \sum_{A_j\in\alpha} \left(\sum_{x\in A_j\cap X} \lambda'_x - \sum_{y\in A_j\cap Y} \lambda'_y \right) k_{A_j} \\
&= 0,
\end{aligned}
$$

and (ii) has dimension equal to

$$\dim(\Phi(\alpha)) + \dim(I(\alpha) = \#\alpha + \#(X+Y) - \#\alpha = \#(X+Y).$$

This proves the proposition. $\square$

We have thus shown we do indeed have a functor $S\colon \mathrm{Corel} \to \mathrm{LagrRel}$. In the next section we shall see that this functor provides the engine of our black box functor, playing the key role in showing that we may indeed treat circuit components as black boxes: that is, that circuits that behave the same compose the same. Before we get there, we quickly make two relevant observations.

Example 4.49 (Symplectification of functions)**.** Let $f : X \to Y$ be a function. In this example we show that Sf has the form

$$Sf = \big\{(\phi_X, i_X, \phi_Y, i_Y) \,\big|\, \phi_X = f^*\phi_Y, i_Y = f_* i_X\big\} \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*,$$

where f^* is the pullback map

$$
\begin{aligned}
f^*\colon \mathbb{F}^Y &\longrightarrow \mathbb{F}^X; \\
\phi &\longmapsto \phi \circ f,
\end{aligned}
$$

and f_* is the pushforward map

$$
\begin{aligned}
f_*\colon (\mathbb{F}^X)^* &\longrightarrow (\mathbb{F}^Y)^*; \\
i(-) &\longmapsto i(- \circ f).
\end{aligned}
$$

(We shall also write f_* for the more general map from functions on $\mathbb{F}^X$ to functions on $\mathbb{F}^Y$ that takes a function $i(-)$ on $\mathbb{F}^X$ to the function $i(- \circ f)$.) The claim is then that these pullback and pushforward constructions express Kirchhoff's laws.

Recall that the corelation corresponding to f partitions $X+Y$ into $\#Y$ parts, each of the form $f^{-1}(y) \cup \{y\}$. The linear relation $\Phi(f)$ requires that if $x \in X$ and $y \in Y$ lie in the same part of the partition, then they have the same potential: that is, $\phi_X(x) = \phi_Y(y)$. This is precisely the arrangement imposed by $f^*\phi_Y = \phi_X$:

$$\phi_X(x) = \phi_Y(f(x)) = \phi_Y(y).$$

On the other hand, the linear relation $I(f)$ requires that if $i_X = \sum_{x\in X} \lambda_x dx$, $i_Y = \sum_{y \in Y} \lambda_y dy$, then for each $y \in Y$ we have

$$\sum_{x \in f^{-1}(y)} \lambda_x = \lambda_y.$$

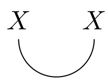
This is precisely what is required by f_*: given any $\phi \in \mathbb{F}^Y$, we have

$$f_* i_X(\phi) = f_* \sum_{x \in X} \lambda_x dx(\phi) = \sum_{x\in X} \lambda_x dx(\phi \circ f) = \sum_{y \in Y} \left(\sum_{x \in f^{-1}(y)} \lambda_x \right) dy.$$

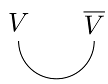
This gives us the above representation of Sf when f is a function.

Remark 4.50. We make a remark on our conventions for string diagrams representing the cups and caps of the dualities in Corel and LagrRel.

Note that by the cap duality diagram



in Corel we mean the corelation $\cup_X \colon (X + X \xrightarrow{[1,1]} X \xleftarrow{!} \varnothing)$, whereas by the cap

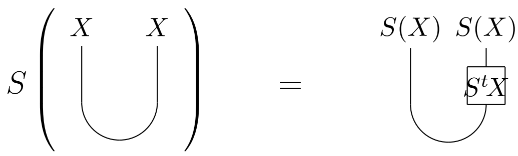


in LagrRel we mean the Lagrangian relation $\cup_V \colon V \oplus \overline{V} \to 0$ given by the Lagrangian subspace $\{(v,v) \in \overline{V \oplus \overline{V}} \mid v \in V\}$. Although we represent them similarly, the functor S does not map these directly onto each other: the image of $\cup_X$ under S is the Lagrangian relation $S(\cup_X)\colon \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to 0$ given by the Lagrangian subspace

$$\{(\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^*\} \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^*},$$

and in particular a Lagrangian relation $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to 0$ and not $\mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \to 0$. As the cup diagrams in these categories simply denote the dagger image of these caps, the analogous statements apply to them too.

This is an expression of the fact that in cospan categories and Corel there is a canonical self-duality, while for Lagrangian relations a basis must be picked before there is an isomorphism between a symplectic vector space and its conjugate, the conjugate being a canonical dual object. Nonetheless as the functor S uses the set X to generate a symplectic vector space, each object in the image of S has a canonical symplectomorphism with its dual $S^tX : \mathbb{F}^X \oplus (F^X)^* \to \overline{\mathbb{F}^X \oplus (F^X)^*}$, with name the Lagrangian subspace $\{(\phi, i, \phi, -i) \mid \phi \in \mathbb{F}^X, i \in (\mathbb{F}^X)^*\}$.[5] The image of the canonical self-duality Corel is thus given by the composite of this symplectomorphism with canonical duality in LagrRel; that is



4.9 The Black Box Functor

We have now developed enough machinery to prove Theorem 4.1: there is a symmetric monoidal dagger functor, the black box functor

$$\blacksquare : \mathrm{Circ} \to \mathrm{LagrRel}$$

taking passive linear circuits to their behaviors. To recap, we have so far developed two categories: Circ, in which morphisms are passive linear circuits, and LagrRel, which captures the external behavior of such circuits. We now define a functor that maps each circuit to its behavior, before proving it is indeed a symmetric monoidal dagger functor.

4.10 Definition of the black box functor

The role of the functor we construct here is to identify all circuits with the same external behavior, making the internal structure of the circuit inaccessible. Circuits

[5] While we could treat S^tX as an atomic notation, we write it here to evoke the idea of the concept of 'twisted' symplectification introduced in the introduction.

treated this way are frequently referred to as 'black boxes', so we call this functor the **black box functor**,

$$\blacksquare \colon \mathrm{Circ} \to \mathrm{LagrRel}.$$

In this section we first provide the definition of this functor, and then check that our definition really does map a circuit to its behavior.

4.10.1 Definition

It should be no surprise that the black box functor maps a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$ of potentials and currents that may be placed on that set. The challenge is to provide a succinct statement of its action on the circuits themselves. To do this, we take advantage of four processes we developed in Parts 4.2 and 4.5.

Let $\Gamma \colon X \to Y$ be a circuit, represented by the decorated cospan

$$\big(X \xrightarrow{i} N \xleftarrow{o} Y,\, \Gamma\big).$$

Recall that this means that X and Y are finite sets, Γ is a $\mathbb{F}$-graph (N, E, s, t, r), and we have a cospan of finite sets

$$\begin{tikzcd} & \Gamma & \\ X \arrow[ur, "i"] & & Y. \arrow[ul, "o"'] \end{tikzcd}$$

To define the image of Γ under our functor $\blacksquare$, by definition a Lagrangian relation $\blacksquare(\Gamma) : \blacksquare(X) \to \blacksquare(Y)$, we must specify a Lagrangian subspace

$$\blacksquare(\Gamma) \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Recall that to each $\mathbb{F}^+$-graph Γ we can associate a Dirichlet form, the extended power functional

$$\begin{aligned} P_\Gamma \colon \mathbb{F}^N &\longrightarrow \mathbb{F}; \\ \phi &\longmapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} \big(\phi(t(e)) - \phi(s(e))\big)^2, \end{aligned}$$

and to this Dirichlet form we associate a Lagrangian subspace

$$\mathrm{Graph}(dP_\Gamma) = \{(\phi, d(P_\Gamma)_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*.$$

We consider this Lagrangian subspace as a Lagrangian relation $\{0\} \to \mathbb{F}^N \oplus (\mathbb{F}^N)^*$.

From the legs of the cospan Γ, the symplectification functor S gives the Lagrangian relation

$$S[i,o]^\dagger \colon \mathbb{F}^N \oplus (\mathbb{F}^N)^* \longrightarrow \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Writing $Y : Y \to Y$ for the identity morphism on the finite set Y, S also provides a way of writing the identity morphism $SY : \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \to \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$.

Lastly, we have the symplectomorphism

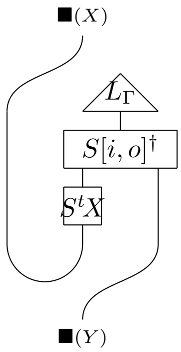
$$\begin{aligned} S^tX \colon \mathbb{F}^X \oplus (\mathbb{F}^X)^* &\longrightarrow \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*}; \\ (\phi, i) &\longmapsto (\phi, -i). \end{aligned}$$

The black box functor maps a circuit Γ to the Lagrangian relation

$$(S^tX \oplus SY) \circ S[i,o]^\dagger \circ \mathrm{Graph}(P_\Gamma).$$

As isomorphisms of cospans of $\mathbb{F}^+$-graphs amount to no more than a relabelling of nodes and edges, this construction is independent of the cospan chosen as representative of the isomorphism class of cospans forming the circuit.

We picture this as

$\blacksquare(\Gamma) =$ 

To summarize:

Definition 4.51. We define the black box functor

$$\blacksquare \colon \mathrm{Circ} \to \mathrm{LagrRel}$$

on objects by mapping a finite set X to the symplectic vector space

$$\blacksquare(X) = \mathbb{F}^X \oplus (\mathbb{F}^X)^*.$$

and on morphisms by mapping a circuit $\Gamma \colon X \to Y$, represented by the decorated cospan

$$\big(X \stackrel{i}{\longrightarrow} N \stackrel{o}{\longleftarrow} Y,\, \Gamma\big)$$

to the Lagrangian relation

$$\blacksquare(\Gamma) = (S^tX \oplus SY) \circ S[i,o]^\dagger \circ \mathrm{Graph}(dP_\Gamma).$$

The coherence maps are given by the natural isomorphisms

$$\blacksquare(X) \oplus \blacksquare(Y) = \mathbb{F}^X \oplus (\mathbb{F}^X)^* \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^* \cong \mathbb{F}^{X+Y} \oplus (\mathbb{F}^{X+Y})^* = \blacksquare(X+Y)$$

and

$$\{0\} \cong \mathbb{F}^{\varnothing} \oplus (\mathbb{F}^{\varnothing})^* = \blacksquare(\varnothing).$$

Theorem 4.52. *The black box functor is a well-defined symmetric monoidal dagger functor.*

The next, and final, section is devoted to the proof of this theorem. Before we get there, we first assure ourselves that we have indeed arrived at the theorem we set out to prove.

4.10.2 Minimization via composition of relations

At this point the reader might voice two concerns: firstly, why does the *black box* functor refer to the *extended* power functional P and, secondly, since it fails to talk about power minimization, how is it the same functor as that defined in Theorem 4.1? These fears are allayed by the remarkable trinity of minimization, the symplectification of functions, and Kirchhoff's laws.

We have seen that symplectification of functions views the cograph of the function as a picture of ideal wires, governed by Kirchhoff's laws (Example 4.49). We have also seen that Kirchhoff's laws are closely related to the principle of minimium power (Theorems 4.20 and 4.22). The final aspect of this relationship is that we may use symplectification of functions to enact minimization.

Theorem 4.53. *Let $\iota : \partial N \to N$ be an injection, and let P be a Dirichlet form on N. Write $Q = \min_{N\setminus\partial N} P$ for the Dirichlet form on ∂N given by minimization over $N \setminus \partial N$. Then we have an equality of Lagrangian subspaces*

$$S\iota^\dagger\big(\mathrm{Graph}(dP)\big) = \mathrm{Graph}(dQ).$$

Proof. Recall from Example 4.49 that $S\iota^\dagger$ is the Lagrangian relation

$$S\iota^\dagger = \big\{(\phi, \iota_* i, \phi\circ\iota, i) \,\big|\, \phi \in \mathbb{F}^N, i \in (\mathbb{F}^{\partial N})^*\big\} \subseteq \overline{\mathbb{F}^N \oplus (\mathbb{F}^N)^*} \oplus \mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*,$$

where $\iota_* i(\phi) = i(\phi\circ\iota)$, and note that

$$\mathrm{Graph}(dP) = \big\{(\phi, dP_\phi) \,\big|\, \phi \in \mathbb{F}^N\big\}.$$

This implies that their composite is given by the set

$$S\iota^\dagger \circ \mathrm{Graph}(dP) = \big\{(\phi \circ \iota, i) \,\big|\, \phi \in \mathbb{F}^N, i \in (\mathbb{F}^{\partial N})^*, dP_\phi = \iota_* i\big\}.$$

We must show this Lagrangian subspace is equal to $\mathrm{Graph}(dQ)$.

Consider the constraint $dP_\phi = \iota_* i$. This states that for all $\varphi \in \mathbb{F}^N$ we have $dP_\phi(\varphi) = i(\varphi \circ \iota)$. Letting $\chi_n : N \to \mathbb{F}$ be the function sending $n \in N$ to 1 and all other elements of N to 0, we see that when $n \in N \setminus \partial N$ we must have

$$\left.\frac{dP}{d\varphi(n)}\right|_{\varphi=\phi} = dP_\phi(\chi_n) = i(\chi_n \circ \iota) = i(0) = 0.$$

So ϕ must be a realizable extension of $\psi = \phi \circ \iota$. We henceforth write $\tilde{\psi} = \phi$. As ι is injective, $\psi = \phi \circ \iota$ gives no constraint on $\psi \in \mathbb{F}^{\partial N}$.

We next observe that we can write $S\iota^\dagger \circ \mathrm{Graph}(dP) = \mathrm{Graph}(dO)$ for some quadratic form O. Recall that Proposition 4.36 states that a Lagrangian subspace L of $\mathbb{F}^{\partial N} \oplus (\mathbb{F}^{\partial N})^*$ is of the form $\mathrm{Graph}(dO)$ if and only if L has trivial intersection with $\{0\} \oplus (\mathbb{F}^N)^*$. But indeed, if $\psi = 0$ then 0 is a realizable extension of ψ, so $\iota_* i = dP_0 = 0$, and hence $i = 0$.

It remains to check that $O = Q$. This is a simple computation:

$$O(\psi) = dO_\psi(\psi) = dO_\psi(\tilde{\psi} \circ \iota) = \iota_* dQ_\psi(\tilde{\psi}) = dP_{\tilde{\psi}}(\tilde{\psi}) = P(\tilde{\psi}) = Q(\psi),$$

where $\tilde{\psi}$ is any realizable extension of $\psi \in \mathbb{F}^{\partial N}$. □

Write $\iota : \partial N \to N$ for the inclusion of the terminals into the set of nodes of the circuit, and $i|^{\partial N} : X \to \partial N$, $o|^{\partial N} : Y \to \partial N$ for the respective corestrictions of the input and output map to ∂N. Note that $[i, o] = \iota \circ [i|^{\partial N}, o|^{\partial N}]$. Then we have the equalities of sets, and thus Lagrangian relations:

$$\begin{aligned}(S^t X \oplus 1_Y) \circ S[i,o]^\dagger \circ \mathrm{Graph}(dP_\Gamma) &= (S^t X \oplus SY) \circ S[i|^{\partial N}, o|^{\partial N}]^\dagger \circ S\iota^\dagger \circ \mathrm{Graph}(dP_\Gamma) \\ &= (S^t X \oplus SY) \circ S[i|^{\partial N}, o|^{\partial N}]^\dagger \circ \mathrm{Graph}(dQ_\Gamma) \\ &= \bigcup_{v \in \mathrm{Graph}(dQ)} S^t i|^{\partial N}(v) \times So|^{\partial N}(v)\end{aligned}$$

We see now that Theorem 4.52 is a restatement of Theorem 4.1 in the introduction.

4.11 Proof of functoriality

To prove that the black box construction is indeed functorial, we factor it into three functors. These functors are each symmetric monoidal dagger functors, so the black box functor is too.

We first make use, twice, of our results on decorated cospans, showing the existence of categories of cospans decorated by Dirichlet forms and then Lagrangian subspaces, and the existence of functors from the category of circuits to each of these. This proceeds by defining functors Dirich and Lagr to construction decorated cospan categories, and then by defining the relevant natural transformations to construct the desired decorated cospan functors.

The third functor takes cospans decorated by Lagrangian subspaces, and black boxes them to give Lagrangian relations between the feet of such cospans. The functoriality of this process relies on interpreting corelations as Lagrangian relations.

This gives a factorization

$$\blacksquare\colon \mathrm{Circ} \longrightarrow \mathrm{DirichCospan} \longrightarrow \mathrm{LagrCospan} \longrightarrow \mathrm{LagrRel}$$

The following subsections deal with these functors in sequence, defining and proving the existence of each one using the techniques of Part 4.5.

4.11.1 From circuits to Dirichlet cospans

To begin, we define a category of cospans of finite sets decorated by Dirichlet forms. This might be seen as a solution to our sought after composition rule for Dirichlet forms—but not quite, as it also requires the additional data of a cospan. Nonetheless, it has the property that there is a functor from the category of circuits to this so-called category of Dirichlet cospans.

The category of Dirichlet cospans

Proposition 4.54. *Let*

$$\mathrm{Dirich}\colon (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

map a finite set X to the set $\mathrm{Dirich}(X)$ *of Dirichlet forms on X, and map a function $f\colon X \to Y$ between finite sets to the pushforward function*

$$\mathrm{Dirich}(f)\colon \mathrm{Dirich}(X) \longrightarrow \mathrm{Dirich}(Y);$$
$$Q \longmapsto f_*Q,$$

where f_*Q maps $\phi \in \mathbb{F}^Y$ to $Q(\phi \circ f))$. This defines a functor.

Moreover, equipping this functor with the family of maps

$$\begin{aligned}\delta_{N,M}\colon \mathrm{Dirich}(N) \times \mathrm{Dirich}(M) &\longrightarrow \mathrm{Dirich}(N+M); \\ (Q_N, Q_M) &\longmapsto Q_N(-\circ \iota_N) + Q_M(-\circ \iota_M),\end{aligned}$$

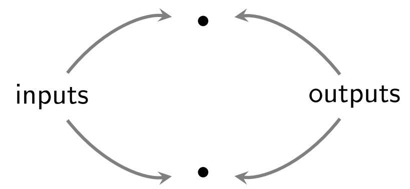
where $\iota_N\colon N \to N+M$, $\iota_M\colon M \to N+M$ are the injections for the coproduct, and with unit

$$\begin{aligned}\delta_1\colon 1 &\longrightarrow \mathrm{Dirich}(\varnothing); \\ \bullet &\longmapsto (\mathbb{F}^{\varnothing} \to \mathbb{F}; \varnothing \mapsto 0)\end{aligned}$$

defines a lax symmetric monoidal functor.

Proof. Functoriality is just the associativity of composition of functions; lax symmetric monoidality is the associativity, unitality, and commutativity of addition of Dirichlet forms. □

By decorated cospans, we thus obtain a category DirichCospan where a morphism is a cospan of finite sets whose apex is equipped with a Dirichlet form. In particular, note that we have overcome our inability to define a category whose morphisms are Dirichlet forms. An identity morphism in DirichCospan is a circuit whose set of inputs is equal to its set of outputs, with all of its nodes being terminals, and no edges:



The functor Circ → DirichCospan

We have now constructed two symmetric monoidal functors

$$(\mathrm{Circuit}, \rho), (\mathrm{Dirich}, \delta)\colon (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

that describe the circuit structures and Dirichlet forms we may put on the set respectively. We have also seen, motivating our discussion of Dirichlet forms, that from any circuit we can obtain a Dirichlet form describing the power usage of that circuit. Now we shall see that this process respects the tensor product. More precisely, it specifies a monoidal natural transformation between Circuit and Dirich. By the decorated cospan construction, this gives us a strict symmetric monoidal dagger functor Circ → DirichCospan.

Proposition 4.55. *Let*

$$\alpha\colon (\mathrm{Circuit}, \rho) \Longrightarrow (\mathrm{Dirich}, \delta)$$

be the collection of functions

$$\begin{aligned} \alpha_N\colon \mathrm{Circuit}(N) &\longrightarrow \mathrm{Dirich}(N); \\ (N, E, s, t, r) &\longmapsto \left(\phi \in \mathbb{F}^N \mapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} \big(\phi(s(e)) - \phi(t(e))\big)^2 \right). \end{aligned}$$

Then α is a monoidal natural transformation.

Proof. Naturality requires that the square

$$\begin{tikzcd} \mathrm{Circuit}(N) \arrow[r, "\alpha_N"] \arrow[d, "\mathrm{Circuit}(f)"'] & \mathrm{Dirich}(N) \arrow[d, "\mathrm{Dirich}(f)"] \\ \mathrm{Circuit}(M) \arrow[r, "\alpha_M"'] & \mathrm{Dirich}(M) \end{tikzcd}$$

commutes. Let (N, E, s, t, r) be an $\mathbb{F}^+$-graph on N and $f\colon N \to M$ be a function N to M. Then both $\mathrm{Dirich}(f) \circ \alpha_N$ and $\alpha_M \circ \mathrm{Circuit}(f)$ map (N, E, s, t, r) to the Dirichlet form

$$\begin{aligned} \mathbb{F}^M &\longrightarrow \mathbb{F}; \\ \psi &\longmapsto \frac{1}{2} \sum_{e \in E} \frac{1}{r(e)} \big(\psi(f(s(e))) - \psi(f(t(e)))\big)^2. \end{aligned}$$

Thus both methods of constructing a power functional on a set of nodes M from a circuit on N and a function $N \to M$ produce the same power functional.

To show that α is a monoidal natural transformation, we must check that the square

$$\begin{tikzcd} \mathrm{Circuit}(N) \times \mathrm{Circuit}(M) \arrow[r, "\alpha_N \times \alpha_M"] \arrow[d, "\rho_{N,M}"'] & \mathrm{Dirich}(N) \times \mathrm{Dirich}(M) \arrow[d, "\delta_{N,M}"] \\ \mathrm{Circuit}(N+M) \arrow[r, "\alpha_{N+M}"'] & \mathrm{Dirich}(N+M) \end{tikzcd}$$

and the triangle

$$\begin{tikzcd} & 1 \arrow[dl, "\rho_\varnothing"'] \arrow[dr, "\delta_\varnothing"] & \\ \mathrm{Circuit}(\varnothing) \arrow[rr, "\alpha_\varnothing"'] & & \mathrm{Dirich}(\varnothing) \end{tikzcd}$$

commute. It is readily observed that both paths around the square lead to taking two graphs and summing their corresponding Dirichlet forms, and that the triangle commutes immediately as all objects in it are the one element set. □

From decorated cospans, we thus obtain a strict symmetric monoidal dagger functor

$$Q\colon \mathrm{Circ} = \mathrm{CircuitCospan} \longrightarrow \mathrm{DirichCospan}.$$

Roughly, this says that the process of composition for circuit diagrams is the same as that of composition for Dirichlet cospans. Note that although this functor preserves much of the information in circuit diagrams, it is not a faithful functor. For example, applying Q to a circuit with edges e, e' from the node m to the node n of resistance r_e and $r_{e'}$ respectively, we obtain the same result as for the circuit with just one edge e'' from m to n whose resistance $r_{e''}$ is given by

$$\frac{1}{r_{e''}} = \frac{1}{r_e} + \frac{1}{r_{e'}}.$$

4.11.2 From Dirichlet cospans to Lagrangian cospans

The next step is to show that our process of turning a Dirichlet form into a Lagrangian subspace—by taking the graph of its differential—is functorial.

The category of Lagrangian cospans

We begin by describing a category where morphisms are cospans decorated by Lagrangian subspaces.

Proposition 4.56. *Define*

$$\mathrm{Lagr}\colon (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times)$$

as follows. For objects let Lagr *map a finite set X to the set* $\mathrm{Lagr}(X)$ *of Lagrangian subspaces of the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$. For morphisms, recall that a function $f\colon X \to Y$ between finite sets may be considered as a corelation, and the symplectification functor S thus maps this corelation to some Lagrangian relation $Sf\colon \mathbb{F}^X \oplus (\mathbb{F}^X)^* \to \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*$. As Lagrangian relations map Lagrangian subspaces to Lagrangian subspaces (Proposition 4.38), this gives a map:*

$$\begin{aligned} \mathrm{Lagr}(f)\colon \mathrm{Lagr}(X) &\longrightarrow \mathrm{Lagr}(Y); \\ L &\longmapsto Sf(L). \end{aligned}$$

Moreover, equipping this functor with the family of maps

$$\begin{aligned} \lambda_{N,M}\colon \mathrm{Lagr}(N) \times \mathrm{Lagr}(M) &\longrightarrow \mathrm{Lagr}(N+M); \\ (L_N, L_M) &\longmapsto L_N \oplus L_M, \end{aligned}$$

and unit

$$\begin{aligned}\lambda_1 \colon 1 &\longrightarrow \mathrm{Lagr}(\varnothing);\\ \bullet &\longmapsto \{0\}\end{aligned}$$

defines a lax symmetric monoidal functor.

Proof. The functoriality of this construction follows from the functoriality of S; the lax symmetric monoidality from the relevant properties of the direct sum of vector spaces. □

We thus obtain a dagger compact category LagrCospan.

The functor DirichCospan $\to$ LagrCospan

We now wish to construct a strict symmetric monoidal dagger functor DirichCospan $\to$ LagrCospan. For this we need a monoidal natural transformation

$$(\mathrm{Dirich}, \delta), (\mathrm{Lagr}, \lambda) \colon (\mathrm{FinSet}, +) \longrightarrow (\mathrm{Set}, \times).$$

Proposition 4.57. *Let*

$$\beta \colon (\mathrm{Dirich}, \delta) \Longrightarrow (\mathrm{Lagr}, \lambda)$$

be the collection of functions

$$\begin{aligned}\beta_N \colon \mathrm{Dirich}(N) &\longrightarrow \mathrm{Lagr}(N);\\ Q &\longmapsto \{(\phi, dQ_\phi) \mid \phi \in \mathbb{F}^N\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*.\end{aligned}$$

Then β is a monoidal natural transformation.

Proof. Naturality requires that the square

$$\begin{tikzcd} \mathrm{Dirich}(N) \arrow[r, "\beta_N"] \arrow[d, "\mathrm{Dirich}(f)"'] & \mathrm{Lagr}(N) \arrow[d, "\mathrm{Lagr}(f)"] \\ \mathrm{Dirich}(M) \arrow[r, "\beta_M"'] & \mathrm{Lagr}(M) \end{tikzcd}$$

commutes for every function $f \colon N \to M$. This is primarily a consequence of the fact that the differential commutes with pullbacks. As we did in Example 4.49, write f^* for the pullback map and f_* for the pushforward map. Then $\mathrm{Dirich}(f)$ maps a Dirichlet form Q on N to the form f_*Q, and β_M in turn maps this to the Lagrangian subspace

$$\left\{(\psi, d(f_*Q)_\psi) \,\middle|\, \psi \in \mathbb{F}^M\right\} \subseteq \mathbb{F}^M \oplus (\mathbb{F}^M)^*.$$

On the other hand, β_N maps a Dirichlet form Q on N to the Lagrangian subspace

$$\big\{(\phi, dQ_\phi) \,\big|\, \phi \in \mathbb{F}^N\big\} \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*,$$

before $\mathrm{Lagr}(f)$ maps this to the Lagrangian subspace

$$\big\{(\psi, f_* dQ_\phi) \,\big|\, \psi \in \mathbb{F}^M, \phi = f^*(\psi)\big\} \subseteq \mathbb{F}^M \oplus (\mathbb{F}^M)^*.$$

But

$$f_* dQ_{f^*\psi} = d(f_* Q)_\psi,$$

so these two processes commute.

Monoidality requires that the diagrams

$$\begin{tikzcd}
\mathrm{Dirich}(N) \times \mathrm{Dirich}(M) \arrow[r, "\beta_N \times \beta_M"] \arrow[d, "\delta_{N,M}"'] & \mathrm{Lagr}(N) \times \mathrm{Lagr}(M) \arrow[d, "\lambda_{N,M}"] \\
\mathrm{Dirich}(N+M) \arrow[r, "\beta_{N+M}"'] & \mathrm{Lagr}(N+M)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & 1 \arrow[dl, "\delta_\varnothing"'] \arrow[dr, "\lambda_\varnothing"] & \\
\mathrm{Dirich}(\varnothing) \arrow[rr, "\beta_\varnothing"'] & & \mathrm{Lagr}(\varnothing)
\end{tikzcd}$$

commute. These do: the Lagrangian subspace corresponding to the sum of Dirichlet forms is equal to the sum of the Lagrangian subspaces that correspond to the summand Dirichlet forms, while there is only a unique map $1 \to \mathrm{Lagr}(\varnothing)$. □

We thus obtain a strict symmetric monoidal dagger functor

$$\mathrm{DirichCospan} \to \mathrm{LagrCospan},$$

which simply replaces the decoration on each cospan in DirichCospan with the corresponding Lagrangian subspace.

4.11.3 From cospans to relations

At this point we have checked that the process of reinterpreting a circuit as a Lagrangian subspace of behaviors is functorial. Our task is now to integrate this information as more than just a 'decoration' on our morphisms. This process constitutes a monoidal dagger functor

$$\mathrm{LagrCospan} \longrightarrow \mathrm{LagrRel}.$$

This factor of the black box functor is the one that gives it its name; through this functor we finally seal off the inner structure of our circuits, leaving us access only to the behavior at the terminals. Its purpose is to take a Lagrangian cospan, which captures information about the behaviors of a circuit measured at every node, and restrict it down to a relation detailing the behaviors simply on the terminals.

Proposition 4.58. *We may define a symmetric monoidal dagger functor*

$$\mathrm{LagrCospan} \longrightarrow \mathrm{LagrRel}$$

as follows. On objects let this functor take a finite set X to the symplectic vector space $\mathbb{F}^X \oplus (\mathbb{F}^X)^*$. On morphisms let it take a Lagrangian cospan

$$\big(X \xrightarrow{\;i\;} N \xleftarrow{\;o\;} Y; L \subseteq \mathbb{F}^N \oplus (\mathbb{F}^N)^*\big)$$

to the Lagrangian relation

$$(S^tX \oplus SY) \circ S[i,o]^\dagger \circ L \subseteq \overline{\mathbb{F}^X \oplus (\mathbb{F}^X)^*} \oplus \mathbb{F}^Y \oplus (\mathbb{F}^Y)^*.$$

Proof. The coherence maps here are the usual natural isomorphisms

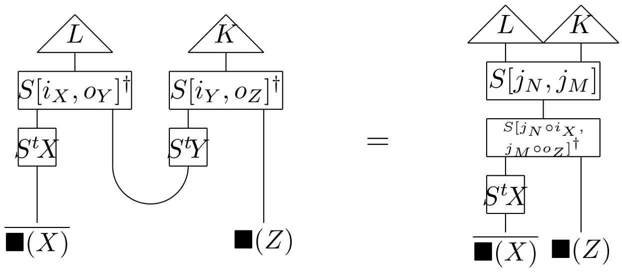
$$\mathbb{F}^X \oplus \left(\mathbb{F}^X\right)^* \oplus \mathbb{F}^Y \oplus \left(\mathbb{F}^Y\right)^* \xrightarrow{\;\sim\;} \mathbb{F}^{X+Y} \oplus \left(\mathbb{F}^{X+Y}\right)^*$$

and

$$\{0\} \xrightarrow{\;\sim\;} \mathbb{F}^{\varnothing} \oplus \left(\mathbb{F}^{\varnothing}\right)^*.$$

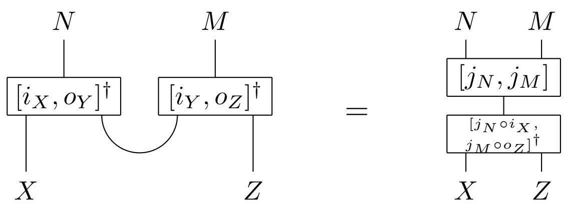
It is now routine to observe the symmetric monoidality and dagger-preserving nature of this construction, as well as that it preserves identities. Finally, we must check that composition is preserved.

Using the concept of names introduced in Section 4.7.5, this comes down to checking this equality of Lagrangian subspaces:



where the left hand side is the composite in LagrRel of the images of the Lagrangian cospans $(X \xrightarrow{i_X} N \xleftarrow{o_Y} Y; L)$ and $(Y \xrightarrow{i_Y} M \xleftarrow{o_Z} Z; K)$, and the right hand side is the image of their composite in LagrCospan. (Recall that we write $j_N : N \to N +_Y M$ and $j_M : M \to N +_Y M$ for the maps given by the pushout.)

Recalling Remark 4.50 pertaining to the functor S and duals for objects, this implies that it is enough to check the equality of corelations



Writing this instead as a commutative diagram, we wish to prove the equality of the two corelations $N+M \to X+Y$:

$$\begin{tikzcd}
 & & N +_Y M & & \\
N+M \arrow[urr, "{[j_N, j_M]}"] & & & & X+Z \arrow[ull, "{[j_N \circ i_X, j_M \circ o_Z]}"'] \arrow[dl, hook', "\mathrm{incl}_{X+Z}"] \\
 & X+Y+Y+Z \arrow[ul, "{[i_X, o_Y]+[i_Y, o_Z]}"] \arrow[rr, "{\mathrm{id}_X+[\mathrm{id}_Y, \mathrm{id}_Y]+\mathrm{id}_Z}"'] & & X+Y+Z &
\end{tikzcd}$$

As the right-hand part admits the factorization

$$\begin{tikzcd}
N +_Y M & & X+Z \arrow[ll, "{[j_N \circ i_X, j_M \circ o_Z]}"'] \arrow[dl, hook', "\mathrm{incl}_{X+Z}"] \\
 & X+Y+Z \arrow[ul, "{[j_N \circ i_X, j_N \circ o_Y, j_M \circ o_Z]}"] &
\end{tikzcd}$$

it is enough to check the following diagram commutes when interpreted as a pair of morphisms from $N+M$ to $X+Y+Z$ in the category of corelations:

$$\begin{tikzcd}
 & N +_Y M & \\
N+M \arrow[ur, "{[j_N, j_M]}"] & & X+Y+Z \arrow[ul, "{[j_N \circ i_X, j_N \circ o_Y, j_M \circ o_Z]}"'] \\
 & X+Y+Y+Z \arrow[ul, "{[i_X, o_Y]+[i_Y, o_Z]}"] \arrow[ur, "{\mathrm{id}_X+[\mathrm{id}_Y, \mathrm{id}_Y]+\mathrm{id}_Z}"'] &
\end{tikzcd}$$

By Lemma 4.45, this is equivalent to checking that it is a pushout square in FinSet. This is so: the square commutes in FinSet as it is the sum along the lower right edges of the three commutative squares

$$\begin{tikzcd}
 & N +_Y M & \\
N+M \arrow[ur, "{[j_N, j_M]}"] & & X \arrow[ul, "j_N \circ i_X"'] \\
 & X \arrow[ul, "i_X"] \arrow[ur, "\mathrm{id}_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N +_Y M & \\
N+M \arrow[ur, "{[j_N, j_M]}"] & & Y \arrow[ul, "j_N \circ o_Y = j_M \circ i_Y"'] \\
 & Y+Y \arrow[ul, "o_Y + i_Y"] \arrow[ur, "{[\mathrm{id}_Y, \mathrm{id}_Y]}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & N +_Y M & \\
N+M \arrow[ur, "{[j_N, j_M]}"] & & Z \arrow[ul, "j_M \circ o_Z"'] \\
 & Z \arrow[ul, "o_Z"] \arrow[ur, "\mathrm{id}_Z"'] &
\end{tikzcd}$$

and given any other object T and maps f, g such that

$$\begin{tikzcd}
 & T & \\
N+M \arrow[ur, "f"] & & X+Y+Z \arrow[ul, "g"'] \\
 & X+Y+Y+Z \arrow[ul, "{[i_X, o_Y]+[i_Y, o_Z]}"] \arrow[ur, "{\mathrm{id}_X+[\mathrm{id}_Y, \mathrm{id}_Y]+\mathrm{id}_Z}"'] &
\end{tikzcd}$$

commutes, there is a unique map $N +_Y M \to T$ defined by sending a in $N +_Y M$ to $f(\hat{a})$, where $\hat{a}$ is a preimage of a in $N + M$ under the coproduct of pushout maps $[j_N, j_M]$. This is well-defined as the preimage of a is either unique or equal to $\{o_Y(y), i_Y(y)\}$ for some element $y \in Y$, and the commutativity of the above square containing T implies that $f(o_Y(y)) = f(i_Y(y))$. This proves the functoriality of the map LagrCospan $\to$ LagrRel defined above. $\square$

The three functors of this section compose to give the black box functor

$$\blacksquare\colon \mathrm{Circ} \to \mathrm{LagrRel}.$$

Since they are each separately symmetric monoidal dagger functors, the black box functor is too.

Chapter 5

Signal flow diagrams

Bibliography

[Fon15] Brendan Fong. Decorated Cospans. *Theory and Applications of Categories*, 30(33):25, August 2015. arXiv: 1502.00872. (Referred to on page 7.)