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Poset-valued sets or how to build models for linear logics

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Abstract

We describe a method for constructing models of linear logic based on the category of sets and relations. The resulting categories are non-degenerate in general; in particular they are not compact closed nor do they have biproducts. The construction is simple, lifting the structure of a poset to the new category. The underlying poset thus controls the structure of this category, and different posets give rise to differently-flavoured models. As a result, this technique allows the construction of models for both, intuitionistic or classical linear logic as desired. A number of well-known models, for example coherence spaces and hypercoherences, are instances of this method.

Keywords: Linear logic; Categorical models

1. Introduction

Models for (fragments of) linear logic can hardly be called scarce—we know the categorical properties required, and a number of examples can be found in the literature. So, why introduce more? In fact, we have been asked in jest whether we would not consider naming this paper *Yet another model for linear logic*.

The answer is twofold. For one, we provide a tool for constructing models for (fragments of) linear logic which gives precise control over such properties as

- classical versus intuitionistic, that is we can choose to have a negation satisfying the usual equations, or not;
- products (&) and coproducts ($\oplus$) coincide on objects, or not, and similarly for their units;
- tensor ($\otimes$) and par ($\invamp$) coincide on objects, or not, and similarly for their units.

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This is achieved by making specific choices for the two parameters in our construction. It is very easy to do this to match a specified design.

From a category-theoretic point of view, our construction starts with the compact closed category of sets and relations and adds enough structure to ensure that the resulting category (in general) is not compact closed, nor does it have biproducts. Whether it is symmetric monoidal closed, $*$-autonomous, has products and co-products depends on the second ingredient, a poset, and its structure as a category.

Finally, a number of existing models for linear logic, such as Girard's coherence spaces and phase spaces as well as Ehrhard's hypercoherences, are instances of our construction. Thus, we obtain new insight into how those models work, and offer an explanation for the 'collapse' that occurs in those, for example.

Looking at models for (classical) linear logic, one finds that many of them split the world into a co-variant and a contra-variant part so that negation can be obtained by exchanging the two. Examples for this are Player versus Opponent in games, [1] the pair of sets for Chu spaces or dialectica categories [13] and morphisms $\mathbf{I} \to A$ versus those $A \to \perp$ in the double glueing construction (see [8,9]) employed by Tan [12]. Structurally somewhat simpler models, on the other hand, such as coherence spaces or hypercoherences, can do without this kind of built-in duality—the negation of any object can be defined just via the structure it carries. [2] Our models definitely belong to this second category. At first sight, the construction may look similar to (a weak form of) glueing, but the categorical structure arises differently.

The paper is split into the following sections: First of all, we motivate our construction by extracting it from some of the intended examples. We then explore how the categorical structure arises discussing how to obtain models for linear logic. We compile a list of examples for our construction which arise in the literature and then employ our methods to describe a model for linear logic similar to coherence spaces but where $\mathbf{I}$ and $\perp$ do not coincide. Finally, we consider the remaining questions and future work.

2. Why poset-valued sets?

Consider one of the best-known models for linear logic, namely coherence spaces [6,7].

Recall that a coherence space X is given by a set $|X|$ (its 'web'), and a reflexive binary relation $\stackrel{\frown}{\smile}$ on $|X|$. We use $\frown$ to denote the relation resulting from removing the diagonal from $\stackrel{\frown}{\smile}$. Viewing this model from a different angle, we encode this structure via a function α from $|X| \times |X|$ to the three element ordered set $\mathbf{3} := \{\smile < \mathrm{i} < \frown\}$ (Fig. 1).

[1] Note that to obtain a model for classical linear logic here, one has to move away from the naive interpretations, see Abramsky and Jagadeesan's work [1].

[2] Coherence spaces at least can also be seen as an example of the 'two-worlds' paradigm, see [9].

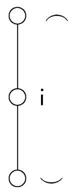

Fig. 1. The poset **3**.

We map $\langle x, x' \rangle$ in $|X| \times |X|$ to i iff $x = x'$, and to $\frown$ iff $x \frown x'$, giving

$$\alpha_X : \langle x, x' \rangle \mapsto \begin{cases} \frown & x \frown x' \\ \mathsf{i} & x = x' \\ \smile & x \smile x' \end{cases}$$

Further recall that a morphism of coherence spaces is given by a relation $R : |X| \nrightarrow |Y|$ satisfying the following condition: Suppose that $x\,R\,y$ and $x'\,R\,y'$. Then $x \mathrel{\underset{\smile}{\frown}} x'$ implies $y \mathrel{\underset{\smile}{\frown}} y'$, and $x \frown x'$ implies $y \frown y'$. In the above representation of coherence spaces as maps from $|X| \times |X|$ to **3**, this is equivalent to the condition:

$$\langle x, x' \rangle (R \times R) \langle y, y' \rangle \quad \text{implies} \quad \alpha_X \langle x, x' \rangle \leqslant \alpha_Y \langle y, y' \rangle.$$

The ordered set **3** can easily be seen to carry a $*$-autonomous structure, which in our chosen representation can be used to explain that same structure on the category of coherence spaces:[3]

$*$	$\frown$	i	$\smile$
$\frown$	$\frown$	$\frown$	$\smile$
i	$\frown$	i	$\smile$
$\smile$	$\smile$	$\smile$	$\smile$

$\rhd$	$\frown$	i	$\smile$
$\frown$	$\frown$	$\smile$	$\smile$
i	$\frown$	i	$\smile$
$\smile$	$\frown$	$\frown$	$\frown$

	$(-)^\perp$
$\frown$	$\smile$
i	i
$\smile$	$\frown$

Since we are interested in coherence spaces as a model of linear logic, this new way of viewing them is only of interest because it captures (at least some of) the corresponding categorical structure. The main idea of this paper is that this structure can be explained in terms of the corresponding structure on the underlying poset **3**, as well as that of the underlying category of sets and relations **Rel**.

To start with, and as a motivation, we will take a look at negation. Clearly, $X^\perp$ is given by

$$|X| \times |X| \xrightarrow{\alpha_X} \mathbf{3} \xrightarrow{(-)^\perp} \mathbf{3}.$$

[3] We use $*$ for the multiplication (or tensor) and $\rhd$ for the linear function space on the posets. Overloading notation does not work well here since we later use the operations on posets to derive the corresponding ones for a proper category. The unit for $*$ is i.

Now for the symmetric monoidal closed structure on the category of coherence spaces. The tensor product of two coherence spaces X and Y, $X \otimes Y$, has the underlying set $|X| \times |Y|$ and its tensor product structure is defined by

$$\langle x, y\rangle \supset\!\!\!\!\smile \langle x', y'\rangle \quad \text{if and only if} \quad x \supset\!\!\!\!\smile x' \text{ and } y \supset\!\!\!\!\smile y'.$$

Given functions $\alpha_X : |X| \times |X| \to \mathbf{3}$ and $\alpha_Y : |Y| \times |Y| \to \mathbf{3}$ encoding the structure on X and Y we can express this as a function

$$\begin{array}{ll} & (|X| \times |X|) \times (|Y| \times |Y|) \to \mathbf{3} \\ \text{via} & \langle\langle x, y\rangle, \langle x', y'\rangle\rangle \mapsto \alpha_X\langle x, x'\rangle * \alpha_Y\langle y, y'\rangle \end{array}$$

Similarly the linear function space of X and Y, $X \multimap Y$, has $|X| \times |Y|$ as its underlying set. In this case $\langle x, y\rangle \supset\!\!\!\!\smile \langle x', y'\rangle$ iff ($x \supset\!\!\!\!\smile x'$ implies $y \supset\!\!\!\!\smile y'$) and ($x \frown x'$ implies $y \frown y'$). Again this can easily be expressed as a function

$$\begin{array}{ll} & (|X| \times |X|) \times (|Y| \times |Y|) \to \mathbf{3} \\ \text{via} & \langle\langle x, y\rangle, \langle x', y'\rangle\rangle \mapsto \alpha_X\langle x, x'\rangle \rhd \alpha_Y\langle y, y'\rangle. \end{array}$$

Thus the tensor of coherence spaces is obtained using the tensor product $*$ of $\mathbf{3}$ and the linear function space of coherence spaces is similarly obtained from the linear function space $\rhd$ in $\mathbf{3}$.

Consider a second example for a model of linear logic, namely that of hypercoherences [4]. Recall that a hypercoherence X is given by a set $|X|$ (also called the 'web'), and a subset $\Gamma(X)$ of the set of finite non-empty subsets of $|X|$ containing all singletons. This can be encoded as a function $\alpha_X : \mathscr{P}_{\text{fne}}|X| \to \mathbf{3}$, where $\mathscr{P}_{\text{fne}}$ denotes the (finite, non-empty) powerset functor: α_X maps a finite subset a of $|X|$ to i iff a is a singleton, and to $\frown$ iff it is an element of $\Gamma(X)$. Again this representation does capture the $*$-autonomous structure of the category of hypercoherences in terms of the corresponding operations on $\mathbf{3}$.

Clearly negation is given in terms of negation on $\mathbf{3}$. The tensor product of two hypercoherences has $|X| \times |Y|$ as its underlying web. The tensor structure is given by $\Gamma(X \otimes Y)$, which is the set of all non-empty finite subsets E of $|X| \otimes |Y|$ such that $\pi_1(E)$ is in $\Gamma(X)$ and $\pi_2(E)$ is in $\Gamma(Y)$. In our chosen representation, we encode this structure by the function $\mathscr{P}_{\text{fne}}(|X| \times |Y|) \to \mathbf{3}$ that maps a set c to $\alpha_X(\pi_1(c)) * \alpha_Y(\pi_2(c))$, using the tensor product $*$ of $\mathbf{3}$. The linear function space of hypercoherences can be represented exactly the same way, with the linear implication operator, $\rhd$, replacing $*$, the tensor product on $\mathbf{3}$.

Morphisms of hypercoherences also fit well into this representation. Recall that a morphism of hypercoherences is a relation $R : |X| \nrightarrow |Y|$ such that for every finite subset E of R, the following conditions are satisfied:

(i) If $\pi_1(E) \in \Gamma(X)$ then $\pi_2(E) \in \Gamma(Y)$.
(ii) If $\pi_2(E)$ is a singleton then $\pi_1(E)$ is a singleton.

We take the finite powerset functor on the category of sets and relations **Rel** to be defined on morphisms by $a\,(\mathscr{P}_{\text{fne}}R)\,b$, where $a \in \Gamma(X)$, $b \in \Gamma(Y)$, iff there is a (finite) subset E of R with $\pi_1(E) = a$ and $\pi_2(E) = b$. Then R being a morphism of hypercoherences is equivalent to the condition $a\,(\mathscr{P}_{\text{fne}}R)\,b$ implying $\alpha_X(a) \leqslant \alpha_Y(b)$.

These observations suggest a pattern. As objects, we will consider functions $\alpha : FA \to P$, where F is a functor on the category of sets and relations, **Rel**, A is a set and P is a poset which, as a category, is symmetric monoidal closed. As morphisms, we consider relations $R : A \nrightarrow B$ such that for a in FA, b in FB, $a(F\ R)b$ implies $\alpha(a) \leqslant \beta(b)$.

To obtain a tensor-product on this category, we need a natural transformation with components $\sigma_{A,B} : F(A \times B) \to FA \times FB$, satisfying suitable properties to make $(A \times B, * \cdot (\alpha \times \beta) \cdot \sigma_{A,B})$ the tensor of (A, α) and (B, β). Similarly, we obtain a linear function space, and we will proceed to demonstrate that the other connectives of Linear Logic can also be handled in this setting.

This construction turns out to be flexible enough to handle both classical and intuitionistic linear logic. It also provides very good control over whether or not various connectives and their units coincide. It suggests that the reason that multiplicative constants are the same in the cases of coherence spaces and hypercoherences is the fact that they do so in the underlying poset **3**. The additive constants work slightly differently.

3. Poset-valued sets

As suggested in the previous section, we want to build categories over **Rel** by considering as objects sets which take values in a poset—in both the examples in the previous section, that was **3**.

All symbols referring to categorical constructs such as $\times$, $+$, $\otimes$ and $⅋$ are meant to be interpreted in terms of the underlying category **Rel** (rather than **Set**). Recall that products and co-products in **Rel** have the disjoint union as underlying set, whereas tensor and par both are obtained via the cartesian product, which also serves as the set underlying the linear function space (**Rel** being compact closed, and every object being its own dual). To remind ourselves of the compact closure, we will use $(-)^*$ to denote the duality.

The traditional use of these symbols regarding our chosen motivating examples, coherence spaces and hypercoherences, is different—it appeals to the category of sets and functions. However, since the underlying category is really **Rel**, it seems less confusing all in all to adopt a different meaning for the various symbols. This has the advantage that the tensor in **Rel** will give rise to the tensor in our category, etc.

There are two more ingredients from which we will build our category: Firstly, we need an endofunctor on **Rel** to play the role of the diagonal functor (for tensor) and the finite powerset functor in the two motivating examples. Secondly, we will need a poset P to take over the rôle of **3**. Any properties we demand for P will be driven by the kind of structure we desire the resulting category to have.

This approach has an odd tension between *functions* and *relations*. Since we work in the category of sets and relations, one might expect to deal with relations exclusively. However, we will show that allowing relations everywhere does in general not lead to a category. Also, when we view P as a category then we expect its structure (multiplication and linear function space, infima (products) and suprema (coproducts)) to be given by functions, too.

Since we are free to think of functions in the category of sets and relations, we can certainly use **Rel** as the underlying universe. It also is not unusual in category theory to impose restrictions on morphisms. We will refer to relations which we know to be functions as 'functional', and we will use the usual style arrow $\rightarrow$ to denote them, whereas maps that may be relations will be denoted by $\nrightarrow$.

3.1. P_F-sets

Having dealt with the preliminaries, we are now ready to formally introduce the category we are interested in—the main idea was described in Section 2.

Definition 3.1. Let F be an endofunctor on **Rel** and let P be a poset. The category of *P_F-sets*, P_F**Set** is defined as follows:
- An **object** is a map $FA \rightarrow P$, a *P_F-set*.
- A **morphism** $(\alpha\colon FA \rightarrow P) \rightarrow (\beta\colon FB \rightarrow P)$ is given by a relation $R\colon A \nrightarrow B$ such that $x\,(FR)\,y$ implies $\alpha(x) \leqslant \beta(y)$.

The set A, via FA, is considered as taking values in the poset P, hence the name 'poset-valued sets'.

Identities in P_F**Set** are identity relations, and composition is also taken from **Rel**. It is not difficult to check that this defines a category. There is an obvious forgetful functor to the category of sets and relations. To map **Rel** to P_F**Set**, on the other hand, one has to map an object A to a constant function $FA \rightarrow P$. Such a functor will not preserve the categorical structure unless P is a singleton.

This is the reason for restricting ourselves to functions $FA \rightarrow P$: For the identity relation on A to give a morphism $(\alpha\colon FA \nrightarrow P) \rightarrow (\alpha\colon FA \nrightarrow P)$ it has to be the case that if $a\,\alpha\,p$ and $a\,\alpha\,p'$ then $p \leqslant p'$ since $a\,\mathsf{id}_{FA}\,a\,\alpha\,p'$, and thus by symmetry $p = p'$. It would be possible to allow *partial functions* (rather than proper relations) here, but that does seem to lead to rather unfamiliar categories overall.

An equivalent definition for the morphisms is given by the demand that such a morphism R satisfy $\alpha \leqslant \beta \cdot FR$ were $\leqslant$ is a pre-order defined on hom-sets with co-domain P as follows:

For $R, S \in \mathbf{Rel}(A, P)$, we say that $R \leqslant S$ iff for all $x \in A$ and all $p, p' \in P$, $x\,R\,p$ and $x\,S\,p'$ implies $p \leqslant p'$. Hence in a situation where we have $\alpha \leqslant \beta \cdot FR$, that is a morphism R of P_F-sets, we will use 'weakly commuting' diagrams drawn as follows:

$$
\begin{tikzcd}
FA \arrow[rr, "\alpha"] \arrow[d, "FR"', "|" marking] \arrow[dr, phantom, "\leqslant"] & & P \\
FB \arrow[urr, "\beta"'] & &
\end{tikzcd}
$$

This shows how to generalize this construction to categories **C** other than **Rel**: What is needed here is an object P in **C** such that all hom-sets with co-domain P are order-enriched. However, for P_F**Set** to have good categorical structure it turns out that we use so much of the structure of **Rel** (the fact that it is compact closed as well as the fact that A negated is naturally isomorphic to A) that it does not seem worthwhile to give the most general version here.

We shall need a number of useful properties of this pre-order which are given in the following lemma.

Lemma 3.2. (i) *Let* $R: B \nrightarrow A$ *and* $S, S': A \nrightarrow P$. *Then* $S \leqslant S'$ *implies* $S \cdot R \leqslant S' \cdot R$.

(ii) *Let* $\alpha, \beta: A \to P$. *Then* $\alpha \leqslant \beta$ *if and only if* α *is less than or equal to* β *in the pointwise order for functions.*

Proof. The proof is straightforward and we omit it here. □

Having given the basic definition, we can concentrate on the question of when P_F**Set** has good categorical structure.

3.2. Tensor products and symmetric monoidal closure

Let F be an endofunctor on **Rel**. Since **Rel** is compact closed, any natural transformation $F(-) \otimes F(-) \to F(- \otimes -)$ making F monoidal yields a natural transformation going in the opposite direction. This is, in fact, the transformation relevant for our considerations. We say that a natural transformation is *functional* if all its components are functions.

Proposition 3.3. *Let* $(P, \mathsf{i}, *)$ *be a symmetric monoidal poset and let* F *be a monoidal functor, where the monoidal structure is given by a functional natural transformation* $\sigma: F(- \otimes -) \to F(-) \otimes F(-)$. *Then* P_F**Set** *is a symmetric monoidal category. The tensor of* $(\alpha: FA \to P)$ *and* $(\beta: FB \to P)$ *is*

$$F(A \otimes B) \xrightarrow{\sigma_{A,B}} F(A) \otimes F(B) \xrightarrow{\alpha \otimes \beta} P \otimes P \xrightarrow{*} P.$$

The unit for tensor is given by $(\iota_{\mathsf{i}}: F\mathbf{I} \to P)$, *where* ι_{i} *is the composition of the* (*only*) *function* $F\mathbf{I} \to \mathbf{I}$ *and the function* $\mathsf{i}: \mathbf{I} \to P$ *that picks the identity* i *for* $*$ *in* P. *The forgetful functor* $P_F\mathbf{Set} \to \mathbf{Rel}$ *preserves the tensor strictly.*

Proof. The proof is simple; all the structure is inherited from **Rel** and P via the following. Let $R, R': A \nrightarrow P$ and $S, S': B \nrightarrow P$. If $R \leqslant R'$ and $S \leqslant S'$ then $* \cdot R \otimes S \leqslant * \cdot R' \otimes S'$. □

If one then asks for P_F**Set** to be symmetric monoidal closed one really starts to make use of the fact that **Rel** has rather special structure. In some ways, we are spoiled for choice here as for how to express the linear function space since isomorphisms abound in **Rel**.

Here is the basic idea. Assume that the poset P is symmetric monoidal closed as a category. This gives us a tensor product $*$ on P as before as well as its adjoint, say $\triangleright$. Then we can use its structure to make $P_F\mathbf{Set}$ a symmetric monoidal category. There is a question of how to view P's structure as it embeds into $\mathbf{Rel}$. If we view $\triangleright$ as a relation $P \otimes P \to P$, we would define the linear function space of $(\alpha : FA \to P)$ and $(\beta : FB \to P)$ as

$$F(A \otimes B) \xrightarrow{\sigma_{A,B}} F(A) \otimes F(B) \xrightarrow{\alpha \otimes \beta} P \otimes P \xrightarrow{\triangleright} P.$$

This looks nice and tidy, but in order to argue that this makes $P_F\mathbf{Set}$ symmetric monoidal closed, we have to use the fact that $\mathbf{Rel}$ is compact closed and that every object is isomorphic to its dual.

On the other hand, we can try to define the same linear function space as

$$F(A^* \otimes B) \xrightarrow{\sigma_{A^*,B}} F(A^*) \otimes F(B) \cong F(A) \otimes F(B) \to P \otimes P \xrightarrow{\triangleright} P,$$

using the same properties. Finally, one might be tempted to instead view $\triangleright$ as a relation $P^* \otimes P \to P$, but that then requires us to view α as going from $(F(A))^*$ to P^*, ultimately making the same assumptions.

Proposition 3.4. *If P is symmetric monoidal closed and F satisfies the conditions from Proposition* 3.3 *the category $P_F\mathbf{Set}$ is symmetric monoidal closed. The forgetful functor $P_F\mathbf{Set} \to \mathbf{Rel}$ preserves the structure strictly.*

Proof. No matter which formulation one ultimately chooses, this is a simple consequence of $\mathbf{Rel}$ being compact closed and P being symmetric monoidal closed. Again we have a useful fact making the connections for us. Let $R, R' : A \nrightarrow P$ and $S, S' : B \nrightarrow P$. If $R \leqslant R'$ and $S \leqslant S'$ then $\triangleright \cdot R \otimes S \leqslant \triangleright \cdot R' \otimes S'$. $\square$

Note that although the base category $\mathbf{Rel}$ is compact closed, $P_F\mathbf{Set}$ will not be degenerate (in general): all that is required is that P not be compact closed (as a category).

Remark 3.5. Attempts to generalize this to non-compact categories fail for the following reason: The obvious underlying objects for tensor and linear functions space are $A \otimes B$ and $A \multimap B$, respectively. This requires the existence of natural transformations

$$F(- \otimes -) \to F(-) \otimes F(-) \quad \text{and} \quad F(- \multimap -) \to F(-) \multimap F(-).$$

In the general case, the existence of the one does not imply that of the other—but the latter implies the existence of a natural transformation

$$F(-) \otimes F(-) \to F(- \otimes -).$$

Adding a couple of innocent conditions linking these transformations with the symmetric monoidal closed structure on the underlying category to make the proposed adjunction work result in the demand that $F(A \otimes B)$ be isomorphic to $F(A) \otimes F(B)$ which seems to be too restrictive to be useful.

3.3. Negation and $*$-autonomous structure

Assume P has a negation, that is an order-reversing function $(-)^{\perp}: P \to P$. This allows us to define a duality on P_F**Set** via

$$(\alpha\colon FA \to P)^{\perp} := ((-)^{\perp} \cdot \alpha\colon FA \to P).$$

Note that this removes the collapse that occurs in **Rel**—there an object and its dual are isomorphic, whereas here this only occurs if P is a singleton. For morphisms, we make use of the duality on **Rel**—the involution on P ensures that the dual of a morphism of P_F-sets is another such.

Such a negation is not of much interest in isolation, and the desired connection is created by the following proposition.

Proposition 3.6. *If P is $*$-autonomous and F satisfies the conditions from Proposition* 3.3 *then the category P_F***Set** *is $*$-autonomous. The forgetful functor P_F***Set** $\to$ **Rel** *preserves the structure strictly.*

Proof. Again this is a simple consequence of the structure on **Rel** together with the fact that P is $*$-autonomous. $\square$

3.4. Products and co-products

The next question we wish to answer is that of what it takes for P_F**Set** to have at least some limits and co-limits. Let us have a look at co-products first. We have a candidate for the underlying set of $(\alpha\colon FA \to P) + (\beta\colon FB \to P)$, namely $A + B$, and candidates for the embeddings, namely inl and inr from **Rel**. It remains to determine the structure, that is a function $F(A+B) \to P$ that will give us the desired universal property.

For that, we will make use of the fact that there are many 'weakly universal' objects in P_F**Set** in the following sense:

Lemma 3.7. '*Fill-in property*': *Let P be a complete lattice, let $R\colon A \nrightarrow B$ and let $(\alpha\colon FA \to P)$ be a P_F-set. Then there is a P_F-set $\beta\colon FB \to P$ based on B such that*

(i) *The relation R is a morphism $(\alpha\colon FA \to P) \to (\beta\colon FB \to P)$ of P_F-sets, that is the following diagram commutes weakly*:

$$\begin{tikzcd}
FA \arrow[rr, "\alpha"] \arrow[d, "FR"', "|" marking] \arrow[dr, phantom, "\geq"] & & P \\
FB \arrow[urr, "\beta"'] & &
\end{tikzcd}$$

(ii) *If* $S: B \nrightarrow C$ *such that* $S \cdot R$ *is a morphism from* $(\alpha: FA \to P)$ *to* $(\gamma: FC \to P)$ *in* P_F**Set** *then* S *is a morphisms from* $(\beta: FB \to P)$ *to* $(\gamma: FC \to P)$, *that is the weak commutativity of*

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "FR"', "|" marking] \arrow[dr, phantom, "\wedge\!|"] & P \\
FB \arrow[d, "FS"', "|" marking] & \\
FC \arrow[uur, "\gamma"'] &
\end{tikzcd}$$

implies that of

$$\begin{tikzcd}
FB \arrow[r, "\beta"] \arrow[d, "FS"', "|" marking] \arrow[dr, phantom, "\wedge\!|"] & P \\
FC \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

.

(iii) *The function* β *is uniquely defined by* (i) *and* (ii).

Proof. Let $\beta(b) := \bigvee_{a\,(FR)\,b} \alpha(a)$. Obviously, this makes $R: A \nrightarrow B$ a morphism of P_F-sets $(\alpha: FA \to P) \nrightarrow (\beta: FB \to P)$. To see that those two conditions determine β uniquely, use (ii) applied to the identity, which proves that any two functions satisfying (i) and (ii) have to be equal. □

We will apply this to get the universal property required for co-products. Consider the following diagrams:

$$\begin{tikzcd}
FA \arrow[r, "\alpha"] \arrow[d, "F\mathsf{inl}"', "|" marking] \arrow[dr, phantom, "\wedge\!|"] & P \\
F(A+B) \arrow[ur, dashed, "\varphi_\iota"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
FB \arrow[r, "\beta"] \arrow[d, "F\mathsf{inr}"', "+" marking] \arrow[dr, phantom, "\wedge\!\mid"] & P \\
F(A+B) \arrow[ur, dashed, "\varphi_r"'] &
\end{tikzcd}$$

The desired co-product is then given by $(\varphi_l \vee \varphi_r : F(A+B) \to P)$ where the structure is the pointwise join of those two functions. Obviously, products can be obtained via the dual of this process. Also, note that we do not make use finiteness here, so we have in fact the following proposition.

Proposition 3.8. *If P is a complete lattice then any category of P_F-sets has all products and co-products.*

Proof. We will just prove that $(\varphi_l \vee \varphi_r : F(A+B) \to P)$ as constructed above has the necessary universal property for co-products—it is obvious how to extend this to infinite co-products. The proof for products works dually. What we really need to show is that if R and S are morphisms of P_F-sets, then the same is true for $[R,S]$ which we know exists in **Rel** (and thus satisfies the desired equations and uniqueness condition).

Suppose that we have morphisms $R : (\alpha : FA \to P) \nrightarrow (\gamma : FC \to P)$, $S : (\beta : FB \to P) \nrightarrow (\gamma : FC \to P)$, so $\alpha \leqslant \gamma \cdot FR = \gamma \cdot F[R,S] \cdot F$ inl. By the definition of φ_l, this implies $\varphi_l \leqslant \gamma \cdot F[R,S]$, and analogously we can prove that $\varphi_r \leqslant \gamma \cdot F[R,S]$. Hence we obtain $\varphi_l \vee \varphi_r \leqslant \gamma \cdot F[R,S]$.

The empty co-product is the empty set, the structure is the function mapping every element of $F\emptyset$ to the least element of P. If $F\emptyset$ is empty then the structure map is the empty function. $\square$

Note that even if we are only asking for finite products and co-products, P will have to be complete, unless we restrict ourselves to a constructive universe of sorts:

If we only allow *finitary* relations as morphisms, that is ones $R : A \nrightarrow B$ such that for all $a \in A$, the set of all $b \in B$ that are R-related to a is finite, and such that for all $b \in B$, the number of elements of A which are R-related to b is finite as well. If F restricts and co-restricts to the resulting category, then we can use that subcategory of **Rel**, and under those circumstances, the existence of finite joins and finite meets in P is sufficient to obtain finite products and co-products in P_F**Set**, respectively. Or, alternatively, if F is such that for each c in some $F(A+B)$, the number of elements related to c by F inl or F inr is finite, then finite suprema are sufficient to guarantee the existence of finite coproducts, and the dual statement is true for products.

Products and coproducts will in general not coincide unless P is a singleton. An obvious exception to this is choosing F to be the identity functor on **Rel**, in which case binary products coincide with binary coproducts. (However, terminal and initial objects will be different.)

3.5. Comonads, comonoids, and linear exponentials

Since ultimately we want to demonstrate how we can combine all the structures we talked about so far to obtain a model for linear logic, we will also have to concern ourselves with comonads and comonoids which are used to model linear exponentials.

We wish to use the notion given in some detail in [2]; see it for a comparison with other definitions. If a symmetric monoidal category has a monoidal comonad satisfying the conditions given in [2] then we say that it has a *linear exponential comonad*. Assume we have a monoidal monad on **Rel** whose free Eilenberg–Moore algebras carry a commutative monoid structure. This could be, for example, the free commutative monoid, or the finite powerset monad. If we turn around the direction of all the morphisms which form such a structure, we obtain a monoidal comonad $(!, \varepsilon, \delta, m_{\mathbf{I}}, m)$ where all free algebras are commutative comonoids via natural transformations with components $d_A : !A \to !A \otimes !A$ and $e_A : !A \to \mathbf{I}$. We will abuse notation and refer to the desired monoidal comonad on $P_F\mathbf{Set}$ again as $(!, \varepsilon, \delta, m_{\mathbf{I}}, m)$.

Since talking about comonoids in this setting only makes sense if $P_F\mathbf{Set}$ has a tensor product, we will assume for the remainder of this section that P is, in fact, a monoid, and that F satisfies the conditions which ensure that $P_F\mathbf{Set}$ is a symmetric monoidal category as per Proposition 3.3.

The underlying set of $!(\alpha : FA \to P)$ is to be $!A$, but it will be a bit more troublesome to obtain a suitable structure on that set. The idea, however, is simple: Since morphisms are given by relations, we can re-use the comonad on **Rel**, including the comonoidal structure on the algebras, as long as we can ensure that the resulting relations satisfy the condition for morphisms of P_F-sets. Therefore, the problem can be reduced to defining a suitable structure on $!(\alpha : FA \to P)$.

This gives us seven 'universal' inequalities we want to hold, one each for ε, δ, $m_{\mathbf{I}}$, m, d, e, and morphisms of the type $!R$. Given the generality of this approach (after all, F is just an arbitrary endofunctor on **Rel**), we cannot expect a constructive definition of the desired structure $F!A \to P$. It is rather surprising that we can get away without putting any restrictions on so general a situation.

The main idea for tackling this problem is to view $!A$ as a comonoid co-generated by A in some sense. This suggests that three of the seven inequalities mentioned above are more central than the remaining ones.

Definition 3.9. A *!-candidate for* $(\alpha : FA \to P)$ is a function $t : F!A \to P$ that satisfies the following inequalities:

$$\begin{tikzcd}
F!A \arrow[rr, "t"] \arrow[d, "Fe_A"'] \arrow[dr, phantom, "\leq"] & & P \\
F\mathbf{I} \arrow[urr, "\alpha"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F!A \arrow[rr, "t"] \arrow[d, "F\epsilon_A"'] \arrow[dr, phantom, "\leq"] & & P \\
FA \arrow[urr, "\alpha"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
F!A \arrow[rrr, "t"] \arrow[d, "Fd_A"'] & & \arrow[d, phantom, "\rotatebox{90}{$\geq$}"] & P \\
F(!A\otimes!A) \arrow[r, "\sigma_{A,A}"] & F!A \otimes F!A \arrow[rr, "t\otimes t"] & & P \otimes P \arrow[u, "*"']
\end{tikzcd}$$

In other words, a !-candidate t for $(\alpha : FA \to P)$ makes $(t : F!A \to P)$ a comonoid 'co-generated' by $(\alpha : FA \to P)$. Among all the !-candidates there is a canonical one that will yield the desired object, namely the (pointwise) join of all !-candidates—if it exists. To ensure that, we assume from now on that P has all joins, that is a complete lattice.

It should be noted that for certain F and !, this condition can be relaxed to just demanding the existence of finite joins and finite meets in P as long as everything takes place in a 'finitary' category as described in the section about products and coproducts. In that case, which is more in the constructive spirit, it is possible to give a constructive definition for the largest !-candidate (that is without quantifying over all !-candidates).

Let us begin the proof that P_F**Set** has the desired comonad by noting that this will indeed give us another !-candidate, namely the largest one:

Proposition 3.10. *The pointwise join of all* !*-candidates for a* P_F*-set is another such.*

There is one lemma we have to present before we can tackle the theorem we are looking for; it explains how to obtain !-candidates for the tensor-product of two P_F-sets.

Lemma 3.11. *If t and u are* !*-candidates for the* P_F*sets* $(\alpha : FA \to P)$ *and* $(\beta : FB \to P)$ *respectively, then the function obtained via the fill-in property* (*Lemma* 3.7) *from the following diagram is a* !*-candidate for*

$$(\alpha : FA \to P) \otimes (\beta : FB \to P) = (* \cdot t \otimes u \cdot \sigma_{A,B} : A \otimes B \to P).$$

$$\begin{tikzcd}
F(!A\otimes!B) \arrow[r, "\sigma_{A,B}"] \arrow[d, "Fm_{A,B}"'] & F!A \otimes F!B \arrow[r, "t\otimes u"] \arrow[d, phantom, "\rotatebox{90}{$\geq$}"] & P \otimes P \arrow[r, "*"] & P \\
F!(A \otimes B) \arrow[rrru, dashed, "v"'] & & &
\end{tikzcd}$$

Proof. We have to prove that the arrow obtained from the fill-in property, behaves well with respect to e, ε, and d. To do that, we will make extensive use of property (ii) from Lemma 3.7. Basically, what we will do is prove the commutativity of the 'outer' diagram from which the desired inequality will follow as the lower sub-diagram.

The weak commutativity of the following diagrams is fairly straight forward, although the last one uses some coherence properties (the morphism indicated by $\cong$ is given by $\alpha^{-1}_{A,A,B\otimes B} \cdot \mathsf{id}_A \otimes \alpha_{B,A,B} \cdot \mathsf{id}_A \otimes (\gamma_{A,B} \cdot \mathsf{id}_B) \cdot \mathsf{id}_A \otimes \alpha^{-1}_{A,B,B} \cdot \alpha_{A,A,B\otimes B}$):

$$
\begin{tikzcd}
 & F(!A\otimes !B) \arrow[r, "\sigma_{!A,!B}"] \arrow[dl, "Fm_{A,B}"', "\shortmid" marking] \arrow[d, "F(e_A\otimes e_B)", "\shortmid" marking] & F!A\otimes F!B \arrow[rr, "t\otimes u"] \arrow[d, "Fe_A\otimes Fe_B", "\shortmid" marking] \arrow[drr, phantom, "\wedge|"] & & P\otimes P \arrow[d, "*"] \\
F!(A\otimes B) \arrow[dr, "Fe_A"', "\shortmid" marking] & F(\mathbf{I}\otimes\mathbf{I}) \arrow[r, "\sigma_{\mathbf{I},\mathbf{I}}"'] \arrow[d, "F\lambda_{\mathbf{I}}"] & F\mathbf{I}\otimes F\mathbf{I} \arrow[r, "\iota_{¡}\otimes\iota_{¡}"] & P\otimes P \arrow[r, "*"] & P \\
 & F\mathbf{I} \arrow[urrr, "\iota_{¡}"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & F(!A\otimes !B) \arrow[r, "\sigma_{!A,!B}"] \arrow[dl, "Fm_{A,B}"', "\shortmid" marking] \arrow[dd, "F(\epsilon_A\otimes\epsilon_B)", "\shortmid" marking] & F!A\otimes F!B \arrow[r, "t\otimes u"] \arrow[dd, "F\epsilon_A\otimes F\epsilon_B", "\shortmid" marking] \arrow[ddr, phantom, "\wedge|"] & P\otimes P \arrow[dr, "*"] & \\
F!(A\otimes B) \arrow[dr, "F\epsilon_{A\oplus B}"', "\shortmid" marking] & & & & P \\
 & F(A\otimes B) \arrow[r, "\sigma_{A,B}"'] & FA\otimes FB \arrow[r, "\alpha\otimes\beta"'] & P\otimes P \arrow[ur, "*"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(!A\otimes !B) \arrow[rr, "Fm_{A,B}"] \arrow[d, "\sigma_{A,B}"'] \arrow[dr, "F(d_A\otimes d_B)"] & & F!(A\otimes B) \arrow[dr, "Fd_{A\otimes B}"] & \\
F!A\otimes F!B \arrow[dr, "Fd_A\otimes Fd_B"] \arrow[dddd, "t\otimes u"'] & F((!A\otimes !A)\otimes(!B\otimes !B)) \arrow[d, "\sigma_{!A\otimes !A,!B\otimes !B}"] & \cong F((!A\otimes !B)^2) \arrow[r, "F(m_{A,B}", "\otimes m_{A,B})"'] \arrow[d, "\sigma_{!A\otimes !B,!A\otimes !B}"] & F(!(A\otimes B)^2) \arrow[d, "\sigma_{!(A\otimes B),!(A\otimes B)}"] \\
& F(!A\otimes !A)\otimes F(!B\otimes !B) \arrow[d, "\sigma_{!A,!A}\otimes\sigma_{!B,!B}"] & (F(!A\otimes !B))^2 \arrow[r, "Fm_{A,B}", "\otimes Fm_{A,B}"'] \arrow[d, "\sigma^2_{!A,!B}"] & (F!(A\otimes B))^2 \arrow[ddd, "v\otimes v"] \\
& (F!A\otimes F!A)\otimes(F!B\otimes F!B) \arrow[d, "(t\otimes t)\otimes(u\otimes u)"] & \cong(F!A\otimes F!B)^2 \arrow[d, "(t\otimes u)^2"] \arrow[r, phantom, "\leq"] & \\
{} \arrow[r, phantom, "\leq"] & (P\otimes P)\otimes(P\otimes P) \arrow[d, "*\otimes *"] & =(P\otimes P)^2 \arrow[d, "*\otimes *"] & \\
P\otimes P \arrow[dr, "*"] & P\otimes P \arrow[d, "*"] & P\otimes P \arrow[dl, "*"] & P\otimes P \arrow[dll, "*"] \\
& P & &
\end{tikzcd}
$$

□

Proposition 3.12. *If P is symmetric monoidal and a complete lattice then any linear exponential comonad on* **Rel** *gives a linear exponential comonad on P_F***Set**.

Proof. All that remains to be shown is that the join of all !-candidates for any P_F-set $(\alpha\colon FA \to P)$ satisfies the seven desired inequalities. The ones for e, ε and d are dealt with in Proposition 3.10, which leaves us with four inequalities to take care of.

Let us start with $m_{\mathbf{I}}$. To prove that this relation is a morphism of P_F-sets we have to prove that ι_{i} is less than or equal to $Fm_{\mathbf{I}}$ followed by the largest !-candidate for $(\iota_{\mathsf{i}} : F\mathbf{I} \to P)$. Let ι' be the function defined via the fill-in property (Lemma 3.7) in the following diagram:

$$\begin{tikzcd}
F\mathbf{I} \arrow[rr, "\iota_{\mathsf{i}}"] \arrow[d, "m_{\mathbf{I}}"', "|" marking] \arrow[dr, phantom, "\leqslant"] & & P \\
F!\mathbf{I} \arrow[urr, dashed, "\iota'"'] & &
\end{tikzcd}$$

We can prove that ι' is a !-candidate and therefore we obtain $\iota_{\mathsf{i}} \leqslant \iota' \cdot Fm_{\mathbf{I}}$, and by inserting the largest !-candidate for $(\iota_{\mathsf{i}} : F\mathbf{I} \to P)$ we get the desired inequality.

To do so, we keep making use of Part (ii) of Lemma 3.7—in fact, we will only state the commutativity of the 'outer' diagram since the conclusion is obvious from there—the same method was employed in the proof of Lemma 3.11. To show that ι' is indeed a !-candidate, the commutativity of the following diagrams is thus sufficient:

$$\begin{tikzcd}
F\mathbf{I} \arrow[dd, "\mathsf{id}_{F\mathbf{I}}"'] \arrow[dr, "Fm_{\mathbf{I}}"', "|" marking] \arrow[drr, "\iota_i"] & & \\
& F!\mathbf{I} \arrow[dl, "Fe_{\mathbf{I}}"', "|" marking] \arrow[r, phantom, "\leqslant"] & P \\
F\mathbf{I} \arrow[urr, "\iota_{\mathsf{i}}"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
F\mathbf{I} \arrow[dd, "\mathsf{id}_{F\mathbf{I}}"'] \arrow[dr, "Fm_{\mathbf{I}}"', "|" marking] \arrow[drr, "\iota_i"] & & \\
& F!\mathbf{I} \arrow[dl, "\epsilon_{\mathbf{I}}"', "|" marking] \arrow[r, phantom, "\leqslant"] & P \\
F\mathbf{I} \arrow[urr, "\iota_{\mathsf{i}}"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
& F\mathbf{I} \arrow[dl, "Fm_{\mathbf{I}}"', "|" marking] \arrow[d, "F\lambda^{-1}"] \arrow[drrr, "\iota_i"] \arrow[r, phantom, "\leqslant"] & & & \\
F!\mathbf{I} \arrow[dr, "Fd_{\mathbf{I}}"', "|" marking] & F(\mathbf{I} \otimes \mathbf{I}) \arrow[r, "\sigma_{\mathbf{I},\mathbf{I}}"] \arrow[d, "F(m_{\mathbf{I}} \otimes m_{\mathbf{I}})", "|" marking] & F\mathbf{I} \otimes F\mathbf{I} \arrow[r, "\iota_{\mathsf{i}} \otimes \iota_{\mathsf{i}}"] \arrow[d, "Fm_{\mathbf{I}} \otimes Fm_{\mathbf{I}}", "|" marking] \arrow[r, phantom, shift right=6, "\leqslant"] & P \otimes P \arrow[r, "*"'] & P \\
& F(!\mathbf{I} \otimes !\mathbf{I}) \arrow[r, "\sigma_{!\mathbf{I},!\mathbf{I}}"'] & F!\mathbf{I} \otimes F!\mathbf{I} \arrow[r, "\iota' \otimes \iota'"'] & P \otimes P \arrow[ur, "*"'] &
\end{tikzcd}$$

All sub-diagrams above commute (weakly) for fairly obvious reasons.

The natural transformation m is being taken care of in a Lemma 3.11, so we can move on to δ. Again we will make use of Lemma 3.7. Let $t : F!A \to P$ be a !-candidate for $(\alpha : FA \to P)$, and let $u : F!!A \to P$ be the fill-in from the following diagram:

$$\begin{tikzcd}
F!A \arrow[rr, "t"] \arrow[d, "F\delta_A"'] \arrow[dr, phantom, "\rotatebox{90}{$\geq$}"] & & P \\
F!!A \arrow[urr, dashed, "u"'] & &
\end{tikzcd}$$

We prove (with similar techniques as employed above) that u is a !-candidate for $(t : F!A \to P)$. Applied to $(\alpha_! : F!A \to P)$ (where $\alpha_!$ denotes the largest !-candidate for $(\alpha : FA \to P)$), this then results in a !-candidate u for the P_F-set $(\alpha_! : F!A \to P)$ with $a_! \leqslant u \cdot F\delta_A$, which implies the desired inequality which has the largest !-candidate for $(\alpha_! : F!A \to P)$ in the place of u.

To see why two of the following diagrams commute weakly, note that δ_A is a morphism of free !-co-algebras (in **Rel**) $(!A, \delta) \nrightarrow (!!A, \delta_{!A})$ and therefore a morphism of comonoids.

$$\begin{tikzcd}
& F!A \arrow[r, "t"] \arrow[dl, "F\delta_A"'] \arrow[dd, "Fe_A"] \arrow[dr, phantom, "\rotatebox{90}{$\geq$}"] & P \\
F!!A \arrow[dr, "Fe_{!A}"'] & & \\
& F\mathbf{I} \arrow[uur, "u"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& F!A \arrow[r, "t"] \arrow[dl, "F\delta_A"'] \arrow[dd, "\mathsf{id}_A"] \arrow[dr, phantom, "\rotatebox{90}{$\geq$}"] & P \\
F!!A \arrow[dr, "F\delta_{!A}"'] & & \\
& F!A \arrow[uur, "\alpha_!"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & F!A \arrow[dl, "F\delta_A"'] \arrow[d, "Fd_A"] \arrow[rrrd, "t"] \arrow[drr, phantom, "\geq"] & & & \\
F!!A \arrow[dr, "Fd_{!A}"'] & F(!A\otimes !A) \arrow[r, "\sigma_{!A,!A}"] \arrow[d, "F(\delta_A\otimes\delta_A)"] & F!A\otimes F!A \arrow[r, "t\otimes t"] \arrow[d, "F\delta_A\otimes F\delta_A"] \arrow[dr, phantom, "\geq"] & P\otimes P \arrow[r, "*"'] & P \\
 & F(!!A\otimes !!A) \arrow[r, "\sigma_{!!A,!!A}"'] & F!!A\otimes F!!A \arrow[r, "u\otimes u"'] & P\otimes P \arrow[ur, "*"] &
\end{tikzcd}
$$

This leaves us with one last thing to show, namely that $!R$ is a morphism if we impose the structure on $!(\alpha\colon FA\to P)$ that we propose. For that, it is sufficient to prove that if t is a !-candidate for $(\alpha\colon FA\to P)$ then the map obtained from the following diagram via the fill-in property is a !-candidate for $(\beta\colon FB\to P)$:

$$
\begin{tikzcd}
F!A \arrow[r, "t"] \arrow[d, "F!R"'] \arrow[dr, phantom, "\geq"] & P \\
F!B \arrow[ur, dashed, "u"'] &
\end{tikzcd}
$$

Again that leaves us with three diagrams to prove weakly commutative. We will use the fact that, as a morphism of free co-algebras, $!R$ is a morphism of comonoids.

$$
\begin{tikzcd}
 & F!A \arrow[r, "t"] \arrow[dl, "FR"'] \arrow[dd, "Fe_A"'] \arrow[dr, phantom, "\geq"] & P \\
F!B \arrow[dr, "Fe_B"'] & & \\
 & F\mathbf{I} \arrow[uur, "!"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& F!A \arrow[dl, "F!R"', "|" marking] \arrow[d, "F\epsilon_A", "|" marking] \arrow[dr, "t"] \arrow[d, phantom, shift left=8ex, "\leq"] & \\
F!B \arrow[dr, "F\epsilon_B"', "|" marking] & FA \arrow[r, "\alpha"] \arrow[d, "FR", "|" marking] \arrow[d, phantom, shift left=8ex, "\leq"] & P \\
& FB \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& F!A \arrow[dl, "FR"', "|" marking] \arrow[d, "Fd_A", "|" marking] \arrow[drrr, "t"] \arrow[dr, phantom, "\leq"] & & & \\
F!B \arrow[dr, "Fd_B"', "|" marking] & F(!A \otimes !A) \arrow[r, "\sigma_{!A,!A}"] \arrow[d, "F(!R \otimes !R)", "|" marking] & F!A \otimes F!A \arrow[r, "t \otimes t"] \arrow[d, "F!R \otimes F!R", "|" marking] & P \otimes P \arrow[r, "*"'] \arrow[d, phantom, "\leq"] & P \\
& F(!B \otimes !B) \arrow[r, "\sigma_{!B,!B}"'] & F!B \otimes F!B \arrow[r, "u \otimes u"'] & P \otimes P \arrow[ur, "*"'] &
\end{tikzcd}
$$

□

3.6. Putting it all together—models for linear logic

We use here the following terminology.

Definition 3.13. A *categorical model of intuitionistic linear logic* consists of a category which
- is symmetric monoidal closed;
- has finite products;
- is equipped with a linear exponential comonad.

To model the classical calculus we additionally require a strong duality. So a *model for classical linear logic* consists of a category which
- is $*$-autonomous;
- has finite products and (so) finite coproducts;
- is equipped with a linear exponential comonad and (so) a linear exponential monad.

Summarizing all the results we have the following.

Theorem 3.14. *If P is $*$-autonomous and a complete lattice and F satisfies the conditions from Proposition* 3.3 *then P_F***Set** *is a model for classical linear logic. In addition, it has arbitrary products and coproducts.*

If P is symmetric monoidal closed and a complete lattice and F satisfies the conditions from Proposition 3.3 *then P_F***Set** *is a model for intuitionistic linear logic. In addition, it has arbitrary products and coproducts.*

Unlike **Rel**, our category P_F**Set** will in general not suffer collapse and thus result in a degenerate model. P_F**Set** will be compact closed only if P is. The units for tensor and par will be different as long as they are in P. It will not have biproducts unless P is a singleton, or F is trivial. The initial and terminal objects coincide if P is a singleton or if F maps the empty set to itself.

Note that if the general aim is to model larger fragments of intuitionistic linear logic, the construction presented here is capable of coping with that as well. For suggestions of a suitable structure on the underlying poset see [3].

4. Examples for poset-valued sets

4.1. Phase spaces and completeness

Let us consider what categories we obtain if we take away some of the degrees of freedom we have with P_F-sets. First of all, there is the possibility of F being a constant functor, mapping everything to a one element set. In that case the structure map α picks out an element of P, and the operations are as defined on P—apart from !, obviously, which is a derived operation, and in that case Proposition 3.12 tells us how to define a modality for P.

One particular example of this would be that of phase spaces: Recall that a phase space M consists of a commutative monoid and a subset $\perp$ of M. For subsets X of M, negation is defined via

$$X^{\perp} := \{m \in M \mid \forall n \in X.mn \in \perp\}.$$

In that case let $P := \{X \subseteq M \mid X = X^{\perp\perp}\}$—Girard [5] calls these sets 'facts'. They form a complete lattice with respect to $\subseteq$ since facts are closed under arbitrary intersection. The tensor is given via $X \otimes Y := \{mn \mid m \in X, n \in Y\}^{\perp\perp}$. We will not repeat here how the other connectives are defined. It seems, however, interesting to point out how Girard's linear exponentials compare to the ones obtained from our method. There are, in fact, more than just one interpretation for ! to be found in the literature, but we are interested in the one described in [6], where $!X = (X \cap I)^{\perp\perp}$. Here $I = 1^{\perp\perp}$ is the unit for $\otimes$, where 1 is the unit of the monoid M.

Our definition, on the other hand, makes ! the largest function $t : P \to P$ satisfying $t(X) \leqslant I$, $t(X) \leqslant X$ and $t(X) \leqslant t(X) \otimes t(X)$. (Note that due to the nature of F, it does not matter with which definition of ! on **Rel** we start.)

Obviously, Girard's definition satisfies the first two of those inequalities, but not the third. However, if a formula ϕ is provable in Linear Logic, then 1 is an element of its interpretation (no matter which phase space we are looking at), and for facts X containing 1, the desired inequality is true. Hence, whereas we make sure that $!X \leqslant !X \otimes !X$ is true for all elements of P, Girard restricts himself to those elements which can possibly be interpretations of formulae. Thus he obtains a nice explicit definition for ! which, however, has a bit of an ad hoc nature. We obtain the somewhat less appealing formula

$$!X := \bigcup_{n \in \mathbb{N}} \{m_1 \cdots m_n \mid m_1, \ldots, m_n \in X\}^{\perp\perp}$$

The above discussion also answers the question whether our semantics is complete, since phase spaces are known to be.

Another model of a similar nature which we can view as a category of P_F-sets are Mitchell's IE-quantales [11]—they also fit the case where F is a constant functor of the kind described above.

The other possibility for obtaining a degenerate model is to have P as a singleton. In the case where F is the identity, this will give us the category of sets and relations with the usual connectives (and collapses, of course), and with whatever definition of ! we start.

4.2. Coherence spaces and hypercoherences revisited

As we have seen in Section 2, coherence spaces can be encoded as P_F-sets, by choosing P to be $\mathbf{3}$, and F to be the diagonal functor Δ for the tensor product on **Rel**. If we denote by G the functor we obtain from mapping a coherent space to the corresponding $\mathbf{3}_\Delta$-set, we get the following result:

Proposition 4.1. *The functor G is full and faithful and preserves the monoidal closed structure on the category of coherence spaces as well as products and co-products.*

The image of the embedding G consists of all $\mathbf{3}_\Delta$-sets whose structure map takes the value 1 exactly on the diagonal.

As some minor calculations show, the modalities we obtain for $\mathbf{3}_\Delta$-sets if we take ! to be the finite powerset functor on **Rel** are similar to the ones described in [6]. If we use the finite multiset functor instead, we obtain linear exponentials similar to the ones in [5]. However, the construction we introduced in the last section has one major difference: The underlying set for $!(A, \alpha)$ is always $!A$. In other words α cannot be used to determine a subset of $!A$ as the underlying set instead, the way it is done in the usual version of the modalities for coherence spaces. However, we can provide for that to some degree:

Under the assumptions of Theorem 3.14, let there be a subset $!_\alpha A$ of $!A$ for every (A, α). Further assume that the restrictions and co-restrictions of the linear structure on **Rel** to these subsets, and the equations between those are still valid. If F preserves inclusion of relations, then this results in another linear category. We will not go into the details of the proof here—basically, it consists on showing that the notion of !-candidate can be adapted to those circumstances.

The other example we looked at in Section 2 was that of hypercoherences. To encode those, we chose $P=\mathbf{3}$ and $F=\mathscr{P}_{\text{fne}}$, the finite powerset functor on **Rel**.

By defining the resulting functor on morphisms the same way we defined G above, we again obtain a full and faithful embedding that preserves the monoidal closed structure. Its image in the category of $\mathbf{3}_{\mathscr{P}_{\text{fine}}}$-sets is given by those (X,α_X) which take the value 1 if and only if the argument is a singleton. However, the image of that embedding is not closed under products and co-products (the property of the structure taking the value 1 exactly on singletons is not preserved under these constructions). The modalities, however, can be expressed in the category of $\mathbf{3}_{\mathscr{P}_{\text{fine}}}$-sets as described for coherence spaces.

4.3. Lamarche's $Q^n_{\mathscr{A}}$-coherences

Lamarche's attempt to find a generalization of models for linear logic such as hypercoherences and coherence spaces led him to the introduction of what he calls $Q^n_{\mathscr{A}}$-coherences [10], where Q is a $*$-autonomous poset. These can be viewed as $Q_{\mathscr{P}_n}$-sets, where $\mathscr{P}_n$ is a powerset functor which only considers sets up to cardinality n. (The additional parameter $\mathscr{A}$ specifies a subset of Q which is used to 'mark' the singleton sets.) Any category of $Q^n_{\mathscr{A}}$-coherences can be embedded into the category of $Q_{\mathscr{P}_n}$-sets, and this embedding preserves all multiplicative and additive connectives as well as being full and faithful.

4.4. New models

Let us assume that we want to build a model for (classical) linear logic such that the units for tensor and par do not coincide. Other than that we would like to keep it simple, say in the spirit of coherence spaces. The framework developed in this paper tells us that this can be achieved as long as (unlike in the three-element poset **3**) the two constants i (the unit for multiplication) and $\perp = \mathsf{i}^\perp$ have different interpretations in the underlying poset model. This suggests using a binary relation with values in **4** given below: (Fig. 2)

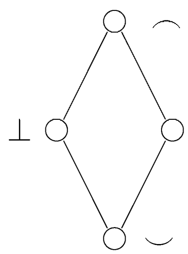

Fig. 2. The poset **4**.

Objects in our category will be functions $\alpha : A \otimes A \to \mathbf{4}$, where A is an object in **Rel**. Morphisms are relations $R : A \nrightarrow B$ such that $a\,R\,b$ and $a'\,R\,b'$ imply that b and b' are at least as related as a and a', in other words, $\alpha\langle a, a'\rangle \leqslant \beta\langle b, b'\rangle$.

The poset **4** is equipped with the obvious negation given in the table below, so the negation of $\alpha : A \otimes A \to \mathbf{4}$ assigns the 'opposite' value to a pair $\langle a, a'\rangle$. Similar to our treatment of coherence spaces, we use the following abbreviations:

$$
\begin{array}{lll}
[= a \smile a' & \text{iff} & \alpha\langle a, a'\rangle = \smile, \\
a \sim a' & \text{iff} & \alpha\langle a, a'\rangle = \mathsf{i}, \\
a \# a' & \text{iff} & \alpha\langle a, a'\rangle = \perp, \\
a \frown a' & \text{iff} & \alpha\langle a, a'\rangle = \frown .
\end{array}
$$

The relation on the tensor of two objects is given by[4]

$$
\begin{array}{lll}
\langle a, b\rangle \smile \langle a', b'\rangle & \text{iff} & a \smile a' \text{ or } b \smile b', \\
\langle a, b\rangle \sim \langle a', b'\rangle & \text{iff} & a \sim a' \text{ and } b \sim b', \\
\langle a, b\rangle \# \langle a', b'\rangle & \text{iff} & (a \sim a' \text{ and } b \# b') \text{ or } (a \# a' \text{ and } b \sim b'), \\
\langle a, b\rangle \frown \langle a', b'\rangle & \text{iff} & \text{none of the above.}
\end{array}
$$

This is justified by the $*$-autonomous structure on **4** given by

$*$	$\frown$	$\perp$	i	$\smile$
$\frown$	$\frown$	$\frown$	$\frown$	$\smile$
$\perp$	$\frown$	$\frown$	$\perp$	$\smile$
i	$\frown$	$\perp$	i	$\smile$
$\smile$	$\smile$	$\smile$	$\smile$	$\smile$

	$(-)^{\perp}$
$\frown$	$\smile$
$\perp$	i
i	$\perp$
$\smile$	$\frown$

Note that it does not make sense to read $\alpha\langle a, a'\rangle = \mathsf{i}$ as $a = a'$, which is what happens for coherence spaces, since the rules for tensor do not fit with that interpretation. The unit for tensor is the singleton set whose one element is $\sim$-related with itself, whereas the unit for par is the singleton set whose one element is #-related with itself.

Product and co-product are formed in a way similar to coherence spaces: The underlying set of the product of $\alpha : A \to \mathbf{4}$ and $\beta : B \to \mathbf{4}$ is $A \times B$, the disjoint union of A and B. Then

$$
(\alpha \times \beta)\langle c, c'\rangle = \begin{cases} \alpha\langle a, a'\rangle & c = \mathsf{inl}(a), c' = \mathsf{inl}(a'), \\ \beta\langle b, b'\rangle & c = \mathsf{inl}(b), c' = \mathsf{inl}(b'), \\ \frown & \text{otherwise.} \end{cases}
$$

Co-products work similarly, only in the last case, the value of $(\alpha + \beta)$ is $\smile$. Note that the unit for product coincides with that for co-product since the underlying set is empty

[4] At first sight the definition of tensor looks rather odd. Here is an example where one would expect this kind of behaviour: Assume that the elements of A are products of intervals in some $\mathbb{R}^{\mathbb{N}}$. We might then be interested in whether two such intervals have empty intersection (are $\smile$-related), are equal (are $\sim$-related), have a union which is a product of intervals, but are not equal (are #-related), or for which none of these hold (are $\frown$-related). Then there is an obvious way to define the tensor of two such objects: To find how $\langle a, b\rangle$ is related with $\langle a', b'\rangle$, study the products $a \times b$ and $a' \times b'$. This evaluates to the tensor given above.

and the tensor product of the empty set with itself is again the empty set, allowing only one choice for a function to **4**. If we wanted to separate them, we would have to use another functor (rather than $\otimes$), one who maps the empty set to a non-empty set. The finite powerset functor is an option here, giving a model similar to hypercoherences. We will not spell out a description of the exponentials for these specific models here.

5. Conclusions and future work

The method for constructing models of linear logic described above can be seen as a manual for a 'do-it-yourself' approach to constructing models for (fragments of) linear logic. By choosing suitable building blocks the resulting model can be tailored to specific requirements. Such models do not have to be more complicated than hypercoherences or coherence spaces. Section 4.4 gives an example for how one might employ the techniques described here to obtain a model with specified properties.

There is the question of whether other models which are mainly 'one-sided' (in that the dual is given internally rather than by *a priori* making every object a pair, so that the dual can be taken by switching components) can be considered within this framework, like event structures, join-complete semilattices, and other domain theoretic models. One possible obstacle is that in the domain theoretic case, morphisms will in general not be relations of some kind, but there is plenty of work in that field suggesting that they can be understood as such, as long as the domains considered satisfy some 'finiteness' or 'approximability' condition. Similar considerations apply to event structures. Our techniques are not designed to cope with what we call 'two-sided' models such as games, Chu-spaces, dialectica and others.

Another thrust for future work would be to find a categorical setting in which this kind of construction could live rather than restrict everything to the category of sets and relations. Our proof of symmetric monoidal closure does make use of the fact that the base category is compact closed, and we currently cannot see an obvious way of generalizing this method to, say, $*$-autonomous or symmetric monoidal closed categories.

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