1. Formal stacks

We follow Conrad here in [1]. Let $\mathfrak{X}$ be a locally Noetherian algebraic stack and $X_0 \subseteq |\mathfrak{X}|$ a closed subset. We assume that $\mathfrak{X}$ is smooth and proper over an algebraically closed field of characteristic 0.

Remark 1.1. The notation $|\mathfrak{X}|$ corresponds to the topological space associated to the algebraic stack $\mathfrak{X}$. This is what we use for the Zariski topology.See [4, tag 04XE] in the stacks project, or look up "points of algebraic stacks there".

Now we consider the system of all coherent ideals $\mathcal{I}_\alpha$ with zero locus X_0. In the Deligne-Mumford situation perhaps we can just use $\{I^n\}$ assume that X_0 is irreducible. In any case for a coherent sheaf $F \in \mathrm{Coh}(\mathfrak{X})$ we define

$$\widehat{F} := \lim_{\leftarrow, \alpha} \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\alpha F$$

Taking $F = \mathcal{O}_{\mathfrak{X}}$ we obtain

$$\mathcal{O}_{\widehat{\mathfrak{X}}} := \widehat{\mathcal{O}_{\mathfrak{X}}}$$

This gives us a ringed topos with structure sheaf $\mathcal{O}_{\widehat{\mathfrak{X}}}$, the underlying site is the lis-etale site. In our situation as we are working with a Noetherian DM stack, Conrad notes we can use the etale site of $\mathfrak{X}$. We from now on denote $\widehat{\mathfrak{X}}$ as the completion of $\mathfrak{X}$ along X_0.

Definition 1.2 (The first Lefschetz codition)**.** Here we follow [2, Chapter IV]. Let $\mathfrak{X}$ be a DM stack.(We could do Artin stack as well, but we probably need locally Noetherian to apply Conrad) Let $\mathcal{Y} \hookrightarrow \mathfrak{X}$ be a closed substack. Let $\widehat{\mathfrak{X}}$ be the completion of $\mathfrak{X}$ along the closed set $|\mathfrak{Y} \subseteq |$. (By [4, tag 0H20] this is a closed subset). We say $(\mathfrak{X}, \mathcal{Y})$ satisfies the Lefscetz condition, written $\mathrm{Lef}(\mathfrak{X}, \mathcal{Y})$ if for every open substack $\mathcal{U}$ of $\mathfrak{X}$ with $\mathcal{Y} \to \mathcal{U} \to \mathfrak{X}$ and every locally free sheaf $\mathcal{E}$ on $\mathcal{U}$ there is an open substack $\mathcal{U}'$ with $\mathcal{Y} \to \mathcal{U}' \to \mathcal{U}$ such that the natural map

$$H^0(\mathcal{U}', \mathcal{E}) \to H^0(\widehat{\mathfrak{X}}, \widehat{\mathcal{E}})$$

is an isomorphism.

Definition 1.3 (The effective Lefschetz condition)**.** If $\mathfrak{X}$ and $\mathcal{Y}$ are as in definition 1.2. We say that $(\mathfrak{X}, \mathcal{Y})$ satisfy the effective Lefschetz condition written $\mathrm{Leff}(\mathfrak{X}, \mathcal{Y})$ if $\mathrm{Lef}(\mathfrak{X}, \mathcal{Y})$ and in addition for every locally free sheaf $\mathcal{E}$ on $\widehat{\mathfrak{X}}$ there is an open substack $\mathcal{Y} \to \mathcal{U} \to \mathfrak{X}$ and a locally free sheaf E on $\mathcal{U}$ with $\widehat{E} \cong \mathcal{E}$.

Remark 1.4. Here we use section 2 of [1, Section 2] to pull-back sections.

Definition 1.5. Let $\mathfrak{X}$ be a dm stack. The cohomological dimension of $\mathfrak{X}$ (denoted by $\mathrm{cd}(\mathfrak{X})$)is the smallest element of $\{0, 1, 2, \ldots\} \cup \{\infty\}$ such that if $i > \mathrm{cd}(\mathfrak{X})$ then $H^i(\mathfrak{X}, F) = 0$ for all coherent sheaves F on $\mathfrak{X}$.

2. The Basics

We want to prove the following:

Proposition 2.1. *Let $\mathfrak{X}$ be a smooth tame DM stack of dimension n with projective coarse moduli space. Let $\mathcal{Y} \hookrightarrow \mathfrak{X}$ be a closed substack. Suppose that $\mathrm{cd}(\mathfrak{X} - \mathcal{Y}) < n - 1$. Then $\mathrm{Lef}(\mathfrak{X}, \mathcal{Y})$ is true and every effective Cartier divisor on $\mathfrak{X}$ meets $\mathcal{Y}$.*

Proof. Let $\mathcal{Y} \to \mathcal{U}$ and let E be a locally free sheaf on $\mathcal{U}$. By [3, Theorem 1] the canonical sheaf $omega := \omega_{\mathfrak{X}}$ is a dualizing sheaf. Now consider the sheaf $F = \mathrm{hom}_{\mathcal{O}_{\mathcal{U}}}(E, \omega|_{\mathcal{U}})$. ☐

References

[1] Brian Conrad. Formal gaga for artin stacks. *preprint*, 2005.

[2] Robin Hartshorne. *Ample subvarieties of algebraic varieties*, volume Vol. 156 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1970. Notes written in collaboration with C. Musili.

[3] Denis Levchenko. Serre duality for tame Deligne-Mumford stacks. *Res. Math. Sci.*, 9(4):Paper No. 67, 5, 2022.

[4] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2024.