



Percolation

Jonathan Marriott

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Supervised by Dr Edward Crane  
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# Percolation

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## 1 Introduction

A project on Percolation

## 2 The Percolation Model

### 2.1 Definitions and theorems

We start with some basic definitions for Percolation on cubic lattices, specifically bond percolation where we consider the edges on the graph to be either open or closed.

**Definition 2.1.**  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  and  $\mathbb{Z}^d = \{(x_1, x_2, \dots, x_d) : x_i \in \mathbb{Z}\}$

**Definition 2.2.** For  $x, y \in \mathbb{Z}^d$ , define the distance from  $x$  to  $y$ , denoted  $\delta(x, y)$ , by

$$\delta(x, y) := \sum_{i=1}^d |x_i - y_i|$$

**Definition 2.3** (d-dimensional cubic lattice). We construct the lattice with vertices  $\mathbb{Z}^d$  and edges between vertices when the distance between them is one.

$$E(\mathbb{Z}^d) = \{u, v \in V(\mathbb{Z}^d) : \delta(u, v) = 1\}$$

We will often refer to this lattice as simply  $\mathbb{Z}^d$  without specifying the edge set.

### 2.2 The Model

We now take some  $p \in [0, 1]$  which will be our parameter which specifies the probability a given edge is open. Setting  $q = 1 - p$  we say an edge is open with probability  $p$  and closed with probability  $q$ . We can think of the open and closed edges defining a subgraph of  $\mathbb{Z}^d$  where only edges set to open are retained.

### 3 Existence of a critical value

#### 3.1 Existence of a critical value on $\mathbb{Z}$

Trivially the critical value is  $p = 1$ . Consider the event  $X_n = \{\text{There is an open self-avoiding path of length } n \text{ starting at the origin}\}$ . Then  $X_n \supseteq X_{n+1}$  and so

$$\lim_{n \rightarrow \infty} \mathbb{P}(X_n) = \theta(p)$$

And since  $\mathbb{P}(X_n) = 2p^n$ , as the path can go left or right from the origin. We have for all  $p < 1$ ,  $\theta(p) = 0$ . Thus  $\theta(p) > 0$  iff  $p = 1$ .

#### 3.2 Existence of a critical value on $\mathbb{Z}^2$

We show the existence of the critical value in this case by bounding it from above and below.

**Theorem 3.1.** *If  $p < 1/3$ ,  $\theta(p) = 0$ .*

*Proof.* Let  $X_n = \{\text{There is an open self-avoiding path of length } n \text{ starting at the origin}\}$  as in Section 3.1. Then the probability for a path of length  $n$  to be open on every edge is  $p^n$ . The number of paths of length  $n$  from the origin is at most  $4(3^{n-1})$  since there are 4 edges to choose from at the origin, then for each next step in the path there are at most 3 edges we can pick as the path is self-avoiding. Hence we get  $\mathbb{P}(X_n) \leq 4(3^{n-1})p^n$ . Then we take the limit since  $\lim_{n \rightarrow \infty} \mathbb{P}(X_n) = \theta(p)$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P}(X_n) &\leq \lim_{n \rightarrow \infty} 4(3^{n-1})p^n \\ &\leq 4 \cdot 3^{-1} \lim_{n \rightarrow \infty} (3p)^n \end{aligned}$$

Since  $p < 1/3$  we have  $\lim_{n \rightarrow \infty} \mathbb{P}(X_n) = 0$  □

**Theorem 3.2.** *For  $p$  close to 1, we have  $\theta(p) > 1$*

*Proof.* We introduce the dual graph  $(\mathbb{Z}^2)^*$  which has vertices in  $(\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2}))$ , as edges as you would expect between vertices at distance 1. Then we can see there is a clear correspondence between the edges of  $\mathbb{Z}^2$  and its dual, since each edge in the dual intersects a unique edge in the original graph. Thus we can create a mapping from the open and closed edges of  $\mathbb{Z}^2$  to the dual graph, where the edge in the dual is open iff the intersecting edge in  $\mathbb{Z}^2$  is open.

Then we notice that if there exists a cycle of closed edges in the dual graph enclosing the origin then the size of the open cluster at the origin is finite.

**Lemma 3.3.**  $|C| < \infty \iff \exists \text{ a cycle of closed edges in } (\mathbb{Z}^2)^* \text{ enclosing the origin}$

*Proof.* todo or leave out?



### 3.3 Existence of a critical value on $\mathbb{Z}^d$