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Percolation

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1 Introduction

A project on Percolation

2 The Percolation Model

2.1 Initial Definitions

We start with some basic definitions for Percolation on cubic lattices, specifically bond percolation where we consider the edges on the graph to be either open or closed.

Definition 2.1. $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $\mathbb{Z}^d = \{(x_1, x_2, ..., x_d) : x_i \in \mathbb{Z}\}$

Definition 2.2. For $x, y \in \mathbb{Z}^d$, define the distance from x to y, denoted $\delta(x, y)$, by

$$\delta(x, y) := \sum_{i=1}^{d} |x - y|$$

Definition 2.3 (d-dimensional cubic lattice)**.** We construct the lattice with vertices in $\mathbb{Z}^d$ and edges where the distance between vertices is one.

$$E(\mathbb{Z}^d) = \{u, v \in V(\mathbb{Z}^d) : \delta(u, v) = 1)\}$$

We will often refer to this lattice by the vertex set $\mathbb{Z}^d$ without specifying the edge set. We also denote the origin by 0.

2.2 The Model

We now take some $p \in [0, 1]$ which will be our parameter which specifies the probability a given edge is open. Setting $q = 1 - p$ we say each edge is independently open with probability p and closed with probability q. We can think of the open and closed edges defining a random subgraph of $\mathbb{Z}^d$ where only edges set to open are retained.

Definition 2.4. Let $C(x)$ denote the open cluster (component) containing x, which is the set of vertices in $\mathbb{Z}^d$ which are connected to x by a path of open edges. We abbreviate the open cluster containing the origin $C(0)$ by C

2.3 Probability Space

We now introduce the Measure thoery basics required to define the probability measure for our percolation model.

Definition 2.5 (σ-algebra)**.** For some set X, we call $\mathcal{A} \subseteq \mathcal{P}(X)$ a σ-algebra if:

1. $\emptyset, X \in \mathcal{A}$
2. $A \in \mathcal{A} \implies A^c := X \setminus A \in \mathcal{A}$ (Closed under complement)
3. For $A_i \in \mathcal{A}, i \in \mathbb{N}$ we have that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ (Closed under countable unions)

We call the elements of $\mathcal{A}$ measurable sets.

We note by De Morgan's Laws that a σ-algebra is also closed under countable intersections.

Definition 2.6 (Percolation function)**.** We define the percolation function $\theta(p)$ as follows

$$\theta(p) = \mathbb{P}(|C| = \infty)$$

In words the percolation function is simply the probability that we can reach an infinite number of vertices from the origin by open edges. We also note that this is the same as asking what is the probability of an having an infinite length self-avoiding path of open edges starting at the origin.

3 Existence of a critical value

3.1 Existence of a critical value on $\mathbb{Z}$

Trivially the critical value is p = 1. Consider the event $X_n = \{$There is an open self-avoiding path of length n starting at the origin$\}$ Then $X_n \supseteq X_{n+1}$ and so

$$\lim_{n\to\infty} \mathbb{P}(X_n) = \theta(p)$$

And since $\mathbb{P}(X_n) = 2p^n$, as the path can go left or right from the origin. We have for all $p < 1$, $\theta(p) = 0$. Thus $\theta(p) > 0$ iff $p = 1$.

3.2 Existence of a critical value on $\mathbb{Z}^2$

We show the existence of the critical value in this case by bounding it from above and below. We follow the proofs given in [1]

Theorem 3.1. *If $p < 1/3$, $\theta(p) = 0$.*

Proof. Let $X_n = \{$There is an open self-avoiding path of length n starting at the origin$\}$ as in Section 3.1. Then the probability for a path of length n to be open on every edge is p^n. The number of paths of length n from the origin is at most $4(3^{n-1})$ since there are 4 edges to choose from at the origin, then for each next step in the path there are at most 3 edges we can pick as the path is self-avoiding. Hence we get $\mathbb{P}(X_n) \leq 4(3^{n-1})p^n$. Then we take the limit since $\lim_{n\to\infty} \mathbb{P}(X_n) = \theta(p)$.

$$\begin{aligned}\lim_{n\to\infty} \mathbb{P}(X_n) &\leq \lim_{n\to\infty} 4(3^{n-1})p^n \\ &\leq 4 \cdot 3^{-1} \lim_{n\to\infty} (3p)^n\end{aligned}$$

Since $p < 1/3$ we have $\lim_{n\to\infty} \mathbb{P}(X_n) = 0$ □

Theorem 3.2. *For p close to 1, we have $\theta(p) > 0$*

Proof. We introduce the dual graph $(\mathbb{Z}^2)^*$ which has vertices in $(\mathbb{Z}^2 + \binom{1/2}{1/2})$, and edges as you would expect between vertices at distance 1. Then we can see there is a clear correspondence between the edges of $\mathbb{Z}^2$ and its dual, since each edge in the dual intersects a unique edge in the original graph. Thus we can create a mapping from the open and closed edges of $\mathbb{Z}^2$ to the dual graph, where the edge in the dual is open iff the intersecting edge in $\mathbb{Z}^2$ is open.
Then we notice that if there exists a cycle of closed edges in the dual graph enclosing the origin then the size of the open cluster at the origin is finite.

Lemma 3.3. *$|C| < \infty \iff \exists$ a cycle of closed edges in $(\mathbb{Z}^2)^*$ enclosing the origin*

Proof. Thinking visually since there is a ring of closed edges in the dual, the open edges from the origin in the original graph cannot extend beyond this ring. A more formal pure graph theory proof can be made but is ommitted here. □

Let $X_n = \{$There is a length n cycle of closed edges in $(\mathbb{Z}^2)^*$ which surrounds the origin$\}$ Then using Lemma 3.3 we see

$$\mathbb{P}(|C| < \infty) = \mathbb{P}(\bigcup_{n=4}^{\infty} \mathbb{P}(X_n)) \leq \sum_{n=4}^{\infty} \mathbb{P}(X_n) \leq \sum_{n=4}^{\infty} n \cdot 4(3^{n-1})q^n$$

This sum is finite when $q < 1/3$, which is when $p > 2/3$. We can make the sum arbitrarily small when $p \to 1$, when the sum is smaller than 1 this implies $\theta(p) > 0$ □

Hence the critical value $p_c \in (1/3, 1)$

3.3 Existence of a critical value on $\mathbb{Z}^d$

References

[1] Jeffrey E Steif. "A mini course on percolation theory". In: *Jyväskylä lectures in mathematics* 3 (2011).