COSC363 Computer Graphics

6

Mathematical Preliminaries

Internals of a graphics engine

R. Mukundan (mukundan@canterbury.ac.nz)
Department of Computer Science and Software Engineering
University of Canterbury, New Zealand.

Homogeneous Coordinates

- A point with Cartesian coordinates (x, y, z) can be expressed in homogeneous coordinates as (hx, hy, hz, h) where h is a **non-zero** real number.

glVertex3f (10, 2, –3);
glVertex4f (10, 2, –3, 1);
glVertex4f (60, 12, –18, 6)
glVertex4f (–20, –4, 6, –2)

Different representations of the same point

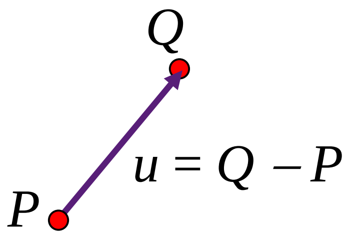
- To convert from homogeneous coordinates to Cartesian coordinates, divide the first three components by the fourth element: $(a, b, c, d) \equiv (a/d, \; b/d, \; c/d)$

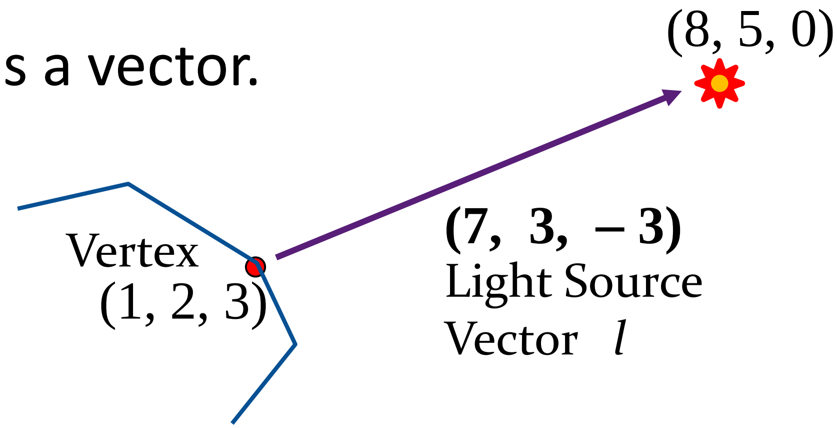
Example: The *xyz* coordinates of the point (12, –16, 1, 4) are (3, –4, 0.25)

- A vector with components (x, y, z) is represented in homogeneous coordinates as $(x, y, z, 0)$.

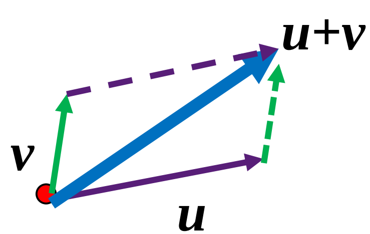
Point and Vector Operations

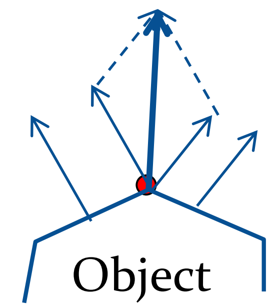
The difference between two points is a vector.





The sum of two vectors is a vector (obtained using parallelogram law).





If $\boldsymbol{v} = (x, y, z)$ denotes a vector, its magnitude (or length) is given by $|\boldsymbol{v}| = \sqrt{x^2 + y^2 + z^2}$ **Normalization** is the process of converting a vector to a unit vector by dividing each of its components by its magnitude.

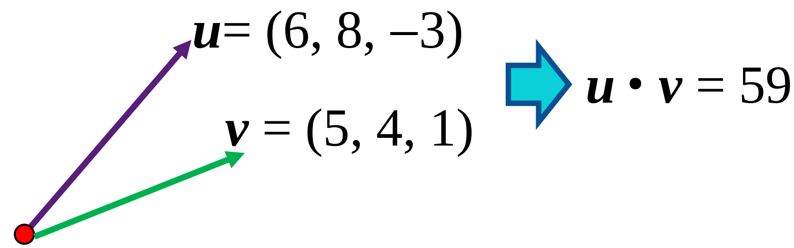
Example: $\boldsymbol{v} = (3, 2, 6)$. $|v| = \sqrt{9+4+36} = 7$

Unit Vector
$\boldsymbol{u} = (3/7,\ 2/7,\ 6/7)$

Vector Dot Product

- The **dot product** of two vectors $\boldsymbol{v}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{v}_2 = (x_2, y_2, z_2)$ is given by

  $$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$



- The dot product is a *scalar value*, not a vector.
- If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote **unit** vectors, then $\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = \cos(\phi)$, where ϕ is the angle between the two vectors. $\phi = \cos^{-1}(\boldsymbol{v}_1 \bullet \boldsymbol{v}_2)$

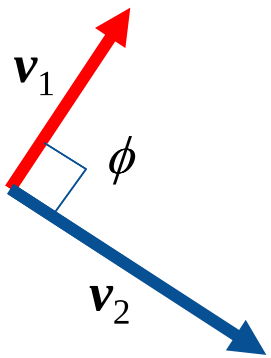
<u>Example</u>:

Compute the angle between vectors (2, 3, 3) and (1,1, 0):

- Normalize both vectors: (0.426, 0.64, 0.64), (0.707, 0.707, 0)
- Compute the dot product: 0.754 (= $\cos\phi$)
- $\phi = \cos^{-1}(0.754) =$ 41.06 Degs.

Orthogonality of Vectors

- If two vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are perpendicular (orthogonal) to each other, then, $\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = 0$.

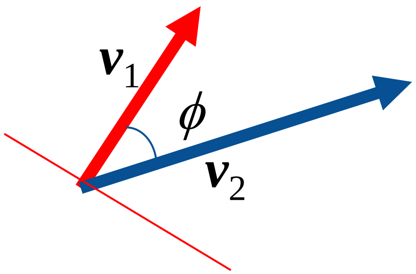


$$\phi = \pm 90 \text{ degs} \quad \Rightarrow \quad \cos\phi = 0 \quad \Rightarrow \quad \boldsymbol{v}_1 \bullet \boldsymbol{v}_2 = 0$$

- <u>Example</u>: Show that vectors (5, 2, –8) and (2, 7, 3) are perpendicular.
  - Compute the dot product: 10 + 14 – 24 = 0
  - Since the dot product is 0, the vectors are orthogonal to each other. (There is no need to normalize the vectors)

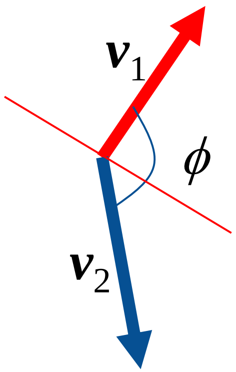
Relative Orientation of Two Vectors

It is often required to know if two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are separated by an acute angle ($\phi < \pi/2$) or obtuse angle ($\phi > \pi/2$).



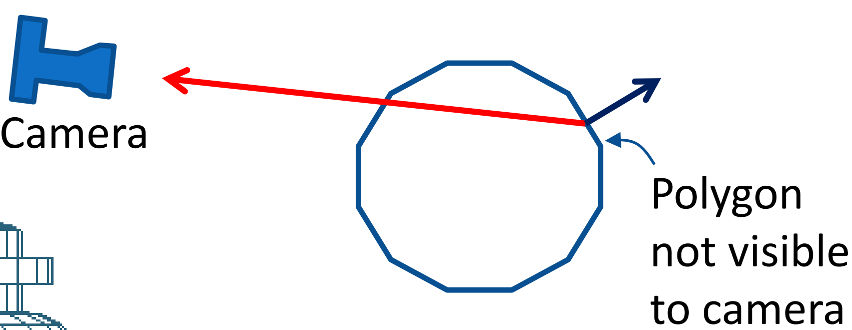
$\cos\phi > 0$

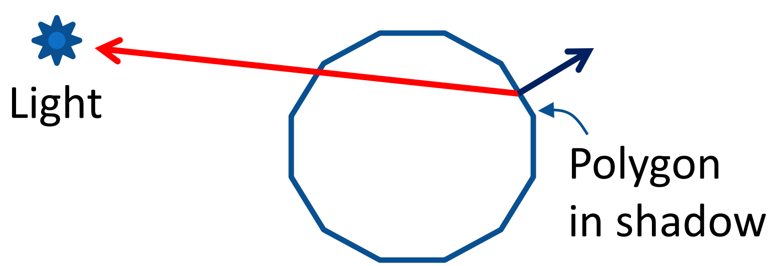
$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 > 0$



$\cos\phi < 0$

$\boldsymbol{v}_1 \bullet \boldsymbol{v}_2 < 0$

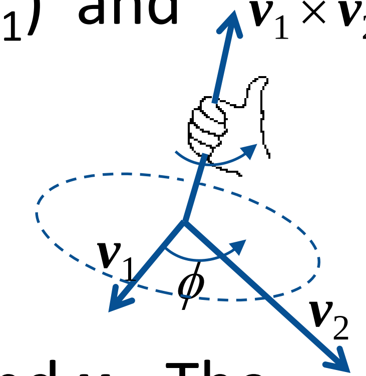




Vector Cross Product

- The cross product of two vectors $\boldsymbol{v}_1 = (x_1, y_1, z_1)$ and $\boldsymbol{v}_2 = (x_2, y_2, z_2)$ is a *vector* given by

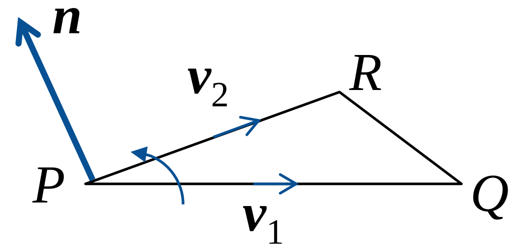
$$\boldsymbol{v}_1 \times \boldsymbol{v}_2 = (y_1 z_2 - y_2 z_1\ ,\ z_1 x_2 - z_2 x_1\ ,\ x_1 y_2 - x_2 y_1).$$



- The above vector is perpendicular to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The direction of $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ is given by the right-hand rule.
- If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote **unit** vectors, then $|\boldsymbol{v}_1 \times \boldsymbol{v}_2| = \sin(\phi)$, where ϕ is the angle between the two vectors.
- If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are parallel vectors, $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = 0$.

Surface Normal Vector: Triangle

- Triangle: $P = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $R = (x_3, y_3, z_3)$.



- We form two vectors at Q: $\boldsymbol{v}_1 = Q - P$, and $\boldsymbol{v}_2 = R - P$.

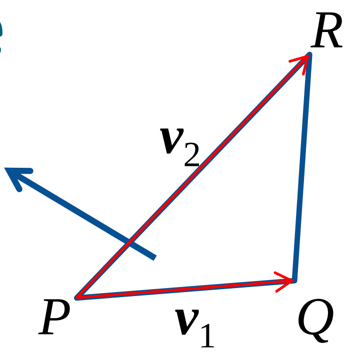
$\boldsymbol{v}_1 = (x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$, $\boldsymbol{v}_2 = (x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1)$

- The cross product $\boldsymbol{v}_1 \times \boldsymbol{v}_2$ gives the normal vector $\boldsymbol{n}$:

$$\begin{aligned}\boldsymbol{n} = (\quad & (y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1), \\ & (z_2 - z_1)(x_3 - x_1) - (z_3 - z_1)(x_2 - x_1)\ , \\ & (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \quad)\end{aligned}$$

$$\begin{aligned}= (\quad & y_1(z_2 - z_3) + y_2(z_3 - z_1) + y_3(z_1 - z_2), \\ & z_1(x_2 - x_3) + z_2(x_3 - x_1) + z_3(x_1 - x_2), \\ & x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \quad)\end{aligned}$$

Surface Normal Vector: Triangle



Input: 3 vertices of a triangle.

```
void normal(float x1, float y1, float z1,
            float x2, float y2, float z2,
            float x3, float y3, float z3 )
{
    float nx, ny, nz;
    nx = y1*(z2-z3)+ y2*(z3-z1)+ y3*(z1-z2);
    ny = z1*(x2-x3)+ z2*(x3-x1)+ z3*(x1-x2);
    nz = x1*(y2-y3)+ x2*(y3-y1)+ x3*(y1-y2);
    glNormal3f(nx, ny, nz);
}
```

Matrices

- OpenGL uses 4x4 matrices for representing transformations.
- A 4x4 matrix may be stored in a two-dimensional array *a*[i][j]: i = row index (0..3), j = column index (0..3).
- Alternatively, the matrix can be stored in a single array m[k], k = 0..15, in either row-major order or column-major order. OpenGL always stores matrices in column-major order.

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} m_0 & m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 & m_7 \\ m_8 & m_9 & m_{10} & m_{11} \\ m_{12} & m_{13} & m_{14} & m_{15} \end{bmatrix} \qquad \begin{bmatrix} m_0 & m_4 & m_8 & m_{12} \\ m_1 & m_5 & m_9 & m_{13} \\ m_2 & m_6 & m_{10} & m_{14} \\ m_3 & m_7 & m_{11} & m_{15} \end{bmatrix}$$

(General form) (Row Major Order) (Column Major Order)

OpenGL

Matrices

- Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For any matrix **A**, **AI = IA = A**
- OpenGL Example (using C array):

```
float matrix[16]={0.5, 3.0, 0.1, 0, 0, 10., 6.0, 0,
                 8.0, 1.0,-4.2, 0, -2.0, 0, 9.0, 1.0};
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
glMultMatrixf(matrix);
```

$$\begin{bmatrix} 0.5 & 0 & 8 & -2 \\ 3 & 10 & 1 & 0 \\ 0.1 & 6 & -4.2 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Product

Transformation of a point as a matrix product:

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} a_{00}x + a_{01}y + a_{02}z + a_{03} \\ a_{10}x + a_{11}y + a_{12}z + a_{13} \\ a_{20}x + a_{21}y + a_{22}z + a_{23} \\ a_{30}x + a_{31}y + a_{32}z + a_{33} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 3 & 0 & 1 & 1 \\ -2 & 1 & 5 & 0 \\ 1 & -1 & 2 & 1 \\ 0 & 4 & 1 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ -8 \\ 23 \end{bmatrix}$$

$$M\,P = Q$$

Matrix Product

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} . & . & . & . \\ . & . & \boxed{.} & . \\ . & . & . & . \\ . & . & . & . \end{bmatrix}$$

$$c_{12} = a_{10}\, b_{02} + a_{11}\, b_{12} + a_{12}\, b_{22} + a_{13}\, b_{32}$$

General formula: $c_{ij} = \sum_{k=0}^{3} a_{ik} b_{kj}$

Example:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0 & 0.7 & 0 \\ 0 & 1 & 0 & 0 \\ -0.7 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.4 & 2 \\ 0 & 1 & 0 & 0 \\ 1.4 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix multiplication is **non-commutative**. In general, $\boldsymbol{AB \neq BA}$

Transformation Matrix

The transformation of a point (x, y, z, 1) to another point (x' , y', z', 1) can be expressed as a matrix-vector multiplication:

$$\begin{bmatrix} x'_p \\ y'_p \\ z'_p \\ 1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

Transformed point

A general transformation matrix

Input point

Translation Matrix

- The translation of a point $(x, y, z, 1)$ by (a, b, c) yields another point $(x+a, y+b, z+c, 1)$

$$\begin{bmatrix} x+a \\ y+b \\ z+c \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation Matrix}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- OpenGL function: `glTranslatef(a, b, c)`

Translation Matrix

The translation matrix has no effect on a vector (*x*, *y*, *z*, 0):

$$\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Translation Matrix}} \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$$

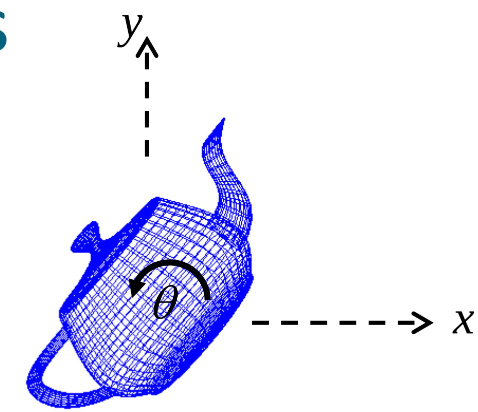
Scale Matrix

- The scaling of a point (x, y, z, 1) by factors (a, b, c) yields another point (xa , yb , zc , 1)

$$\begin{bmatrix} xa \\ yb \\ zc \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Scale Matrix}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- OpenGL function: `glScalef(a, b, c)`

Rotation About the Z-axis



- Equations:

  x' = x cosθ – y sinθ

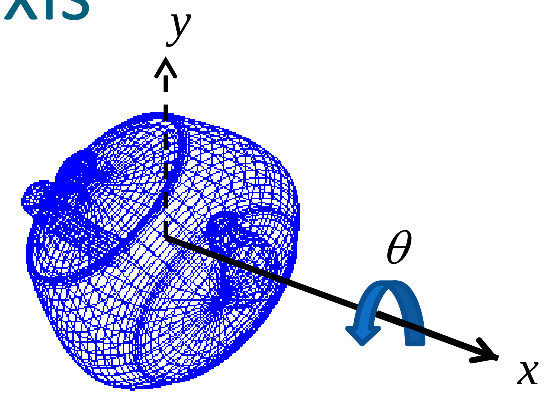
  y' = x sinθ + y cosθ

  z' = z

- Matrix Form:
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- OpenGL function: `glRotatef(theta, 0, 0, 1)`

Rotation About the X-axis



- Equations:

  $x' = x$

  $y' = y\cos\theta - z\sin\theta$

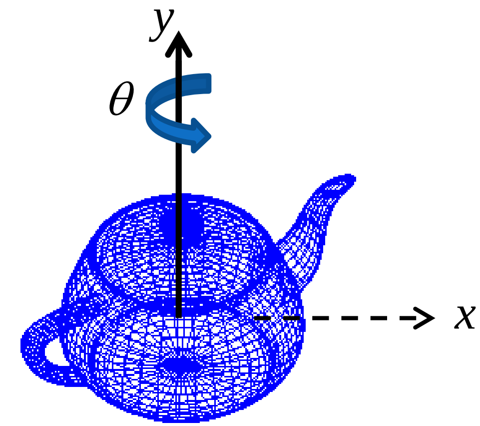
  $z' = y\sin\theta + z\cos\theta$

- Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- OpenGL function: `glRotatef(theta, 1, 0, 0)`

Rotation About the Y-axis



- Equations:

  $x' = x\cos\theta + z\sin\theta$

  $y' = y$

  $z' = -x\sin\theta + z\cos\theta$

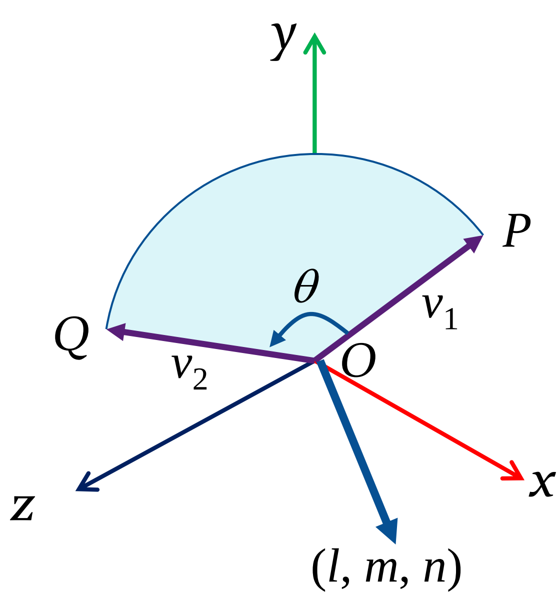
- Matrix Form:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- OpenGL function: `glRotatef(theta, 0, 1, 0)`

General Rotation

- Suppose a point *P* needs to be transformed to *Q*, where *P*, *Q* are at the same distance from the origin *O*.
- If we can find the angle of rotation θ from *P* to *Q*, and the axis of rotation (l, m, n), then we can apply a rotational transformation glRotatef(θ, l, m, n).



$$\theta = \cos^{-1}(v_1 \bullet v_2)$$

$$(l, m, n) = v_1 \times v_2$$

Custom Transformations

User-defined transformations can be represented in matrix form and applied with other transforms.

```
float myMatrix[16]={0.5, 3.0, 0.1, 0,
                      0, 10., 6.0, 0,
                      8.0, 1.0,-4.2, 0,
                     -2.0, 0, 9.0, 1.0};
glMatrixMode(GL_MODELVIEW);
glLoadIdentity();
gluLookAt(...)
glPushMatrix();
  glTranslatef(5, 2, -3);
  glMultMatrixf(myMatrix);
  glRotatef(25, 0, 1, 0);
  glutSolidTeapot(1);
glPopMatrix();
```

$$\begin{bmatrix} 0.5 & 0 & 8 & -2 \\ 3 & 10 & 1 & 0 \\ 0.1 & 6 & -4.2 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Teapot rotated→transformed using myMatrix →translated

Affine Transformation

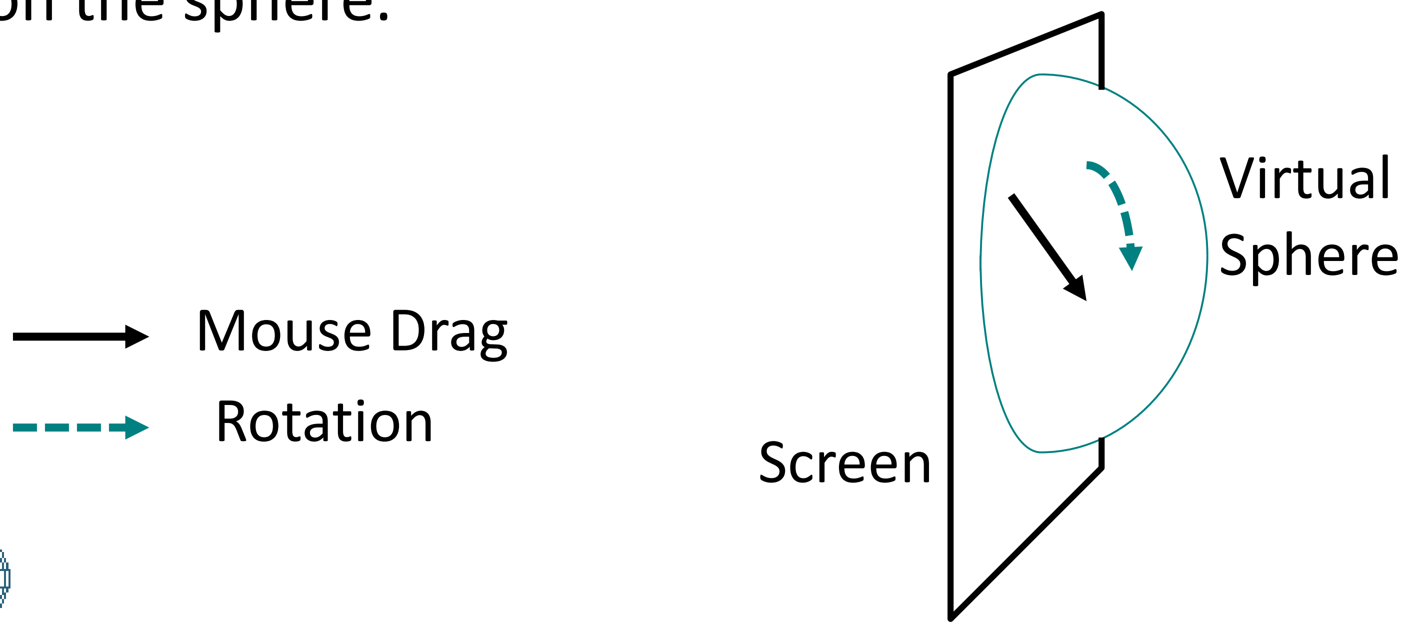
- A general linear transformation followed by a translation is called an affine transformation.
- Matrix form:

$$\begin{bmatrix} x'_p \\ y'_p \\ z'_p \\ 1 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_p \\ y_p \\ z_p \\ 1 \end{bmatrix}$$

- Translation, rotation, scaling and shear transformations are all affine transformations.
- Under an affine transformation, line segments transform into line segments, and parallel lines transform into parallel lines.

Virtual Trackball

- A user interface for drag-rotating an object.
- Assume that the objects displayed on the screen are attached to a virtual sphere.
- When the mouse is dragged from one point to another on the screen, a corresponding path of rotation is generated on the sphere.

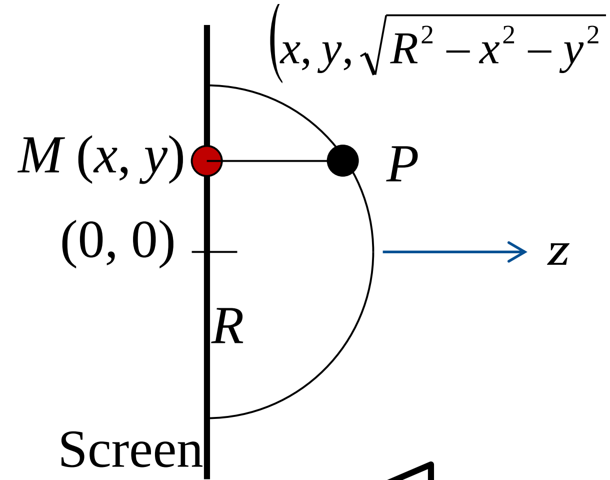


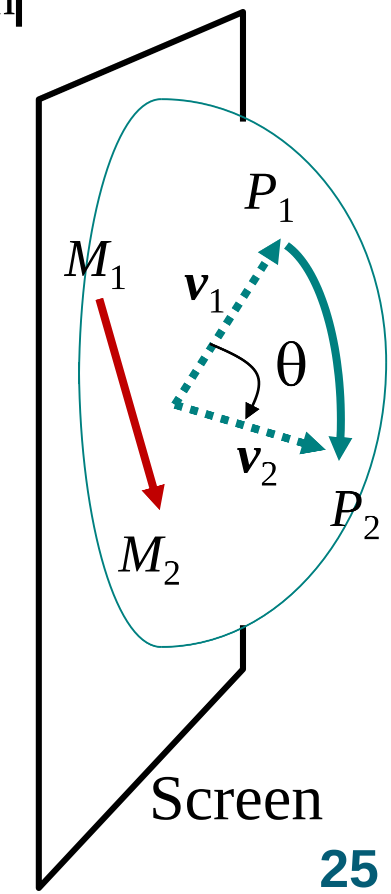
Virtual Trackball

- Let M_1M_2 be the path through which the mouse is dragged, and P_1 , P_2, the corresponding points on the virtual sphere.
- The angle of rotation is the angle between unit vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

$$\theta = \cos^{-1}(v_1 \bullet v_2)$$

- The axis of rotation is the axis perpendicular to both $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, given by $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = (l, m, n)$
- Use glRotatef(θ, l, m, n) to rotate the object.

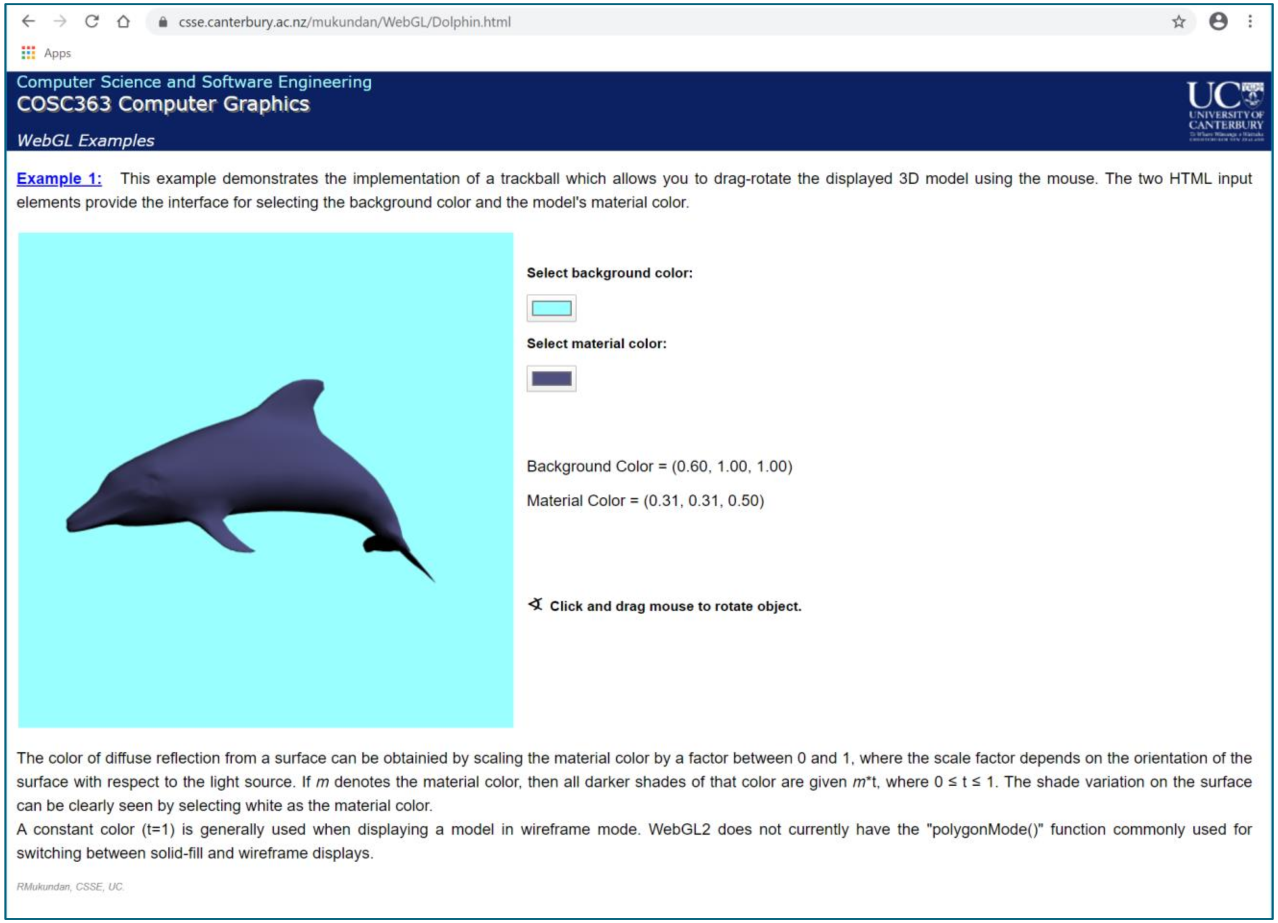




Virtual Trackball: WebGL Example

https://www.csse.canterbury.ac.nz/mukundan/WebGL/Dolphin.html

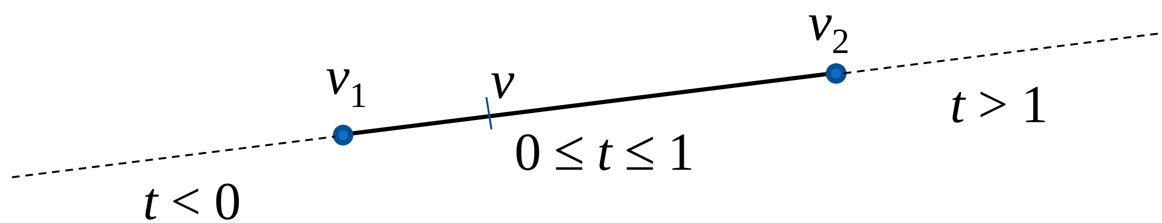
Chrome or Firefox



Linear Interpolation

Linear interpolation is useful in computing an in-between value, given the values v_1, v_2 of some attribute at the end points of a path.

$$v = \mathbf{(1-t)}\, v_1 + \boldsymbol{t}\, v_2\,, \qquad 0 \le t \le 1.$$



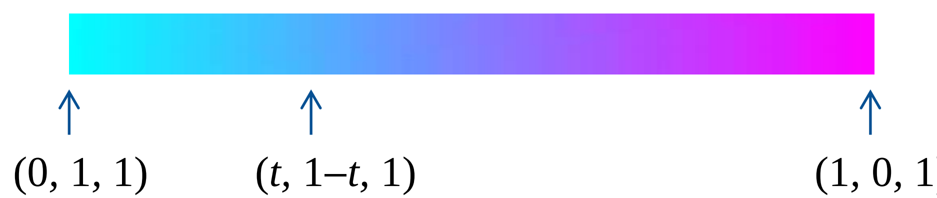
Example:

$v_1 = (0, 1, 1)$

$v_2 = (1, 0, 1)$

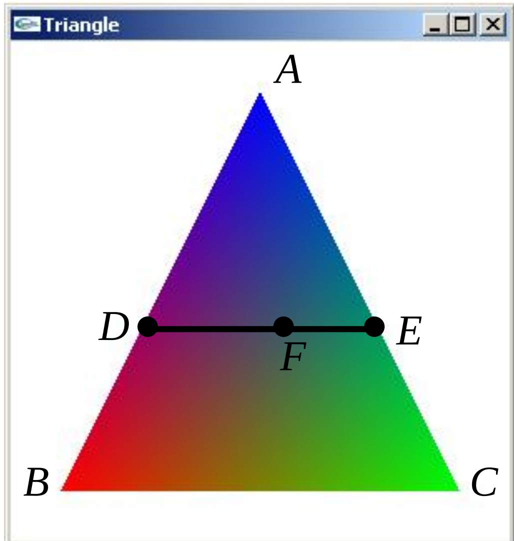
$v \;= (1-t)(0, 1, 1) \;+\; t\,(1, 0, 1)$

$\;\;\; = (\,t,\;\; 1-t,\;\; 1)$



Bi-Linear Interpolation

- Given the values of an attribute (such as colour) at the vertices of a triangle, bi-linear interpolation is used to obtain the values at the interior points.



$$\boldsymbol{D} = (1-t)\,\boldsymbol{A} + t\,\boldsymbol{B}$$
$$\boldsymbol{E} = (1-t)\,\boldsymbol{A} + t\,\boldsymbol{C}$$
$$0 \le t \le 1$$

$$\boldsymbol{F} = (1-s)\,\boldsymbol{D} + s\,\boldsymbol{E}$$
$$0 \le s \le 1$$

$$\boldsymbol{F} = (1-k_1-k_2)\,\boldsymbol{A} + k_1\boldsymbol{B} + k_2\boldsymbol{C}$$

Convex Combination

- Interpolate along the two edges *AC, BC* using a single parameter *t* , to get *D, E.*
- Interpolate along *DE* using a second parameter *s*, to get *F*