Family Name ____________________

First Name ____________________

Student Number |__|__|__|__|__|__|__|__|

Venue ____________________

Seat Number ________

UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

No electronic/communication devices are permitted.

No exam materials may be removed from the exam room.

Mathematics and Statistics

EXAMINATION

End-of-year Examinations, 2017

EMTH211-17S2 (C) Engineering Linear Algebra and Statistics

For Examiner Use Only

Question	Mark
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	

Total ________

Examination Duration: 180 minutes

Exam Conditions:

Closed Book exam: Students may not bring in any written or printed materials.

Calculators with a 'UC' sticker approved.

Materials Permitted in the Exam Venue:

None.

Materials to be Supplied to Students:

1 x Write-on question paper/answer book

Instructions to Students:

Write your answers in the spaces provided.

There is a total of 80 points.

Use black or blue ink. Do not use pencil.

There is no formula sheet for this test.

Show all working. Write neatly. Marks can be lost for poorly presented answers.

Question 1 [14 points]

(a) Consider the matrix $A = \begin{bmatrix} 4 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 1 & 5 \end{bmatrix}$.

(i) Draw the row based Gerschogorin disks for A.

(ii) How do you deduce from part (i) that A is invertible?

(iii) Note that the matrix A is symmetric. What extra information can we deduce about the location of the eigenvalues?

(b) Consider the system $\begin{cases} 2x + y = 1 \\ x + 3y = -2. \end{cases}$

(i) Using the zero vector as an initial approximation, carry out two iterations of the Jacobi method to approximate the solution of this system.

(ii) Again using the zero vector as initial approximation, carry out two iterations of the Gauss-Seidel method to approximate the solution of this system.

(iii) Why can we be sure that the iterative processes will converge for this system?

Question 2 [8 points]

A farmer's herd of cows can be divided into three age classes: calves (aged 0 – 1 year), juveniles (aged 1 – 2 years), and adults (aged 2 – 3 years). Each year, each juvenile produces an average of 0.8 female calves, and each adult produces an average of 0.6 female calves. Calves do not reproduce until they are juveniles. Each year 10% of calves and 20% of juveniles either die or are sold. All adult cows either die or are sold at the end of their third year.

(a) Write down the Leslie matrix L for this population.

Using the command [P,D]=eig(L) in Matlab with the matrix L from part (a) gives the following output:

$$P = \begin{bmatrix} -0.68568 & 0.15547 - 0.37310i & 0.15547 + 0.37310i \\ -0.58132 & -0.47449 + 0.31631i & -0.47449 - 0.31631i \\ -0.43808 & 0.71514 & 0.71514 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1.06158 & 0 & 0 \\ 0 & -0.53079 + 0.35385i & 0 \\ 0 & 0 & -0.53079 - 0.35385i \end{bmatrix}$$

(b) Will the size of the herd of cows in the long run grow, decline, or remain stable? Explain your answer.

(c) What is the long term distribution of cows among the three age classes?

Question 3 [10 points]

(a) Use the power method to find estimates for the dominant eigenvalue **and** an associated eigenvector for

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}.$$

Use $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as the initial vector and calculate two iterations. Normalise your iteration using the ∞-norm.

(b) Let A be an $m \times n$ matrix with columns $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$. Express the product $A\mathbf{x}$ in terms of these columns.

(c) (i) Use block multiplication to calculate

$$\begin{bmatrix} A^{-1} & O \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix} \begin{bmatrix} A & O \\ C & D \end{bmatrix}.$$

(ii) What does the result of part (i) tell you about $\begin{bmatrix} A & O \\ C & D \end{bmatrix}^{-1}$? Justify your answer.

(iii) Using the results of the previous parts calculate

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{bmatrix}^{-1}.$$

Question 4 [12 points]

(a) Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}.$$

Use the Gram–Schmidt process to find an orthonormal basis for $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

(b) The matrix $B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 3 & 0 \\ 2 & -1 & -2 \\ 2 & 3 & 2 \end{bmatrix}$ has an economy QR decomposition with $Q = \dfrac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$.

(i) Find R.

(ii) Use this economy QR decomposition to solve the system $B\mathbf{x} \approx \begin{bmatrix} 8 \\ 4 \\ 4 \\ 4 \end{bmatrix}$ by least squares.

Question 5 [12 points]

(a) Consider $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$. Given that $A^T A$ has eigenvalues $\mu_1 = 3$ and $\mu_2 = 1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find a singular value decomposition of A.

(b) The pixel values of a greyscale image of a certain sports team's lineout calls are stored in an $m \times n$ matrix A. An SVD is then performed on the matrix giving

$$A = \sum_{j=1}^{r} \sigma_j \mathbf{u}_j \mathbf{v}_j^T$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the nonzero singular values, and $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ are the left and right singular vectors. A spy wishes to transmit the image to a galaxy far away, using only a very limited amount of data. Can you suggest how he/she might do so?

Question 6 [7 points]

Suppose we have two data vectors $\mathbf{x}$ and $\mathbf{y}$ of length n.
Note: In this exam the notation $\mathbf{x}$ and $\mathbf{y}$ will be used for $\overrightarrow{x}$ and $\overrightarrow{y}$.

(a) Give the definition of covariance and correlation for the data vectors.

(b) Assume that $\mathbf{x}$ is temperature in Fahrenheit. How will the covariance $cov(\mathbf{x}, \mathbf{y})$ and the correlation $cor(\mathbf{x}, \mathbf{y})$ change, when $\mathbf{x}$ is transformed into degree Celsius? The transformation is given by

$$\frac{5(\mathbf{x} - 32)}{9}.$$

(c) Let $\mathbf{y}$ be the amount of carbon dioxide dissolved in water and $\mathbf{x}$ the water temperature. The warmer the water gets the smaller is the amount of dissolved carbon dioxide. What is the sign of $cov(\mathbf{x}, \mathbf{y})$ and $cor(\mathbf{x}, \mathbf{y})$?

(d) Assume we run a simple linear regression

$$\mathbf{y} = b_0 + b_1\mathbf{x} + \mathbf{e}.$$

Which sign do you expect for b_1?

Question 7 [10 points]

Let $\mathbf{x} = (1, 2, 4, 5)^\top$ and $\mathbf{y} = (1, 2, 3, 10)^\top$. We consider the regression

$$\mathbf{y} = b_0 + b_1\mathbf{x} + \mathbf{e}.$$

(a) Compute the slope $\widehat{b_1}$ of the least squares regression line.

(b) Compute the intercept $\widehat{b}_0$ of the regression line.

(c) Use the value $t_2(0.975) = 4.30$ to give a 95% confidence interval for b_1 from $\widehat{b}_1$.

EXTRA PAGE FOR WORKING

Question 8 [7 points]

For a multiple linear regression model

$$\mathbf{y} = b_0 + b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_p\mathbf{x}_p + \mathbf{e}$$

we can consider the vector of estimators $\widehat{\mathbf{b}} = (\widehat{b}_0, \widehat{b}_1, \widehat{b}_2, \ldots, \widehat{b}_p)^\top$ and the matrix

$$X = [\mathbf{1}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p]\,.$$

(a) Give the normal equation for the vector of estimators $\widehat{\mathbf{b}}$ using this notation.

(b) A researcher regresses carbon dioxide dissolved in water y on oxygen dissolved in water $\mathbf{x}_1$, temperature in Celsius $\mathbf{x}_2$, and temperature in Fahrenheit $\mathbf{x}_3$. But the researcher gets results that seem to be odd.

(i) What might have gone wrong in this regression?

(ii) How can you check mathematically that this was the reason for the unusual results?

(iii) How can this be fixed?

END OF PAPER