What is this course about?

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

Beauty & th beast

Qs: When is $A\underset{\sim}{x} = \underset{\sim}{b}$ solvable

How big is th set of solutions

Spanning sets & Linear Independence

Burning question

When is a given vector a linear combination of other given vectors?

Motivating example

Is the vector $\begin{bmatrix}1\\3\end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}-1\\1\end{bmatrix}$?

Want to find scalars x and y such that

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

Considering the system

$$x - y = 1$$
$$x + y = 3$$

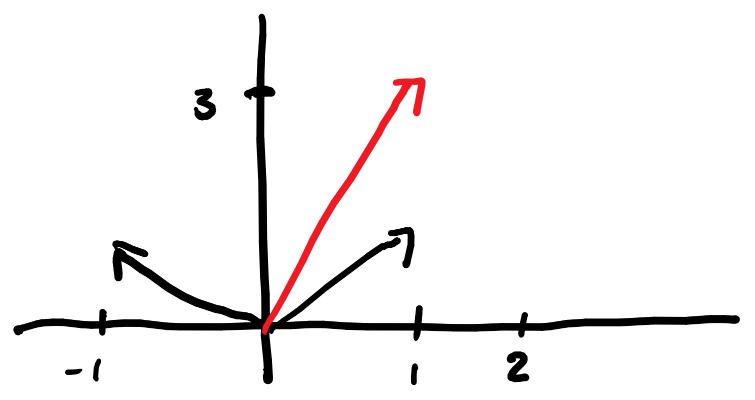
or $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 1 & 3 \end{array}\right]$ • solve any way you can

$= \ldots$

This has a unique solution

$$x = 2$$
$$y = 1$$

$$\therefore \quad 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$



Another example

$$x - y = 2$$
$$2x - 2y = 4$$

(Do the same procedure as the last example)

This system has infinitely many solution

$$x = 2 + t$$
$$y = t$$

This implies that

$$(2+t)\begin{bmatrix}1\\2\end{bmatrix} + t\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \quad (1)$$

last example

$$x - y = 1$$
$$x - y = 3$$

Has no solutions

No solutions that satisfy

$$x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$$

• We will be interested in the collect. of all linear combinations of a given set of vectors.

Big Idea

$A\underset{\sim}{x} = \underset{\sim}{b}$ is consistent (solvable) exactly when $\underset{\sim}{b}$ is a linear combination of the column vector of A

$$\underset{\sim}{b} = \underset{\sim}{a_1} x_1 + \underset{\sim}{a_2} x_2 + \ldots \underset{\sim}{a_n} x_n$$

$\Rightarrow$ next page

Def

The set of a linear combination of a set of vectors

$$\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_n}\}$$

is called the span of that set

? How big is the span?

example

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Given a plane!

$$\underset{\sim}{a_2} = \frac{1}{2}(\underset{\sim}{a_1} + \underset{\sim}{a_3})$$

Some columns may be redundant - they might be expressible in terms of other columns

We say there is a dependence relation among vectors

$$\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_n}$$

If we can express one of the vectors in terms of the vectors in term of the others (i.e, linear combination)

Every such dependence verification can be arranged into a more symetrical form:

$$\frac{1}{2}\underset{\sim}{a_1} + (-1)\underset{\sim}{a_2} + \frac{1}{2}\underset{\sim}{a_3} = 0$$

Def

A set of vectors $\{\underset{\sim}{v_1}, \dots, \underset{\sim}{v_n}\}$ is linearly dependent if there are scalars $c_1, \dots, c_k$ (Not all zero) such that

$$c_1\underset{\sim}{v_1} + c_2\underset{\sim}{v_2} + \dots + c_k\underset{\sim}{v_k} = 0$$

otherwise the set is called linearly independant

Example

① $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\4\\6\end{pmatrix}$ L.D

② $\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}4\\5\\6\end{pmatrix}\begin{pmatrix}7\\8\\9\end{pmatrix}$ L.D

③ $\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$ L.I

④ What about

$\underset{\sim}{v_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \underset{\sim}{v_2} = \begin{pmatrix}2\\3\\0\end{pmatrix}, \underset{\sim}{v_3} = \begin{pmatrix}4\\5\\6\end{pmatrix}$

Looking for a dependence relation

$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} = \underset{\sim}{0}$

This is the same as solving

$$\begin{bmatrix} \underset{\sim}{v_1} & \underset{\sim}{v_2} & \underset{\sim}{v_3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underset{\sim}{0}$$

and asking for a non-trivial solution (not all c_i's $= 0$)

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \underset{\sim}{c} = \underset{\sim}{0}$$

Back-subsitution shows that $c_3 = 0, c_2 = 0, c_1 = 0$

Hence $\underset{\sim}{v_1}, \underset{\sim}{v_2}, \underset{\sim}{v_3}$ are linearly independant

Linearly independent is all zero

⑤ $\underset{\sim}{v_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}, \underset{\sim}{v_2} = \begin{pmatrix}0\\1\\-1\end{pmatrix}, \underset{\sim}{v_3} = \begin{pmatrix}1\\0\\1\end{pmatrix}$

$c_1 \underset{\sim}{v_1} + c_2 \underset{\sim}{v_2} + c_3 \underset{\sim}{v_3} = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \underset{\sim}{c} = \underset{\sim}{0}$$

$= \ldots$

$\Rightarrow c_2 = 1, c_1 = -1, c_3 = 1$

$-\underset{\sim}{v_1} + \underset{\sim}{v_2} + \underset{\sim}{v_3} = \underset{\sim}{0}$ (!)