UNIVERSITY OF CANTERBURY

Test

Prescription Number: **EMTH211-18S2**

Time allowed: 60 minutes.

Write your answers in the spaces provided.

There is a *total* of 34 points.

Use black or blue ink. Do not use pencil.

Only UC approved calculators are allowed.

There is no formula sheet for this test.

Show all working. Write neatly. Marks can be lost for poorly presented answers.

MARKS Office Use Only	
Q1	
Q2	
Q3	
Q4	
Total	

Family name:	
Given names:	
Student ID:	

Question 1 [9 points]

Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 6 \\ 4 & 5 & 3 \end{bmatrix}.$$

The reduced row echelon form for A is given by

$$RREF = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Give a basis for the row space of A.

[1 MARK]

$[1\ 0\ 2]$
$[0\ 1\ -1]$

(b) Give a basis for the column space of A.

[1 MARK]

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 5 \end{bmatrix}$

(c) What is the rank of A?

[1 MARK] 2

(d) Give a basis for the null space of A.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = x_3 \end{cases}$$

[2 MARKS]

→ 1 for solution
→ 1 for basis

solution $\bar{x} \in \left\{ \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} \middle| s \in \mathbb{R} \right\}$

basis: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$

(e) What is the nullity of A?

[1 MARK] 1

(f) Give a formula that, for a general $m \times n$-matrix, relates its rank and its nullity.

[1 MARK] rank + nullity = n

(g) What is the rank of A^T? *You should not calculate A^T in order to solve this question !*

[1 MARK] = rank A = 2

(h) What is the nullity of A^T? *You should not calculate A^T in order to solve this question !*

[1 MARK] rank A^T + nullity A^T = 4

so nullity A^T = 2

Question 2 [7 points]

(a) The matrix

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 6 \\ -1 & 2 & 3 \end{bmatrix}$$

can be reduced to the echelon form

$$EF = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

by executing the row operations
$R_2 \to R_2 - 3R_1$ $m_{21} = 3$
$R_3 \to R_3 + R_1$ $m_{31} = -1$
$R_3 \to R_3 - R_2$. $m_{32} = 1$

Write down the LU-decomposition for B. (*Hint: use the multiplier method*)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

[2 MARKS]
↳ 1 for L
↳ 1 for U

(b) Let C be a 3×3 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ be an eigenvector of A with associated eigenvalue 1, let $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ be an eigenvector of C with associated eigenvalue 2 and let $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ be an eigenvector of C with associated eigenvalue 3.

(i) Diagonalise C (i.e. write down matrices P and D such that $C = PDP^{-1}$ and D is diagonal).

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

[2 MARKS]
↳ 1 for P
↳ 1 for D

(ii) Use (i) to calculate C^{2018}. *(You can of course leave large powers of real numbers in your answer.)*

$$C^{2018} = PD^{2018}P^{-1}$$

$$D^{2018} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{2018} & 0 \\ 0 & 0 & 3^{2018} \end{bmatrix}$$

[3 MARKS]
↳ 1 for $PD^{2018}P^{-1}$ and D^{2018} correct
↳ 1 for P^{-1} (extra)
↳ 1 for C^{2018} completely correct (extra)

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(note that P is an elementary matrix, corresponding to the row operation $R_2 + R_1$)

$$C^{2018} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{2018} & 0 \\ 0 & 0 & 3^{2018} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2^{2018} & 2^{2018} & 0 \\ 0 & 0 & 3^{2018} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1-2^{2018} & 2^{2018} & 0 \\ 0 & 0 & 3^{2018} \end{bmatrix}$$

Question 3 [11 points]

(a) *Remember that a subspace W of a vector space V is a set that is closed under taking linear combinations (that is, if $\mathbf{v}_1, \mathbf{v}_2 \in W$ and $k_1, k_2 \in \mathbb{R}$ then $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 \in W$).*

Let $M_{3,3}$ be the vector space of 3×3 matrices (with entries in $\mathbb{R}$).

[1 MARK] (i) What is the dimension of $M_{3,3}$? *(no proof required)* 9

[1 MARK] (ii) Recall a matrix A is called symmetric if it is equal to its transpose, i.e. if $A^T = A$. Let W be the set of all symmetric 3×3-matrices. W is a subspace of $M_{3,3}$. What is the dimension of W? *(no proof required)* 6

[1 MARK] (iii) Let T be the vector space consisting of all $n \times n$ symmetric matrices. What is the dimension of T? *(no proof required)* $\frac{n \times (n+1)}{2}$

[2 MARKS] (iv) Let U be the set of all 3×3 symmetric matrices whose diagonal elements are all zero. Show that U is a subspace of $M_{3,3}$.

×) If $\bar{u}_1$ and $\bar{u}_2$ are in U, then $\bar{u}_1 + \bar{u}_2 \in U$:

$$\begin{bmatrix} 0 & a_1 & b_1 \\ a_1 & 0 & c_1 \\ b_1 & c_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 & b_2 \\ a_2 & 0 & c_2 \\ b_2 & c_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a_1+a_2 & b_1+b_2 \\ a_1+a_2 & 0 & c_1+c_2 \\ b_1+b_2 & c_1+c_2 & 0 \end{bmatrix} \hookrightarrow \in U$$

×) If $\bar{u} \in U$, then $k\bar{u} \in U$:

$$k\begin{bmatrix} 0 & a_1 & b_1 \\ a_1 & 0 & c_1 \\ b_1 & c_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ka_1 & kb_1 \\ ka_1 & 0 & kc_1 \\ kb_1 & kc_1 & 0 \end{bmatrix} \hookrightarrow \in U$$

[2 MARKS] (v) Give a basis for U. *(no proof required)*

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) Let $E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The inverse of E is given by $E^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$.

(i) Calculate $\|E\|_1$, $\|E\|_\infty$ and the condition number $k(E)$ using the ∞-norm.

[2 MARKS]
0.5 each

$\|E\|_1 = \max\{4, 6\} = 6$

$\|E\|_\infty = \max\{3, 7\} = 7$

$\|E^{-1}\|_\infty = \max\{3, 2\} = 3$

$\Rightarrow k(E) = 21$

(ii) Describe **briefly** how the condition number of a matrix E may affect the accuracy of a solution to $E\mathbf{x} = \mathbf{b}$. *A formula relating the condition number to the error of a solution might be relevant.*

[2 MARKS]

We have that $\dfrac{\|\bar{e}\|}{\|\bar{x}\|} \leq k(E) \dfrac{\|A\bar{e}\|}{\|A\bar{x}\|}$.

✓

relative error in the solution.

If $k(E)$ is large, the accuracy of the solution can be really bad.

Question 4 [7 points]

(a) Let A be a matrix such that there is a vector $\mathbf{v}$ with

$$A\mathbf{v} = 2\mathbf{v}.$$

[1 MARK]

Why is $(2-\lambda)$ a divisor of the characteristic polynomial of A?

2 is an eigenvalue and all roots of the characteristic polynomial are eigenvalues ($(2-\lambda)$ is a factor of a polynomial iff 2 is a root).

(b) Let B be a 4×4-matrix with eigenvalues $1, 2, -1, -2$.

[1 MARK]

(i) What is the determinant of B? $1 \times 2 \times -1 \times -2 = 4$

(ii) Is B invertible? Explain your answer.

[1 MARK]

yes, 0 is not an eigenvalue.

(iii) Use the theorem of Cayley-Hamilton to deduce that $B^4 = 5B^2 - 4I$.

[2 MARKS]
↳ 1 for $p(\lambda)$
↳ 1 for rest

The characteristic polynomial is

$(\lambda-1)(\lambda-2)(\lambda+1)(\lambda+2)$

$= (\lambda^2-1)(\lambda^2-4) = \lambda^4 - 5\lambda^2 + 4$

A matrix B satisfies its own characteristic equation, so $B^4 - 5B^2 + 4I = 0$,

hence,

$B^4 = 5B^2 - 4I$.

(c) Let C be a 6×6-matrix and suppose that C has eigenvalues 1, 2, 3, 4. Suppose that one of the eigenvalues has geometric multiplicity 3. Is C diagonalisable? Explain why or why not.

[2 MARKS]

For all 4 eigenvalues, we have

$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity ≤ 6

Since the sum of all algebraic multiplicities is 6, the only possibility for the algebraic multiplicities is $1+1+1+3=6$.

This implies, since we have an eigenvalue with geometric multiplicity 3, that the geometric mult. are 1,1,1,3 too.

Hence, for all eigenvalues, the algebraic and geometric multiplicity coincide, and thus C is diagonalisable.

Page for rough working

Page for rough working

Good luck!

END OF PAPER