# EMTH211–Tutorial 7

## Attempt the following problems before the tutorial.

1. (Use Matlab) Use the power method to approximate the dominant eigenvalue and eigenvector of

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$$

using $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the starting vector and carrying out 6 iterations.

*Solution:*

Here's the first iteration in full: Let

$$\mathbf{y} = A\mathbf{x}_0 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

Scaling by its $\infty$-norm (`y/norm(y,inf)`) gives the next approximate eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0.2857 \end{bmatrix}.$$

As estimation for the eigenvalue, we use the Rayleigh quotient

$$(A.\mathbf{x}_1)\mathbf{x}_1/\mathbf{x}_1.\mathbf{x}_1.$$

The remaining iterations are:

| $k$ | $\mathbf{y} = A\mathbf{x}_k$ | *estimated eigenvalue* |
|---|---|---|
| 0 | $[1, 0]^T$ | 7 |
| 1 | $[1, 0.2857]^T$ | 7.547 |
| 2 | $[1, 0.3774]^T$ | 7.8227 |
| 3 | $[1, 0.4039]^T$ | 7.8280 |
| 4 | $[1, 0.4113]^T$ | 7.8284 |
| 5 | $[1, 0.4134]^T$ | 7.8284 |
| 6 | $[1, 0.4140]^T$ | 7.8284 |

## In-tutorial problems

2. (Use Matlab) Use the power method to approximate the dominant eigenvalue and eigenvector of

$$A = \begin{bmatrix} 3.5 & 1.5 \\ 1.5 & -0.5 \end{bmatrix}$$

using $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the starting vector and carrying out 6 iterations. Use the shifted power method to find the second eigenvalue of $A$ again using the same starting vector. Calculate a few iterations and see what happens.

*Solution:* Here's the first iteration: Let

$$\mathbf{y} = A\mathbf{x}_0 = \begin{bmatrix} 3.5 \\ 1.5 \end{bmatrix}$$

The largest entry (in absolute value) is 3.5. Scaling by 3.5 gives the approximate eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0.4286 \end{bmatrix}.$$

The remaining iterations are:

| $k$ | $\mathbf{x}_k$ | *estimated eigenvalue* |
|---|---|---|
| 0 | $[1, 0]^T$ | 3.5 |
| 1 | $[1, 0.4286]^T$ | 3.9655 |
| 2 | $[1, 0.3103]^T$ | 3.9978 |
| 3 | $[1, 0.3391]^T$ | 3.9999 |
| 4 | $[1, 0.3319]^T$ | 4.0000 |
| 5 | $[1, 0.3337]^T$ | 4.0000 |
| 6 | $[1, 0.3332]^T$ | |

After we have found the eigenvalue 4, we are now looking for the eigenvalues of

$$B = A - 4I = \begin{bmatrix} -0.5 & 1.5 \\ 1.5 & -4.5 \end{bmatrix}$$

in a similar fashion. The iterations are

| $k$ | $\mathbf{x}_k$ | *estimated eigenvalue* |
|---|---|---|
| 0 | $[1, 0]^T$ | $-0.5$ |
| 1 | $[-0.333, 1]^T$ | $-5$ |
| 2 | $[0.333, -1]^T$ | $-5$ |
| 3 | $[-0.333, 1]^T$ | $-5$ |
| 4 | $[0.333, 1]^T$ | $-5$ |

We see that we always get the same vector as eigenvector (but with different sign), and that the approximation for the eigenvalue $-5$ stays the same. The reason is that $[-1/3, 1]^T$ is an eigenvector of $A$ and we do not expect to find a better approximation for $1/3$ than 0.3333 in Matlab.

We conclude that $A-4I$ has eigenvalue $-5$, which means that we find the eigenvalue $-5+4=-1$ for $A$.

3. (Use Matlab) Use the inverse power method to approximate an eigenvector corresponding to the least eigenvalue of

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$$

using $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the starting vector and carrying out 6 iterations. Recall that for the inverse power method, we first find an $LU$-decomposition for $A$ and then solve the system $A\mathbf{z} = \mathbf{y}^k$.

*Solution:*

The LU-decomposition is

$$\begin{bmatrix} 1.0000 & 0.0000 \\ 0.2857 & 1.0000 \end{bmatrix} \begin{bmatrix} 7.0000 & 2.0000 \\ 0.0000 & 2.4286 \end{bmatrix}.$$

We could find this by the command `[L,U]=lu(A)`.
In each iteration, we solve $L\mathbf{w} = \mathbf{y}^{(k)}$, $U\mathbf{z} = \mathbf{w}$, and scale the solution $z$. We then compute the Rayleigh quotient:

```
w=L\y
z=U\w
y=z/norm(z,inf)
transpose(A*y)*y/(transpose(y)*y)
```

We find

| $k$ | $\mathbf{y}_k$ | *estimated eigenvalue* |
|---|---|---|
| 0 | $[1, 0]^T$ | |
| 1 | $[1.0000, -0.6667]^T$ | 3.9231 |
| 2 | $[0.6500, -1.0000]^T$ | 2.3603 |
| 3 | $[0.4760, -1.0000]^T$ | 2.1866 |
| 4 | $[0.4311, -1.0000]^T$ | 2.1727 |
| 5 | $[0.4188, -1.0000]^T$ | 2.1717 |
| 6 | $[0.41550, -1.0000]^T$ | 2.1716 |

4. Consider the matrix

$$A = \begin{bmatrix} 6 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0.5 \\ 0 & -2 & 12 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix}.$$

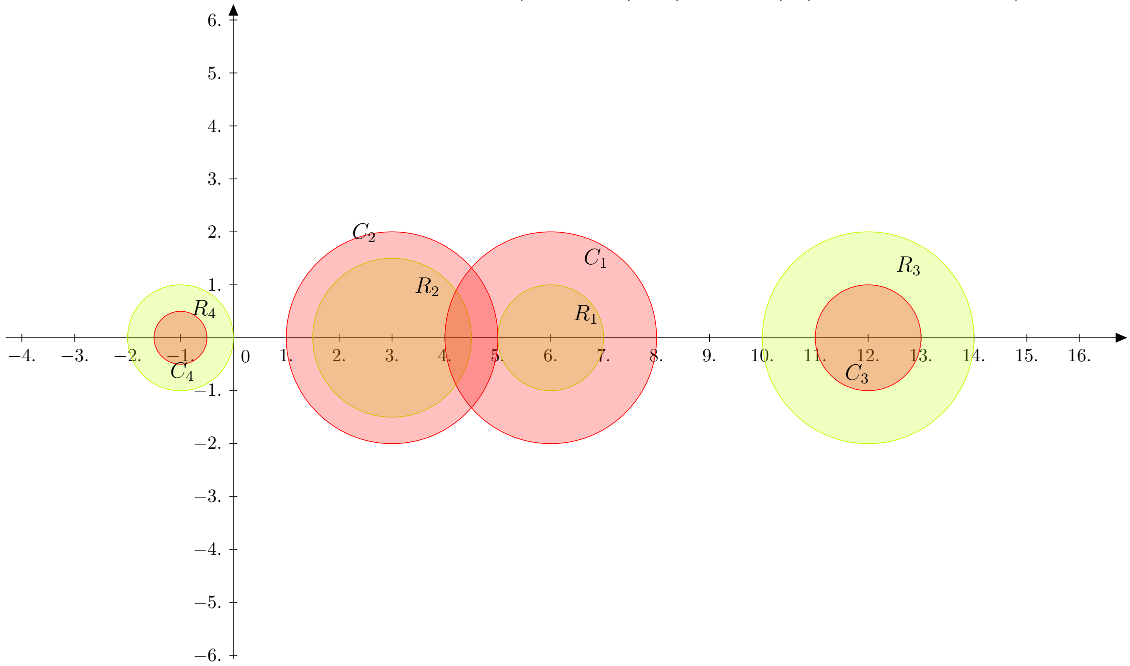
(a) Draw the Gerschgorin Discs of $A$. (Column and row based). Mark the area that the eigenvalues of $A$ must lie in.

(b) Show that $A$ is invertible, using the information gained from the previous part of the exercise.

(c) Calculate the eigenvalues of $A$ (use Matlab) and check whether they are indeed contained in the area you indicated in (a).

*Solution:*

(a) Using the rows, we find a disk $R_1$ of radius 1 with center 6, a disk $R_2$ of radius 1.5 with center 3, a disk $R_3$ with radius 2 and center 12 and a disk $R_4$ with radius 1 and center $-1$. Similarly, using the columns, we find a disk $C_1$ with center 6 and radius 2, a disk $C_2$ with center 3 and radius 2, a disk $C_3$ with center 12 and radius 1 and a disk $C_4$ with center $-1$ and radius 0.5. We see that every eigenvalue has to be contained in $(R_1 \cup R_2) \cap (C_3 \cup C_4)$ (the brown area).




(b) As we can see from the picture, $(R_1 \cup R_2) \cap (C_3 \cup C_4)$ does not contain the origin. Hence 0 is not an eigenvalue, which means that $A$ must be invertible.

(c) We find $\lambda_1 = 12.0353$, $\lambda_2 = 5.8951$, $\lambda_3 = 3.0669$ and $\lambda_4 = -0.9973$. All values are contained in the brown area of the picture.

5. Recall that a matrix is *strictly diagonally dominant* if the absolute value of each diagonal entry is strictly larger than the sum of the absolute values of the remaining entries in that row. Use Gerschgorin's disk theorem to prove that a strictly diagonally dominant matrix must be invertible.

   *Solution:* If the matrix $A$ is strictly diagonally dominant, we have that

$$r_i = \sum_{j \neq i} |a_{ij}| < |a_{ii}|$$

   for all $i$. This means that the row based Gerschgorin disk $R_i$, defined by $z \in \mathbf{C} : |z - a_{ii}| \leq r_i\}$ does not contain the origin. This holds for all rows $i$, and since we know that all eigenvalues of $A$ are contained in at least one Gerschgorin disk, we have that 0 not cannot be an eigenvalue of $A$. This implies that $A$ is invertible.