UNIVERSITY OF CANTERBURY

Exam

Prescription Number: **EMTH211-18S2**

Time allowed: 180 minutes.

Write your answers in the spaces provided.

There is a *total* of 90 points.

Use black or blue ink. Do not use pencil.

Only UC approved calculators are allowed.

There is no formula sheet for this test.

Show all working. Write neatly. Marks can be lost for poorly presented answers.

Family name:	
Given names:	
Student ID:	

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Total	

Question 1 [15 points]

The matrix $A = \begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix}$ has dominant eigenvalue $\lambda_1 = 1$.

(a) Calculate the trace of A.

$\mathrm{Tr}(A) = 0.8 + 0.6 = 1.4$

(b) Use (a) to find the second eigenvalue λ_2 of A.

$\lambda_1 + \lambda_2 = 1.4$, so $\lambda_2 = 0.4$

(c) Find the LU decomposition of A.

$$\begin{bmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{bmatrix} \to \begin{bmatrix} 0.8 & 0.4 \\ 0 & 0.5 \end{bmatrix}$$

where we have used the row operation $R_2 - 1/4 * R_1$. Hence, we have

$$L = \begin{bmatrix} 1 & 0 \\ 0.25 & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 0.8 & 0.4 \\ 0 & 0.5 \end{bmatrix}.$$

(d) Use (c) to carry out one iteration of the inverse power method with initial vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to approximate an eigenvector for λ_2. Use the ∞-norm to normalise your vector. *You do not need to approximate the eigenvalue* λ_2.

We need to calculate

$$\mathbf{z} = A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

by solving

$$A\mathbf{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now $A\mathbf{z} = LU\mathbf{z}$. We first solve

$$L\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This yields $y_1 = 1, y_2 = -0.25$. We then solve

$$U\mathbf{z} = \begin{bmatrix} 1 \\ -0.25 \end{bmatrix}$$

yielding

$$\mathbf{z} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}.$$

Normalising $\mathbf{z}$ with respect to the ∞-norm gives us the first approximation: $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

Question 2 [15 points]

A population of female *internet sharks* is distributed into three age classes:

- *baby sharks* are between 0 and 1 year old
- *mummy sharks* are between 1 and 2 year old, and
- *grandma sharks* are between 2 and 3 year old.

Baby sharks are not able to give birth. No internet shark gets older than 3 years. The average number of babies born to a mummy shark is 9.93 and the average number of babies born to a grandma shark is 0.1. Only 1 out of 5 baby sharks survives their first year, but 7 out of 10 mummy sharks become grandma sharks.

(a) Write down the Leslie matrix L describing this population.

$$L = \begin{bmatrix} 0 & 9.93 & 0.1 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}.$$

(b) Suppose that in a certain year y_0, there are equally many baby, mummy and grandma sharks. What is the fraction of mummy sharks within the total population in year $y_0 + 1$?

$$L \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10.03 \\ 0.2 \\ 0.7 \end{bmatrix}.$$

So the fraction of mummy sharks is

$$\frac{0.2}{10.93},$$

which is about 1.8%.

(c) When using the matrix L from (a) in MatLab, we obtain:

```
eig(L)
ans=
     1.4127667
    -1.4057171
    -0.0070495
```

What is the destiny of the population of internet sharks? Explain your reasoning.

The dominant eigenvalue is 1.413, which is strictly larger than 1. Hence, the population will keep on expanding.

(d) Suppose we are harvesting a fraction $h < 1$ of baby sharks at the end of each growth year. We are not harvesting any mummy or grandma sharks. Write down the matrix H such that, if $\mathbf{x}$ is the population in year z_0, $(I - H)L\mathbf{x}$ is the population after harvesting. (That is, the population at the start of year $z_0 + 1$).

$$H = \begin{bmatrix} h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(e) Let h be as in (d), namely, the fraction of baby sharks that are harvested. Determine the smallest value h_0 such that if $h > h_0$, we are sure that the population of internet sharks eventually goes extinct. *Hint: h_0 is the fraction for which the population remains stable; express finding h_0 as an eigenvalue problem.*

The population is stable if

$$(I - H)L\mathbf{x} = \mathbf{x}.$$

Hence, $(I - H)L$ should have eigenvalue 1. Now

$$\begin{bmatrix} 1-h & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 9.93 & 0.1 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & (1-h)0.93 & (1-h)0.1 \\ 0.2 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}.$$

$\lambda = 1$ needs to be a solution of the characteristic equation. This happens if and only if

$$det \begin{bmatrix} -1 & (1-h)0.93 & (1-h)0.1 \\ 0.2 & -1 & 0 \\ 0 & 0.7 & -1 \end{bmatrix} = 0.$$

This equation gives us

$$\begin{aligned} -1 - 0.2((h-1)0.93 + (h-1)(0.07)) &= 0 \\ -1 - 2(h-1) &= 0 \\ 1 &= 2h \\ h &= 1/2 \end{aligned}$$

We conclude that $h_0 = 1/2$. That is, if we harvest more than half of the baby sharks each year, eventually, the population will go extinct.

Question 3 [15 points]

(a) Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

with $W = \text{span}\,(\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3)$. Find an orthogonal basis for W. Furthermore, find the projection matrix that projects onto $W^\perp$.

SOLUTION:

$$\mathbf{v}_1 = \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_1 - \frac{\mathbf{x}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The projection onto W is given by

$$\mathrm{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

and so the projection onto $W^\perp$ will be

$$\mathrm{I} - \mathrm{P} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

(b) Let $A = P\,R$ with

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 1 \\ 0 & 1/2 \end{bmatrix}.$$

Find the least squares solution to

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution:

The columns of P are orthogonal but are not normalized. Thus we write

$$\mathrm{P} = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{6} \\ -\frac{1}{3}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & 0 \\ \frac{1}{3}\sqrt{3} & -\frac{1}{6}\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \end{bmatrix} = \mathrm{Q\,D}$$

say. The least square solution will be the solution of

$$\mathrm{D\,R}\,\mathbf{x} = \mathrm{Q}^{\mathrm{T}}\,\mathbf{b};$$

that is

$$\begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \frac{1}{2}\sqrt{6} \end{bmatrix} \mathbf{x} = \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}.$$

Therefore the least square solution is

$$\mathbf{x} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}.$$

Question 4 [15 points]

(a) Orthogonally diagonalize

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

SOLUTION:
The eigenvalues of A are 2, 2, 2 and 4. Now

$$[A - 2I] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the geometric multiplicity of 2 is 3 as expected. The eigenspace E_2 is given by

$$s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We need to choose an orthonormal basis (or use Gram-Schmidt). An orthonormal basis is given by

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since A is symmetric, the remaining eigenvector must be orthogonal to these three eigenvectors. Therefore we can choose

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

to obtain an orthonormal basis. Therefore

$$Q = \begin{bmatrix} 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

with

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

(b) Find a SVD for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Solution:
We begin by computing the singular values.

$$B = A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues are 0 (the columns are linearly dependent) and 2 (trace formula). Thus the singular values of A are

$$\sigma_1 = \sqrt{2} \qquad \text{and} \qquad \sigma_2 = 0$$

and

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues are distinct and so the eigenvectors of B will be orthogonal. Thus $\mathbf{v}_1$ will be the unit eigenvector of B associated with 2. Now

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since $\sigma_1 \neq 0$, we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since $\sigma_2 = 0$, $\mathbf{u}_2$ is any vector such that $\mathbf{u}_1$, $\mathbf{u}_2$ forms an orthonormal basis for $\mathbf{R}^2$. Therefore we choose

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and so

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which completes the SVD.

Question 5 [10 points]

Suppose we have two data vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^\top$ of length n. We denote by $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ the sample means and by $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ the centered vectors. The column vector of n-ones is denoted by $\mathbf{1}$.

(a) Give the definitions of the sample mean $\overline{\mathbf{x}}$ and the centered vector $\widetilde{\mathbf{x}}$. Show that $\widetilde{\mathbf{x}}$ and $\mathbf{1}$ are orthogonal.

The (sample) mean is defined as

$$\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and the centered data as $\widetilde{\mathbf{x}} = \mathbf{x} - \mathbf{1}\overline{\mathbf{x}}$. We need to show for orthogonality that $\widetilde{\mathbf{x}} \cdot \mathbf{1} = 0$.

$$\widetilde{\mathbf{x}} \cdot \mathbf{1} = (\mathbf{x} - \mathbf{1}\overline{\mathbf{x}}) \cdot \mathbf{1} = \sum_{i=1}^{n}(x_i - \overline{\mathbf{x}}) = -n\overline{\mathbf{x}} + \sum_{i=1}^{n} x_i = -n\overline{\mathbf{x}} + n\overline{\mathbf{x}} = 0.$$

(b) Give the definition of the sample covariance $cov(\mathbf{x}, \mathbf{y})$ and show that centering does not change the sample covariance, i.e. $cov(\mathbf{x}, \mathbf{y}) = cov(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}})$.

The (sample) covariance is defined as $cov(\mathbf{x}, \mathbf{y}) = \dfrac{1}{n-1}\widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{y}}$.

`Option 1`

The covariance is location invariant, i.e. for all $a, b \in \mathbb{R}$ we have $cov(\mathbf{x}+\mathbf{1}a, \mathbf{y}+\mathbf{1}b) = cov(\mathbf{x}, \mathbf{y})$. Hence,

$$cov(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = cov(\mathbf{x} - \mathbf{1}\overline{\mathbf{x}}, \mathbf{y} - \mathbf{1}\overline{\mathbf{y}}) = cov(\mathbf{x}, \mathbf{y}).$$

`Option 2`

For any vector $\mathbf{x}$ we have $\overline{\widetilde{\mathbf{x}}} = 0$ and therefor $\widetilde{\widetilde{\mathbf{x}}} = \widetilde{\mathbf{x}}$. Hence,

$$cov(\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}) = \frac{1}{n-1}\widetilde{\widetilde{\mathbf{x}}} \cdot \widetilde{\widetilde{\mathbf{y}}} = \frac{1}{n-1}\widetilde{\mathbf{x}} \cdot \widetilde{\mathbf{y}} = cov(\mathbf{x}, \mathbf{y}).$$

(c) Assume that $\mathbf{x}$ and $\mathbf{y}$ report temperatures in degree Celsius. How will the covariance $cov(\mathbf{x}, \mathbf{y})$ **and** the correlation $corr(\mathbf{x}, \mathbf{y})$ change, when $\mathbf{x}$ and $\mathbf{y}$ are transformed into degree Fahrenheit? The transformation from Celsius to Fahrenheit is $9/5\mathbf{x} - 32$.

We use the scale invariance of the covariance

$$cov(9/5\mathbf{x} - 32, 9/5\mathbf{y} - 32) = cov(9/5\mathbf{x}, 9/5\mathbf{y}) = 9/5cov(\mathbf{x}, 9/5\mathbf{y}) = \frac{81}{25}cov(\mathbf{x}, \mathbf{y}).$$

The correlation does not change. It is invariant to adding a constant or multiplying by a positive number.

`Optional`

$$corr(9/5\mathbf{x} - 32, 9/5\mathbf{y} - 32) = corr(9/5\mathbf{x}, 9/5\mathbf{y}) = sign(9/5)^2corr(\mathbf{x}, \mathbf{y}) = corr(\mathbf{x}, \mathbf{y}).$$

(d) Assume that $\mathbf{x}$ and $\mathbf{y}$ report the same temperature, $\mathbf{x}$ in Celsius and $\mathbf{y}$ in Fahrenheit. What is the value of $corr(\mathbf{x}, \mathbf{y})$? Give a reason for your answer.

The correlation will be 1 since Celsius and Fahrenheit have a perfect linear relationship and with positive slope.

Question 6 [20 points]

1. Let $\mathbf{x} = (1, 2, 4, 5)^\top$ and $\mathbf{y} = (1, 2, 3, 10)^\top$. We consider the regression

$$\mathbf{y} = b_0 + b_1\mathbf{x} + \mathbf{e}.$$

(a) Compute the slope $\widehat{b}_1$ of the least squares regression line.

$$\widehat{b}_1 = \frac{\tilde{\mathbf{x}} \cdot \tilde{\mathbf{y}}}{\|\tilde{\mathbf{x}}\|^2}$$

$\overline{\mathbf{x}} = 3$ and $\overline{\mathbf{y}} = 4$,
$\tilde{\mathbf{x}} = (-2, -1, 1, 2)$,
$\tilde{\mathbf{y}} = (-3, -2, -1, 6)$,
$\tilde{\mathbf{x}} \cdot \tilde{\mathbf{y}} = 6 + 2 - 1 + 12 = 19$, and
$\|\tilde{\mathbf{x}}\|^2 = 4 + 1 + 1 + 4 = 10$.
Hence, $\widehat{b}_1 = 19/10 = 1.9$.

(b) Compute the intercept $\widehat{b}_0$ of the regression line.
$\widehat{b}_0 = \overline{\mathbf{y}} - \widehat{b}_1\overline{\mathbf{x}} = 4 - 1.9 \times 3 = -1.7$.

(c) Compute a 90% confidence interval for b_1 from $\widehat{b}_1$. A table with the critical values of the t-distribution is given at the end of this booklet.

$$\begin{aligned} SSR &= (1-1.9+1.7)^2 + (2-1.9\times 2+1.7)^2 + (3-1.9\times 4+1.7)^2 + (10-1.9\times 5+1.7)^2 \\ &= 0.64 + 0.01 + 8.41 + 4.84 \\ &= 13.9 \end{aligned}$$

$$se(\widehat{b}_1) = \sqrt{\frac{1}{n-2}\frac{SSR}{\|\tilde{\mathbf{x}}\|^2}} = \sqrt{\frac{1}{2}\frac{13.9}{10}} \approx 0.834$$

Hence, $b_1 = \widehat{b}_1 \pm t_2(0.95)se(\widehat{b}_1) \approx 1.9 \pm 2.92 \times 0.834 \approx 1.9 \pm 2.435$
Or equivalently, $b_1 \in [-0.535, 4.335]$.

(d) Compute the goodness-of-fit measure R^2 for the regression.

$$R^2 = 1 - \frac{SSR}{SST}$$

We already know that $SSR = 13.9$ from the last question. $SST = \|\tilde{\mathbf{y}}\|^2 = 9+4+1+36 = 50$. Hence,

$$R^2 = 1 - \frac{13.9}{50} = 0.722.$$

2. For a multiple linear regression model

$$\mathbf{y} = b_0 + b_1\mathbf{x}_1 + b_2\mathbf{x}_2 + \cdots + b_p\mathbf{x}_p + \mathbf{e}$$

we denote the vector of least squares estimators by $\widehat{\mathbf{b}} = (\widehat{b}_0, \widehat{b}_1, \widehat{b}_2, \ldots, \widehat{b}_p)^\top$.

(a) Give the definition of the model matrix and the normal equation for the vector of estimators $\widehat{\mathbf{b}}$.

The first column of the model matrix is the vector $\mathbf{1}$. The second to $(p+1)$-th columns are given by the data vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$

$$X = [\mathbf{1}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p] \, .$$

The normal equation is

$$X^\top X\widehat{\mathbf{b}} = X^\top \mathbf{y} \qquad \text{or} \qquad \widehat{\mathbf{b}} = (X^\top X)^{-1} X^\top \mathbf{y}$$

(b) Some researchers want to investigate the energy loss in power lines of same diameter and different lengths. They regress the lost energy on voltage, and current, as well as on resistance, and length of the power line. But when they repeat their measurements under very similar conditions the OLS estimator produces quite different results.

Explain what might have gone wrong in this regression?

Resistance is a linear function of the length of a power line, when diameter (and temperature) are constant. We will have multicollinearity in the data. This causes instability for the solution of the least squares problem which results in instable estimators.

(c) Try to find evidence that nothing is wrong with the regression by using a statistical test. Use the level $\alpha = 0.05$ for the test. In the regression above $p = 4$. Assume that $n = 20$ and $det(X^\top X) = 0.5$. Interpret your result in one or two sentences. We can apply a Haitovsky test. The test statistic is

$$H = \left(1 + \frac{2p+5}{6} - n\right) \ln\left(1 - det(X^\top X)\right) = \left(-16\frac{5}{6}\right) \ln(0.5) \approx 11.668.$$

We need to compare the value of the test statistic with the critical value of the Chi-square distribution with $p(p-1)/2$ degrees of freedom $\chi^2_{0.05}(6) = 12.592$.

The test fails to reject the hypothesis that we do not have multicollinearity in the regressors. This is not evidence that there is multicollinearity, we just just cannot rule it out.

(d) Suggest a way how this can be fixed and give a reason why this would help.

`Option 1`

We need to drop either resistance or length from the regression since these two are the reason for the multicollinearity.

`Option 2`

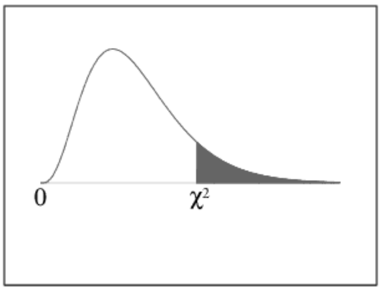
We can collect more date. This will give more precise information about the regressors which can compensate the instabilities caused by multicollinearity.

Page for rough working

Page for rough working

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	**0.50**	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.001**	**0.0005**
two-tails	**1.00**	**0.50**	**0.40**	**0.30**	**0.20**	**0.10**	**0.05**	**0.02**	**0.01**	**0.002**	**0.001**
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	18.549	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	7.042	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.041	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.195	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.916	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.953	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.425	104.215
80	51.172	53.540	57.153	60.391	64.278	96.578	101.879	106.629	112.329	116.321
90	59.196	61.754	65.647	69.126	73.291	107.565	113.145	118.136	124.116	128.299
100	67.328	70.065	74.222	77.929	82.358	118.498	124.342	129.561	135.807	140.169