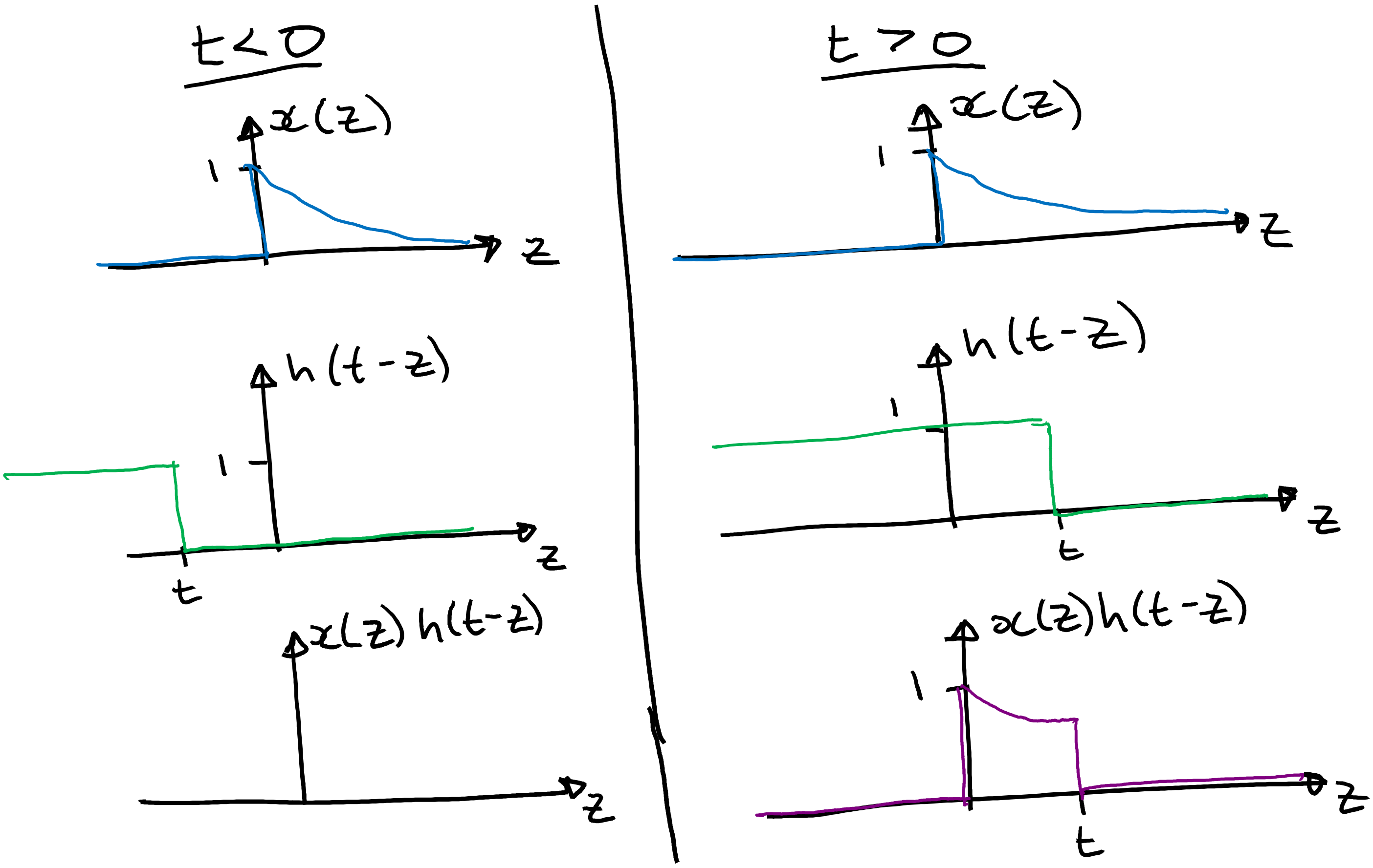
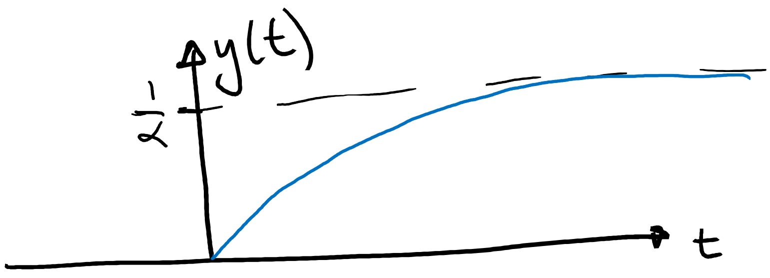
Graphically:



$$y(t) = \begin{cases} 0 & t<0 \\ \int_0^t e^{-\alpha z}\,dz & t>0 \end{cases}$$

$$= \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) u(t)$$



Properties of Convolution

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. $f * (g + h) = (f * g) + (f * h)$
4. $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

If $x(t) = 3u(t)$ and $h(t) = 4u(t) - 2u(t-2)$, what is $y(t)$? What is $y(t)$ if $x(t) = 3u(t) + 3u(t-10)$?

a) $X(s) = \frac{3}{s}$ $\quad H(s) = \frac{4}{s} - \frac{2e^{-2s}}{s}$

$$Y(s) = X(s)H(s)$$
$$= \frac{3}{s}\left(\frac{4}{s} - \frac{2e^{-2s}}{s}\right)$$
$$= \frac{12}{s^2} - \frac{6e^{-2s}}{s^2}$$

$$y(t) = 12tu(t) - 6(t-2)u(t-2)$$

b) $X(s) = \frac{3}{s} + \frac{3}{s}e^{-10s}$

time shift:
$f(t-a)u(t-a) \Leftrightarrow e^{-as}F(s)$
$f(t-a) = 3u(t-10)$
$\therefore f(t) = 3u(t)$
$F(s) = \frac{3}{s}$

$$Y(s) = X(s)H(s)$$
$$= \left(\frac{3}{s} + \frac{3}{s}e^{-10s}\right)\left(\frac{4}{s} - \frac{2e^{-2s}}{s}\right)$$
$$= \frac{12}{s^2} - \frac{6e^{-2s}}{s^2} + \frac{12e^{-10s}}{s^2} - \frac{6e^{-12s}}{s^2}$$

$$y(t) = 12tu(t) - 6(t-2)u(t-2) + 12(t-10)u(t-10) - 6(t-12)u(t-12)$$

Why $f(t-a) \neq 3$? If $f(t-a) = 3$ there is no '$t-a$' $\Rightarrow 3$ applies for all t.

To make it work, multiply 3 by $u(t-10)$ NB: $u(t-10)u(t-10) = u(t-10)$

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

If $x(t) = 7tu(t) + 8e^{-9t}u(t) + 6u(3t)$ and $h(t) = e^{-7t}u(t)$, what is $y(t)$?

$$X(s) = \frac{7}{s^2} + \frac{8}{s+9} + \frac{6}{3} \cdot \frac{1}{s/3}$$

$\Longrightarrow$ scaling in tables!

$f(at) \Longleftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$

$+ \; u(t) \Longleftrightarrow \frac{1}{s}$

$6u(3t)$:

$f(at) = u(3t) \;\therefore a = 3$

$\Longleftrightarrow \frac{1}{3} F\left(\frac{s}{3}\right)$

$f(t) = u(t)$

$F(s) = \frac{1}{s}$

$\therefore \frac{1}{3}\left(\frac{1}{s/3}\right)$

$$= \frac{14s^2 + 61s + 63}{s^2(s+9)}$$

$$H(s) = \frac{1}{s+7}$$

$$Y(s) = X(s)H(s)$$

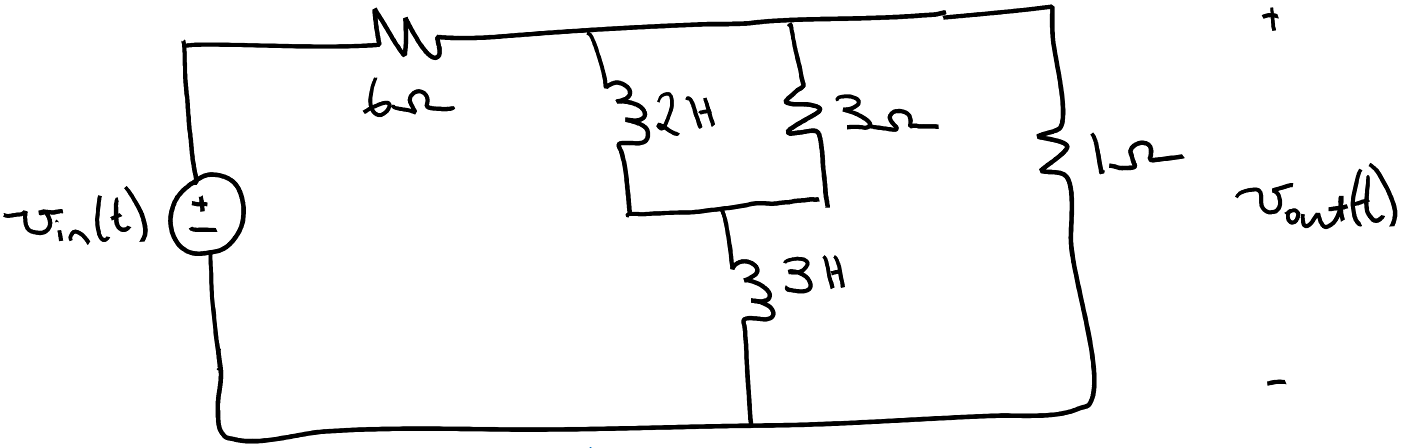
$$= \left(\frac{14s^2 + 61s + 63}{s^2(s+9)}\right)\left(\frac{1}{s+7}\right)$$

$$= \frac{1}{s^2} + \frac{5/7}{s} - \frac{4}{s+9} + \frac{23/7}{s+7}$$

$$y(t) = \left(t + 5/7 - 4e^{-9t} + \frac{23}{7}e^{-7t}\right)u(t)$$

Example

For the circuit below, if $v_{in}(t) = 6u(t) + e^{-3t}u(t)$ V, what is the transfer function, **H(s)**, and the output of the circuit, $v_{out}(t)$? Plot the pole-zero diagram for **H(s)**, and from it determine the shape of the system response.



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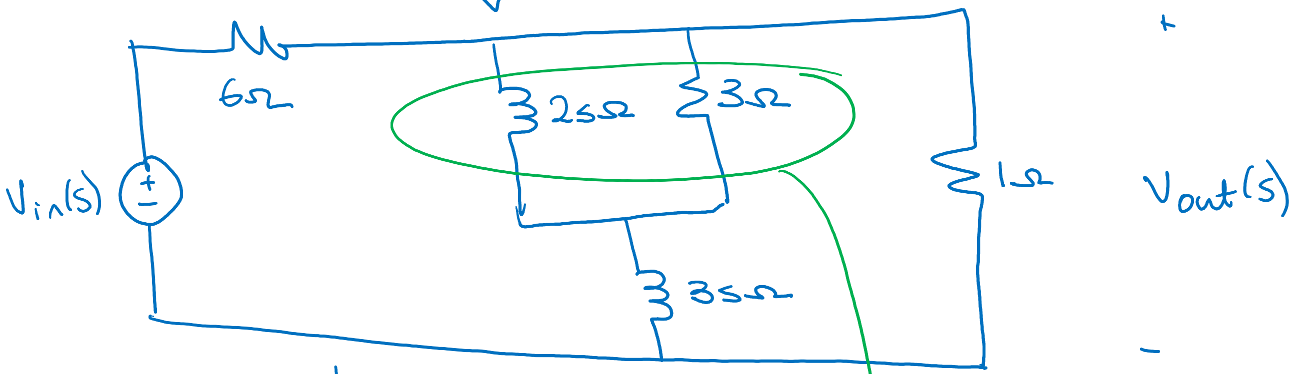


$$V_{in}(s) = \frac{6}{s} + \frac{1}{s+3}$$

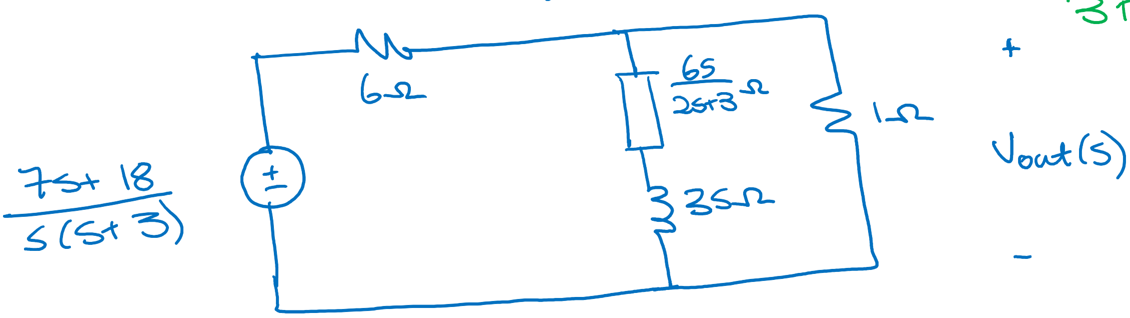
$$= \frac{7s+18}{s(s+3)}$$

$$Y(s) = \frac{1}{2s} + \frac{1}{3}$$

$$= \frac{3+2s}{6s}$$

$$Z(s) = \frac{6s}{3+2s}$$

$\Downarrow$



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} \qquad \Sigma I_{in} = \Sigma I_{out} \quad (KCL)$$

$$\frac{V_{in} - V_{out}}{6} = \frac{V_{out}}{3s + \frac{6s}{2s+3}} + \frac{V_{out}}{1}$$

$$\frac{V_{in}}{6} = V_{out}\left(\frac{2s+3}{6s^2+9s+6s} + 1 + \frac{1}{6}\right)$$

$$V_{in} = 6V_{out}\left(\frac{2s+3}{6s^2+15s} + \frac{7}{6}\right)$$

$$\frac{V_{in}}{V_{out}} = \cancel{6}\left(\frac{12s+18+42s^2+105s}{\cancel{6}(6s^2+15s)}\right)$$

$$= \frac{42s^2+117s+18}{6s^2+15s}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{6s^2+15s}{42s^2+117s+18}$$

$$= \frac{6s^2+15s}{42(s+0.16)(s+2.6)}$$

$$V_{out}(s) = V_{in}(s)H(s)$$

$$= \frac{7s+18}{\cancel{s}(s+3)} \cdot \frac{\cancel{s}(6s+15)}{42(s+0.16)(s+2.6)}$$

$$= \frac{(7s+18)(6s+15)}{42(s+3)(s+0.16)(s+2.6)}$$

$$= \frac{0.19}{s+3} + \frac{0.81}{s+0.16} + \frac{0.002}{s+2.6}$$

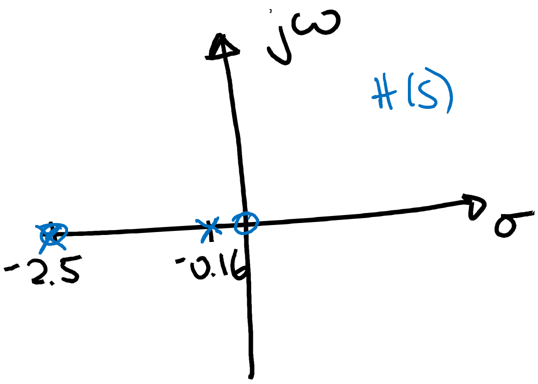
$$v_{out}(t) = \left(0.19e^{-3t} + 0.81e^{-0.16t} + 0.002e^{-2.6t}\right)u(t)\ V$$

zeros of $H(s) \Rightarrow s(6s+15)=0$

$s = 0, -2.5$

poles of $H(s) \Rightarrow (s+0.16)(s+2.6) = 0$

$s = -0.16, -2.6$



$\Rightarrow$ shape of system response is a negative exponential (all poles on LHS of σ axis)

$\Rightarrow$ matches $v_{out}(t)$

$\Downarrow$