EMTH211-19S2 LABORATORY 3 SOLUTIONS

JULY 29 - AUGUST 2, 2019

These problems should be done by hand (you can check your answers using MATLAB).

3.1 Suppose that the product $A\,B$ is defined. Show that

$$(A\,B)^T = B^T\,A^T.$$

SOLUTION:
The (i, j) entry in $A\,B$ is

$$\mathbf{A}_i^T\,\mathbf{b}_j$$

and so the (j, i) entry in $(A\,B)^T$ is also $\mathbf{A}_i^T\,\mathbf{b}_j$. The (j, i) entry in $B^T\,A^T$ is

$$\mathbf{b}_j^T\,\mathbf{A}_i$$

since the rows of B^T are $\mathbf{b}_i^T$ and the columns of A^T are $\mathbf{A}_i$. Since $\mathbf{x}^T\mathbf{y} = \mathbf{y}^T\,\mathbf{x}$, these entries are the same and so

$$(A\,B)^T = B^T\,A^T.$$

3.2

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E_{21}(\alpha) \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{bmatrix} = E_{42}(\beta).$$

Compute $A\,B$ and $B\,A$.

SOLUTION:

$$A\,B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \beta & 0 & 1 \end{bmatrix}, \qquad B\,A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha\,\beta & \beta & 0 & 1 \end{bmatrix}.$$

Thus we need to be careful about the order of multiplying elementary matrices.

3.3 Since we can row reduced any non-singular matrix A to the identity matrix (this is Gauss-Jordan)

$$\left[A \mid I\right] \to \left[I \mid A^{-1}\right]$$

we can write A and A^{-1} as a product of elementary matrices. Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Find a sequence of elementary matrices $L_1, L_2, \ldots, L_k$ such that

$$L_k \cdots L_2 L_1 A = I.$$

Hence write both A and A^{-1} as products of these elementary matrices.

SOLUTION:

We have

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} = L_1 A$$

with

$$L_1 = E_{21}(-2) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Next

$$L_1 A \xrightarrow{R_1 \to \frac{1}{2} R_1} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -1 \end{bmatrix} = L_2 L_1 A$$

with

$$L_2 = L_1(\tfrac{1}{2}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Continuing we have

$$L_2 L_1 A \xrightarrow{R_1 \to R_1 + \frac{3}{2} R_2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = L_3 L_2 L_1 A$$

with

$$L_3 = E_{12}(\tfrac{3}{2}) = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}.$$

Finally

$$L_3 L_2 L_1 A \xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = L_4 L_3 L_2 L_1 A$$

with

$$L_4 = E_2(-1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now

$$L_4 L_3 L_2 L_1 A = I$$

and so

$$A^{-1} = L_4 L_3 L_2 L_1 = E_2(-1)\, E_{12}(\tfrac{3}{2})\, E_1(\tfrac{1}{2})\, E_{21}(-2)$$

and

$$A = \left(E_2(-1)\, E_{12}(\tfrac{3}{2})\, E_1(\tfrac{1}{2})\, E_{21}(-2)\right)^{-1} = E_{21}(2)\, E_1(2)\, E_{12}(-\tfrac{3}{2})\, E_2(-1).$$

Note that the elementary matrices in these products are not unique; changing the order of the row reduction will change them.

3.4 Find the LU decomposition of

$$A = \begin{bmatrix} 2 & 2 & 2 & 1 \\ -2 & 4 & -1 & 2 \\ 4 & 4 & 7 & 3 \\ 6 & 9 & 5 & 8 \end{bmatrix}.$$

Note you do not have to use partial pivoting. Hence solve $A\mathbf{x} = \mathbf{b}$ with

$$\mathbf{b} = \begin{bmatrix} 0 \\ 9 \\ 9 \\ 0 \end{bmatrix}.$$

SOLUTION:

We have

$$A \to \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 3 & -1 & 5 \end{bmatrix}$$

by the row operations

$$\begin{aligned} R_2 &\to R_2 + R_1 \\ R_3 &\to R_3 - 2R_1 \\ R_4 &\to R_4 - R_1 \end{aligned}$$

Thus $L_1 = E_{41}(-1)\,E_{31}(-2)\,E_{21}(1)$ and so

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & * & 1 & 0 \\ 1 & * & * & 1 \end{bmatrix} = L_1^{-1} \cdots = E_{21}(-1)\,E_{31}(2)\,E_{41}(1) \cdots$$

For the second column

$$A \to \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -\frac{3}{2} & \frac{7}{2} \end{bmatrix}$$

by the row operation

$$R_4 \to R_4 - \tfrac{1}{2}R_2.$$

Therefore $L_2 = E_{42}(-\frac{1}{2})$ and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & * & 1 \end{bmatrix} = L_1^{-1}\,L_2^{-1} \cdots = E_{21}(-1)\,E_{31}(2)\,E_{41}(1)\,E_{42}(\tfrac{1}{2}) \cdots$$

Finally. the third column gives

$$A \to \begin{bmatrix} 2 & 2 & 2 & 1 \\ 0 & 6 & 1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U$$

with

$$R_4 \to R_4 + \tfrac{1}{2} R_3.$$

We have $L_3 = E_{43}(\frac{1}{2})$ and

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = L_1^{-1}\, L_2^{-1}\, L_3^{-1} = E_{21}(-1)\, E_{31}(2)\, E_{41}(1)\, E_{42}(\tfrac{1}{2})\, E_{43}(-\tfrac{1}{2}).$$

First we solve $L\,\mathbf{y} = \mathbf{b}$:

$$[L \mid \mathbf{b}] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 9 \\ 2 & 0 & 1 & 0 & 9 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \end{array}\right] \qquad \mathbf{y} = \begin{bmatrix} 0 \\ 9 \\ 9 \\ 0 \end{bmatrix}.$$

We now solve

$$[U \mid \mathbf{y}] = \left[\begin{array}{cccc|c} 2 & 2 & 2 & 1 & 0 \\ 0 & 6 & 1 & 3 & 9 \\ 0 & 0 & 3 & 1 & 9 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right] \qquad \mathbf{x} = \begin{bmatrix} -4 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

3.5 Use the LU decomposition with partial pivoting to solve the system $A^2\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 6 & 3 & -8 \\ 3 & -1 & 5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}.$$

SOLUTION:

We first must swap rows 1 and 2:

$$E_{1\leftrightarrow 2}\, A = \begin{bmatrix} 6 & 3 & -8 \\ 2 & 1 & -3 \\ 3 & -1 & 5 \end{bmatrix}.$$

Now

$$E_{1\leftrightarrow 2}\, A \to \begin{bmatrix} 6 & 3 & -8 \\ 0 & 0 & -\frac{1}{3} \\ 0 & -\frac{5}{2} & 9 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{2} & * & 1 \end{bmatrix}.$$

We now must swap rows 2 and 3. We also need to reorder L. Thus

$$E_{2\leftrightarrow 3}\, E_{1\leftrightarrow 2}\, A \to \begin{bmatrix} 6 & 3 & -8 \\ 0 & -\frac{5}{2} & 9 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & * & 1 \end{bmatrix}.$$

This is now in LU factorisation with

$$\begin{aligned} R_1 &\to R_2 \to R_3 \\ R_2 &\to R_1 \\ R_3 &\to R_2 \end{aligned}$$

Thus

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -\frac{5}{2} & 9 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}.$$

We now solve $P\,A\mathbf{y} = P\,\mathbf{b}$ followed by $A\mathbf{x} = \mathbf{y}$.

$$\left[L \mid P\,\mathbf{b}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 5 \\ \frac{1}{3} & 0 & 1 & 0 \end{array}\right] \qquad \tilde{\mathbf{y}} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

and

$$\left[U \mid \tilde{\mathbf{y}}\right] = \left[\begin{array}{ccc|c} 6 & 3 & -8 & 0 \\ 0 & -\frac{5}{2} & 9 & 5 \\ 0 & 0 & -\frac{1}{3} & 0 \end{array}\right] \qquad \mathbf{y} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}.$$

Finally

$$\left[L \mid P\,\mathbf{y}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ \frac{1}{2} & 1 & 0 & -2 \\ \frac{1}{3} & 0 & 1 & 0 \end{array}\right] \qquad \tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ -\frac{5}{2} \\ -\frac{1}{3} \end{bmatrix}$$

and

$$\left[U \mid \tilde{\mathbf{x}}\right] = \left[\begin{array}{ccc|c} 6 & 3 & -8 & 1 \\ 0 & -\frac{5}{2} & 9 & -\frac{5}{2} \\ 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \qquad \mathbf{x} = \begin{bmatrix} -\frac{4}{5} \\ \frac{23}{5} \\ 1 \end{bmatrix}.$$