EMTH211-19S2 Laboratory 9

September 23-27, 2019

These exercises deal with

- Least-squares solutions to inconsistent linear systems
- Least-squares fits to data points
- Pseudoinverses
- Symmetric matrices

Reading guide (Poole, Linear Algebra)

Sections 7.3 and 5.4.

9.1 Section 7.3, Exercises 11, 17, 27, 29, 32.

9.2 Compute the pseudoinverse for A where

(a)

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

9.3 Orthogonally diagonalize

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

The projection matrix becomes $[P] = AA^+ = AA^{-1} = I$. (What is the geometric interpretation of this equality?)

Theorem 7.12 summarizes the key properties of the pseudoinverse of a matrix. (Before reading the proof of this theorem, verify these properties for the matrix in Example 7.32.)

Theorem 7.12 Let A be a matrix with linearly independent columns. Then the pseudoinverse A^+ of A satisfies the following properties, called the ***Penrose conditions*** for A:

a. $AA^+A = A$
b. $A^+AA^+ = A^+$
c. AA^+ and A^+A are symmetric.

Proof We prove condition (a) and half of condition (c) and leave the proofs of the remaining conditions as Exercises 54 and 55.

(a) We compute

$$\begin{aligned} AA^+A &= A((A^TA)^{-1}A^T)A \\ &= A(A^TA)^{-1}(A^TA) \\ &= AI = A \end{aligned}$$

(c) By Theorem 3.4, A^TA is symmetric. Therefore, $(A^TA)^{-1}$ is also symmetric, by Exercise 46 in Section 3.3. Taking the transpose of AA^+, we have

$$\begin{aligned} (AA^+)^T &= (A(A^TA)^{-1}A^T)^T \\ &= (A^T)^T((A^TA)^{-1})^TA^T \\ &= A(A^TA)^{-1}A^T \\ &= AA^+ \end{aligned}$$

Exercise 56 explores further properties of the pseudoinverse. In the next section, we will see how to extend the definition of A^+ to handle *all* matrices, whether or not the columns of A are linearly independent.

Exercises 7.3

CAS

In Exercises 1–3, consider the data points $(1, 0)$, $(2, 1)$, *and* $(3, 5)$. *Compute the least squares error for the given line. In each case, plot the points and the line.*

1. $y = -2 + 2x$ **2.** $y = -3 + 2x$ **3.** $y = -3 + \frac{5}{2}x$

In Exercises 4–6, consider the data points $(-5, 3)$, $(0, 3)$, $(5, 2)$, *and* $(10, 0)$. *Compute the least squares error for the given line. In each case, plot the points and the line.*

4. $y = 2 - x$ **5.** $y = \frac{5}{2}$ **6.** $y = 2 - \frac{1}{5}x$

In Exercises 7–14, find the least squares approximating line for the given points and compute the corresponding least squares error.

7. $(1, 0), (2, 1), (3, 5)$

8. $(1, 5), (2, 3), (3, 2)$

9. $(0, 4), (1, 1), (2, 0)$

10. $(0, 2), (1, 2), (2, 5)$

11. $(-5, -1), (0, 1), (5, 2), (10, 4)$

12. $(-5, 3), (0, 3), (5, 2), (10, 0)$

13. $(1, 1), (2, 3), (3, 4), (4, 5), (5, 7)$

14. $(1, 10), (2, 8), (3, 5), (4, 3), (5, 0)$

In Exercises 15–18, find the least squares approximating parabola for the given points.

15. $(1, 1), (2, -2), (3, 3), (4, 4)$

16. $(1, 8), (2, 7), (3, 5), (4, 2)$

17. $(-2, 4), (-1, 7), (0, 3), (1, 0), (2, -1)$

18. $(-2, 0), (-1, -11), (0, -10), (1, -9), (2, 8)$

In Exercises 19–22, find a least squares solution of $A\mathbf{x} = \mathbf{b}$ by constructing and solving the normal equations.

19. $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

20. $A = \begin{bmatrix} 3 & -2 \\ 1 & -2 \\ 2 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

21. $A = \begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \\ 3 \end{bmatrix}$

In Exercises 23 and 24, show that the least squares solution of $A\mathbf{x} = \mathbf{b}$ is not unique and solve the normal equations to find all the least squares solutions.

23. $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ -1 \\ 1 \end{bmatrix}$

In Exercises 25 and 26, find the best approximation to a solution of the given system of equations.

25.
$$\begin{aligned} x + y - z &= 2 \\ -y + 2z &= 6 \\ 3x + 2y - z &= 11 \\ -x + z &= 0 \end{aligned}$$

26.
$$\begin{aligned} 2x + 3y + z &= 21 \\ x + y + z &= 7 \\ -x + y - z &= 14 \\ 2y + z &= 0 \end{aligned}$$

In Exercises 27 and 28, a QR factorization of A is given. Use it to find a least squares solution of $A\mathbf{x} = \mathbf{b}$.

27. $A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}, R = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ -1 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix},$

$R = \begin{bmatrix} \sqrt{6} & -\sqrt{6}/2 \\ 0 & 1/\sqrt{2} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

29. A tennis ball is dropped from various heights, and the height of the ball on the first bounce is measured. Use the data in Table 7.3 to find the least squares approximating line for bounce height b as a linear function of initial height h.

Table 7.3

h (cm)	20	40	48	60	80	100
b (cm)	14.5	31	36	45.5	59	73.5

30. Hooke's Law states that the length L of a spring is a linear function of the force F applied to it. (See Figure 7.17 and Example 6.92.) Accordingly, there are constants a and b such that

$$L = a + bF$$

Table 7.4 shows the results of attaching various weights to a spring.

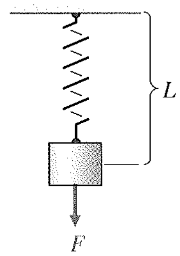

Figure 7.17

Table 7.4

F (oz)	2	4	6	8
L (in.)	7.4	9.6	11.5	13.6

Table 7.5

Year of Birth	1920	1930	1940	1950	1960	1970	1980	1990
Life Expectancy (years)	54.1	59.7	62.9	68.2	69.7	70.8	73.7	75.4

Source: World Almanac and Book of Facts. New York: World Almanac Books, 1999

(a) Determine the constants a and b by finding the least squares approximating line for these data. What does a represent?
(b) Estimate the length of the spring when a weight of 5 ounces is attached.

31. Table 7.5 gives life expectancies for people born in the United States in the given years.
(a) Determine the least squares approximating line for these data and use it to predict the life expectancy of someone born in 2000.
(b) How good is this model? Explain.

32. When an object is thrown straight up into the air, Newton's Second Law of Motion states that its height $s(t)$ at time t is given by

$$s(t) = s_0 + v_0 t + \tfrac{1}{2}gt^2$$

where v_0 is its initial velocity and g is the constant of acceleration due to gravity. Suppose we take the measurements shown in Table 7.6.

Table 7.6

Time (s)	0.5	1	1.5	2	3
Height (m)	11	17	21	23	18

(a) Find the least squares approximating quadratic for these data.
(b) Estimate the height at which the object was released (in m), its initial velocity (in m/s), and its acceleration due to gravity (in m/s^2).
(c) Approximately when will the object hit the ground?

33. Table 7.7 gives the population of the United States at 10-year intervals for the years 1950–2000.
(a) Assuming an exponential growth model of the form $p(t) = ce^{kt}$, where $p(t)$ is the population at time t, use least squares to find the equation for the growth rate of the population. [*Hint:* Let $t = 0$ be 1950.]
(b) Use the equation to estimate the U.S. population in 2010.

Table 7.7

Year	Population (in millions)
1950	150
1960	179
1970	203
1980	227
1990	250
2000	281

Source: U.S. Bureau of the Census

34. Table 7.8 shows average major league baseball salaries for the years 1970–2005.
(a) Find the least squares approximating quadratic for these data.
(b) Find the least squares approximating exponential for these data.
(c) Which equation gives the better approximation? Why?
(d) What do you estimate the average major league baseball salary will be in 2010 and 2015?

Table 7.8

Year	Average Salary (thousands of dollars)
1970	29.3
1975	44.7
1980	143.8
1985	371.6
1990	597.5
1995	1110.8
2000	1895.6
2005	2476.6

Source: Major League Baseball Players Association

35. A 200 mg sample of radioactive polonium-210 is observed as it decays. Table 7.9 shows the mass remaining at various times.

Assuming an exponential decay model, use least squares to find the half-life of polonium-210. (See Section 6.7.)

Table 7.9

Time (days)	0	30	60	90
Mass (mg)	200	172	148	128

36. Find the plane $z = a + bx + cy$ that best fits the data points $(0, -4, 0)$, $(5, 0, 0)$, $(4, -1, 1)$, $(1, -3, 1)$, and $(-1, -5, -2)$.

In Exercises 37–42, find the standard matrix of the orthogonal projection onto the subspace W. Then use this matrix to find the orthogonal projection of **v** *onto W.*

37. $W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

38. $W = \text{span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

39. $W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

40. $W = \text{span}\left(\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

41. $W = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

42. $W = \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right), \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

43. Verify that the standard matrix of the projection onto W in Example 7.31 (as constructed by Theorem 7.11) does not depend on the choice of basis. Take

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}\right\}$$

as a basis for W and repeat the calculations to show that the resulting projection matrix is the same.

44. Let A be a matrix with linearly independent columns and let $P = A(A^TA)^{-1}A^T$ be the matrix of orthogonal projection onto col(A).

(a) Show that P is symmetric.
(b) Show that P is idempotent.

In Exercises 45–52, compute the pseudoinverse of A.

45. $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

46. $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

47. $A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$

48. $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}$

49. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

50. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

51. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

52. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

53. (a) Show that if A is a square matrix with linearly independent columns, then $A^+ = A^{-1}$.
(b) If A is an $m \times n$ matrix with orthonormal columns, what is A^+?

54. Prove Theorem 7.12(b).

55. Prove the remaining part of Theorem 7.12(c).

56. Let A be a matrix with linearly independent columns. Prove the following:

(a) $(cA)^+ = (1/c)A^+$ for all scalars $c \neq 0$.
(b) $(A^+)^+ = A$ if A is a square matrix.
(c) $(A^T)^+ = (A^+)^T$ if A is a square matrix.

57. Let n data points $(x_1, y_1), \ldots, (x_n, y_n)$ be given. Show that if the points do not all lie on the same vertical line, then they have a unique least squares approximating line.

58. Let n data points $(x_1, y_1), \ldots, (x_n, y_n)$ be given. Generalize Exercise 57 to show that if at least $k + 1$ of $x_1, \ldots, x_n$ are distinct, then the given points have a unique least squares approximating polynomial of degree at most k.