EMTH211-19S2 Laboratory 9 Solutions

September 23-27, 2019

These exercises deal with

- Least-squares solutions to inconsistent linear systems
- Least-squares fits to data points
- Pseudoinverses
- Symmetric matrices

Reading guide (Poole, Linear Algebra)

Sections 7.3 and 5.4.

9.1 Section 7.3, Exercises 11, 17, 27, 29, 32.

Solution:

Poole, Section 7.3

11. Find the least squares approximating line for the points
$(-5,-1)$, $(0,1)$, $(5,2)$, $(10,4)$
and compute the least squares error.

Fitting the line
$$a_0 + a_1 x = y$$
to the points gives the equations
$$\left.\begin{aligned} a_0 - 5a_1 &= -1 \\ a_0 + 0a_1 &= 1 \\ a_0 + 5a_1 &= 2 \\ a_0 + 10a_1 &= 4 \end{aligned}\right\}$$
so we want the least squares solution to the system
$$A\underline{a} = \underline{y}$$
where
$$A = \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \quad \text{and} \quad \underline{y} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \text{ are given}$$
and where $\underline{a} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$ is to be found.

We can find the least squares solution by solving the corresponding normal equations
$$A^T A \underline{a} = A^T \underline{y}$$
that is,
$$\begin{bmatrix} 4 & 10 \\ 10 & 150 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6 \\ 55 \end{bmatrix}$$
For tutorial work you can use Matlab to solve this system:
$$\left.\begin{aligned} a_0 &= 0.7 \\ a_1 &= 0.32 \end{aligned}\right\}$$
Indeed, in Matlab, the command $\underline{a} = A \backslash \underline{y}$ will give the least squares solution without seeming to use the normal equations. In an exam you would need to use row reduction.

Hence the least squares approximating line is

Section 7.3 (ctd)

$$y = 0.7 + 0.32x.$$

11. (ctd) The least squares error in the corresponding value of

$$\|A\underline{a} - \underline{y}\| = \left\| \begin{bmatrix} 1 & -5 \\ 1 & 0 \\ 1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.32 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -0.9 \\ 0.7 \\ 2.3 \\ 3.9 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 2 \\ 4 \end{bmatrix} \right\| = 0.447$$

17. Fitting a quadratic

$$a_0 + a_1 x + a_2 x^2 = y$$

to the points

$(-2,4)$, $(-1,7)$, $(0,3)$, $(1,0)$, $(2,-1)$

gives the system of linear equations

$$A\underline{a} = \underline{y}$$

where

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}, \quad \underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \text{ and } \underline{y} = \begin{bmatrix} 4 \\ 7 \\ 3 \\ 0 \\ -1 \end{bmatrix}.$$

Proceeding as in the previous problem, we construct the normal equations

$$A^T A \underline{a} = A^T \underline{y}$$

or

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 13 \\ -17 \\ 19 \end{bmatrix}.$$

You could use Matlab again, or you can use row reduction:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2.6 \\ -1.7 \\ -7 \end{bmatrix}$$

giving the solution

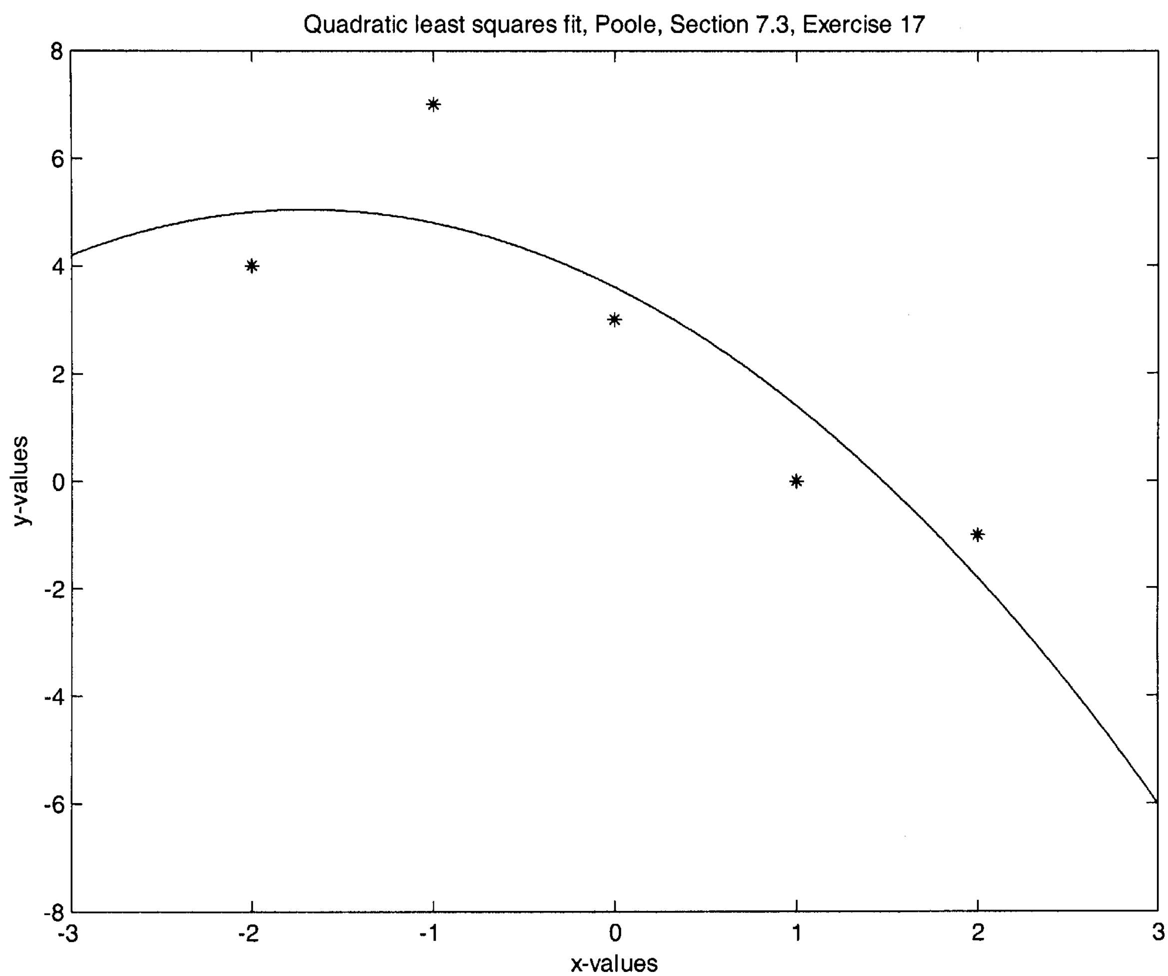
$$a_0 = 3.6$$
$$a_1 = -1.7$$
$$a_2 = -0.5$$

So that the least squares quadratic fit to the points is

$$y = 3.6 - 1.7x - 0.5x^2.$$

The resulting least squares error is again calculated as:

$$\|A\underline{a} - \underline{y}\| = \left\| A \begin{bmatrix} 3.6 \\ -1.7 \\ -0.5 \end{bmatrix} - \underline{y} \right\| = \left\| \begin{bmatrix} 5 \\ 4.8 \\ 3.6 \\ 1.4 \\ -1.8 \end{bmatrix} - \begin{bmatrix} 4 \\ 7 \\ 3 \\ 0 \\ -1 \end{bmatrix} \right\| = 2.966$$


Quadratic least squares fit, Poole, Section 7.3, Exercise 17
y-values
x-values
8
6
4
2
0
-2
-4
-6
-8
-3
-2
-1
0
1
2
3

Section 7.3

27. To solve $A\underline{x} = \underline{b}$ given a QR factorization $A = QR$ we just need to solve

$$R\underline{x} = Q^T\underline{b}$$

or $$\begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Solving this triangular system gives $\underline{x} = \begin{bmatrix} 1.6667 \\ -2 \end{bmatrix}$.

As this clearly does not satisfy $A\underline{x} = \underline{b}$, we must have found a least squares solution, rather than an exact solution. (The original system must have been inconsistent.)

5

No. OF QUESTION

Section 7.3

29. Expressing the bounce height b as a linear function of the initial height h, means fitting a formula

$$b = a_0 + a_1 h$$

to the data

h	20	40	48	60	80	100
b	14.5	31	36	45.5	59	73.5

This gives a system of linear equations

$$A \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \underline{b}$$

where $\underline{b}$ is the vector of b-values and A is the matrix

$$A = \begin{bmatrix} 1 & 20 \\ 1 & 40 \\ 1 & 48 \\ 1 & 60 \\ 1 & 80 \\ 1 & 100 \end{bmatrix}.$$

The corresponding normal equations

$$A^T A \, \underline{a} = A^T b$$

will give the least squares solution to the linear fit problem:

$$\begin{bmatrix} 6 & 348 \\ 348 & 24304 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 260 \\ 18058 \end{bmatrix}.$$

Using Matlab gives

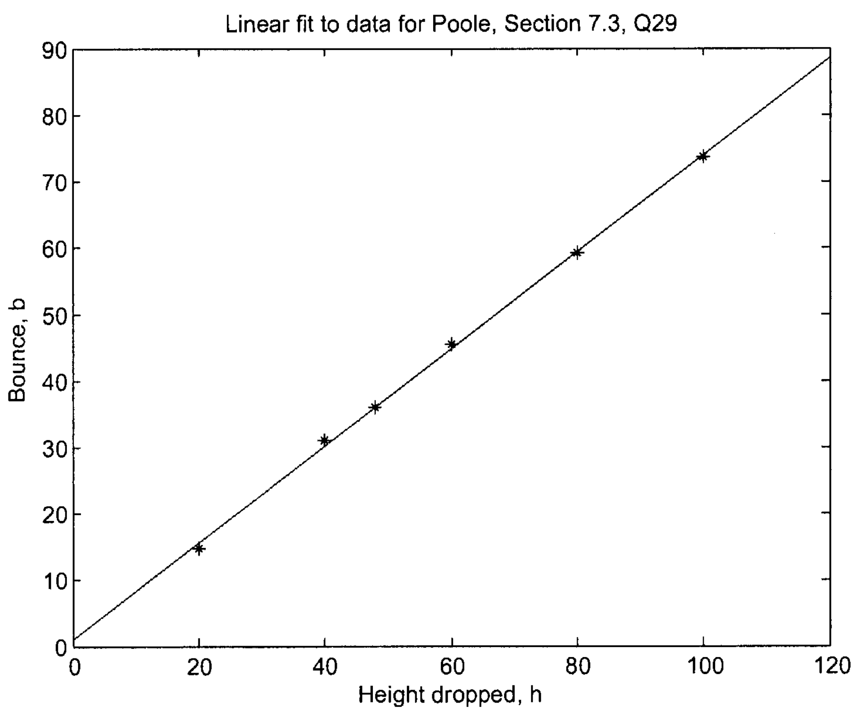
$$a_0 = 0.9184$$
$$a_1 = 0.7299$$

So that the least squares linear fit is the line

$$b = 0.9184 + 0.7299 h.$$

(As the experimental data has only 2 or 3 significant digits, it would make sense to say

$$b = 0.92 + 0.73 h \qquad \text{instead.})$$


Linear fit to data for Poole, Section 7.3, Q29
Bounce, b
Height dropped, h
0
10
20
30
40
50
60
70
80
90
0
20
40
60
80
100
120

Section 7.3

32. Poole seems to expect us to treat all three coefficients as unknown,
(a) even though one of them is $\frac{1}{2}g$.

So we are trying to fit a quadratic

$$a_0 + a_1 t + a_2 t^2 = s$$

to the data

t	0.5	1	1.5	2	3
s	11	17	21	23	18

This gives a system of linear equations

$$A\underline{a} = \underline{s}$$

where $\underline{s}$ is the vector of s-values, and

$$A = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \\ 1 & 1.5 & 2.25 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \quad \text{and} \quad \underline{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

The least squares fit is given by the solution to the normal equations

$$A^T A \underline{a} = A^T \underline{s}$$

or

$$\begin{bmatrix} 5 & 8 & 16.5 \\ 8 & 16.5 & 39.5 \\ 16.5 & 39.5 & 103.125 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 90 \\ 154 \\ 321 \end{bmatrix}$$

Using Matlab to solve this system gives

$$a_0 = 1.9175$$
$$a_1 = 20.3063$$
$$a_2 = -4.9720$$

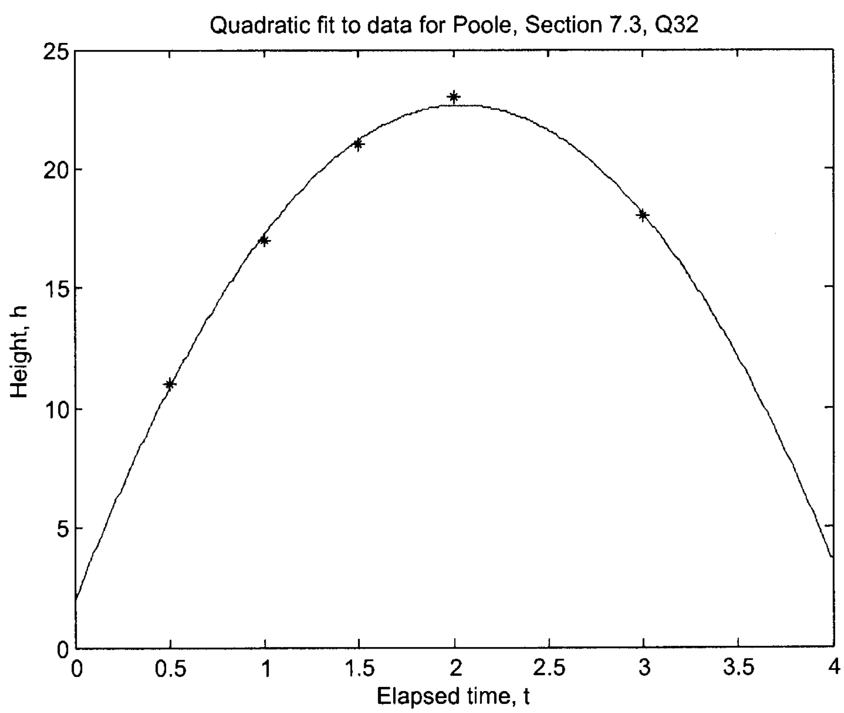
so that the least squares quadratic fit to the data is

$$s = 1.9 + 20t - 5.0t^2$$

(using 2 significant figures).

(b) Height released $= s(0) = 1.9$ metres.
Initial velocity $= s'(0) = 20$ metres/sec.
Acceleration $= 2 \times (\text{coefficient of } t^2) = 9.9$ m/sec^2

(c) Object hits the ground when $s = 0$.
Solving the quadratic gives $t = 4.2$ seconds
(in good agreement with the attached Matlab plot).


Quadratic fit to data for Poole, Section 7.3, Q32
Height, h
Elapsed time, t
0
5
10
15
20
25
0
0.5
1
1.5
2
2.5
3
3.5
4

9.2 Compute the pseudoinverse for A where

(a)

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}$$

(c)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

SOLUTION:

(a)

$$A^T A = 5$$

and so

$$A^+ = (A^T A)^{-1} A^T = \tfrac{1}{5} \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

(b)

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 14 \end{bmatrix}$$

and so

$$A^+ = \tfrac{1}{6} \begin{bmatrix} 2 & -4 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(c)

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and so

$$A^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = A^{-1}.$$

9.3 Orthogonally diagonalize

$$A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

SOLUTION:

The eigenvalues of A are 2, 2, 2 and 4. Now

$$[A - 2I] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the geometric multiplicity of 2 is 3 as expected. The eigenspace E_2 is given by

$$s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We need to choose an orthonormal basis (or use Gram-Schmidt). An orthonormal basis is given by

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since A is symmetric, the remaining eigenvector must be orthogonal to these three eigenvectors. Therefore we can choose

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

to obtain an orthonormal basis. Therefore

$$Q = \begin{bmatrix} 0 & 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

with

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$