* What is a Subspace?
* What is the null space?
* What is a column space?
* What is a row space?

* How big are thos spaces?
(Rank, dimensions, nullity)

- Closed addition multiplication
Sub set $\mathbb{R}^n$
- $A\underset{\sim}{x} = \underset{\sim}{0}$
- Consistent RHS: $A\underset{\sim}{x} = \underset{\sim}{b}$
All linear combos of the cols of A
- All linear combinations of the Rows of A.

Examples of these matrix subspaces

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

* Find the Null space of A null(A)
Space of all solutions to

$A\underset{\sim}{x} = 0$

that is, to the system of linear equations

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$
$$5x_1 + 6x_2 + 7x_3 + 8x_4 = 0$$
$$9x_1 + 10x_2 + 11x_3 + 12x_4 = 0$$

This is a subspace of $\mathbb{R}^4$ (!)

* column space of A: col(A)

$$c_1\begin{bmatrix}1\\5\\9\end{bmatrix} \quad c_2\begin{bmatrix}2\\6\\10\end{bmatrix} \quad c_3\begin{bmatrix}3\\7\\11\end{bmatrix} \quad c_4\begin{bmatrix}4\\8\\12\end{bmatrix}$$

and since this is just all vectors of the form

$$A\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}$$

So it is the space of all RHS'S $\underset{\sim}{b}$ which make

$A\underset{\sim}{x} = \underset{\sim}{b}$ consistent (solvable)

This is a subspace of $\mathbb{R}^3$

* row(A) All linear combinations of our rows

$$c_1(1,2,3,4)$$
$$+\ c_2(5,6,7,8)$$
$$+\ c_3(9,10,11,12)$$

Subspace of $\mathbb{R}^4$ (!)

Extra Example

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$$

a) Is $\underset{\sim}{b} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ in the column space of A?

Column space consists of all linear combinations of the columns of A

$\underset{\sim}{b}$ is 'Linear combination' of the cols of A

$A\underset{\sim}{x} = \underset{\sim}{b}$ consistent

$$\overset{A}{\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}} \overset{\underset{\sim}{b}}{\begin{bmatrix}1\\2\\3\end{bmatrix}} \overset{(!)}{\Rightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix}3\\2\\0\end{bmatrix}$$

yes, system is consistant! ✓

$\underset{\sim}{b}$ is in column space of A $\therefore$ col(A)

b) is $\underset{\sim}{w} = [4,5]$ in a row(A)?

Elementary row operations just create linear combinations of the rows of a matrix

So consider $\begin{bmatrix} A \\ \hline \underset{\sim}{w} \end{bmatrix} \xRightarrow[\text{Elementary row operations}]{\text{Apply}} \begin{bmatrix} U \\ \hline \underset{\sim}{0} \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{bmatrix} \xRightarrow[R_4 - 4R_1]{R_3 - 3R_1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{bmatrix} \xRightarrow{R_4 - 9R_2} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{bmatrix}$$

$\underset{\sim}{w}$ is a linear combination ✓

$\underset{\sim}{w} = 4[1\ -1] + 9[0, 1]$

Basis and Dimension

This answers th "how big" questions.

- How big are these spaces?
- We can measure th size of these spaces by counting th smallest of vectors needed to span that space.

Definition Let S be a subspace of n-space

A basis for S is a set B of vectors

i) B spans th Subspace

ii) B is linearly independant

(so B is a spanning set for th supspace & B doesn't contain any redundent vectors).

Example ① $\{\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\}$ is a basis for $\mathbb{R}^3$ "standard basis" linearly independant

② $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

- What is th basis for th column space?

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$ is a basis for $Col(A)$

In this case 2 columns are just scalar multiples of another. only (1) column is needed

③ What is basis for th row space of A

$\{(1, 2, 3)\}$

④ What about a basis for th null space of A?

Let S be a subspace of n-space.

A __basis__ for S is a set of vectors B such that

loads of bases for any given subspace.
But # of vectors Always remains th same

(*) B spans th subspace

(*) B is linearly independant

Throw away redundant vectors

① $\{\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\}$ is a basis for R^3

② $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ Basis for col(A) is $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$

③ What is null(A)?

③ What about null(A)? $(A\underset{\sim}{x} = \underset{\sim}{0})$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{0}$$

$$x_1 + 2x_2 + 3x_3 = 0$$
$$\cancel{2x_1 + 4x_2 + 6x_3 = 0}$$

→ not needed redundant !!!

Here x_2 & x_3 are free

$$x_2 = s$$
$$x_3 = t$$

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

by inspection

Span ✓

linearly indedendent ?

That is every vector $\underset{\sim}{x} \in$ null(A) is a linear combination of th vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

These are linearly independent (look at th 0's & 1's)

Basis for null(A) is $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right\}$

Dimension & Rank

(How 'big' are these bases)

__Def:__ Th dimension of a subspace S, dim(S), is th number of vectors in its basis.

Let A be an $m \times n$ matrix

→ Th row rank of A is th __dimension__ of Row(A)

→ " column " " " " " ... " " col(A)

→ Th nullity of A " " ... " " null(A)

① $\mathbb{R}^3$ has dimension 3 # must contain 3 vectors

② $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ $\left.\begin{matrix} \dim(\mathrm{col}(A)) = 1 \\ \dim(\mathrm{row}(A)) = 1 \end{matrix}\right\}$ rank of A

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Basis & dimension for null(A)? $A\underset{\sim}{x} = \underset{\sim}{0}$

$$A\underset{\sim}{x} = \underset{\sim}{0}$$

Row operations! pivots

$$U = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

row zeroes

"rref".
"Reduces Row echelon form" (pivots)

free varibles

Reduced system has 2 free varibles

$$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \dots = \begin{bmatrix} s+2t \\ -2s-3t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

__Basis__ for null(A) (as those 2 vectors l.i.) are

$$\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right\}$$

__Dimension__ 2 (nullity)

= 2 number of free varibles in $A\underset{\sim}{x} = \underset{\sim}{0}$

• __Basis__ and __dimension__ for row(A)

row(A) = row(U)

$A \xrightarrow{\text{rref}} U$

$$U = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

__Basis__ for row(A)
$\{(1, 0, -1, -2), (0, 1, 2, 3)\}$

__Dimension__ is 2

seeds of a big idea

If A is an $m \times n$ matrix

$$\mathrm{rank}(A) + \dim(\mathrm{null}(A)) = n$$

Thm poole 3.26

Example

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix}$$

* what is th nullity?

Our theorem bypass to save work

Cols of M are linearly independent (!)

$$\mathrm{rank}(M) = 2$$

rank M + nullity of M = n

$$\text{nullity} = n - \mathrm{rank}(M)$$
$$= 2 - 2$$
$$= 0$$