EMTH211-19S2 Tutorial 8 Solutions

September 16-20, 2019

These exercises deal with

- Orthogonal vectors
- Orthogonal matrices
- Orthogonal complements and projections
- Gram-Schmidt process

Reading guide (Poole, Linear Algebra)

Sections 5.1, 5.2 and 5.3.

All references below are to this book. The exercises listed below are the same in the 2nd, 3rd and 4th editions.

8.1 Section 5.1, Exercises 3, 5, 9, 11, 13, 19, 27.

Solution:

Section 5.1

3. $(3,1,-1)\cdot(-1,2,1) = -2 \neq 0$ so this set is not orthogonal.

5. $(2,3,-1,4)\cdot(-2,1,-1,0) = 0$
$(2,3,-1,4)\cdot(-4,-6,2,7) = 0$
$(-2,1,-1,0)\cdot(-4,-6,2,7) = 0$
} So this set of vectors is orthogonal.

9. If $\underline{v}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\underline{v}_3 = \begin{bmatrix}1\\-1\\1\end{bmatrix}$

then $\underline{v}_1\cdot\underline{v}_2 = 0 = \underline{v}_1\cdot\underline{v}_3 = \underline{v}_2\cdot\underline{v}_3$ so $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ form an orthogonal set in $\mathbb{R}^3$.
Since these vectors are therefore linearly independent, and since $\mathbb{R}^3$ has dimension 3, these vectors give a basis for $\mathbb{R}^3$.
From the notes (or Poole's Theorem 5.2) we can express $\underline{w} = (1,1,1)^T$ in terms of this basis as follows

$$\underline{w} = \frac{\underline{v}_1\cdot\underline{w}}{\underline{v}_1\cdot\underline{v}_1}\underline{v}_1 + \frac{\underline{v}_2\cdot\underline{w}}{\underline{v}_2\cdot\underline{v}_2}\underline{v}_2 + \frac{\underline{v}_3\cdot\underline{w}}{\underline{v}_3\cdot\underline{v}_3}\underline{v}_3.$$

Calculating the dot products gives

$$\underline{w} = \frac{0}{2}\underline{v}_1 + \frac{4}{6}\underline{v}_2 + \frac{1}{3}\underline{v}_3 = \frac{2}{3}\underline{v}_2 + \frac{1}{3}\underline{v}_3.$$

(Check that this really does equal $\underline{w}$!)

11. $(3/5, 4/5)\cdot(-4/5, 3/5) = 0$ (so the vectors are orthogonal) and both vectors have length 1 since $(3/5)^2 + (4/5)^2 = \frac{9+16}{25} = 1$.
Hence we have an orthonormal set.

13. If $\underline{v}_1 = (1/3, 2/3, 2/3)^T$, $\underline{v}_2 = (2/3, -1/3, 0)^T$, $\underline{v}_3 = (1, 2, -5/2)^T$
Then $\underline{v}_1\cdot\underline{v}_2 = 0 = \underline{v}_1\cdot\underline{v}_3 = \underline{v}_2\cdot\underline{v}_3$ so we have an orthogonal set.
But although $\|\underline{v}_1\| = 1$, the other two vectors do not have length 1:

$$\|\underline{v}_2\| = \sqrt{\tfrac{4}{9} + \tfrac{1}{9} + 0} = \tfrac{1}{3}\sqrt{5}$$

and $$\|\underline{v}_3\| = \sqrt{(1 + 4 + \tfrac{25}{4})} = \tfrac{1}{2}\sqrt{45} = \tfrac{3}{2}\sqrt{5}$$

So we do not have an orthonormal set.
For an orthonormal set we divide each vector by its length:

$$\underline{v}_1,\ \frac{3}{\sqrt{5}}\underline{v}_2,\ \frac{2}{3\sqrt{5}}\underline{v}_3$$

Section 5.1

19. To show that a matrix $Q = \begin{bmatrix} \underline{v}_1 & \vdots & \underline{v}_2 & \vdots & \underline{v}_3 \end{bmatrix}$ is orthogonal,

you can either show that $QQ^T = I$ (the identity matrix),
or show that $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is an orthonormal set.
The actual calculations are identical, whichever way you do it!
Here

$$\underline{v}_1 = \begin{bmatrix} \cos\theta \sin\theta \\ \cos^2\theta \\ \sin\theta \end{bmatrix}, \quad \underline{v}_2 = \begin{bmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad \underline{v}_3 = \begin{bmatrix} -\sin^2\theta \\ -\cos\theta\sin\theta \\ \cos\theta \end{bmatrix}.$$

It is easy to see that

$$\underline{v}_1 \cdot \underline{v}_2 = 0 = \underline{v}_2 \cdot \underline{v}_3$$

and almost as easy to see that

$$\begin{aligned} \underline{v}_1 \cdot \underline{v}_3 &= -\cos\theta\sin^3\theta - \sin\theta\cos^3\theta + \sin\theta\cos\theta \\ &= -\cos\theta\sin\theta(\sin^2\theta + \cos^2\theta) + \sin\theta\cos\theta \\ &= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0. \end{aligned}$$

So the vectors are orthogonal.
It is easy to see that $\|\underline{v}_2\| = 1$. For $\underline{v}_1, \underline{v}_3$ the calculations are a little messier:

$$\begin{aligned} \|\underline{v}_1\|^2 &= (\cos\theta\sin\theta)^2 + \cos^4\theta + \sin^2\theta \\ &= \cos^2\theta(\sin^2\theta + \cos^2\theta) + \sin^2\theta \\ &= \cos^2\theta + \sin^2\theta = 1 \end{aligned}$$

and similarly $\|\underline{v}_3\|^2 = 1$. So $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ is an orthonormal set. Hence Q is an orthogonal matrix.

Hence $Q^{-1} = Q^T$.

27. The angle between $\underline{x}$ and $\underline{y}$ is determined by the equation

$$\underline{x} \cdot \underline{y} = \|\underline{x}\| \, \|\underline{y}\| \cos\theta$$

In other words

$$\cos\theta = \frac{\underline{x}\cdot\underline{y}}{\|\underline{x}\| \, \|\underline{y}\|}$$ gives the angle θ between $\underline{x}, \underline{y}$.

So the angle ϕ between $Q\underline{x}$ and $Q\underline{y}$ is given by

$$\cos\phi = \frac{(Q\underline{x})\cdot(Q\underline{y})}{\|Q\underline{x}\| \, \|Q\underline{y}\|} \underset{\uparrow \text{ since } Q \text{ is orthogonal}}{=} \frac{\underline{x}\cdot\underline{y}}{\|\underline{x}\| \, \|\underline{y}\|} = \cos\theta$$

So the angles ϕ and θ must be equal.

8.2 Let W be the subspace of $\mathbf{R}^3$ spanned by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}.$$

Apply the Gram Schmidt Process to obatin an orthogonal basis for W. Furthermore find the orthogonal decomposition of

$$\mathbf{v} = \begin{bmatrix} 4 \\ -4 \\ 3 \end{bmatrix}$$

with respect to W.

SOLUTION:
Using GS, we have

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 2 \end{bmatrix}$$

Note that we can tidy this basis up by choosing

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

We have an orthogonal basis for W (it does not matter which $\mathbf{v}_2$ we use). Thus

$$\mathbf{v}_\parallel = \frac{1}{\mathbf{v}_1^T\mathbf{v}_1}\,\mathbf{v}_1\,\mathbf{v}_1^T\,\mathbf{v} + \frac{1}{\mathbf{v}_2^T\mathbf{v}_2}\,\mathbf{v}_2\,\mathbf{v}_2^T\,\mathbf{v}$$
$$= 0\,\mathbf{v}_1 + \tfrac{2}{9}\,\mathbf{v}_2 = \tfrac{1}{9}\begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$$

using

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

The perpendicular component (that is the component in $W^\perp$) is

$$\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = \tfrac{19}{9}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

8.3 Find an orthogonal basis for $\mathbf{R}^4$ that contains the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}.$$

SOLUTION:

First we notice that the two vectors

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

are actually orthogonal. It is easy to see that together with $\mathbf{x}_3 = [0\ 0\ 1\ 0]^T$ and $\mathbf{x}_4 = [0\ 0\ 0\ 1]^T$ they form a basis. So we can start GS (skipping the first two steps, since we already have to orthogonal vectors).

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

which, after scaling is

$$\mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ 6 \end{bmatrix}.$$

And

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$$

which, after scaling leads to

$$\mathbf{v}_4 = \begin{bmatrix} 2 \\ -5 \\ 0 \\ -1 \end{bmatrix}.$$

8.4 Use Gram Schmidt to find an orthogonal basis for the column space of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

SOLUTION:

Denote the columns of the matrix A by $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. Orthogonalise using Gram Schmidt. Set $\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Then set

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

To avoid fractions in this hand calculation we take

$$\mathbf{v}_2' = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Now take

$$\begin{aligned} \mathbf{v}_3 &= \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2'}{\mathbf{v}_2' \cdot \mathbf{v}_2'}\mathbf{v}_2' \\ &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{5}{6} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix}. \end{aligned}$$

Then $\{\mathbf{v}_1, \mathbf{v}_2', \mathbf{v}_3\}$ is the required orthogonal basis.

8.5 Let $\mathbf{v}$ be any non-zero column vector in $\mathbf{R}^n$. Show that

$$H = I - \frac{2}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T$$

is an orthogonal matrix where I is the $n \times n$ identity matrix. Let $W = \text{span}(\mathbf{v})$. Show that

$$H\mathbf{x} = \begin{cases} -\mathbf{x} & \text{if } \mathbf{x} \in W \\ \mathbf{x} & \text{if } \mathbf{x} \in W^\perp. \end{cases}$$

SOLUTION:

$$\begin{aligned} H^T H &= \left(I - \frac{2}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T\right)\left(I - \frac{2}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T\right) \\ &= I - \frac{4}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T + \frac{4}{(\mathbf{v}^T \mathbf{v})^2} \mathbf{v} \mathbf{v}^T \mathbf{v} \mathbf{v}^T \\ &= I \end{aligned}$$

Therefore H is orthogonal. Now

$$H\mathbf{x} = \mathbf{x} - \frac{2}{\mathbf{v}^T \mathbf{v}} \mathbf{v} \mathbf{v}^T \mathbf{x}.$$

If $\mathbf{x} \in W$ then $\mathbf{x} = \alpha \mathbf{v}$ and

$$H\mathbf{x} = \mathbf{x} - 2\alpha \mathbf{v} = -\mathbf{x}.$$

If $\mathbf{x} \in W^\perp$ then $\mathbf{v}^T \mathbf{x} = 0$ and so

$$H\mathbf{x} = \mathbf{x}.$$

8.6 Compute the QR-factorisation of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

and use this factorisation to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = [1, 2, 3, 4]^T$.

SOLUTION:

Orthogonalising (and scaling so that there are no fractions) we get the vectors

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 3 \\ 3 \end{bmatrix}$$

These have norms 4, 20, 20, so normalising (roots) of these we get

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2} & \frac{3}{2\sqrt{5}} & -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2} & \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{bmatrix}$$

Since $A = QR$ is equivalent to $R = Q^T A$ we get

$$R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{10}{2\sqrt{5}} & \frac{12}{2\sqrt{5}} \\ 0 & 0 & \frac{6}{2\sqrt{5}} \end{bmatrix}$$

Hence to solve $A\mathbf{x} = \mathbf{b}$ we can solve the simpler $R\mathbf{x} = Q^T\mathbf{b}b$. That is

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{10}{2\sqrt{5}} & \frac{12}{2\sqrt{5}} \\ 0 & 0 & \frac{6}{2\sqrt{5}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ \frac{16}{2\sqrt{5}} \\ \frac{18}{2\sqrt{5}} \end{bmatrix}$$

By back substitution we get

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 3 \end{bmatrix}.$$