EMTH211 — Exam Questions

Topics: Linear Systems, Matrices, Vector Spaces, and Norms

1. (a) *Briefly* describe the technique of partial pivoting and state its purpose when numerically solving linear systems.

Partial pivoting is the process of performing row swaps to maximise pivot elements in the process of solving a linear system. It is used to reduce round-off error.

(b) Suppose that, instead of the usual definition, the dot product of any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$ were defined as the product of the (Euclidean) lengths of the vectors. Are each of the following statements true or false? For each, either provide a brief reason why the statement is true, or a counterexample if the statement is false.

For any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and any real number k,

(i) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(ii) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

(iii) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$ (where k is a scalar)

(iv) $\mathbf{u} \cdot \mathbf{u} \geq 0$ and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

(i) True; follows from commutativity of multiplication in $\mathbb{R}$.

(ii) False; counterexample e.g. $\mathbf{u} = [0, 1]$, $\mathbf{v} = [0, 1]$, $\mathbf{w} = [1, 0]$. Then

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = 1 \times (\sqrt{2}) = \sqrt{2} \qquad \text{but} \qquad \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} = 1 \times 1 + 1 \times 1 = 2.$$

(iii) False. If $k < 0$ and $\mathbf{u}, \mathbf{v}$ are nonzero (e.g. $k = -1, \mathbf{u} = \mathbf{v} = [1, 1]$) then $(k\mathbf{u}) \cdot \mathbf{v} > 0$ but $k(\mathbf{u} \cdot \mathbf{v}) < 0$.

(iv) True; follows from the nonnegativity and zero of the real function $f(x) = x^2$.

(c) The line $\mathcal{L}$ passes through the point $(1, 1, 1)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$.

For each of the following planes $\mathcal{P}$, explain briefly whether $\mathcal{L}$ and $\mathcal{P}$ are parallel, perpendicular, or neither.

(i) $x + 5y + 2z = 3$, (ii) $2x - 3y + z = 4$, (iii) $-6x - 2y + 8z = 2$.

(i) The normal to $\mathcal{P}$ is $\mathbf{n} = [1, 5, 2]^T$. Since $\mathbf{n} \cdot \mathbf{d} = 0$, $\mathbf{n}$ and $\mathbf{d}$ are perpendicular, so $\mathcal{L}$ is parallel to $\mathcal{P}$.

(ii) The normal to $\mathcal{P}$ is $\mathbf{n} = [2, -3, 1]^T$. Since $\mathbf{n} \cdot \mathbf{d} \neq 0$ and $\mathbf{n} \neq k\mathbf{d}$ for all k, $\mathcal{L}$ is neither parallel nor perpendicular to $\mathcal{P}$.

(iii) The normal to $\mathcal{P}$ is $\mathbf{n} = [-6, -2, 8]^T$. Since $\mathbf{n} = -2\mathbf{d}$, $\mathbf{n}$ and $\mathbf{d}$ are parallel, so $\mathcal{L}$ is perpendicular to $\mathcal{P}$.

(d) Prove that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are linearly independent vectors and $c_1, c_2, \ldots, c_n$ are nonzero constants, then

$$\{c_1\mathbf{u}_1, c_2\mathbf{u}_2, \ldots, c_n\mathbf{u}_n\}$$

is a linearly independent set.

Consider the equation

$$b_1(c_1\mathbf{u}_1) + b_2(c_2\mathbf{u}_2) + \cdots + b_n(c_n\mathbf{u}_n) = 0.$$

Since we can rewrite this as

$$(b_1c_1)\mathbf{u}_1 + (b_2c_2)\mathbf{u}_2 + \cdots + (b_nc_n)\mathbf{u}_n = 0,$$

the first equation holds if and only if $b_ic_i = 0$ for all i, by linear independence of the $\mathbf{u}_i$. Since each c_i is nonzero, this implies that each $b_i = 0$, and thus that the $c_i\mathbf{u}_i$ are also linearly independent.

(e) Let

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 4 & 1 & 2 \\ 8 & 0 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 15 \\ -4 \\ 1 \end{bmatrix}.$$

(i) Find the P^TLU decomposition of A.

(ii) Use the P^TLU decomposition of A to solve the system $A\mathbf{x} = \mathbf{b}$.

(i) First we row reduce A. Any row permutations required will be captured by the permutation matrix P.

$$\begin{bmatrix} 0 & 0 & 5 \\ 4 & 1 & 2 \\ 8 & 0 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 1 & 2 \\ 0 & 0 & 5 \\ 8 & 0 & 3 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 4 & 1 & 2 \\ 0 & 0 & 5 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

We observe we needed $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$ so the corresponding permutation matrix P is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \text{so} \qquad P^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus we row reduce

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 5 \\ 4 & 1 & 2 \\ 8 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 2 \\ 8 & 0 & 3 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix} = U$$

and so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$A = P^T LU = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

(ii) Observe that $PA\mathbf{x} = LU\mathbf{x} = P\mathbf{b}$ so that we can write

$$U\mathbf{x} = \mathbf{y} \qquad \text{and} \qquad L\mathbf{y} = P\mathbf{b}.$$

Forward substitution on $L\mathbf{y} = P\mathbf{b}$ gives

$$\mathbf{y} = \begin{bmatrix} -4 \\ 9 \\ 15 \end{bmatrix}$$

and back substitution on $U\mathbf{x} = \mathbf{y}$ then gives

$$\mathbf{x} = \begin{bmatrix} -1 \\ -6 \\ 3 \end{bmatrix}$$

2. (a) Let V be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$ with the usual definition of addition and multiplications by a constant. If E is the subset of even functions (that is, functions for which $f(-x) = f(x)$), and O is the subset of odd functions (for which $f(-x) = -f(x)$), prove that:

(i) E and O are subspaces of V; and

(ii) $E \cap O = \{\mathbf{0}\}$.

(i) If $f, g \in E$ and c is a scalar, then

$$(f + cg)(-x) = f(-x) + cg(-x) = f(x) + cg(x) = (f + cg)(x),$$

so $f + cg \in E$; that is, E is a subspace.

Likewise if f, g are both odd; so O is also a subspace.

(ii) If $f \in E \cap O$, then $-f(x) = f(-x) = f(x)$, so $f(x) = 0$.

(b) (i) Find a basis for

$$\operatorname{span}\left\{1 - x, x - x^2, 1 - x^2, 1 - 2x + x^2\right\}$$

in P_2. Then find the coordinates of $p(x) = 2 - 5x + 3x^2$ relative to this basis.

(ii) Let

$$A = \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 6 & 1 & 2 \\ 3 & 0 & -2 & 4 \end{bmatrix}.$$

Give bases for $\operatorname{row}(A)$ and $\operatorname{col}(A)$. Find, also, the rank and nullity of A.

(iii) If B is a general 3×4 matrix, what are the possible values of the nullity of B? (Give a *brief* reason for your answer.)

(i) Begin by constructing a matrix whose rows are the components of these polynomials. Then reduce it to echelon form. So we have

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that a basis can be taken as $\{1 - x, x - x^2\}$. (There are other possibilities.) We want to write

$$p(x) = 2 - 5x + 3x^2 = a(1 - x) + b(x - x^2)$$

which, equating coefficients, leads to

$$\begin{aligned} a \quad\;\; &= 2 \\ -a + b &= -5 \\ -b &= 3 \end{aligned}$$

which has solution $a = 2, b = -3$. So the coordinate vector of $p(x)$ relative to $\{1 - x, x - x^2\}$ is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(ii) Start by row reducing A.

$$\begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 6 & 1 & 2 \\ 3 & 0 & -2 & 4 \end{bmatrix} \xrightarrow{R_3 - 3R_1} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 6 & 1 & 2 \\ 0 & -12 & -2 & -5 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 6 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So a basis for row(A) is

$$\left\{ \begin{bmatrix} 1 & 4 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix} \right\};$$

and a basis for col(A) is the columns of A corresponding to columns with pivots in the row echelon form of A:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right\}.$$

Furthermore, $\operatorname{rank}(A) = 3$ and $\operatorname{nullity}(A) = 1$ (since $\operatorname{rank}(A) + \operatorname{nullity}(A) = 4$).

(iii) $\operatorname{rank}(B)$ cannot be greater than 3 (the minimum of the number of rows and columns); that is,

$$\operatorname{rank}(B) = 0, 1, 2, \text{ or } 3.$$

Since $\operatorname{rank}(B) + \operatorname{nullity}(B) = 4$,

$$\operatorname{nullity}(B) = 1, 2, 3, \text{ or } 4.$$

(c) Let $A = \begin{bmatrix} 1 & -1 \\ -t & 1 \end{bmatrix}$, with $0 < t < 1$.

(i) Find the condition number $k(A)$ using the ∞-norm.

(ii) What happens to $k(A)$ as $t \to 1$? Describe col(A) in the two cases $t \neq 1$ and $t = 1$.

(iii) Consider the equation $A\mathbf{x} = \mathbf{b}$. Explain *briefly* the role of $k(A)$ in solving this equation when $\mathbf{b}$ may be subject to error. (No proofs are required; an equation may help.)

(iv) Solve the equation with $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and $t \neq 1$). Using (b) above, explain why you would not expect a solution when $t = 1$.

(i) $A^{-1} = \dfrac{1}{1-t}\begin{bmatrix} 1 & 1 \\ t & 1 \end{bmatrix}$ so that

$$\begin{aligned} \|A\|_\infty &= \max\{2, 1+t\} = 2, \text{ and} \\ \|A^{-1}\|_\infty &= \frac{1}{1-t}\max\{2, 1+t\} = \frac{2}{1-t}, \text{ so that} \\ k(A) &= \frac{4}{1-t}. \end{aligned}$$

(ii) As $t \to 1$, $k(A) \to \infty$.
If $t \neq 1$, the columns of A are linearly *independent* and so $\mathrm{col}(A) = \mathbb{R}^2$.
If $t = 1$, the columns of A are linearly *dependent* and so $\mathrm{col}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

(iii) If $A\mathbf{x} = \mathbf{b}$ and also $A\mathbf{x}' = \mathbf{b}'$ then

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq k(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

where $\delta\mathbf{x} = \mathbf{x}' - \mathbf{x}$ (and similarly for $\mathbf{b}$).
I.e. the relative error in $\mathbf{x}$ can be as large as $k(A)$ times the relative error in $\mathbf{b}$.

(iv) We have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{1-t}\begin{bmatrix} 1 & 1 \\ t & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{1-t}\begin{bmatrix} 1 \\ t \end{bmatrix}.$$

When $t = 1$, $\mathbf{b} \notin \mathrm{col}(A)$ and so $A\mathbf{x} = \mathbf{b}$ cannot have a solution in that case.