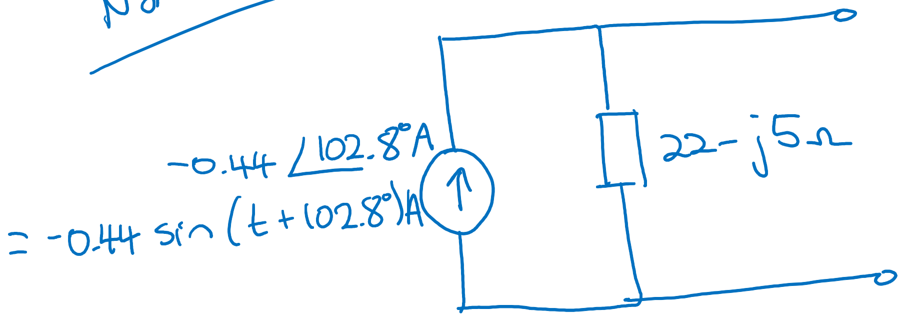
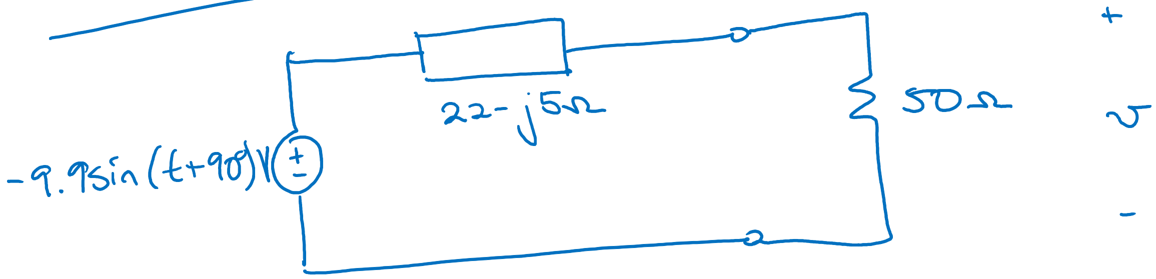
Norton Equiv.



source transform:



$$\bar{V}_{TH} = \bar{I}\bar{Z}$$
$$= -0.44 \angle 102.8^\circ \times 22 - j5$$
$$= -0.44 \angle 102.8^\circ \times 22.56 \angle -12.8^\circ$$
$$= -9.9 \angle 90^\circ \ V$$

$$\bar{V} = \frac{50}{22 - j5 + 50} \times -9.9 \angle 90^\circ$$
$$= \frac{-495 \angle 90^\circ}{72.17 \angle -3.97^\circ}$$
$$= -6.86 \angle 93.97^\circ \ V$$

$$v(t) = -6.86 \sin(t + 93.97^\circ) V$$

Complex Frequency; Definition of the Laplace Transform

Readings: Sections 14.1, 14.2

Complex Frequency

If we have a damped sinusoid for a source, $x(t) = Ae^{\sigma t}\cos(\omega t + \emptyset)$, then things get even more complicated. We can represent various types of waveforms, depending on the values of σ and ω.

1. $\sigma = \omega = 0$

   $x(t) = A\cos(\phi)$

   $\Rightarrow$ a constant i.e. DC

2. $\sigma = 0$

   $x(t) = A\cos(\omega t + \phi)$

   $\Rightarrow$ a sinusoid

3. $\omega = 0$

   $x(t) = Ae^{\sigma t}\cos(\phi)$

   $\Rightarrow$ an exponential

4. Neither equal 0

   $x(t) = Ae^{\sigma t}\cos(\omega t + \phi)$

   $\Rightarrow$ a damped sinusoid

Using Euler's formula, we can rearrange the general form to get:

$x(t) = Re\{Ae^{(\sigma + j\omega)t + j\phi}\} = Re\{Ae^{j\phi}e^{st}\}$

Complex frequency $= s = \sigma + j\omega$

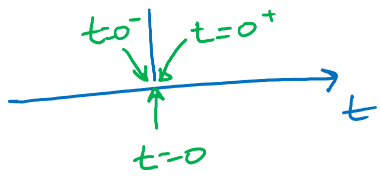
The Laplace Transform

For a general function f(t), the definition of the Laplace Transform is:

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

As you can see from the limits of integration, this covers all of both positive and negative time. However, we're usually only interested in what happens after $t = 0$ s, so we usually use the one-sided Laplace Transform instead:

$$\boxed{F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt}$$



We use 0^- rather than 0 or 0^+, as we want to be able to use the initial conditions of the circuit.

Remember:

$t(0^-) \neq t(0) \neq t(0^+)$

$\therefore$ $v(0^-)$, $v(0)$, $v(0^+)$ could all be different, as can $i(0^-)$, $i(0)$, $i(0^+)$

Voltage across a capacitor, v_c, and current through an inductor, i_L, are special cases. Other voltages and currents can change instantly.

Notation (hand-written):

$$\mathcal{L}[f(t)] = F(s)$$

Notation (typed):

$$\mathcal{L}[f(t)] = \mathbf{F}(\mathbf{s})$$

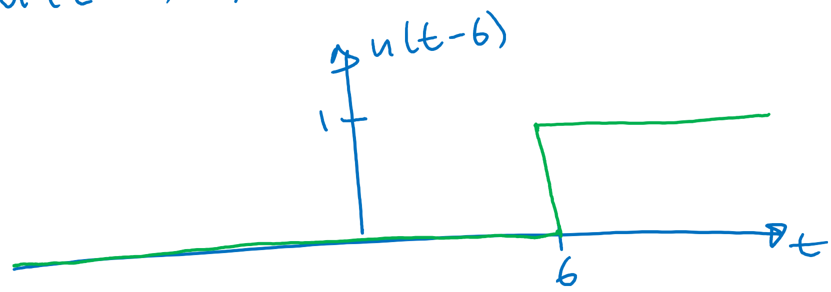
Example:

Work out the LT of $f(t) = 3u(t-6)$ using the LT formula.

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$

$$= \int_{0^-}^{\infty} e^{-st} \times 3u(t-6)\, dt$$

$u(t-6)$ looks like:



$$\therefore \quad F(s) = 3 \int_{6}^{\infty} e^{-st}\, dt$$

$$= 3 \left[-\frac{1}{s} e^{-st} \right]_6^{\infty}$$

$$= 3 \left[0 - \frac{-1}{s} e^{-6s} \right]$$

$$= \frac{3}{s} e^{-6s}$$

Example:

What is the one-sided LT of $f(t) = e^{-3t}$?

$$F(s) = \int_{0^-}^{\infty} e^{-st} e^{-3t}\, dt$$

$$= \left[\frac{-1}{s+3} e^{-(s+3)t} \right]_{0^-}^{\infty}$$

$$= 0 - \left(-\frac{1}{s+3} \right)$$

$$= \frac{1}{s+3}$$

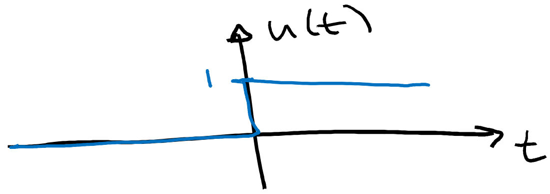
Laplace transforms of simple time functions; Basic theorems for the Laplace transform

Readings: Sections 14.3,14.5

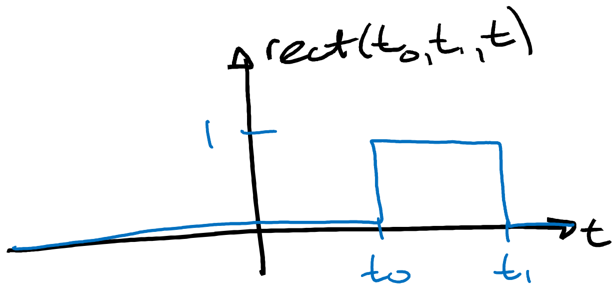
More Useful Functions

Reminder: in Term 2 we looked at the unit step function, $u(t)$, and the rectangular pulse function, $rect(t_0, t_1, t)$:

$u(t) \Rightarrow$

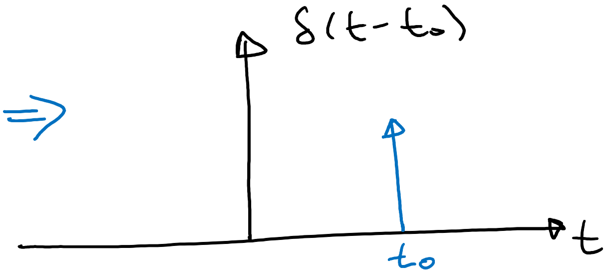


$rect(t_0, t_1, t)$



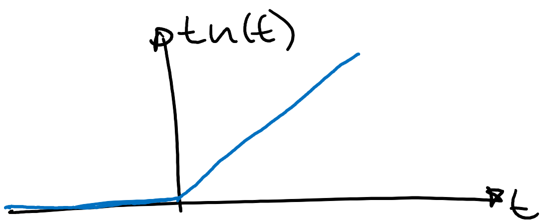
We will add to that the unit impulse, $\delta(t - t_0)$, and the ramp function, $tu(t)$.

$\delta(t - t_0) \Rightarrow$



The unit impulse has *infinite* amplitude at one instant in time. The area "under" the unit impulse is defined as equal to 1. A really useful function.

$tu(t) \Rightarrow$



The unit ramp has zero amplitude for t < 0 s, but is equal to t for positive time.

Properties of the Laplace transform

To be able to use the LT on a circuit, we need to know a few more things… Equations for circuits with capacitors and inductors in them usually look something like this (assuming initial conditions are zero):

$$v(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i(t)\,dt$$

So – can we add LTs? How do we deal with differentials and integrals?

1. Linearity: Since the LT is calculated using integration, and we know that $\int (a+b) = \int a + \int b$, then it makes sense that:

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$
$$= F_1(s) + F_2(s)$$

Similarly, if b is a constant, then:

$$\mathcal{L}[bf(t)] = b\mathcal{L}[f(t)]$$
$$= bF(s)$$

2. Differentiation and Integration in the Time Domain: Using integration by parts you can show that:

$$\mathcal{L}[df/dt] = sF(s) - f(0^-)$$
$$\mathcal{L}[d^2f/dt^2] = s^2F(s) - sf(0^-) - f'(0^-)$$

$$\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{1}{s}F(s)$$

3. Time Shift: This helps us deal with situations where we might have different switches turning on at different points in time.

$$\mathcal{L}[f(t-a)u(t-a)] = \int_{0^-}^{\infty} e^{-st} f(t-a)\,u(t-a)\,dt$$

$$= \int_{a^-}^{\infty} e^{-st} f(t-a)\,dt$$

set $\tau = t-a$

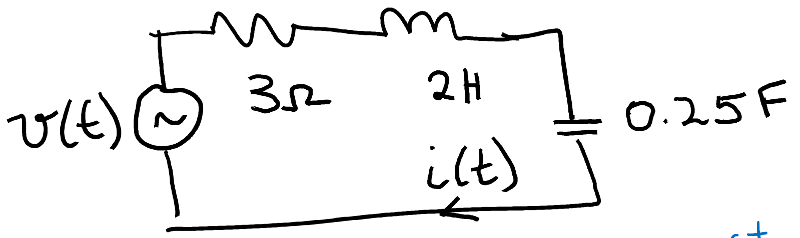
$$= \int_{0^-}^{\infty} e^{-s(\tau+a)} f(\tau)\,d\tau$$

$$= e^{-as}\int_{0^-}^{\infty} e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-as}F(s)$$

Example:

For the circuit below, find **I(s)** in terms of **V(s)** if $i(0^-) = 0$ A.



$$v(t) = 3i(t) + 2\frac{di}{dt} + 4\int_0^t i(\tau)\,d\tau$$

$$V(s) = 3I(s) + 2\left(sI(s) - \cancel{i(0^-)}^{\,0}\right) + \frac{4}{s}I(s)$$

$$= 3I(s) + 2sI(s) + \frac{4}{s}I(s)$$

$$I(s) = \frac{V(s)}{3 + 2s + 4/s}$$

$$I(s) = \frac{sV(s)}{3s+2s^2+4}$$

What does **I(s)** look like if $\mathrm{v(t) = u(t)}$?

$$V(s) = \int_{0^-}^{\infty} e^{-st} u(t)\, dt$$

$$= \int_{0}^{\infty} e^{-st}\, dt$$

$$= \frac{1}{s}$$

$$I(s) = \frac{s\,(1/s)}{2s^2+3s+4}$$

$$= \frac{1}{2s^2+3s+4}$$

We can use tables of LT pairs rather than integrating – this is normally faster and easier. The tables are at the start of this book, and are the same ones you will get in the exam.

General Process

1. Write an equation for $\mathrm{i(t)}$ and/or $\mathrm{v(t)}$.
2. Take the Laplace Transform and get **I(s)** and/or **V(s)**
3. Rearrange the equation(s) as desired
4. Take the inverse LT to get back into the time domain (see next section)