**UNIVERSITY OF CANTERBURY**

# Final Exam

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| **Prescription Number:** | **EMTH211-16S2** |

Time allowed: 180min

Attempt ALL 6 questions.

Write your answers in the spaces provided.

There is a *total* of 60 points.

Use black or blue ink. Do not use pencil except for diagrams.

Only UC approved calculators are allowed.

Show all working. Write neatly. Marks will be lost for poorly presented answers.

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| Family name: | |
| Given names: | |
| Student ID: | |

| **MARKS** Office Use Only | |
|---|---|
| Q1 | |
| Q2 | |
| Q3 | |
| Q4 | |
| Q5 | |
| Q6 | |
| **Total** | |

**Question 1** [10 points]

Assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are linearly independent.

1. Are the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_3$ also linearly independent?
2. Are the vectors $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{v}_1$ also linearly independent?

**Solution:**

1. These are linearly independent, since

$$\begin{aligned} & a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3 = \mathbf{0} \\ \Longrightarrow \quad & a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_2 + \mathbf{v}_3) + c(\mathbf{v}_1 + \mathbf{v}_3) = \mathbf{0} \\ \Longrightarrow \quad & a + c = a + b = b + c = 0 \end{aligned}$$

   by the linear independence of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. This last system has only the trivial solution

$$a = b = c = 0,$$

   so the vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are linearly independent.

2. These are not linearly independent, since

$$\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = \mathbf{0}.$$

   That is the zero vector can be written as a linear combination with some nonzero coefficients.

**Question 2** [10 points]

Consider the following matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 3 & 0 & 8 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 5 & k \end{bmatrix}$$

where $k \in \mathbb{R}$.

(a) Determine the row rank (that is the dimension of the row space) of the matrix $A$ depending on $k$

(b) Find a basis for the null-space

$$\text{null}(A) = \{\mathbf{x} \in V : A\mathbf{x} = \mathbf{0}\}$$

of $A$, in the case that $k = -4$

(c0 How are the row rank and the nullity (the dimension of the null space) of a general matrix $B$ related in general?

**Solution**

(a) The row echelon form is

$$A \rightsquigarrow \begin{bmatrix} 1 & 3 & 1 & 4 \\ 0 & -3 & -2 & 0 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 0 & k+4 \end{bmatrix}.$$

That means that, if $k = -4$ three rows are linearly independent, and therefore form a basis for the row-space. If $k \neq -4$ all four rows are linearly independent and therefore form a basis for the row-space.

(b) To find the null-space we can also continue from the row echelon form and use back-substitution to solve the homogenous system: $x_4 = s$ (free parameter), $x_3 = \frac{4}{5}s$, $x_2 = -\frac{8}{15}s$, and $x_1$, so $x_1 = \frac{-16}{5}s$. Thus the general solution of the homogeneous system is

$$\mathcal{N} = \left\{ s \begin{bmatrix} -\frac{16}{5} \\ \frac{-8}{15} \\ -\frac{4}{5} \\ 1 \end{bmatrix} : s \in \mathbb{R} \right\}.$$

with the vector above being a basis for the null-space $\mathcal{N}$.

(c) If $B$ is $m \times n$ then

$$\text{rank}(B) + \text{nullity}(B) = n = \quad \text{the dimension of the domain.}$$

**Question 3** [10 points]

(a) Show that the *parallelogram-law* holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2$$

(b) Give examples that show that it fails for the 1-norm and for the $\infty$-norm.

**Solution:**

(a) Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then

$$\begin{aligned}
\|x+y\|_2^2 + \|x-y\|_2^2 &= \sum_{i=1}^n (x_i + y_i)^2 + \sum_{i=1}^n (x_i - y_i)^2 \\
&= \sum_{i=1}^n (x_i^2 + y_i^2 + 2x_i y_i) + \sum_{i=1}^n (x_i^2 + y_i^2 - 2x_i y_i) \\
&= 2\sum_{i=1}^n x_i^2 + 2\sum_{i=1}^n y_i^2 \\
&= 2\|x\|_2^2 + 2\|y\|_2^2 .
\end{aligned}$$

(b) Let $x = (1, 0)$ and $y = (0, 1)$ vectors in $\mathbb{R}^2$. Then $x + y = (1, 1)$ and $x - y = (1, -1)$. Therefore

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 1 + 1 = 2 \neq 4 = 2 + 2 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2 .$$

Let $x = (1, 1)$ and $y = (1, -1)$ vectors in $\mathbb{R}^2$. Then $x + y = (2, 0)$ and $x - y = (0, 2)$. Therefore

$$\|x+y\|_1^2 + \|x-y\|_1^2 = 4 + 4 = 8 \neq 16 = 8 + 8 = 2\|x\|_1^2 + 2\|y\|_1^2 .$$

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**Question 4** [10 points]

(a) What is an eigenvector? What is an eigenvalue?

**Solution:** If $A$ is an $n \times n$ matrix then any scalar $\lambda$ such there is a nonzero vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$, is called an eigenvalue of $A$. Then, any nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an eigenvector corresponding to the eigenvalue $\lambda$.

(b) Let

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}.$$

i) Find the eigenvalues and eigenvectors of $A$.

**Solution:** To find the eigenvalues solve

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 4 & 2-\lambda \end{vmatrix} = 0, \quad \text{for } \lambda.$$

Hence

$$(3-\lambda)(-2-\lambda) + 4 = 0 \implies \lambda^2 - \lambda - 2 = 0 \implies (\lambda-2)(\lambda+1) = 0.$$

Hence, $\lambda_1 = 2$ and $\lambda_2 = -1$ are eigenvalues. Solving $(A - \lambda I)\mathbf{v} = \mathbf{0}$ gives the corresponding eigenspaces.

$$(A - 2I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix} \mathbf{v} = \mathbf{0},$$

so one eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Similar reasoning shows $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda_2 = -1$.

ii) Diagonalise $A$.

**Solution:** For each of the eigenvalues

*geometric multiplicity=algebraic multiplicity.*

Hence, there is a complete system of $n$ linearly independent eigenvectors. Then

$$P^{-1}AP = D \text{ where } P = [\mathbf{v}_1 : \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \text{ and } D = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.$$

iii) Find the general solution of the linear sustem of differential equations

$$\mathbf{y}' = A\mathbf{y}.$$

**Solution:** Since $A$ is diagonalisable we can solve by rewrieing the system in diagonal, or seperated, form. This yields the general solution

$$\mathbf{y}(t) = Pe^{Dt}\mathbf{c}$$

Which is in this case

$$\mathbf{y} = \left\{ \begin{array}{c} c_1e^{2t} + c_2e^{-t} \\ c_1e^{2t} + 4c_2e^{-t}. \end{array} \right.$$

**Question 5** [10 points]

(a) A carhire business has a fleet of 1000 cars based in three towns: Christchurch ($c$), Queenstown ($q$), and Dunedin ($d$). Experience has shown that the distribution of cars satisfies

$$\mathbf{v}_{k+1} = A\mathbf{v}_k$$

where $\mathbf{v}_k = (c_k, q_k, d_k)^T$ is a vector whose components give the number of cars in each town at the end of the $k$-th week and where $B$ is the matrix:

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

i) What is the probability that a car based in Christchurch this week will still be based in Christchurch next week?

**Solution:** Car in Christchurch implies $\mathbf{v} = [1, 0, 0]^T$. Therefore

$$A\mathbf{v} = \begin{bmatrix} 0.6 \\ 0.2 \\ 0.2 \end{bmatrix}.$$

Thus the probability the car remains in Christchurch is 0.6.

ii) What is the probability that a car based in Queenstown this week will be based in Dunedin next week?

**Solution:** For the car initially in Queenstown $vv = [0, 1, 0]^T$, and

$$A\mathbf{v} = \begin{bmatrix} 0.1 \\ 0.7 \\ 0.2 \end{bmatrix}.$$

Thus the probabiity that the car is in Dunedin next week is 0.2.

iii) Using the command $[\mathsf{E},\mathsf{D}] = \mathsf{eig}(\mathsf{A})$ in Matlab with the above matrix A gives the following output:

$$E = \begin{bmatrix} -0.4082 & -0.7071 & -0.2673 \\ -0.8165 & 0.7071 & -0.5345 \\ -0.4082 & -0.0000 & 0.8018 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 0.5000 & 0 \\ 0 & 0 & 0.2000 \end{bmatrix}$$

What is the long term distribution of cars among the three towns?

**Solution:** Look at the dominant eigenvector corresponding to eigenvalue $\lambda = 1$. One such eigenvector is the first column of $E$. Scale to get an eigenvector whose components sum to 1 in order to get probabilities. Thus

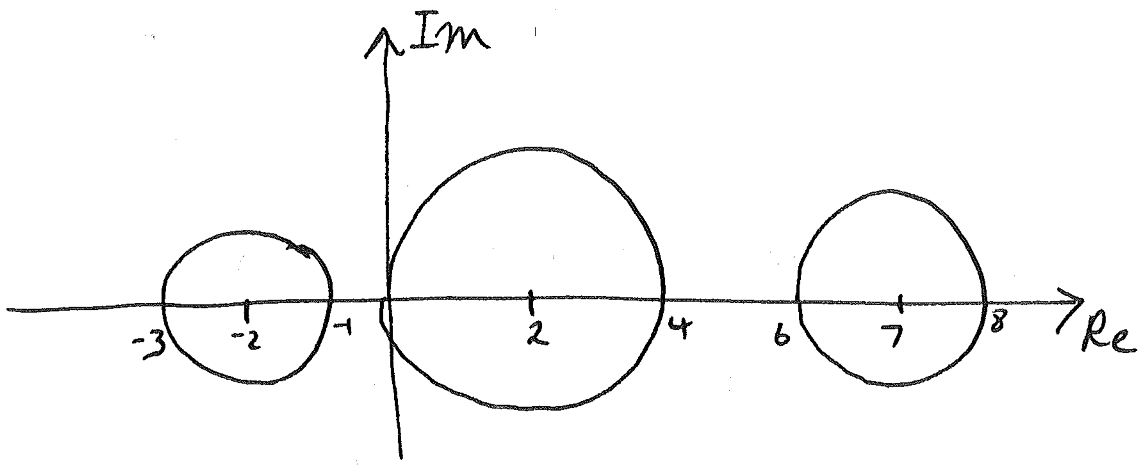
$$\mathbf{v} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}.$$

This vector gives the long term probabilities. Thus in the long term 25% of the cars are in Christchurch.

b) Consider the matrix:

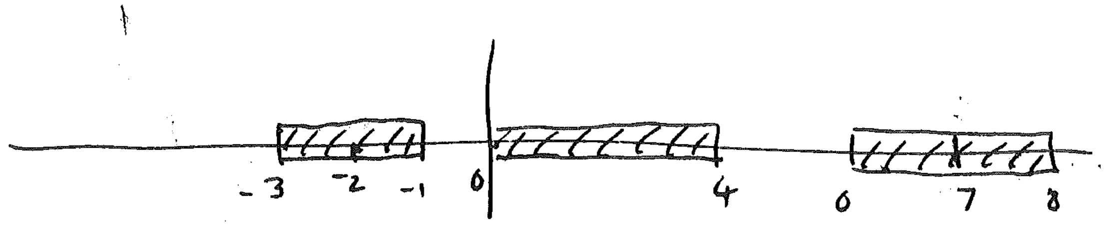
$$B = \begin{bmatrix} 7 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

i) On a diagram draw the Gerschgorin row-related disks of $B$ on the complex plane.




ii) Given that B is symmetric, refine your estimation for the location of the eigenvalues. Explain your reasoning.

Symmetric ⇒ all eigenvalues are real



(c) i) Carry out two steps of the power method for the matrix
$G = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$, using the starting vector $\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**Solution:** First step

$$\mathbf{y} = G\mathbf{x}^{(0)} = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}.$$

The largest absolute entry is $-5$. So the first approximation to the (presumed) dominant eigenvalue is $\lambda^{(1)} = -5$. Scale $\mathbf{y}$ by dividing by this yielding $\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}$, as our new approximation to an eigenvector corresponding to the (presumed) dominant eigenvalue.

Second step

$$\mathbf{y} = G\mathbf{x}^{(1)} = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0.2 \end{bmatrix}.$$

Scale and get next approximation to an eigenvector corresponding to a dominant eigenvalue $\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We are back where we started!

ii) Would you expect the power method to converge in this case? Explain your answer.

**Solution:** No!. The calculations will clearly keep repeating, and never converge. The reason the power method is failing to converge is that $g$ has eigenvalues $\pm 1$. So there is no dominant eigenvalue.

**Question 6** [10 points]

(a) (i) Use the Gram-Schmidt process to convert the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -1 \\ -9 \end{bmatrix}, \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} -3 \\ -8 \\ 5 \end{bmatrix}$$

to an orthogonal basis for $\mathbb{R}^3$.

**Solution:**

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix},$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_2}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \mathbf{x}_2 - \frac{-49}{49}\mathbf{v}_1 = \mathbf{x}_2 + \mathbf{v}_1 = \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix},$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \left(\frac{\mathbf{v}_1 \cdot \mathbf{x}_3}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{x}_3}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2$$

$$= \cdots = \mathbf{x}_3 - 0\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}.$$

(ii) Hence find a $QR$ factorisation for the matrix:

$$M = \begin{bmatrix} 2 & 4 & -3 \\ 3 & -1 & -8 \\ 6 & -9 & 5 \end{bmatrix}.$$

**Solution:** The columns of $M$ are the three vectors we orthogonalised in the previous part. Hence, we can find $Q$ by normalising the orthogonal vectors found in the previous part. That is $Q = [\mathbf{q}_1 : \mathbf{q}_2 : \mathbf{q}_3]$ where $\mathbf{q}_i = \mathbf{v}_i/\|\mathbf{v}_i\|$ is the normalised orthogonal vector. Hence

$$Q = \frac{1}{7}\begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}.$$

Now

$$M = QR \implies R = Q^{-1}M = Q^T M.$$

Therefore,

$$R = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 & -3 \\ 3 & -1 & -8 \\ 6 & -9 & 5 \end{bmatrix} = \begin{bmatrix} 7 & -7 & 0 \\ 0 & 7 & -7 \\ 0 & 0 & 7 \end{bmatrix}.$$

(iii) Use the resulting factorisation to solve the system of equations

$$M\mathbf{x} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}.$$

**Solution:**

$$M\mathbf{x} = \mathbf{b} \Leftrightarrow QR\mathbf{x} = \mathbf{b} \Leftrightarrow Q^TQR\mathbf{x} = Q^T\mathbf{b} \Leftrightarrow R\mathbf{x} = Q^T\mathbf{b}.$$

Hence we can solve by first computing $Q^T\mathbf{b}$ and then solving $R\mathbf{x} = Q^T\mathbf{b}$ by back substitution. Proceeding in this manner

$$Q^T\mathbf{b} = \frac{1}{7}\begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}.$$

Then

$$R\mathbf{x} = \mathbf{b}$$

$$\Leftrightarrow \quad \begin{bmatrix} 7 & -7 & 0 \\ 0 & 7 & -7 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -1 \end{bmatrix}$$

and solving by back substitution $x_3 = -\frac{1}{7}$, $x_2 = \frac{4}{7}$ and $x_1 = \frac{15}{7}$.

(b) Find a singular value decomposition for $W = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}$

**Solution:**

$$W^TW = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

The eigenvalues of this matrix **in decreasing order** are $\mu_1 = 9$ and $\mu_2 = 4$. Thus, the singular values of $W$ are $\sigma_1 = 3$ and $\sigma_2 = 2$. Corresponding orthonormal eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$\mathbf{u}_1 = \frac{1}{\sigma_1}A\mathbf{v}_1 = \frac{1}{3}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sigma_2}A\mathbf{v}_2 = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Choose $\mathbf{u}_3$ as any vector which together with $\mathbf{u}_1$ and $\mathbf{u}_2$ forms an orthonormal basis for $\mathbb{R}^3$ A suitable choice is $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Then the desired singular value decomposition is

$$W = U\Sigma V^T,$$

where $U = [\mathbf{u}_1 : \mathbf{u}_2 : \mathbf{u}_3]$, $V = [\mathbf{v}_1 : \mathbf{v}_2]$ and

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

**The statistics questions have deliberately been omitted. Work the 2014 Statistics questions instead. They are closer what has been taught this year.**