Inverse transform techniques; The initial-value and final-value theorems

Readings: Sections 14.4, 14.6

The Inverse Laplace Transform

Analysing something in the **s**-domain isn't terribly useful if we can't get back into the time domain. The formula for taking the inverse LT is:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} e^{st}F(s)\,ds$$

Because it's complicated, we always use tables in this course. The notation used is:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

$$f(t) \Longleftrightarrow F(s)$$

The tables don't include all possible options, so we usually need to simplify the equation to get functions that are close to those in the tables.

We usually have polynomials of the form:

$$I(s),\ V(s) = \frac{N(s)}{D(s)} \quad \begin{array}{l} \leftarrow \text{zeros} \\ \leftarrow \text{poles} \end{array}$$

The values of **s** which result in **N(s)** = 0 are called zeros. The values of **s** which result in **D(s)** = 0 are called poles. We will do more on these later!

Note that both **N(s)** and **D(s)** could have multiple powers of **s** – in our example on the previous page **N(s)** = 1 (no powers of **s**), but **D(s)** = $2s^2 + 3s + 4$ (two powers of **s**). We often use partial fractions to get our equation into a form that works with the tables.

Example

If $\mathbf{V(s)} = \frac{2}{s^2+6s-7}$, what is $v(t)$?

$$V(s) = \frac{2}{(s+7)(s-1)}$$ ← two distinct poles $s = 1, -7$

$$= \frac{A}{s+7} + \frac{B}{s-1} = \frac{A(s-1)+B(s+7)}{(s+7)(s-1)}$$

$$\therefore A(s-1) + B(s+7) = 2$$

option 1

$s=1$

$B(8) = 2$

$B = 1/4$

$s=-7$

$A(-8) = 2$

$A = -1/4$

option 2

$$s(A+B) + (-A+7B) = 2$$

$A+B=0$

$A=-B$

$-A+7B=2$

$B+7B=2$

$8B=2$

$B=1/4, \quad A=-1/4$

$$V(s) = \frac{-1/4}{s+7} + \frac{1/4}{s-1}$$

$$v(t) = -\frac{1}{4}e^{-7t}u(t) + \frac{1}{4}e^{t}u(t)$$ ⇐ from tables

$$= \frac{1}{4}\left(e^{t} - e^{-7t}\right)u(t) \text{ V}$$

Example:

Find $v(t)$ if $\mathbf{V(s)} = \frac{s+6}{s^2-4s+4}$.

$$V(s) = \frac{s+6}{(s-2)^2}$$

$$= \frac{A}{(s-2)^2} + \frac{B}{s-2}$$

$$\therefore \quad s+6 = A + B(s-2)$$

$\underline{s=2}$

$$8 = A$$

$$\therefore \quad s+6 = 8 + B(s-2)$$

$$Bs = s$$
$$B = 1$$

$$V(s) = \frac{8}{(s-2)^2} + \frac{1}{s-2}$$

$$v(t) = 8te^{2t}u(t) + e^{2t}u(t)$$
$$= (8t+1)e^{2t}u(t) \text{ V}$$

Example:

Factorise $\mathbf{I}(\boldsymbol{s}) = \frac{2(s+3)}{(s+1)(s^2+2)}$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$A(s^2+2) + (Bs+c)(s+1) = 2(s+3)$$

$s=-1$

$$3A = 4$$
$$A = 4/3$$

$$\frac{4}{3}s^2 + \frac{8}{3} + Bs^2 + Bs + Cs + C = 2s+6$$

$$\frac{4}{3} + B = 0 \qquad \frac{8}{3} + C = 6$$
$$B = -\frac{4}{3} \qquad C = \frac{10}{3}$$

$$I(s) = \frac{4/3}{s+1} + \frac{(-4/3)s + 10/3}{s^2+2}$$

Initial-Value and Final-Value Theorems

These two theorems mean we can evaluate the whatever we're interested in at $t = 0^+$ and at $t = \infty$.

Initial-Value Theorem: $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\{s\mathbf{F}(\mathbf{s})\}$

Final-Value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0}\{s\mathbf{F}(\mathbf{s})\}$

NOTE: The poles of **F(s)** *must* all have a real part < 0 for the FVT to work. A single pole at the original *might* be ok, but we will avoid examples of this type.

Poles and Zeros

Remember that if we have $\mathbf{F}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})}$, then the zeros of the function are the values of **s** that lead to **N(s)** = 0, and the poles of the function are the values of **s** that lead to **D(s)** = 0.

Example

If we have $\mathrm{V}(\mathrm{s}) = \frac{\mathrm{s}+2}{(\mathrm{s}+1)(\mathrm{s}+3)}$, what are the poles are zeroes of the function? Could we use the FVT?

zeros $\Rightarrow$ $s+2=0$

$s=-2$

poles $\Rightarrow$ $(s+1)(s+3)=0$

$s=-1,-3$

In this case, all the poles are < 0, therefore it's ok to use the FVT.

Example

If $\mathrm{v}(\mathrm{t}) = e^{-3t}\mathrm{u}(\mathrm{t})$, find $\mathrm{v}(0^+)$ and $\mathrm{v}(\infty)$ using the IVT and the FVT.

$$V(s) = \frac{1}{s+3}$$

$$v(0^+) = \lim_{s\to\infty} sV(s)$$

$$= \lim_{s\to\infty} \frac{s}{s+3}$$

$$= \lim_{s\to\infty} \frac{1}{1+3/s}$$

$$= 1\ \mathrm{V}$$

For FVT, check poles:

$$s+3=0$$
$$s=-3<0 \quad \therefore \text{ ok to use FVT}$$

$$v(\infty)=\lim_{s\to 0}\left[\frac{s}{s+3}\right]$$
$$=0\text{ V}$$

In this instance we knew this without using the IVT or FVT, since we know what a negative exponential looks like. But, this won't always be the case.

Example

If $v(t)=e^{-3t}u(t)+9\delta(t)$, find $v(0^+)$ and $v(\infty)$ using the IVT and the FVT.

$$V(s)=\frac{1}{s+3}+9$$
$$=\frac{9s+28}{s+3}$$

$$v(0^+)=\lim_{s\to\infty} sV(s)$$
$$=\lim_{s\to\infty}\frac{9s^2+28s}{s+3}$$
$$=\infty\text{ V}$$

poles $\Rightarrow s+3=0,\ s=-3<0 \quad \therefore$ FVT ok

$$v(\infty)=\lim_{s\to 0}\frac{9s^2+28s}{s+3}$$
$$=0\text{ V}$$

Makes sense: