UNIVERSITY OF CANTERBURY

End-of-Year Examination 2016

Course Code: EMTH211-16S2
Course Title: Engineering Linear Algebra and Statistics

Time allowed: 180min

Attempt ALL 9 questions.

Write your answers in the spaces provided.

There is a *total* of 105 points.

Use black or blue ink. Do not use pencil except for diagrams.

Only UC approved calculators are allowed.

Show all working. Write neatly. Marks will be lost for poorly presented answers.

Family name:	
Given names:	
Student ID:	

MARKS	
Q1	
Q2	
Q3	
Q4	
Q5	
Q6	
Q7	
Q8	
Q9	
Total	

Page for rough working ...

Question 1 [10 points]

Assume that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are linearly independent.

a) Are the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_1 + \mathbf{v}_3$ also linearly independent?

b) Are the vectors $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{v}_1$ also linearly independent?

Question 2 [10 points]

Consider the following matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 3 & 0 & 8 \\ 1 & 0 & 4 & 0 \\ 0 & 0 & 5 & k \end{bmatrix}$$

where $k \in \mathbb{R}$.

a) Determine the row rank (that is the dimension of the row space) of the matrix A depending on k

b) Find a basis for the null-space

$$\text{null}(A) = \{\mathbf{x} \in V : A\mathbf{x} = \mathbf{0}\}$$

of A, in the case that $k = -4$

c) How are the row rank and the nullity (the dimension of the null space) of a general matrix B related?

Question 3 [10 points]

a) Show that the *parallelogram-law* holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\|x+y\|_2^2 + \|x-y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2$$

b) Give an example that shows the parallelogram law fails for the ∞-norm **AND** give an example that shows it fails for the 1-norm.

Question 4 [10 points]

a) What is an eigenvector? What is an eigenvalue?

b) Let

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}.$$

i) Find the eigenvalues and eigenvectors of A.

ii) Diagonalise A

iii) Find the general solution to the linear system of differential equations:

$$\mathbf{y}' = A\mathbf{y}$$

Question 5 [15 points]

a) A carhire business has a fleet of 1000 cars based in three towns: Christchurch (c), Queenstown (q), and Dunedin (d). Experience has shown that the distribution of cars satisfies

$$\mathbf{v}_{k+1} = A\mathbf{v}_k$$

where $\mathbf{v}_k = (c_k, q_k, d_k)^T$ is a vector whose components give the number of cars in each town at the end of the k-th week and where A is the matrix:

$$A = \begin{bmatrix} 0.6 & 0.1 & 0.2 \\ 0.2 & 0.7 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}.$$

i) What is the probability that a car based in Christchurch this week will still be based in Christchurch next week?

ii) What is the probability that a car based in Queenstown this week will be based in Dunedin next week?

iii) Using the command $[\mathsf{E},\mathsf{D}] = \mathsf{eig}(\mathsf{A})$ in Matlab with the above matrix A gives the following output:

$$E = \begin{bmatrix} -0.4082 & -0.7071 & -0.2673 \\ -0.8165 & 0.7071 & -0.5345 \\ -0.4082 & -0.0000 & 0.8018 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 0.5000 & 0 \\ 0 & 0 & 0.2000 \end{bmatrix}$$

What is the long term distribution of cars among the three towns?

b) Consider the matrix:

$$B = \begin{bmatrix} 7 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}.$$

i) On a diagram draw the Gerschgorin row-related disks of B on the complex plane.

ii) Given that B is symmetric, refine your estimation for the location of the eigenvalues. Explain your reasoning.

c) i) Carry out two steps of the power method for the matrix
$G = \begin{bmatrix} 7 & -12 \\ 4 & -7 \end{bmatrix}$, using the starting vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

ii) Would you expect the power method to converge in this case? Explain your answer.

Question 6 [10 points]

a) i) Use the Gram-Schmidt process to convert the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -1 \\ -9 \end{bmatrix}, \text{ and } \quad \mathbf{x}_3 = \begin{bmatrix} -3 \\ -8 \\ 5 \end{bmatrix}$$

to an orthogonal basis for $\mathbb{R}^3$.

ii) Hence find a QR factorisation for the matrix:

$$M = \begin{bmatrix} 2 & 4 & -3 \\ 3 & -1 & -8 \\ 6 & -9 & 5 \end{bmatrix}.$$

iii) Use the resulting factorisation to solve the system of equations

$$M\mathbf{x} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix}$$

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b) Find a singular value decomposition for $W = \begin{bmatrix} 0 & 0 \\ 0 & 3 \\ -2 & 0 \end{bmatrix}$

Satistics section deliberately omitted. The statistics material taught in 2017 is closer to that in the 2014 exam. So use the 2014 exam questions for practice instead.

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