EMTH211–Tutorial 4

Attempt the following problems before the tutorial

1. Determine whether the following sets are a basis for the given vector space:

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ for $\mathbb{R}^2$

(c) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

(d) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$

(e) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^4$

Solution:

(a) In general, to show that a set forms a basis, we need the set to be a spanning set and a linearly independent set. In this case, we see that the number of vectors is equal to the number of elements in the basis and hence, we only need to show one of the two requirements. We show both requirements for the first example to illustrate the techniques.

To show linearly independence, suppose that $\lambda[1, 0]^T + \mu[0, 1]^T = [0, 0]^T$, then $[\lambda, \mu]^T = [0, 0]^T$, so $\lambda = \mu = 0$. This implies that the set is linearly independent. Furthermore, an arbitrary vector $[x, y]^T \in \mathbb{R}^2$ can be written as $[x, y]^T = x[1, 0]^T + y[0, 1]^T$, which show that the set is a spanning set.

(b) The set is not linearly independent: $[1, -1]^T + [1, 1]^T - 2[1, 0]^T = [0, 0]^T$. This also follows from the fact that we have more vectors in the set than the dimension of the vector space which always implies linear dependence.

(c) Like in (a), we see that the vectors are linearly independent. Moreover, we have 3 vectors which equals the dimension of the vector space. Hence, the linear independent set forms a basis.

(d) This is not a basis. The set of vectors is not linearly independent as we have that $[-1, 1, 0]^T - [0, 1, 1]^T + [1, 0, 1]^T = [0, 0, 0]^T$ (it is also not a spanning set).

(e) This is not a basis as we have only three vectors and our vector space is 4-dimensional.

2. Let $x = [1,3,4,5]$, and $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

   (a) Find $\|x\|_1, \|x\|_2, \|x\|_\infty$ by hand and check your solution with MatLab.

Solution:

(a) $\|x\|_1 = 1+3+4+5 = 13$, $\|x\|_2 = \sqrt{1^2+3^2+4^2+5^2} = \sqrt{51}$, $\|x\|_\infty = 5$. You can check your solutions with MatLab using the commands norm(v,p) where `v=[1 3 4 5]` and p=1,p=2 or p=`Inf`. Note that `norm(v)` returns the 2-norm.

In-tutorial problems

3. Find a basis for the following subspaces.

   (a) $U_1 = \left\{ \begin{bmatrix} r \\ r \\ s \end{bmatrix} \middle| r, s \in \mathbb{R} \right\}$;

   (b) $U_2 = \left\{ \begin{bmatrix} r+s \\ r-s \\ r \end{bmatrix} \middle| r, s \in \mathbb{R} \right\}$;

   (c) $U_3 = \left\{ \begin{bmatrix} r \\ r \\ s+t \end{bmatrix} \middle| r, s, t \in \mathbb{R} \right\}$;

   (d) $U_4 = \left\{ \begin{bmatrix} r+s \\ r-t \\ s+t \end{bmatrix} \middle| r, s, t \in \mathbb{R} \right\}$.

Solution:

(a) The vectors $[1,1,0]^T$ and $[0,0,1]^T$ span this subspace and are linearly independent.

(b) The vectors $[1,1,1]^T$ en $[1,-1,0]^T$ span this subspace and are linearly independent.

(c) The vectors $[1,1,0]^T$ en $[0,0,1]^T$ span this subspace are linearly independent. Note that this space is 2-dimensional, even though we have 3 parameters in its definition.

(d) It is clear that $[1,1,0]^T$, $[1,0,1]^T$ and $[0,-1,1]^T$ span this subspace. The second vector is the sum of the first and the third, so it is sufficient to take only $[1,1,0]^T$ and $[0,-1,1]^T$ to span U_4. We see that these two vectors are linearly independent.

4. Determine the rank of the following matrices over $\mathbb{R}$.

$$(a)\begin{bmatrix}1 & 2 & 1 & 1\\ 2 & -1 & 1 & 2\\ 4 & 3 & 3 & 4\\ 3 & 1 & 2 & 3\end{bmatrix} \qquad (b)\begin{bmatrix}2 & 1 & 4 & 0 & 0\\ 1 & 0 & -2 & 0 & 1\\ -2 & 0 & 3 & 4 & 1\end{bmatrix}$$

Solution:

(a) We reduce the matrix to echelon form

$$\begin{bmatrix}1 & 0 & \frac{3}{5} & 1\\ 0 & 1 & \frac{1}{5} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

which has two pivots, so the rank is 2.
Alternatively, we could observe that if $v = [1, 2, 1, 1]$ and $w = [2, -1, 1, 2]$ then $[4, 3, 3, 4] = w + 2v$ and $[3, 1, 2, 3] = w + v$. Since v and w are linearly independent, the dimension of the row space is 2.

(b) it is not hard to show that the columns $[1, 0, 0]^T$, $[0, 0, 4]^T$ and $[0, 1, 1]^T$ are linearly independent, so the dimension of the column space is at least 3. The column space is a subspace of $\mathbb{R}^3$ in this case, so its dimension is at most 3. We conclude that the rank of A is 3. Alternatively, you could just us row reduction and count the number of pivots.

5. For which values of $a \in \mathbb{R}$ are the following 3 vectors in $\mathbb{R}^3$ linearly dependent:

$$\begin{bmatrix}1\\ 2\\ a\end{bmatrix}, \quad \begin{bmatrix}a\\ 0\\ 0\end{bmatrix}, \quad \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} \ ?$$

Solution: The three vectors are linearly dependent if there exist $\lambda_1, \lambda_2, \lambda_3$, not all 0, such that

$$\lambda_1 \begin{bmatrix}1\\ 2\\ a\end{bmatrix} + \lambda_2 \begin{bmatrix}a\\ 0\\ 0\end{bmatrix} + \lambda_3 \begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}\lambda_1 + a\lambda_2 + \lambda_3\\ 2\lambda_1\\ a\lambda_1 + \lambda_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix},$$

alternatively, if the system

$$\begin{cases}\lambda_1 + a\lambda_2 + \lambda_3 = 0\\ 2\lambda_1 = 0\\ a\lambda_1 + \lambda_3 = 0\end{cases}$$

has a solution different from $[0, 0, 0]$. When we apply the row reduction algorithmn we see that this only occurs if $a = 0$. We conclude that if $a \neq 0$, the three vectors are linearly independent and if $a = 0$, the three vectors are linearly dependent. (This last fact also followed from the fact that if $a = 0$, the second vector is the zero vector. By the previous exercise, a set containing the zero vector is linearly dependent.)

6. Give a basis for the row space, the column space and the null space of the matrix $\begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$.

Solution: The row operations $R_2 + (1/2)R_1, R_3 - (1/2)R_1, R_3 - R_2$ reduce A to the row echelon form

$$B = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. The row space:

A basis is given by the non-zero rows of B:

$$\left\{ \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 3 & 7/2 \end{bmatrix} \right\}.$$

2. The column space:

The pivots in B are in columns 1 and 3. So the first and the third column of A will form a basis. That is, a basis for col(A) is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

3. The null space:

The matrix equation $A\mathbf{x} = \mathbf{0}$ is reduced to the system of equations

$$\begin{aligned} 2x_1 - 4x_2 + 2x_4 + x_5 &= 0 \\ x_3 + 3x_4 + \frac{7}{2}x_5 &= 0. \end{aligned}$$

This system is consistent as it admits the all-zero solution. The free variables are in the non-pivot columns (columns 2,4,5), corresponding to x_2, x_4, x_5, so we may put e.g. $x_2 = s, x_4 = t, x_5 = u$ and solve by back substitution to obtain

$$\begin{aligned} x_5 &= u \\ x_4 &= t \\ x_3 &= -3t - \frac{7}{2}u \\ x_2 &= s \\ x_1 &= 2s - t - \frac{1}{2}u \end{aligned}$$

and therefore

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s - t - \frac{1}{2}u \\ s \\ -3t - \frac{7}{2}u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$$

so that a basis for null(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

7. Show that for $x \in \mathbb{R}^n$

$$\|x\|_\infty \leqslant \|x\|_1 \leqslant n\|x\|_\infty \ .$$

Hint: for some k between 1 and n we have $|x_k| = \max\{|x_1|, \ldots, |x_n|\}$.

Solution: Let k be as in the hint. Then $\|x\|_\infty = |x_k| \leqslant |x_1| + \cdots + |x_k| + \cdots + |x_n| = \|x\|_1$. For the second inequality note that $|x_i| \leqslant |x_k|$ for all $i = 1 \ldots n$, so $\|x\|_1 = |x_1| + \cdots + |x_n| \leqslant n|x_k| = n\|x\|_\infty$.