EMTH211–Tutorial 6

Attempt the following problem before the tutorial.
Questions 1 and 2 should be solved by hand, for the other questions, the use of Matlab is recommended/required.

1. Find the solution to the following system of differential equations:

$$\begin{aligned} x_1'(t) &= -x_1(t) - x_2(t) + 3x_3(t) \\ x_2'(t) &= x_1(t) + x_2(t) - x_3(t) \\ x_3'(t) &= -x_1(t) - x_2(t) + 3x_3(t) \end{aligned}$$

where $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

Solution:

This system is of the form $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{bmatrix} -1 & -1 & 3 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}.$$

We want to diagonalise A. In order to do this, we solve the characteristic equation $|A - \lambda I| = 0$ and find the solutions $0, 1, 2$ and the corresponding eigenvectors

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

respectively.

This gives us $A = PDP^{-1}$ with

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and } P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

The solutions of our system are of the form $\mathbf{x} = Pe^{Dt}\mathbf{c}$, where $\mathbf{c} = P^{-1}\mathbf{x}(0) = P^{-1}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$. Instead of calculating $P^{-1}\mathbf{x}_0$, we solve $P\mathbf{c} = \mathbf{x}_0$ for $\mathbf{c}$ and find $\mathbf{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Finally, we get $\mathbf{x} = Pe^{Dt}\mathbf{c} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ which is equal to

$$\begin{bmatrix} 1 + e^t + e^{2t} \\ -1 + e^t \\ e^t + e^{2t} \end{bmatrix}$$

2. Let A be the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$. Use the theorem of Cayley-Hamilton to

   (i) write A^3 as a linear combination of I, A, A^2

   (ii) write A^{-1} as a linear combination of I, A, A^2.

   Solution: The theorem of Cayley-Hamilton says that a matrix satisfies its own characteristic equation. The matrix A is an upper triangular matrix, which means that its eigenvalues are the diagonal elements, $1, 2, 3$. This in turn implies that the characteristic polynomial is $p(\lambda) = (1 - \lambda)(2 - \lambda)(3 - \lambda) = -\lambda^3 + 6\lambda^3 - 11\lambda + 6$. The theorem of Cayley-Hamilton says that $p(A) = O$, where O is the zero matrix. Hence

   $$p(A) = -A^3 + 6A^2 - 11A + 6I = O,$$

   where I is the identity matrix. This gives us

   $$A^3 = 6A^2 - 11A + 6I.$$

   We know that A^{-1} exists since zero is not an eigenvalue of A. Multiplying both sides of the equation $A^3 = 6A^2 - 11A + 6I$ by A^{-1} yields $A^2 = 6A - 11I + 6A^{-1}$, or

   $$A^{-1} = \frac{1}{6}A^2 - A + \frac{11}{6}I.$$

In-tutorial problems

3. A study of pine nut crops in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, and 0.85, respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, and 0.84, respectively.

   (a) Write down the transition matrix for this Markov chain.

(b) If the pine nut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945.

(c) In the long run, what proportion of the crops will be good, fair, and poor?

Solution:

(a) The given data allows you to construct the transition matrix as:

```
>> A = [0.08 0.09 0.11; 0.07 0.11 0.05; 0.85 0.80 0.84]

A =

    0.0800    0.0900    0.1100
    0.0700    0.1100    0.0500
    0.8500    0.8000    0.8400
```

(b) If the crop was good in 1940, we can represent this by the vector

```
>> c1940 = [1; 0; 0]

c1940 =

    1
    0
    0
```

and then the probabilities of each type of crop over the next five years can be found by multiplying by A repeatedly

```
>> c1941 = A * c1940

c1941 =

    0.0800
    0.0700
    0.8500

>> c1942 = A * c1941

c1942 =

    0.1062
    0.0558
    0.8380

>> c1943 = A * c1942

c1943 =

    0.1057
    0.0555
    0.8388

>> c1944 = A * c1943

c1944 =

    0.1057
    0.0554
    0.8388
```

```
>> c1945 = A * c1944

c1945 =

    0.1057
    0.0554
    0.8388
```

The probabilities of good crops in each year are given by the first components of these vectors.

```
>> c1941(1), c1942(1), c1943(1), c1944(1), c1945(1)

ans =

    0.0800

ans =

    0.1062

ans =

    0.1057

ans =

    0.1057

ans =

    0.1057
```

(c) Looking at the matrix A, we see that it is a postive matrix. Moreover, if we ask Matlab to calculate its eigenvalues, we see that they are all different, and hence, the matrix A is diagonalisable. The theory tells us that the long term probabilities are given by the eigenvector belonging to the eigenvalue $\lambda = 1$.

```
>> [E, D] = eig(A)

E =

   -0.1248   -0.7806    0.1830
   -0.0654    0.1830   -0.7806
   -0.9900    0.5976    0.5976

D =

    1.0000         0         0
         0   -0.0253         0
         0         0    0.0553
```

Here the columns of E are eigenvectors of A belonging to the eigenvalues given by the corresponding diagonal entries of D. MATLAB automatically scales the eigenvectors so that they all have length 1. So we need to pick out the first column of E and scale it:

```
>> col1 = E(:, 1)

col1 =
```

```
   -0.1248
   -0.0654
   -0.9900

>> SteadyState = col1/sum(col1)

SteadyState =

    0.1057
    0.0554
    0.8388
```

Thus, in the long term, the probabilities of good, fair and poor crops are (respectively) 0.1057, 0.0554 and 0.8388.

4. Data have been accumulated on the heights of children relative to their parents. Suppose that the probabilities that a tall parent will have a tall, medium- height, or short child are 0.6, 0.2, and 0.2, respectively; the probabilities that a medium-height parent will have a tall, medium-height, or short child are 0.1, 0.7, and 0.2, respectively; and the probabilities that a short parent will have a tall, medium-height, or short child are 0.2, 0.4, and 0.4, respectively.

   (a) Write down the transition matrix for this Markov chain.

   (b) What is the probability that a tall person will have a short grandchild?

   (c) If 20% of the current population is tall, 50% is of medium height, and 30% is short, what will the distribution be in three generations?

   (d) What proportion of the population will be tall, of medium height, and short in the long run?

   Solution:

   (a) The matrix P is given by `P=[0.6 0.1 0.2;0.2 0.7 0.4;0.2 0.2 0.4]`

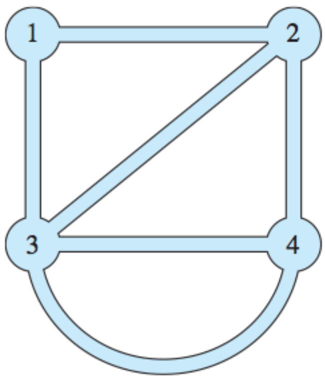
   (b) To focus on the tall population at the beginning we use the initial vector `x=[1;0;0]`. Their fate after two steps is given by the matrix $P^2\mathbf{x}$. We find the column vector $\begin{bmatrix} 0.42 \\ 0.34 \\ 0.24 \end{bmatrix}$. We may conclude that the probability that a tall person has a short grandchild is 24%.

   (c) Put `x=[0.2;0.5;0.3]` and calculate the matrix P^3x. We find $\begin{bmatrix} 0.2457 \\ 0.5039 \\ 0.2504 \end{bmatrix}$, so 24.57% will be tall, 50.39% will be medium-height and 25.04% will be small.

   (d) The steady state vector is the eigenvector of A belonging to the eigenvalue $\lambda = 1$, scaled so that its entries are all positive and sum to one.

   To find the eigenvectors using MATLAB, use `eig` in its two-output form: `[E,D]=eig(P)`. We see that A is diagonalisable as it has 3 different eigenvalues, and A is positive. So the theory tells us that the behaviour in the long run, will be determined by an eigenvector corresponding to the eigenvalue 1. We see that the first column gives us an eigenvector belonging to the eigenvalue $\lambda = 1$, but we still need to scale it to have unit length. Put `col1=E(:,1)` and `col1/sum(col1)`, this gives us the matrix $\begin{bmatrix} 0.25 \\ 0.50 \\ 0.25 \end{bmatrix}$. So in the long run, a quarter of the population will be tall, a quarter small and halve of the population is medium-height.

5. Robots have been programmed to traverse the maze shown below and at each junction randomly choose which way to go.



   (a) Construct the transition matrix for the Markov chain that models this situation.

   (b) Suppose we start with 15 robots at each junction. Find the steady state distribution of robots. (Assume that it takes each robot the same amount of time to travel between two adjacent junctions.)

   Solution:

   (a) This example is modelled by the Markov chain given by the matrix

$$A = \left(\begin{bmatrix} 0 & 1/3 & 1/4 & 0 \\ 1/2 & 0 & 1/4 & 1/3 \\ 1/2 & 1/3 & 0 & 1/3 * 2 \\ 0 & 1/3 & 1/4 * 2 & 0 \end{bmatrix} \right) .$$

   Notice that robots never stay at one of the corners, and that there are two connections between 3 and 4, which will double the probabilities to move between these two nodes.

   (b) If we look at the matrix A, we see it is not positive, however, if we calculate A^2, we see it is positive, which makes A regular. Using MatLab again, we see that A has 4 distinct eigenvalues, hence, it is diagonalisable as well. So we again need to look for an eigenvector for the eigenvalue 1 and we find

```
>> col1 = P(:, 1)

col1 =

   -0.3244
   -0.4867
   -0.6489
   -0.4867
```

Again, we have to scale it according to the starting population of 4*15 =60 robots.

```
>> SteadyState = 60 * col1/sum(col1)

SteadyState =

   10
   15
   20
   15
```

6. A grasshopper has three life stages: egg, nymph, adult. We focus on the female grasshoppers. The population satisfies the following properties:

   - each adult produces 1000 eggs per year;
   - 2% of the eggs survives to be nymphs;
   - 5% of the nymphs survives to adulthood.

   (a) Write down the Leslie model.

   (b) How does the population evolve if there are initially 50 adults and no eggs or nymphs? What if we begin with 50 adults, 50 eggs and 100 nymphs? Give a table representing the number of eggs, nymphs, adults in 25 years.

   (c) Could you explain why this happens looking at the eigenvalues of L?

   Solution:

   (a) $\mathbf{x}_{n+1} = L\mathbf{x}_n$ with $L = \begin{bmatrix} 0 & 0 & 1000 \\ 0.02 & 0 & 0 \\ 0 & 0.05 & 0 \end{bmatrix}$ and $\mathbf{x}_n = \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix}$, where for a year n, a_n is the number of eggs, b_n is the number of nymphs and c_n is the number of adults.

   (b)
   ```
   L=[0 0 1000;0.02 0 0;0 0.05 0];
   x=[0;0;50];
   R=[0;x];
   for k=1:25
   x=L*x;
   r=[k;x];
   R=[R r];
   end;
   ```

```
array2table(transpose(R))
```

We see that the population follows a 3-year cycle.

```
L=[0 0 1000;0.02 0 0;0 0.05 0];
x=[50;100;50];
R=[0;x];
for k=1:25
x=L*x;
r=[k;x];
R=[R r];
end;
array2table(transpose(R))
```

We see that the population follows a 3-year cycle.

(c) The dominant eigenvalue of L is 1. In this case, there is indeed no population growth (this happens if the dominant eigenvalue is larger than 1) nor decline (if the dominant eigenvalue is smaller than 1).

Extra question

7. Suppose that a population is divided into only two classes: children and adults. Let c_n denote the number of children at time step n and a_n the number of adults. The population evolves according to the following rules:

$$\begin{aligned} c_{n+1} &= \frac{1}{8}c_n + 6a_n \\ a_{n+1} &= \frac{1}{5}c_n \end{aligned}$$

(a) Write down the Leslie model for this population.

(b) What is the population starting from an initial population of 10 children and 10 adults after 3 timesteps (you can think of the population being measured in thousands to make these numbers more realistic.) ?

(c) Make a plot that represents the number of children and the number of adults at 25 (consecutive) timesteps. Plot both graphs in the same picture.

(d) Determine the long-time behaviour: in the long run, what percentage of the population would be adults?

Solution:

(a) $\mathbf{x}_{n+1} = L\mathbf{x}_n$, where $L = \begin{bmatrix} 1/8 & 6 \\ 1/5 & 0 \end{bmatrix}$ and $\mathbf{x}_n = \begin{bmatrix} c_n \\ a_n \end{bmatrix}$

(b) We have to calculate $L^3\mathbf{x}_0$, with $\mathbf{x}_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$. So we type:

```
x0=[10;10]
L^3*x
```

which gives us $\begin{bmatrix} 75.9570 \\ 3.9313 \end{bmatrix}$. So we have approximately 76 children and 4 adults.

(c)
```
x = [10; 10];
P = [x];
for k=1:25
x = L*x;
P = [P x];
end
plot(P')
```

(d) `[E D] = eig(L)` gives us the eigenvalues of L. They are 1.1597 and -1.0347, and hence, the dominant value is larger than one which predicts population growth (as we have seen from the graphs already). The long-term ratio between adults and children is determined by the eigenvector corresponding to the positive eigenvalue 1.1597. `E(:,1)/sum(E(:,1))` gives us 0.8529 0.1471, which shows that the population would consist of approx 85% children and 15% adults.