EMTH211-19S2 LABORATORY 10 SOLUTIONS

SEPTEMBER 30-OCTOBER 4, 2019

These exercises deal with

- SVD

Reading guide (Poole, Linear Algebra)

Section 7.4.

10.1 Let

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find a SVD for both $\mathbf{v}$ and $\mathbf{v}^T$.

SOLUTION:
From the lectures, the singular value for $\mathbf{v}$ is $\sigma_1 = \sqrt{3}$. The unit eigenvector of $\mathbf{v}^T\mathbf{v} = \begin{bmatrix}3\end{bmatrix}$ is $\begin{bmatrix}1\end{bmatrix}$ and so

$$\Sigma = \begin{bmatrix} \sqrt{3} \\ 0 \\ 0 \end{bmatrix}, \qquad V = \begin{bmatrix}1\end{bmatrix}.$$

Now

$$\mathbf{u}_1 = \frac{1}{\sigma_1}\mathbf{v}\begin{bmatrix}1\end{bmatrix} = \tfrac{1}{3}\sqrt{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We need to choose an orthonormal basis for $\mathbf{R}^3$ that includes $\mathbf{u}_1$. We could use Gram-Schmidt on, say, $\mathbf{u}_1$, $\mathbf{e}_1$ and $\mathbf{e}_2$. However, we can easily pick a vector orthogonal to $\mathbf{u}_1$ and normalize it. Thus we choose

$$\mathbf{u}_2 = \tfrac{1}{2}\sqrt{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

For $\mathbf{u}_3$ we choose another (linearly independent) vector orthogonal to $\mathbf{v}_1$ and use Gram-Schmidt to orthogonalize it to $\mathbf{u}_2$;

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \cdot \mathbf{u}_2\right)\mathbf{u}_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}.$$

Normalizing this vector, we obtain

$$\mathbf{u}_3 = \tfrac{1}{6}\sqrt{6}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Thus

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & -\frac{1}{3}\sqrt{6} \end{bmatrix}$$

Note that

$$\mathbf{v} = U\,\Sigma\,V^{\mathsf{T}}$$

and so

$$\mathbf{v}^{\mathsf{T}} = V\,\Sigma^{\mathsf{T}}\,U^{\mathsf{T}}.$$

This is a SVD for $\mathbf{v}^{\mathsf{T}}$.

10.2 Find a SVD for

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

SOLUTION:
We begin by computing the singular values.

$$B = A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The eigenvalues are 0 (the columns are linearly dependent) and 2 (trace formula). Thus the singular values of A are

$$\sigma_1 = \sqrt{2} \qquad \text{and} \qquad \sigma_2 = 0$$

and

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues are distinct and so the eigenvectors of B will be orthogonal.Thus $\mathbf{v}_1$ will be the unit eigenvector of B associated with 2. Now

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since $\sigma_1 \neq 0$, we have

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since $\sigma_2 = 0$, $\mathbf{u}_2$ is any vector such that $\mathbf{u}_1$, $\mathbf{u}_2$ forms an orthonormal basis for $\mathbf{R}^2$. Therefore we choose

$$\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and so

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which completes the SVD.

10.3 Compute the pseudoinverse of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

A does not have linearly independent columns and so we need a SVD.

$$B = A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since 0 is an eigenvalue (columns are not linearly independent) and 2 is also an eigenvalue; the third eigenvalue is

$$\operatorname{tr} B - 2 = 4 - 2 = 2.$$

Therefore the singular values are $\sigma_1 = \sigma_2 = \sqrt{2}$, $\sigma_3 = 0$. Eigenvectors associated with 2 are

$$B - 2I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v}_2 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

will form an orthonormal basis for E_2.

$$\mathbf{v}_3 = \frac{\sqrt{2}}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

will be an orthonormal basis for E_0. Thus

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} A \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} A \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We need a third vector to form an orthonormal basis for $\mathbf{R}^3$. Thus we choose

$$\mathbf{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

and so

$$U = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Now

$$\Sigma^+ = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so

$$\begin{aligned} A^+ = V\,\Sigma^+\,U^T &= \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \end{aligned}$$

10.4 Every complex number z can be written in the form

$$z = r\, e^{i\theta}$$

where r is a real non-negative number. With this representation, z has been decomposed into a stretching factor r and a rotation θ. There is an analogous decomposition for *any* square matrix A. Show that a square matrix A may be factored

$$A = R\, Q$$

where R is a symmetric matrix with non-negative eigenvalues (and NOT a upper triangular matrix) and Q is an orthogonal matrix. (*Hint:* Write a SVD of $A = U\,\Sigma\, V^T = \left(U\,\Sigma\, U^T\right)\left(U\, V^T\right)$.) This decomposition is called the **polar decomposition** of A. R represents a scaling and Q represents a rotation.

SOLUTION:
Using the hint, we have $A = R\, Q$ with

$$R = U\,\Sigma\, U^T \qquad \text{and} \qquad Q = U\, V^T.$$

Clearly Q is orthogonal (product of two orthogonal matrices). Since Σ is square, it is diagonal and so R is symmetric. Furthermore R is similar to Σ and so have the same eigenvalues. Thus R has non-negative eigenvalues.

10.5 Compute the polar decomposition for the matrices in questions 2 and 3.

Solution:
For

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

we have

$$R = U\Sigma U^T = \Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$Q = UV^T = V^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

For

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$R = U\Sigma U^T = \tfrac{1}{4} \begin{bmatrix} 0 & 2 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 2 & 0 & \sqrt{2} \\ -\sqrt{2} & 0 & \sqrt{2} \end{bmatrix} = \tfrac{1}{2}\sqrt{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$Q = UV^T = \tfrac{1}{4} \begin{bmatrix} 0 & 2 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ \sqrt{2} & \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} \sqrt{2} & \sqrt{2} & 0 \\ 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \end{bmatrix}.$$