EMTH211–Tutorial 5

Attempt the following problems before the tutorial.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

   (a) Find $\|A\|_1, \|A\|_\infty, \|A\|_{Fr}$ by hand and check your solution with MatLab.

   Solution: $\|A\|_1 = \max\{4, 7, 14\} = 14$, $\|A\|_\infty = \max\{6, 10, 9\} = 10$,
   $\|A\|_{Fr} = \sqrt{1+4+9+4+25+9+1+64} = \sqrt{117}$. You can check your solutions with Matlab using the commands `norm(A,p)`, with p=1 or p=`Inf` and `norm(A,'fro')`.

2. For each of the following matrices:

   - determine all eigenvalues
   - determine for every eigenvalue its eigenspace
   - write down the algebraic and geometric multiplicity for each eigenvalue.

   (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -5 \\ 0 & 3 & -6 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 5 & 0 & 6 \\ 0 & -3 & -1 & -3 \\ 3 & -3 & 0 & -4 \end{bmatrix}$

   You can check your answer using MatLab.

   Solution:

   (a) We determine the characteristic polynomial $p(\lambda)$:

   $$p(\lambda) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = (\lambda-3)(\lambda-1)\ .$$

   The solutions of the equation $p(\lambda) = 0$ are 3 and 1, so the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 1$, with corresponding eigenspaces V_1 and V_2. The eigenspace

V_1 consists of all vectors w such hat $(A-3I)w = 0 \in \mathbb{R}^2$. Using row reduction we find

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \overset{R_2 \to R_2 + R_1}{\sim} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

So $V_1 = \{[k,k]^T \mid k \in \mathbb{R}\} = \mathrm{span}([1,1]^T)$. Analoguously we find $V_2 = \{[k,-k]^T \mid k \in \mathbb{R}\} = \mathrm{span}([1,-1]^T)$. We conclude that both eigenvalues have algebraic and geometric multiplicity one.

(b) The characteristic equation is $(2-\lambda)(\lambda^2+4\lambda+3) = 0$, the eigenvalues are its roots 2,-1, -3 The eigenvalues with corresponding eigenspaces are: $\lambda_1 = 2$ with eigenspace $V_1 = \mathrm{span}([1,0,0]^T)$, $\lambda_2 = -1$ with eigenspace $V_2 = \mathrm{span}([14,-15,-9]^T)$ and $\lambda_3 = -3$ with eigenspace $V_3 = \mathrm{span}([4,-5,-5]^T)$. We conclude that all three eigenvalues have algebraic and geometric multiplicity one.

(c) We determine the characteristic polynomial $p(\lambda)$:

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (2-\lambda)((2-\lambda)^2-1) + (\lambda-3) + (\lambda-3) =$$

$$(2-\lambda)(\lambda-3)(\lambda-1) + 2(\lambda-3) = -\lambda(\lambda-3)^2 .$$

The solutions of $p(\lambda) = 0$ are 3 and 0. Note that the algebraic multiplicity of 3 is equal to 2. We see that the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 0$, with eigenspaces V_1 and V_2. The eigenspace V_1 consists of all vectors w such that $(3I-A)w = 0 \in \mathbb{R}^3$. Using row reduction we find

$$\begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix} \overset{R_2 \to R_2 - R_1, R_3 \to R_3 + R_1}{\sim} \begin{bmatrix} -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

So $V_1 = \{([r,s,r+s]^T \mid r,s \in \mathbb{R}\} = \mathrm{span}([1,0,1]^T, [0,1,1]^T)$. Note that λ_1 also has geometric multiplicity equal to 2. Analoguously, we have that $V_2 = \{[k,k,-k]^T \mid k \in \mathbb{R}\} = \mathrm{span}([1,1,-1]^T)$. We conclude that 3 has algebraic and geometric multiplicity two, whereas 0 has algebraic and geometric multiplicity one.

(d) The characteristic polynomial of A is $(\lambda+1)^2(\lambda-2)^2 = 0$. We find two eigenvalues: $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenspaces are $V_1 = \mathrm{span}([1,1,-1,0]^T, [1,-1,0,1]^T)$ and $V_2 = \mathrm{span}([0,0,1,0]^T, [0,1,0,-1]^T)$. We see that both eigenvalues have algebraic and geometric multiplicity two.

In-tutorial problems

3. Recall the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

where every is the sum of the previous two terms.

We can write this sequence as $F(1)$, $F(2)$, $F(3)$, ..., where $F(1) = 1, F(2) = 1, \ldots,$ and

$$F(n) = F(n-1) + F(n-2),$$

for $n \geq 3$. We can write

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} F(n+1) + F(n) \\ F(n+1) \end{bmatrix}$$

and

$$\begin{bmatrix} F(2) \\ F(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We see that

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F(n+1) \\ F(n) \end{bmatrix},$$

so that

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F(n) \\ F(n-1) \end{bmatrix},$$

and continuing like this

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F(2) \\ F(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Calculate the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and call them λ_1 and λ_2 (take λ_1 to be the largest of the two).

(b) Show that $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue λ_1 and that $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ is an eigenvector for λ_2.

(c) Show that $\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

(d) Calculate $A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ by diagonalising the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.*It is a good idea work with the symbols λ_1 and λ_2 instead of their actual values.*

(e) Use the fact that $\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to show that

$$F(n+2) = \frac{1}{\sqrt{5}} \left(\lambda_1^{n+2} - \lambda_2^{n+2} \right).$$

This implies

$$F(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This explicit formula for the n-th term of the Fibonacci sequence that you just derived is called *Binet's formula.*

Solution:

(a) The characteristic equation is $\lambda^2 - \lambda - 1 = 0$, which has solutions $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. (The value of λ_1 is the *golden ratio.*)

(b) We calculate $A\begin{bmatrix}\lambda_1\\1\end{bmatrix} = \begin{bmatrix}\lambda_1+1\\\lambda\end{bmatrix} = \lambda_1\begin{bmatrix}\lambda_1\\1\end{bmatrix}$, since $\lambda_1^2 = \lambda_1 + 1$ (remember that λ_1 is a solution of the characteristic equation). The same reasoning works for λ_2.

(c) $\begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1-\lambda_2 & 0\\0 & \lambda_1-\lambda_2\end{bmatrix} = I$ since $\lambda_1 - \lambda_2 = \sqrt{5}$.

(d) Write $P = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}$, the matrix with eigenvectors as columns and $D = \begin{bmatrix}\lambda_1 & 0\\0 & \lambda_2\end{bmatrix}$, then we know that $A = PDP^{-1}$. It follows that $A^n = PD^nP^{-1}$. We have that $D^n = \begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}$ and $P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix}$. This implies

$$A^n = \begin{bmatrix}\lambda_1 & \lambda_2\\1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0\\0 & \lambda_2^n\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1 & -\lambda_2\\-1 & \lambda_1\end{bmatrix} =$$

$$\frac{1}{\sqrt{5}}\begin{bmatrix}\lambda_1^{n+1}-\lambda_2^{n+1} & \lambda_1^n-\lambda_2^n\\\lambda_1^n-\lambda_2^{n-1} & \lambda_1^n-\lambda_2^{n-1}\end{bmatrix}$$

(e) From $\begin{bmatrix}F(n+2)\\F(n+1)\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n\begin{bmatrix}1\\1\end{bmatrix}$ we obtain that

$$F(n+2) = \frac{1}{\sqrt{5}}(\lambda_1^{n+1} + \lambda_1^n - \lambda_2^{n+1} - \lambda_2^n)$$

$$= \frac{1}{\sqrt{5}}(\lambda_1^n(\lambda_1+1) - \lambda_2^n(\lambda_2+1)) = \frac{1}{\sqrt{5}}(\lambda_1^{n+2} - \lambda_2^{n+2}).$$

In the last step we used that $\lambda_1 + 1 = \lambda_1^2$ and that $\lambda_2 + 1 = \lambda_2^2$ (since λ_1, λ_2 are solutions of the characteristic equation).

Replacing $n+2$ by n gives us Binet's formula.

4. Diagonalize $A = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\\1 & 1 & 0\end{bmatrix}$, i.e. find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution:

Characteristic equation:

$$
\begin{aligned}
0 = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} & \qquad\qquad \text{(expand det along top row)} \\
= (1-\lambda)\,[(1-\lambda)(-\lambda) - 1] + [0 - (1-\lambda)] & \\
= (1-\lambda)\,\big[(\lambda^2 - \lambda - 1) - 1\big] & \\
= (1-\lambda)(\lambda^2 - \lambda - 2) & \\
= (1-\lambda)(\lambda - 2)(\lambda + 1) &
\end{aligned}
$$

so the eigenvalues of A are $\lambda = 1, -1, 2$.

Since these are all different, the corresponding eigenvectors will be linearly independent. Hence A has three independent eigenvectors, and must be diagonalizable.

To find P we need the eigenvectors:

$\lambda = 1:$ Solve $(A - I)\mathbf{v} = \mathbf{0}$:

$$
A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \xrightarrow[\textit{operations}]{\textit{row}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad \text{so} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} s
$$

where s can be any nonzero scalar.

$\lambda = -1:$ Solve $(A + I)\mathbf{v} = \mathbf{0}$:

$$
A + I = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\textit{operations}]{\textit{row}} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{so} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} s
$$

where again s can be any nonzero scalar.

$\lambda = 2:$ Solve $(A - 2I)\mathbf{v} = \mathbf{0}$:

$$
A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \xrightarrow[\textit{operations}]{\textit{row}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{so} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} s
$$

and again s can be any nonzero scalar.

The matrices D and P come from putting eigenvalues on the diagonal of D, and putting the corresponding eigenvectors into the corresponding columns of P. So using the same order as above, you could get

$$
D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}
$$

Extra questions

5. Let A be a 2×2-matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = 2$ respectively. Put $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Use this to find $A^{10}\mathbf{x}$ and (a formula) for $A^k\mathbf{x}$. What happens if $k \to \infty$?

   Solution:

   We first need to write $\mathbf{x} = a\mathbf{v}_1 + b\mathbf{v}_2$. One can easily see (or solve the system of linear equations) that putting $a = 2$ and $b = 3$ works. That is

   $$A^{10}\mathbf{x} = A^{10}(2\mathbf{v}_1 + 3\mathbf{v}_2) = 2A^{10}\mathbf{v}_1 + 3A^{10}\mathbf{v}_2 = 2(\frac{1}{2})^{10}\mathbf{v}_1 + 3 \cdot 2^{10}\mathbf{v}_2 = \frac{1}{2^9}\mathbf{v}_1 + 3 \cdot 2^{10}\mathbf{v}_2 \ ,$$

   plugging in the values for v_1 and v_2 therefore

   $$A^{10}\mathbf{x} = \begin{bmatrix} \frac{1}{2^9} \\ -\frac{1}{2^9} \end{bmatrix} + \begin{bmatrix} 3 \cdot 2^{10} \\ 3 \cdot 2^{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{2^9} + 3 \cdot 2^{10} \\ -\frac{1}{2^9} + 3 \cdot 2^{10} \end{bmatrix}$$

   For general k we have $A^k\mathbf{x} = 2(\frac{1}{2})^k\mathbf{v}_1 + 3 \cdot 2^k\mathbf{v}_2$, and we see that if k tends to infinity, the resulting vector will be a(very large) multiple of $\mathbf{v}_2$, since the contribution of $\mathbf{v}_1$ will tend to zero.

6. Find a 2×2 matrix A such that:

   (a) A has two distinct real eigenvalues.

   (b) A has exactly one real eigenvalue, with a geometric multiplicity of 2.

   (c) A has exactly one real eigenvalue, with a geometric multiplicity of 1.

   (d) A has no real eigenvalue.

   Solution:

   (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has the eigenvalues 1 and 2, associated to the eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ respectively.

   (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has the eigenvalue 1, with associated eigenspace $E_1 = \text{span}(\mathbf{e}_1, \mathbf{e}_2)$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are linearly independent, E_1 is two-dimensional.

   (c) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has the eigenvalue 1, with associated eigenspace $E_1 = \text{span}(\mathbf{e}_1)$, so E_1 is one-dimensional.

   (d) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no real eigenvalue (the characteristic polynomial is $\lambda^2 + 1$, which has no real root).