ENEL220 Circuits and Signals Term 3

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Office hours: Mon 1.30 – 4pm, Thurs 11am – 4pm (term time only). I will have a break for lunch on Thurs though ☺. Email me if you want to meet outside of this time.

Note that I am a contract lecturer, and not full time staff, so I am not usually at the university. I may be in my office outside my office hours, so feel free to come and check, but no guarantees!

References for these notes:

- W. H. Hayt, Jr., J. E Kemmerly and S. M. Durbin, "Engineering Circuit Analysis", 7th, 8th, or 9th Editions, McGraw-Hill.

Please reference the textbook directly rather than these notes.

I strongly recommend purchasing a copy of the text book. Either Edition is fine, although the readings in this study guide refer to the 9th edition. Second hand is a good option. Library also has copies.

ENEL220 Term 3 Checklist 2019

Chapter 10

By the end of the Chapter 10 notes you should be able to:

- ☐ Do complex algebra.
- ☐ Analyse a circuit using phasors.

Chapter 14

By the end of the Chapter 14 notes you should be able to:

- ☐ Take the Laplace Transform of a function.
- ☐ Take the Inverse Laplace Transform of a function.
- ☐ Analyse a circuit with a damped sinusoidal input, R, L, and C components using the LT
- ☐ Analyse a circuit in the s-domain using techniques already learnt (e.g. mesh analysis, Norton's theorem etc).
- ☐ Work out the transfer function H(s) of a circuit.
- ☐ Work out the poles and zeroes of a circuit.
- ☐ Explain what convolution is.
- ☐ Work out the output of a circuit using convolution and the impulse response.

Exam Content

Remember you can look up old exams on the UC library website. These are a very good guide to the type of questions you are likely to get! Basic things to remember:

- ☐ Always show <u>all</u> working, even if you're doing something in your head, or if you think it's obvious (for example, write "by inspection"). This makes it easy for me to give you carried error marks if you make a silly mistake.
- ☐ Always put units on your answers!

Exam Formulas for Term 3 Material

Laplace Transform Pairs

$\mathbf{f(t) = \mathcal{L}^{-1}\{F(s)\}}$	$\mathbf{F(s) = \mathcal{L}\{f(t)\}}$	$\mathbf{f(t) = \mathcal{L}^{-1}\{F(s)\}}$	$\mathbf{F(s) = \mathcal{L}\{f(t)\}}$
$\delta(t)$	1	$\frac{1}{\beta-\alpha}\left(e^{-\alpha t}-e^{-\beta t}\right)u(t)$	$\frac{1}{(s+\alpha)(s+\beta)}$
$u(t)$	$\frac{1}{s}$	$\sin\omega t\, u(t)$	$\frac{\omega}{s^2+\omega^2}$
$tu(t)$	$\frac{1}{s^2}$	$\cos\omega t\, u(t)$	$\frac{s}{s^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$, n = 1,2,…	$\frac{1}{s^n}$	$\sin(\omega t+\theta)u(t)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\cos(\omega t+\theta)u(t)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$te^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^2}$	$e^{-\alpha t}\sin\omega t\, u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$, n = 1,2,…	$\frac{1}{(s+\alpha)^n}$	$e^{-\alpha t}\cos\omega t\, u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$

Laplace Transform Operations

Operation	$\mathbf{f(t)}$	$\mathbf{F(s)}$
Addition	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
Scalar Multiplication	$kf(t)$	$kF(s)$
Time Differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
Time Integration	$\int_{0^-}^{t} f(t)\,dt$	$\frac{1}{s}F(s)$
	$\int_{-\infty}^{t} f(t)\,dt$	$\frac{1}{s}F(s) + \frac{1}{s}\int_{-\infty}^{0^-} f(t)\,dt$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$
Time Shift	$f(t-a)u(t-a), a \geq 0$	$e^{-as}F(s)$
Frequency Shift	$f(t)e^{-at}$	$F(s+a)$
Frequency Differentiation	$-tf(t)$	$\frac{dF(s)}{ds}$
Frequency Integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(s)\,ds$
Scaling	$f(at), a \geq 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Initial Value	$f(0^+)$	$\lim_{s\to\infty} sF(s)$
Final Value	$f(\infty)$	$\lim_{s\to 0} sF(s)$ All poles of $sF(s)$ in LHP
Time Periodicity	$f(t) = f(f + nT)$, n = 1,2,…	$\frac{1}{1-e^{-Ts}}F_1(s)$ Where $F_1(s) = \int_{0^-}^{T} f(t)e^{-st}\,dt$

Complex Numbers; The Phasor; Impedance and Admittance

Readings: Appendix 5, Sections 10.4, 10.5

Complex Numbers Representations

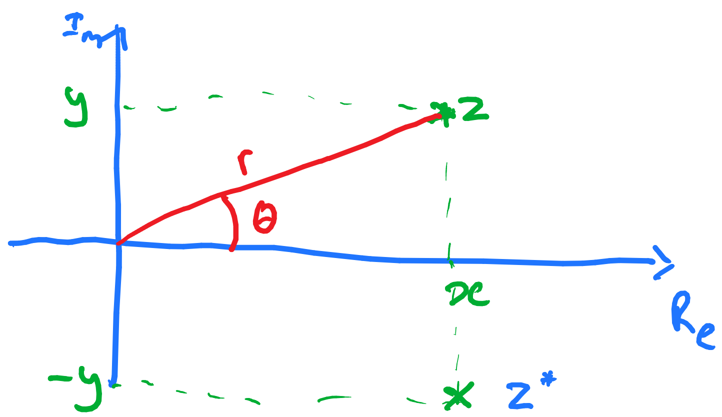
A complex number in Cartesian form is represented as $z = x + jy$.

Where $x = \mathrm{Re}\{z\}$, $y = \mathrm{Im}\{z\}$, and $j = \sqrt{-1}$; x, y are real numbers.

Real Imaginary

The complex conjugate of z is $z^* = x - jy$. The sign for the imaginary part of the number changes, but the real part stays the same.

Complex numbers can be represented on Real/Imaginary axes - the Complex Plane. The terms r and θ are used if representing a complex number in Polar form ($z = re^{j\theta}$).



Maths with Complex Numbers

Addition and subtraction is most easily done in Cartesian form (simply add/subtract the real components and the imaginary components). For multiplication and division, it depends if you want to end up with a Polar or Cartesian answer.

polar

Multiplication: $z_1 = r_1 e^{j\theta_1}$ & $z_2 = r_2 e^{j\theta_2}$

$z_1 \cdot z_2 = r_1 r_2 e^{j(\theta_1 + \theta_2)}$ or we can do cartisian

Division: polar

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$$

or we can do cartisian

It is best to swap between forms using the appropriate functions on your calculator! You can do it using trigonometry, but you will need to be careful to get the angle correct.

Example:

Multiply $z_1 = 3 + j2$ and $z_2 = 8 - j7$ in both Cartesian and Polar forms.

polar: $z_1 = 3.6 e^{j0.6}$, $z_2 = 10.6 e^{-j0.7}$

$z_1 \cdot z_2 = 3.6 \cdot 10.6 \, e^{j(0.6 + 0.7)} \cdots = 38 - j5$

Cartesian:

$z_1 \cdot z_2 = (3 + j2) \cdot (8 - j7)$

$\vdots$

$= 38 - j5$

Example:

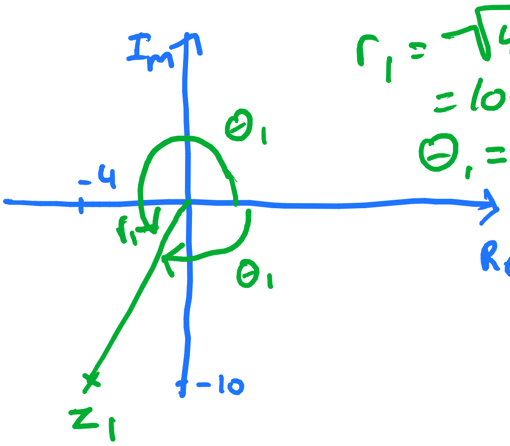
Work out $\frac{z_1}{z_2}$ if $z_1 = -4 - j10$ and $z_2 = 6 - j5$. First use the Cartesian form and the complex conjugate. Then use the Polar form, and do the conversion to Polar form using trigonometry.

$$\frac{z_1}{z_2} = \frac{-4-j10}{6-j5} \times \frac{6+j5}{6+j5}$$ (swap units)

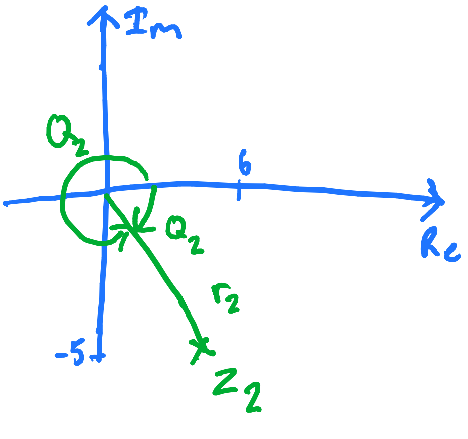
$$= \frac{-24 - j20 - j60 - j^2 \cdot 50}{36 - j^2 25}$$

$$\vdots$$

$$= 0.43 - j1.31$$



$$r_1 = \sqrt{4^2 + 10^2}$$
$$= 10.77$$
$$\theta_1 = \tan^{-1}\left(\frac{10}{4}\right) + \pi$$ right Quadrant
$$= 4.33$$
$$Z_1 = 10.77e^{j4.33}$$



$$r_2 = \sqrt{5^2 + 6^2} = 7.81$$

$$\theta_2 = \tan^{-1}\left(\frac{5}{6}\right) = -0.69$$

$$z_2 = 7.81\, e^{-j0.69}$$

$$\frac{z_1}{z_2} = \frac{10.77}{7.81} e^{j(4.33 - -0.69)} = 1.38\, e^{j5.02} \rightarrow = 0.43 - j1.31 \text{ (using function on calculator)}$$

Notation

1) We use j for imaginary numbers in electrical engineering so we don't get confused with the current (i).

2) The Polar form can also be written as:

$$z = r(\cos\theta + j\sin\theta)$$

(Remember Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$)

3) Sometimes you will see cis (cos i sine) notation, although not in this course.

$$z = r\,\mathrm{cis}\,\theta$$

4) We will use Angular or Phasor notation a lot:

$$z = r\angle\theta$$

Angular notation is quick and easy to write, but is not mathematically equal to the other Polar forms (see following section). It can be useful to change between different Polar forms – you will possibly come across this in other courses.

Phasor Representation

In Term 2 we mainly looked at circuits with DC sources. Phasors are useful for solving circuits with AC sources in the time domain.

If we have a source $v(t) = 5\cos(3t + 10°)$ then simply using the techniques from last term will quickly get fairly complicated. Using phasors simplifies things.

Using Euler's identity:

$$\cos(\omega t) = R_e\{e^{j\omega t}\}$$

$$V_m\cos(\omega t + \phi) = R_e\{V_m e^{j(\omega t + \phi)}\}$$

$$\therefore\ v(t) = 5\cos(3t + 10°) = R_e\{5e^{j(3t + 10°)}\}$$

We simplify further by representing the signal as a complex quantity. We do this by adding an imaginary component to the signal – as this doesn't affect the real component (which is what we care about), then this isn't a problem.

$$v(t) = 5e^{j(3t + 10°)}$$

Finally, we supress the $e^{j\omega t}$ factor to write in Phasor form ($\mathbf{V}$) (a frequency-domain representation):

$$\bar{V} = 5e^{j10°}$$

$$= 5\angle 10°$$

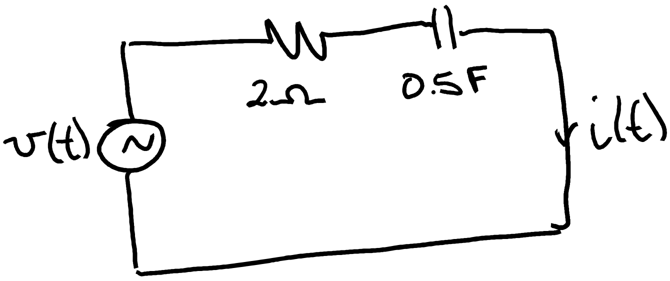
To convert back to a time-domain representation, $v(t)$, we need to know ω. We can supress it while doing the calculations, as capacitors, inductors, and resistors have no effect on it.

Integrating and Differentiating a Phasor

To integrate a phasor, you divide by $j\omega$. To differentiate you multiply by $j\omega$. If interested in the proof, look in the textbook.

Example:

If $v(t) = 5\cos(3t + 10°)$ V, what is $i(t)$ in the circuit below? Assume there is only a steady-state (forced) response.



Using KVL:

$$v(t) = 2i(t) + \frac{1}{0.5}\int i(t)\,dt$$

Converting to phasors:

$$\bar{V} = 5\angle 10° \qquad \bar{I} = I_m \angle \phi$$

$$5\angle 10° = 2\bar{I} + 2\int \bar{I}\,dt$$

$$= 2\bar{I} + \frac{2}{3j}\bar{I}$$

integrating a phasor = divide by ωc

$$\bar{I} = \frac{5\angle 10°}{2 + \frac{2}{3j}}$$

$$= \frac{5\angle 10° \times 3j}{6j + 2}$$

$$= \frac{5\angle 10° \times 3\angle 90°}{6.32\angle 71.6°}$$

$$= 2.37\angle 28.4°$$

$$i(t) = 2.37\cos(3t + 28.4°)\text{ A}$$

Impedance and Admittance

• can be complex values

Impedance, **Z**, is defined as the voltage-current ratio:

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} \quad (\Omega)$$

For resistors, the impedance is the same as the resistance: $\boldsymbol{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \mathrm{R}$.

For capacitors:

$$i(t) = C\frac{dv(t)}{dt}$$

$$\bar{I} = j\omega C\bar{V}$$

$$\bar{Z} = \frac{\bar{V}}{\bar{I}} = \frac{1}{j\omega C}$$

Similarly for inductors, $\boldsymbol{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \mathrm{j\omega L}$.

You can add impedances in the same way as resistors. Because all components are now in Ohms, you can add capacitors to inductors to resistors.

Admittance, **Y,** is the inverse of impedance, and can also be useful.

$$\bar{Y} = \frac{\bar{I}}{\bar{V}} \quad (S)$$

• inverse of Impedance

You can add admittances in the same way that you add capacitors.

Nodal and Mesh Analysis; Superposition, Source Transformations, and Thévenin's Theorem

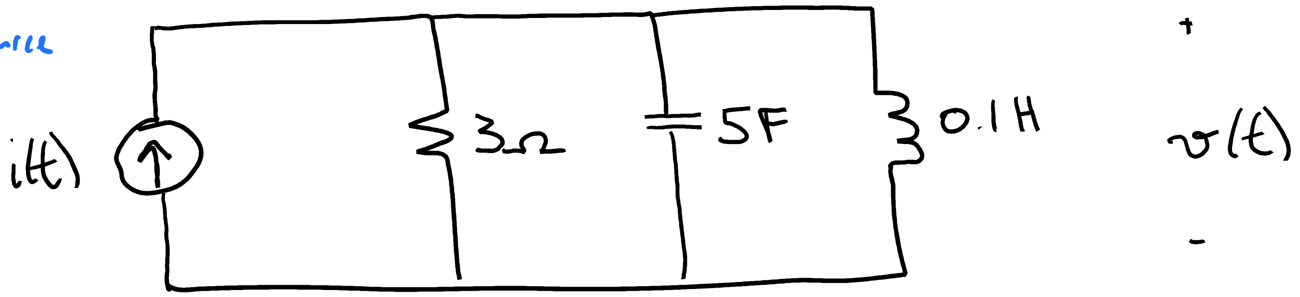
Readings: Sections 10.6, 10.7

The circuits we are looking at are still linear, so we can still apply the techniques from last semester.

Example:

If $i(t) = 30\cos(7t + 90°)$, what is $v(t)$ for the circuit below? Assume there is no natural response.

* AC Source



$\Sigma I_{in} = \Sigma I_{out}$ (nodal analysis)

Transform to phasors
- AC circuit

$$i(t) = \frac{v(t)}{3} + \frac{1}{0.1}\int v(t)\,dt + 5\frac{dv(t)}{dt}$$

$$\bar{I} = \frac{\bar{V}}{3} + \frac{10}{7j}\cdot\bar{V} + 5\cdot 7j\,\bar{V}$$

$$30\angle 90° = \bar{V}\left(\frac{1}{3} + \frac{10}{7j} + 35j\right)$$

$$\bar{V} = \frac{3\angle 90°}{\left(\frac{1}{3} + \frac{10}{7j} + 35j\right)}$$

$$= \frac{21j \times 30\angle 90°}{-705 + 7j} = 0.89\angle 0.6°$$

$$v(t) = 0.89\cos(7t + 0.6°)\ \text{V}$$

As an alternative to using the differential/integral equations for capacitors and inductors, you can use admittances (which are easier when adding in parallel):

Y = Admittance

$$\bar{Y} = \frac{\bar{I}}{\bar{V}}$$

$$\bar{V} = \frac{\bar{I}}{\bar{Y}}$$

$$= \frac{30\angle 90^\circ}{\bar{Y}_R + \bar{Y}_c + \bar{Y}_L}$$

$$= \frac{30\angle 90^\circ}{\frac{1}{3} + j(7.5) + \frac{1}{j(7 \cdot 0.1)}}$$

$$= \frac{30\angle 90^\circ}{\frac{1}{3} + 35j - \frac{j}{0.7}}$$

$$= \frac{30\angle 90^\circ}{33.6\angle 89.4^\circ}$$

$$= 0.89\angle 0.6^\circ$$

$v(t) = \ldots$ (same as before)

$$Y = \frac{1}{Z}$$

$$\therefore\ Y_R = \frac{1}{R}$$

$$Y_C = j\omega C$$

$$Y_c = \frac{1}{jL}$$

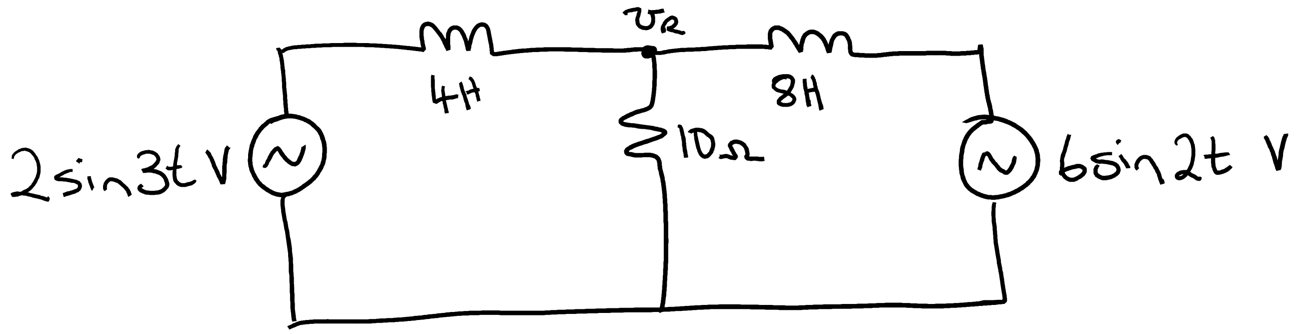
If there are two sinusoidal sources with different frequencies, then we can use superposition to solve this, doing the final addition in the time domain. (Note: superposition can also be used if there are multiple sources with the same frequency.)

Superposition

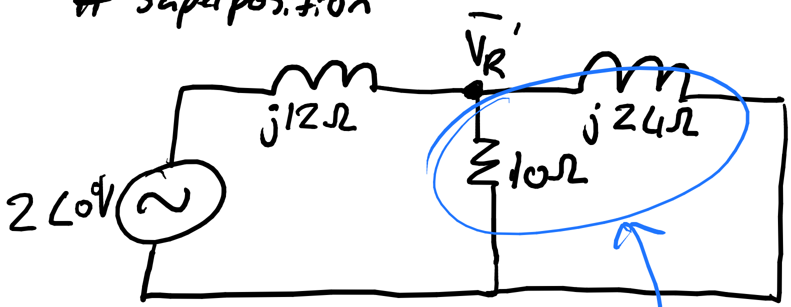
Example:

* having one source on at a time

For the circuit below, determine the voltage across the resistor. Assume there is no natural response.

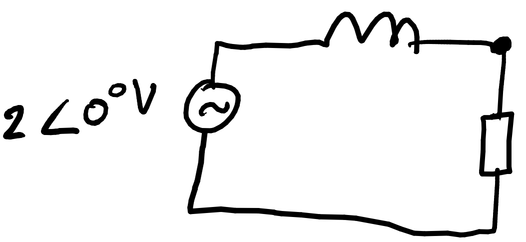


superposition



$(Z_L = j\omega L)$

0V = s.c
0A = o.c



$\dfrac{j240}{10+j24}$ parallel $(R_1 // L_2)$

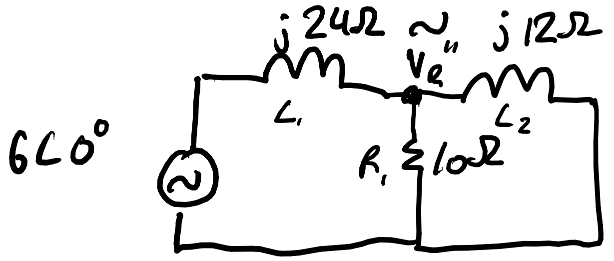
$$\tilde{V}_R' = 2\angle 0^\circ \left(\frac{\frac{j240}{10+j24}}{\frac{j240}{10+j24} + j12} \right)$$

$\vdots$

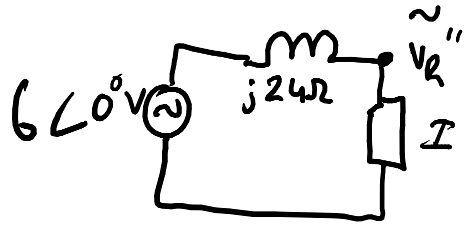
$$= \frac{j480}{-288+j360}$$

$$= \frac{480\angle 90^\circ}{2.902k\angle 172.9^\circ} = 0.17\angle -82.9\ V$$

\# Source 2



$(Z_L = j\omega L)$



$$I = R_1 // L_2 \quad \frac{j80}{10+j8}$$

$$\tilde{V}_R'' = \frac{\frac{j80}{10+j8}}{\frac{j80}{10+j8} + j16} \times 6\angle 0^\circ$$

$$= \frac{480\angle 90^\circ}{272\angle 118.1^\circ}$$

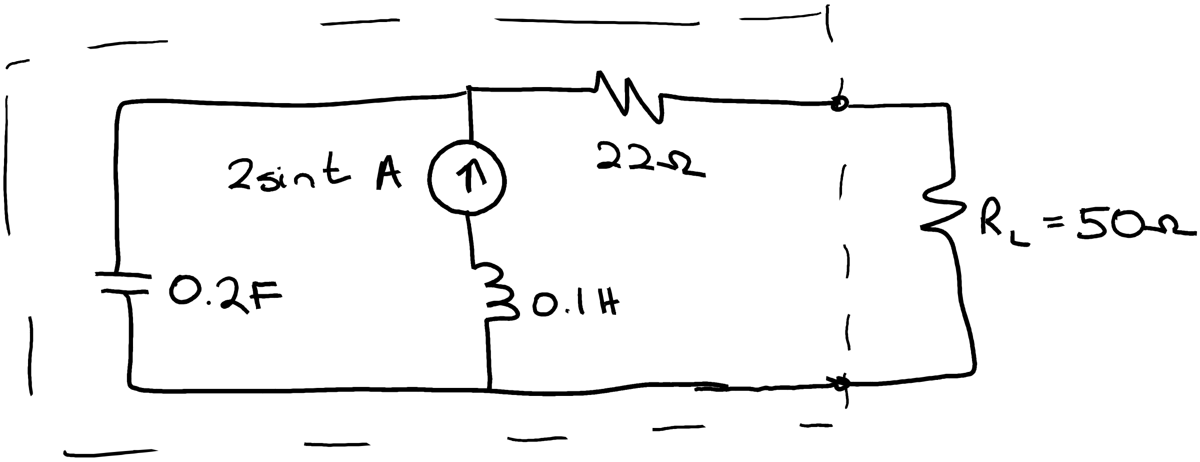
$$= 1.76\angle -28.1^\circ V$$

\# in time domain

$$V_R = V_R' + V_R''$$

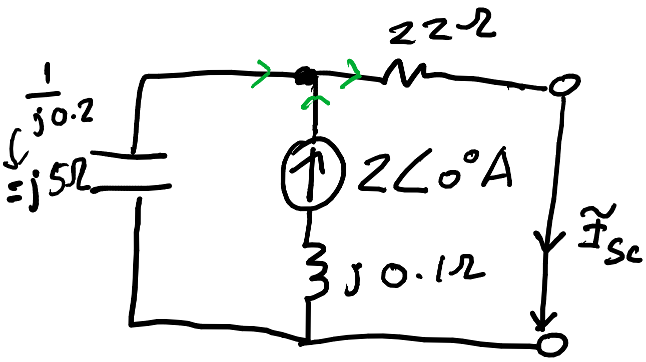
$$= 1.04\sin(3t - 38.7^\circ) + 1.76\sin(2t - 28.1^\circ)\ V$$

Example:
Find the Norton equivalent of circuit A below, then do a source transformation, and find the voltage across the load resistor.



A

Network A



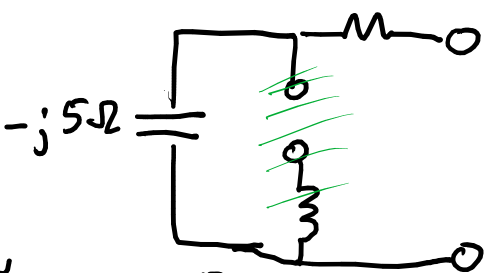
$$\tilde{I}_{sc} = \frac{-j5}{22-j5} \times 2\angle 0^\circ$$ (current divider)

$$= \frac{-5\angle 90^\circ \times 2\angle 0^\circ}{22.56\angle -12.8^\circ}$$

$$= -0.44\angle 102.8^\circ \text{ A}$$

$$\tilde{Z}_n = \tilde{Z}_{Th}$$

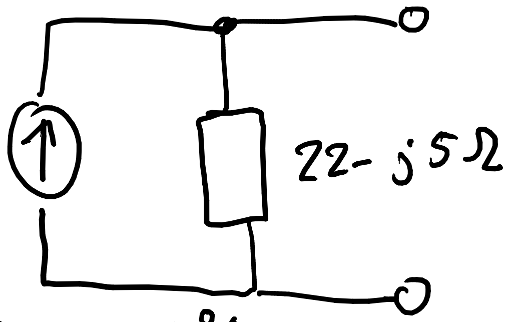
Current source open circuit



find $\tilde{Z}_n$

$$\therefore \tilde{Z}_n = 22 - j5\,\Omega$$

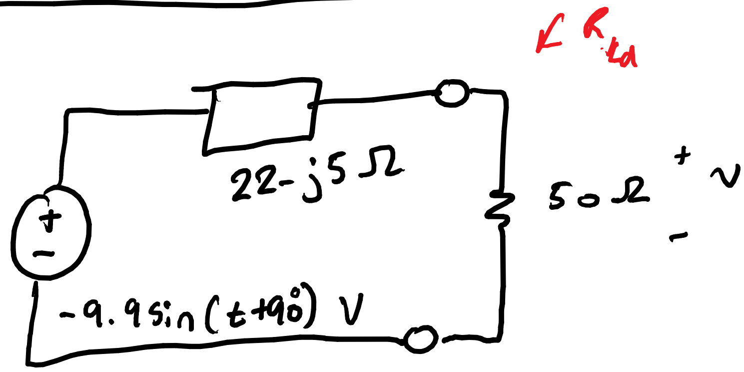
Norton Equivalent



$-0.44\angle 102.8^\circ$ A

$= -0.44\sin(t + 102.8^\circ)$ A

Source transformation :



$$\tilde{V}_{TH} = \tilde{I}\tilde{Z}$$
$$= -0.44\angle 102.8^\circ \times 22 - j5$$
$$= -0.44\angle 102.8^\circ \times 22.56\angle -12.8^\circ$$
$$= -9.9\angle 90^\circ \text{ V}$$

$$\tilde{v} = \frac{50}{22 - j5 + 50} \times -9.9\angle 90^\circ$$
$$= \frac{-495\angle 90^\circ}{72.17\angle -3.97^\circ}$$
$$= -6.86\angle 93.97^\circ \text{ V}$$
$$v(t) = -6.86\sin(t + 93.97^\circ) \text{ V}$$

Complex Frequency; Definition of the Laplace Transform

Readings: Sections 14.1, 14.2

Complex Frequency

If we have a damped sinusoid for a source, $x(t) = Ae^{\sigma t}\cos(\omega t + \emptyset)$, then things get even more complicated. We can represent various types of waveforms, depending on the values of σ and ω.

1. $\sigma = \omega = 0$

   $x(t) = A\cos(\phi)$

   => a constant i.e DC source

2. $\sigma = 0$

   $x(t) = A\cos(\omega t + \phi)$

   => a sinusoid

3. $\omega = 0$

   $x(t) = Ae^{\sigma t}\cos(\phi)$

   => a exponential

4. Neither equal 0

   $x(t) = Ae^{\sigma t}\cos(\omega t + \phi)$

   => damped sinusoid

Using Euler's formula, we can rearrange the general form to get:

$$x(t) = Re\{Ae^{(\sigma + j\omega)t}\} = Re\{Ae^{j\phi}e^{st}\}$$

$s = \sigma + j\omega$

complex frequency

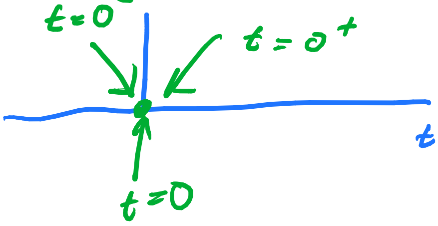
The Laplace Transform

For a general function f(t), the definition of the Laplace Transform is:

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

As you can see from the limits of integration, this covers all of both positive and negative time. However, we're usually only interested in what happens after $t = 0$ s, so we usually use the one-sided Laplace Transform instead:

$$F(s) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt$$



We use 0^- rather than 0 or 0^+, as we want to be able to use the initial conditions of the circuit.

Remember:

$t(0^-) \neq t(0) \neq t(0^+)$

$\therefore$ $v(0^-)$, $v(0)$, $v(0^+)$, could all be different,

as can $i(0^-)$, $i(0)$, $i(0^+)$

Voltage across a capacitor, v_c, and current through an inductor, i_L, are special cases. Other voltages and currents can change instantly.

Notation (hand-written):

$$\mathcal{L}[\underset{\text{time}}{f(t)}] = \overset{\text{frequency}}{F(s)}$$

Notation (typed):

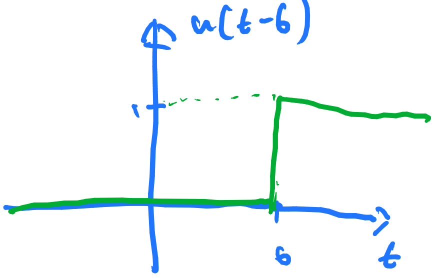
$$\mathcal{L}[f(t)] = \mathbf{F(s)}$$

Example:
Work out the LT of $f(t) = 3u(t-6)$ using the LT formula.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_{6^-}^\infty e^{-st} \times 3u(t-6)\, dt$$

u(t-6) Looks like :



$$\therefore F(s) = 3 \int_t^\infty e^{-st}\, dt$$

$$= 3t -\frac{1}{s} e^{-st} \Big]_6^\infty$$

$$= 3\left[0 - \frac{1}{s} e^{6s}\right]$$

$$= \frac{3}{s} e^{-6s}$$

Example:
What is the one-sided LT of $f(t) = e^{-3t}$?

$\vdots$

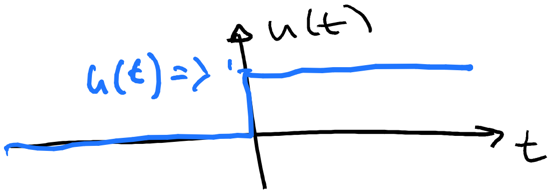
$$= \frac{1}{s+3}$$

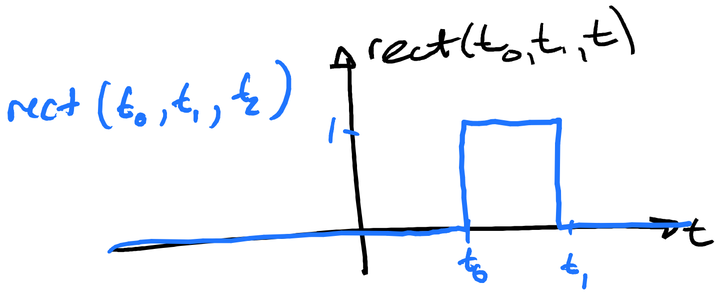
Laplace transforms of simple time functions; Basic theorems for the Laplace transform

Readings: Sections 14.3,14.5

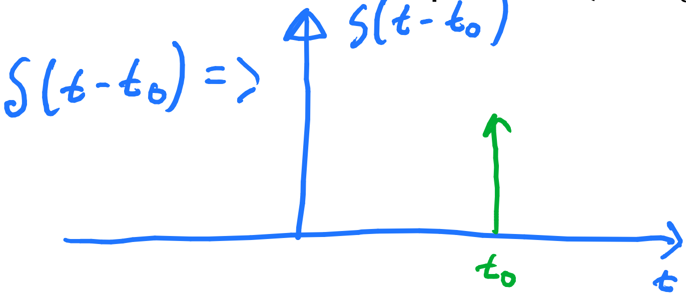
More Useful Functions

Reminder: in Term 2 we looked at the unit step function, $u(t)$, and the rectangular pulse function, $\mathrm{rect}(t_0, t_1, t)$:

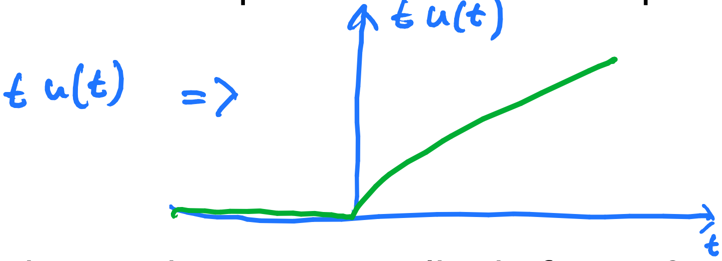




We will add to that the unit impulse, $\delta(t - t_0)$, and the ramp function, $tu(t)$.



The unit impulse has *infinite* amplitude at one instant in time. The area "under" the unit impulse is defined as equal to 1. A really useful function.



The unit ramp has zero amplitude for t < 0 s, but is equal to t for positive time.

Properties of the Laplace transform

To be able to use the LT on a circuit, we need to know a few more things… Equations for circuits with capacitors and inductors in them usually look something like this (assuming initial conditions are zero):

$$v(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int i(t)\,dt$$

So – can we add LTs? How do we deal with differentials and integrals?

1. Linearity: Since the LT is calculated using integration, and we know that $\int (a+b) = \int a + \int b$, then it makes sense that:

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$
$$= F_1(s) + F_2(s)$$

   Similarly, if b is a constant, then:

$$\mathcal{L}[bf(t)] = b\mathcal{L}[f(t)]$$
$$= bF(s)$$

2. Differentiation and Integration in the Time Domain: Using integration by parts you can show that:

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0^-)$$
$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0^-) - f'(0^-)$$
$$\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \frac{1}{s}F(s)$$

3. Time Shift: This helps us deal with situations where we might have different switches turning on at different points in time.

$$\mathcal{L}\left[f(t-a)\,u(t-a)\right] = \int_{0^-}^{\infty} e^{-st} f(t-a)\,u(t-a)\,dt$$

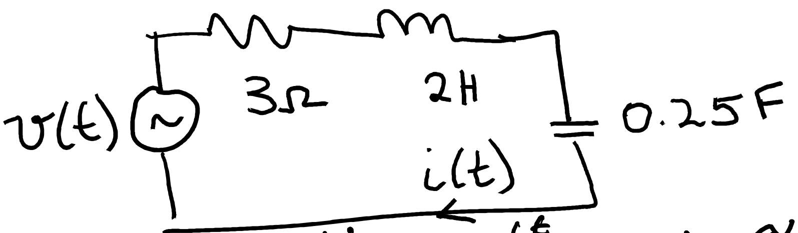
$$= \int_{a^-}^{\infty} e^{-st} f(t-a)\,dt \qquad \text{set } \tau = t-a$$

$$= \int_{0^-}^{\infty} e^{-s(\tau+a)} f(\tau)\,d\tau$$

$$= e^{-as}\int_{0^-}^{\infty} e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-as}F(s)$$

Example:
For the circuit below, find **I(s)** in terms of **V(s)** if $i(0^-) = 0$ A.



$$v(t) = 3i(t) + 2\frac{di}{dt} + 4\int_0^t i(\tau)\,d\tau$$

$$V(s) = 3I(s) + 2\left(sI(s) - i(\overset{0}{\cancel{0^-}})\right) + \frac{4}{s}I(s)$$

$$= 3I(s) + 2sI(s) + \frac{4}{s}I(s)$$

$$I(s) = \frac{V_s}{3 + 2s + 4/s}$$

$$I(s) = \frac{s\,V(s)}{3s + 2s^2 + 4}$$

What does **I(s)** look like if $\mathrm{v(t)} = \mathrm{u(t)}$?

$$V(s) = \int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}$$

$$I(s) = \frac{s\left(\frac{1}{s}\right)}{2s^2 + 3s + 4}$$

$$= \frac{1}{2s^2 + 3s + 4}$$

We can use tables of LT pairs rather than integrating – this is normally faster and easier. The tables are at the start of this book, and are the same ones you will get in the exam.

General Process

1. Write an equation for $\mathrm{i(t)}$ and/or $\mathrm{v(t)}$.
2. Take the Laplace Transform and get **I(s)** and/or **V(s)**
3. Rearrange the equation(s) as desired
4. Take the inverse LT to get back into the time domain (see next section)

Inverse transform techniques; The initial-value and final-value theorems

Readings: Sections 14.4, 14.6

The Inverse Laplace Transform

Analysing something in the **s**-domain isn't terribly useful if we can't get back into the time domain. The formula for taking the inverse LT is:

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_0 - j\omega}^{\sigma_0 + j\omega} e^{st}F(s)\,ds$$

Because it's complicated, we always use tables in this course. The notation used is:

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

$$f(t) \Leftrightarrow F(s)$$

The tables don't include all possible options, so we usually need to simplify the equation to get functions that are close to those in the tables.

We usually have polynomials of the form:

$$I(s),\ V(s) = \frac{N(s)}{D(s)} \begin{array}{l} \leftarrow \text{zero's} \\ \leftarrow \text{poles} \end{array}$$

The values of **s** which result in **N(s)** = 0 are called zeros. The values of **s** which result in **D(s)** = 0 are called poles. We will do more on these later!

Note that both **N(s)** and **D(s)** could have multiple powers of **s** – in our example on the previous page **N(s)** = 1 (no powers of **s**), but **D(s)** = $2s^2 + 3s + 4$ (two powers of **s**). We often use partial fractions to get our equation into a form that works with the tables.

Example

If $\mathbf{V}(\mathbf{s}) = \frac{2}{s^2+6s-7}$, what is $v(t)$?

$$V(s) = \frac{2}{(s+7)(s-1)}$$

two distinct poles
$s = 1, -7$

$$= \frac{A}{s+7} + \frac{B}{s-1} = \frac{A(s-1) + B(s+7)}{(s+7)(s-1)}$$

$$\therefore A(s-1) + B(s+7) = 2$$

① $s = 1$ $\quad B(8) = 2 \quad B = \frac{1}{4}$

$\underline{s = -7}$ $\quad A(-8) = 2 \quad A = -\frac{1}{4}$

② $s(A+B) + (-A+7B) = 2$

$A = -B \qquad -A + 7B = 2$

$B + 7B = 2$

$8B = 2$

$B = 1/4 \;,\; A = -1/4$

$$V(s) = \frac{-1/4}{s+7} + \frac{1/4}{s-1}$$

$$v(t) = -\frac{1}{4}e^{-7t}u(t) + \frac{1}{4}e^{t}u(t)$$ ⇐ from table

$$= \frac{1}{4}\left(e^{t} - e^{-7t}\right)u(t) \text{ V}$$

Example:

Find $v(t)$ if $\mathbf{V}(\mathbf{s}) = \frac{s+6}{s^2-4s+4}$

$$V(s) = \frac{s+6}{(s-2)^2}$$

$$= \frac{A}{(s-2)^2} + \frac{B}{s-2}$$

$$\therefore\ s+6 = A + B(s-2)$$

$\underline{s=2}$

$$8 = A$$

$$\therefore\ s+6 = 8 + B(s-2)$$

$$Bs = s$$

$$B = 1$$

$$V(s) = \frac{8}{(s-2)^2} + \frac{1}{s-2}$$

$$v(t) = 8t\,e^{2t}u(t + e^{2t}u(t)$$

$$= (8t+1)e^{2t}u(t)\ \text{V}$$

Example:

Factorise $\mathbf{I}(\boldsymbol{s}) = \frac{2(s+3)}{(s+1)(s^2+2)}$

$$= \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$$

$$A(s^2+2)+(Bs+C)(s+1) = 2(s+3)$$

$s=-1$

$$\frac{4}{3}s^2 + \frac{8}{3} + Bs^2 + Bs + Cs + C = 2s+6$$

$3A=4$

$A=\frac{4}{3}$

$$\frac{4}{3}+B=0 \qquad B=-\frac{4}{3}$$

$$\frac{8}{3}+C=6 \qquad C=\frac{10}{3}$$

$$I(s) = \frac{4/3}{s+1} + \frac{(-4/3)s + 10/3}{s^2+2}$$

Initial-Value and Final-Value Theorems

These two theorems mean we can evaluate the whatever we're interested in at $t = 0^+$ and at $t = \infty$.

<u>Initial-Value Theorem:</u> $\lim_{t\to 0^+} f(t) = \lim_{s\to\infty}\{s\mathbf{F}(\mathbf{s})\}$

<u>Final-Value Theorem:</u> $\lim_{t\to\infty} f(t) = \lim_{s\to 0}\{s\mathbf{F}(\mathbf{s})\}$

<u>NOTE</u>: The poles of **F(s)** *must* all have a real part < 0 for the FVT to work. A single pole at the original *might* be ok, but we will avoid examples of this type.

Poles and Zeros

Remember that if we have $\mathbf{F}(\mathbf{s}) = \frac{\mathbf{N}(\mathbf{s})}{\mathbf{D}(\mathbf{s})}$, then the zeros of the function are the values of **s** that lead to **N(s)** = 0, and the poles of the function are the values of **s** that lead to **D(s)** = 0.

Example

If we have $\mathrm{V}(\mathrm{s}) = \frac{\mathrm{s}+2}{(\mathrm{s}+1)(\mathrm{s}+3)}$, what are the poles are zeroes of the function? Could we use the FVT?

zeros $\Rightarrow$ $s+2=0$

$s=-2$

poles $\Rightarrow$ $(s+1)(s+3)=0$

$s=-1,-3$

In this case, all the poles are < 0, therefore it's ok to use the FVT.

Example

If $\mathrm{v}(\mathrm{t}) = e^{-3t}\mathrm{u}(\mathrm{t})$, find $\mathrm{v}(0^+)$ and $\mathrm{v}(\infty)$ using the IVT and the FVT.

$$V(s) = \frac{1}{s+1}$$

$$v(0^+) = \lim_{s\to\infty} s\,V(s)$$

$$= \lim_{s\to\infty} \frac{s}{s+3}$$

$$= \lim_{s\to\infty} \frac{1}{1+\frac{3}{s}}$$

$$= 1\,V$$

For FVT, check poles:

$s+3=0$

$s=-3<0$ $\therefore$ okay to use FVT

$$v(\infty)=\lim_{s\to 0}\left[\frac{s}{s+3}\right]$$

$$=0\,\text{V}$$

In this instance we knew this without using the IVT or FVT, since we know what a negative exponential looks like. But, this won't always be the case.

Example

If $\mathrm{v(t)} = \mathrm{e}^{-3\mathrm{t}}\mathrm{u(t)} + 9\delta(\mathrm{t})$, find $\mathrm{v}(0^+)$ and $\mathrm{v}(\infty)$ using the IVT and the FVT.

$$V(s) = \frac{1}{s+3} + 9$$

$$= \frac{9s+28}{s+3}$$

$$v(0^+) = \lim_{s\to\infty} s\,V(s)$$

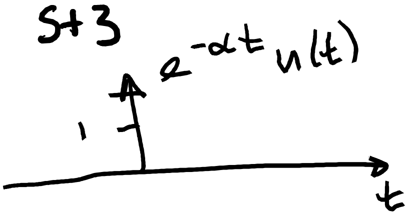
$$= \lim_{s\to\infty} \frac{9s^2+28}{s+3}$$

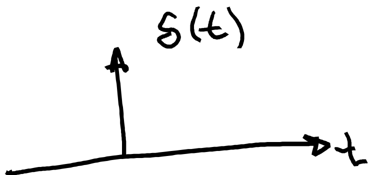
$$=$$

poles $\Rightarrow$ $s+3=0$, $s=-3<0$ $\therefore$ FVT ok

$$v(\infty) = \lim_{s\to 0} \frac{9s^2+28}{s+3}$$

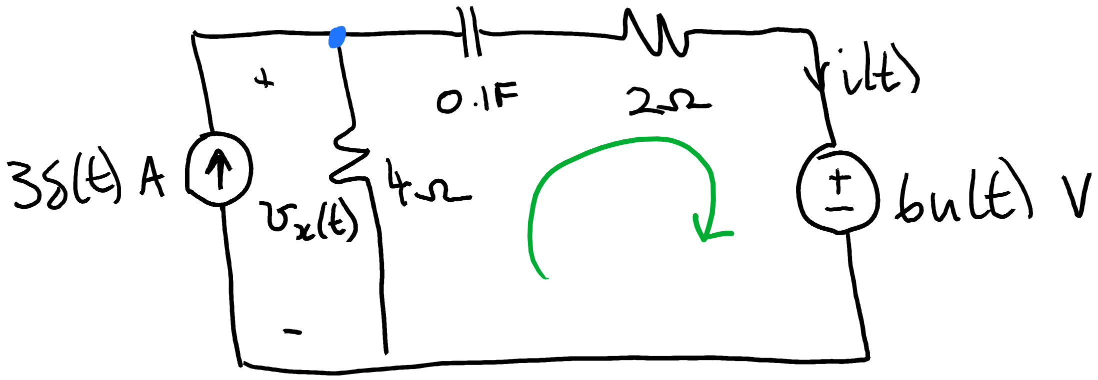
$$=0\,\text{V}$$





Example:

For the circuit below, write an equation with $\mathrm{i(t)}$ as the only unknown. Take the LT, solve for **I(s)**, and then find $\mathrm{i(t)}$ by taking the inverse LT. Assume the initial conditions are zero.



(1) $3\delta(t) = i(t) + \dfrac{V_x(t)}{4}$ (KCL ⇒ $\Sigma I_{in} = \Sigma I_{out}$)

$V_x(t) = 12\delta(t) - 4i(t)$

(2) $-V_x + \dfrac{1}{0.1}\displaystyle\int_{0^-}^{\infty} i(t)\,dt + 2i(t) + 6u(t) = 0$ (KVL)

(1) into (2):

$$-12\delta(t) + 4i(t) + 10\int_{0^-}^{\infty} i(t)\,dt + 2i(t) + 6u(t) = 0$$

$$-12 + 4I(s) + \frac{10}{s}I(s) + 2I(s) + \frac{6}{s} = 0$$

$$I(s)\left(6 + \frac{10}{s}\right) = 12 - \frac{6}{s}$$

$$I(s) = \frac{\frac{12s-6}{s}}{\frac{6s+10}{s}}$$

$$= \frac{12s-6}{6s+10}$$

$$= \frac{12s-6}{6(s+\frac{10}{6})}$$

$$= \frac{A}{6} + \frac{B}{s+\frac{10}{6}}$$

$\Rightarrow 12s-6 = A(s+\frac{10}{6}) + 6B$

⋮

$B = -4.33$, $A = 12$

$$\therefore I(s) = \frac{12}{s} - \frac{4.33}{s+\frac{10}{6}}$$

$$= 2 - \frac{4.33}{s+1.67}$$

$$i(t) = 2\delta(t) - 4.33e^{-1.67t}u(t)\,A$$

Z(s) and Y(s)

Readings: Section 14.7

Admittance and Impedance

Remember from the start of the term:

Impedance => voltage-current ratio

Admittance => current-voltage ratio

These definitions hold true in the **s**-domain too.

$$\mathbf{Z}(\mathbf{s}) = \frac{\mathbf{V}(\mathbf{s})}{\mathbf{I}(\mathbf{s})} \text{ and } \mathbf{Y}(\mathbf{s}) = \frac{\mathbf{I}(\mathbf{s})}{\mathbf{V}(\mathbf{s})} = \frac{1}{Z(s)}$$

Resistors

Very straightforward:

$$v(t) = i(t) R$$

$$V(s) = I(s) R$$

$$Z(s) = \frac{V(s)}{I(s)} = R \quad (\text{unit } \Omega)$$

$$Y(s) = \frac{I(s)}{V(s)} = \frac{1}{R} \quad (\text{unit } S)$$

Inductors

A little more complicated:

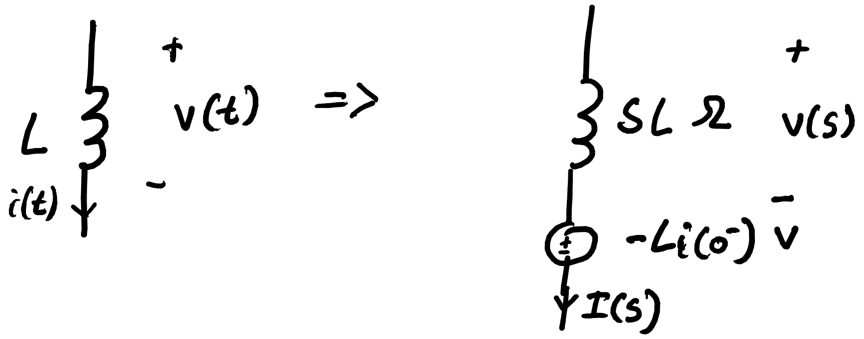
$$v(t) = L \frac{di}{dt}$$

$$V(s) = L(sI(s) - i(0^-))$$

If $i(0^-) = 0 \Rightarrow V(s) = LsI(s)$

$Z(s) = sL \ \Omega$ $\quad i(0^-) = 0 A$

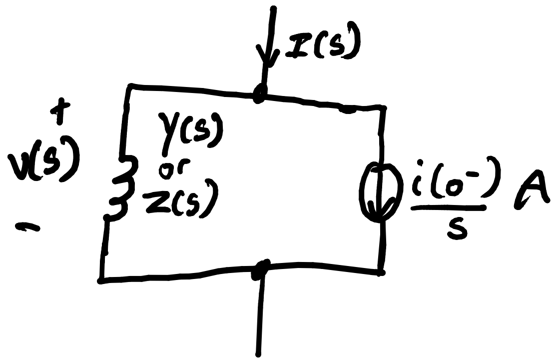
We can now draw an inductor in the **s**-domain:



$\Rightarrow Li(0^-)$ will be a constant, which is why it can be modelled as a DC source

Alternatively, if we want a current-based representation, we can rearrange our formula for **V(s)** to:

$$I(s) = \frac{V(s)}{sL} + \frac{i(0^-)}{s}$$



$$Y(s) = \frac{1}{sL} \ \text{S}$$

$$Z(s) = sL \ \Omega$$

Capacitors

We can do a similar analysis for capacitors:

$$i(t) = C\frac{dv}{dt}$$

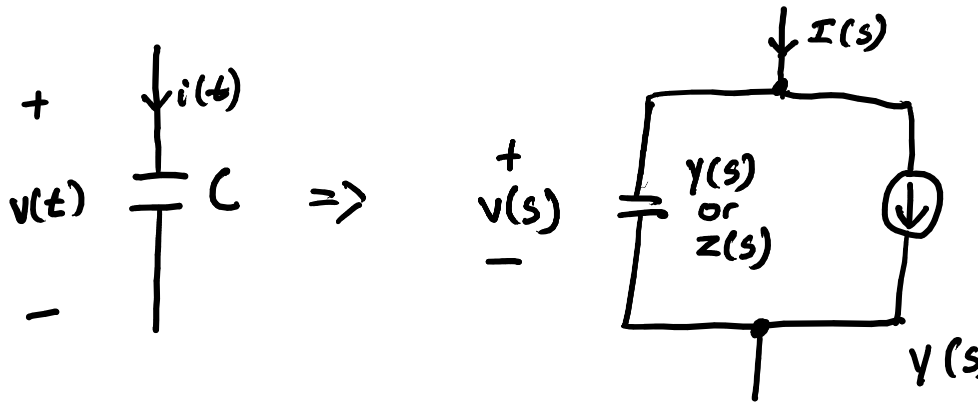
$$I(s) = C\left(sV(s) - v(0^-)\right)$$

If $v(0^-) = 0 \Rightarrow I(s) = CsV(s)$

$$Z(s) = \frac{1}{sC} \ \Omega$$

$$v(0^-) = 0 \text{ V}$$

Representing this as a circuit:

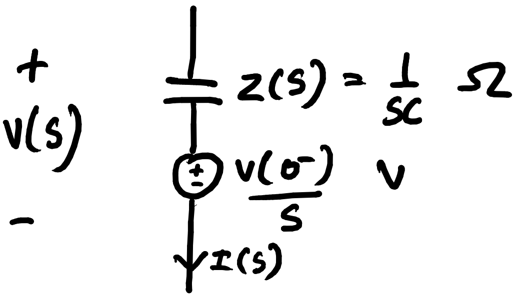


$$Y(s) = sC \quad S$$
$$Z(s) = \frac{1}{sC} \quad \Omega$$

Again, $\mathrm{Cv}(0^-)$ will be a constant, which is why we can represent it as a DC current source.

If we want a series representation, we can rearrange the equation to get:

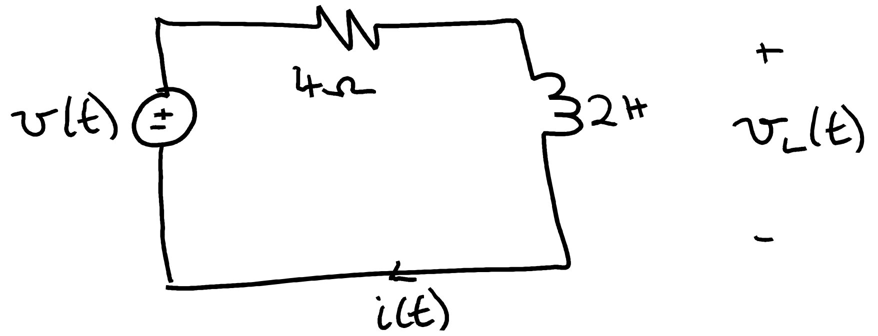
$$V(s) = \frac{I(s) + Cv(0^-)}{sC}$$
$$= \frac{I(s)}{sC} + \frac{v(0^-)}{s}$$



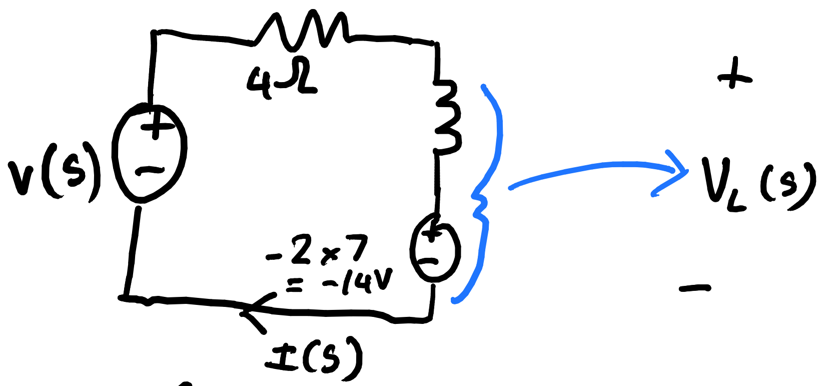
We use these circuit representations to make analysis in the **s**-domain easier. Now we redraw the circuit in the **s**-domain and then analyse it, rather than taking the LT of everything after writing an expression in the time-domain.

Example

For the circuit below, find $v_L(t)$ if $v(t) = 6u(t)$ V, and $i(0^-) = 7$ A.



Redraw circuit in s-domain:



$V(s) = \frac{6}{s}$ V $\quad \left(\frac{6}{u(t)} \right)$

$$-\frac{6}{s} + 4I(s) + 2sI(s) - 14 = 0 \quad (\text{KVL})$$

$$I(s)(4+2s) = 14 + \frac{6}{s}$$

$$I(s) = \frac{14s+6}{s(4+2s)}$$

$$V_L(s) = 2sI(s) - 14$$

$$= 2s\left(\frac{14s+6}{s(4+2s)}\right) - 14$$

$$= \frac{14s+6}{2+s} - 14$$

$$V_c(s) = \frac{14s + 6 - 28 - 14s}{2+s}$$

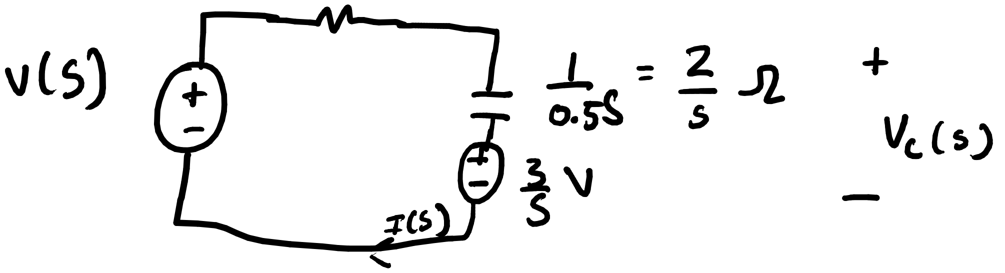
$$= \frac{-22}{s+2}$$

$$V_c(t) = -22e^{-2t}u(t) \text{ V}$$

Example

For the circuit below, what is $\mathrm{i(t)}$ if $\mathrm{v(t)} = 7\mathrm{e}^{-3\mathrm{t}}\mathrm{u(t)}$ V and $\mathrm{v_c(0^-)} = 3$ V?

$\Downarrow$



$$V(s) = \frac{7}{s+3}$$

$$-\frac{7}{s+3} + 3I(s) + \frac{2}{s}I(s) + \frac{3}{s} = 0 \quad \text{(KVL)}$$

$$I(s)\left(3 + \frac{2}{s}\right) = \frac{7}{s+3} - \frac{3}{s}$$

$$\left(\frac{3s+2}{s}\right)I(s) = \frac{7s - 3s - 9}{s(s+3)}$$

$$I(s) = \frac{s(4s-9)}{s(s+3)(3s+2)}$$

$$\vdots$$

#partial fractions

$$= \frac{3}{s+3} - \frac{5}{3s+2}$$

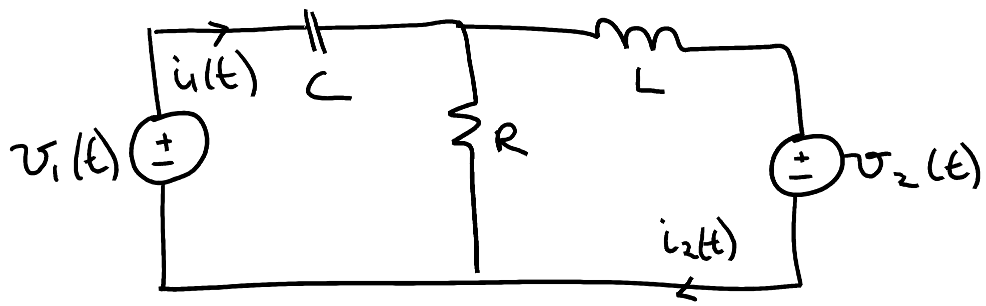
$$i(t) = \left(3e^{-3t} - \frac{5}{3}e^{-\frac{2}{3}t}\right)u(t) \text{ A}$$

Nodal and mesh analysis in the s-domain; Additional circuit analysis techniques

Readings: Sections 14.8, 14.9

Term 1 Techniques in the s-domain
We can do nodal and mesh analysis on **s**-domain circuits, as well as source transformations, superposition, and using Thévenin and Norton's theorems.

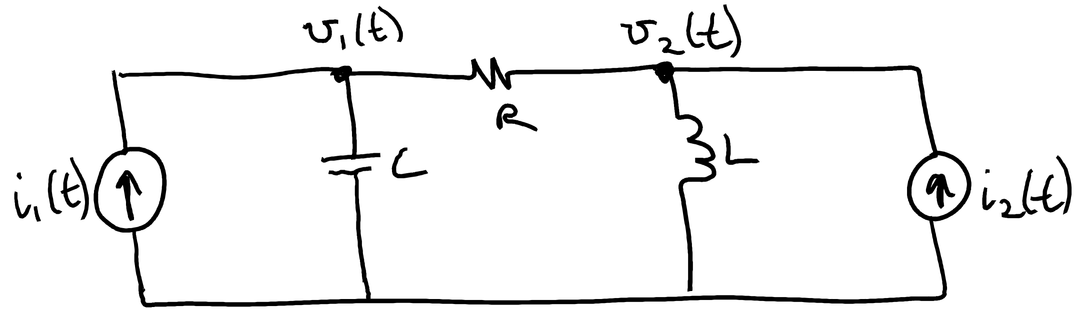
Example
Convert the below circuit into the **s**-domain, and apply mesh analysis to it. Assume initial conditions are zero.



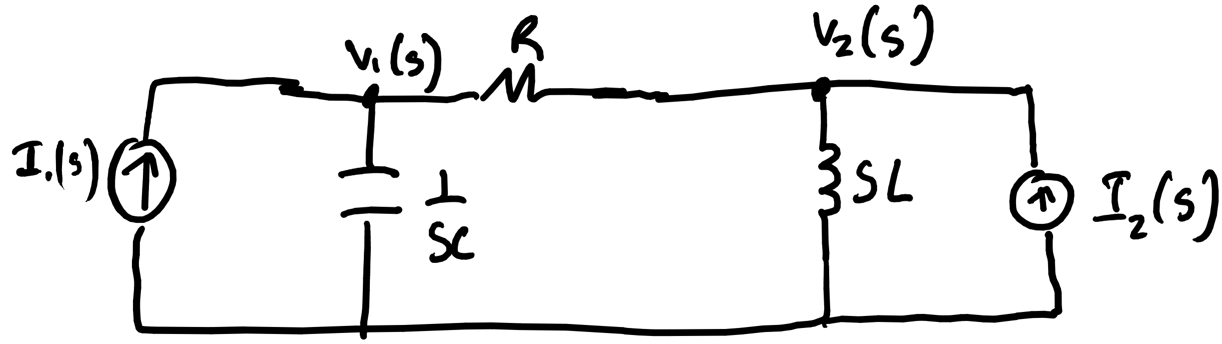
$$-V_S(s) + \frac{I(s)}{sC} + RC\,I_1(s) - I_2(s)) = 0$$

Example

Convert the below circuit into the **s**-domain, and apply nodal analysis to it. Assume the initial conditions are zero.



$\Downarrow$



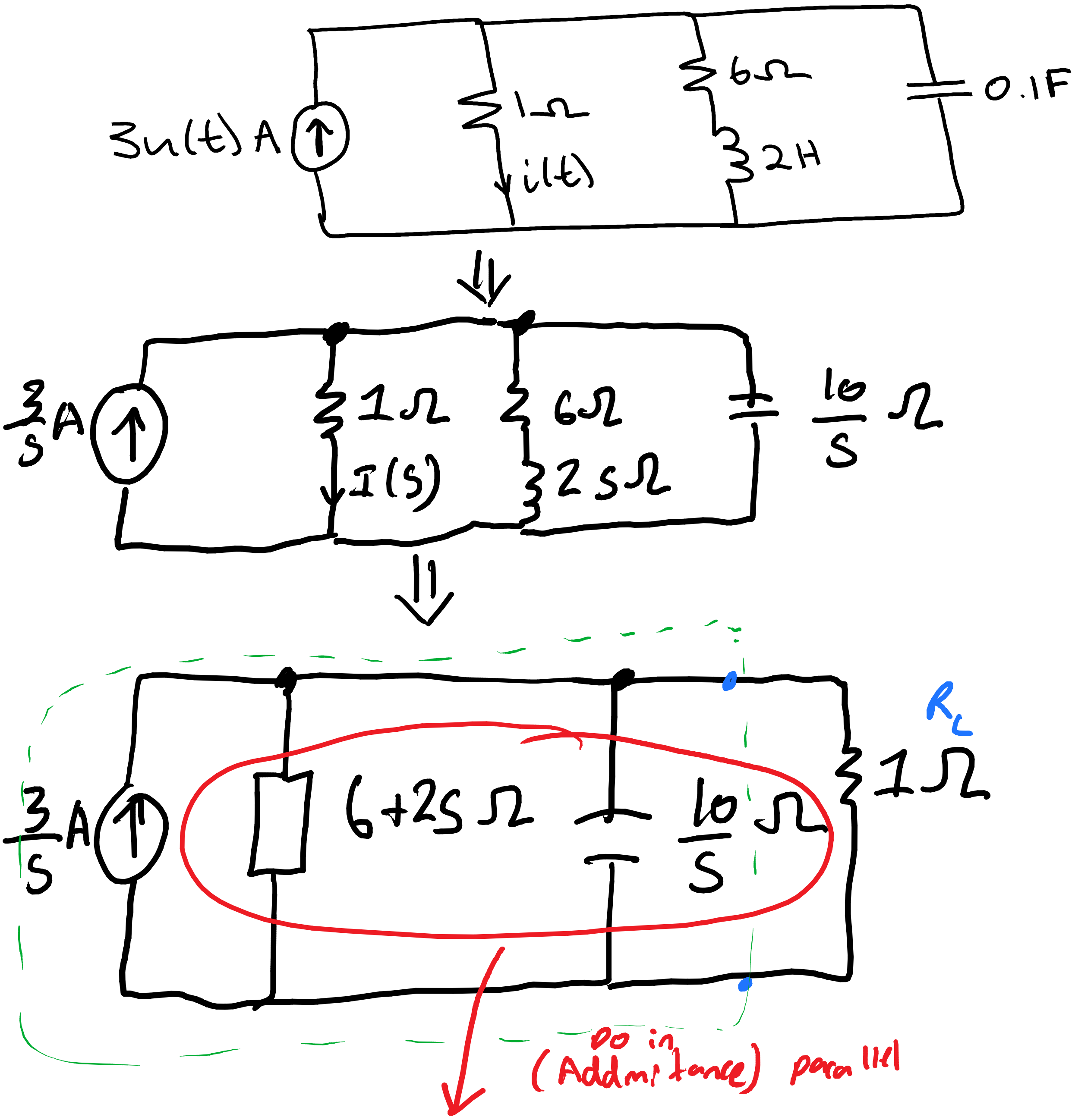
$$I_1(s) = V_1(s) \cdot sC + \frac{V_1(s) - V_2(s)}{R}$$

$$I_2(s) = \frac{V_2(s)}{sL} + \frac{V_2(s) - V_1(s)}{R}$$

$\Rightarrow$ 2 equation, 2 unknown, can solve

Example

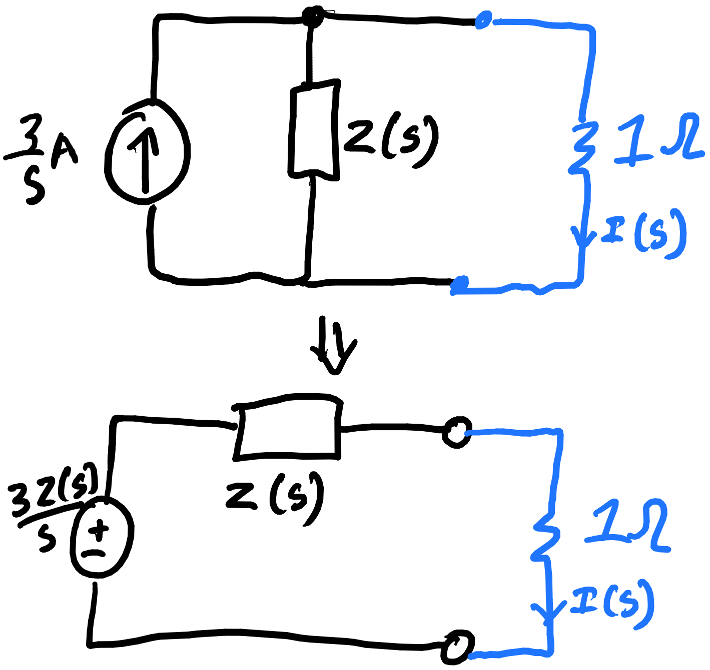
For the circuit below, find the Thévenin equivalent circuit seen by the 1 Ω resistor. Assume the initial conditions are zero.



$$Y(s) = \frac{1}{6+2s} + \frac{s}{10}$$

$$= \frac{10+6s+2s^2}{60+20s}$$

$$= \frac{s^2+3s+5}{10s+30} \quad // \quad Z(s) = \frac{10s+30}{s^2+3s+5}$$



Thevenin Equivalent

Poles, zeros, and transfer functions

RHS = Outputs
LHS = Inputs

Readings: Section 14.10

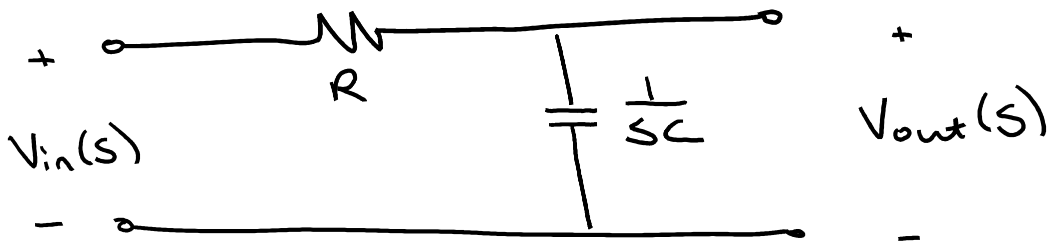
Transfer Function, H(s)

The transfer function **H(s)** is defined as:

$$H(s) = \frac{Y(s) \leftarrow \text{output}}{X(s) \leftarrow \text{input}}$$

Y(s) and **X(s)** can be either voltages or currents, and they don't need to be the same (one could be a voltage and one a current). Always assume the initial conditions are zero when working out the transfer function.

Example



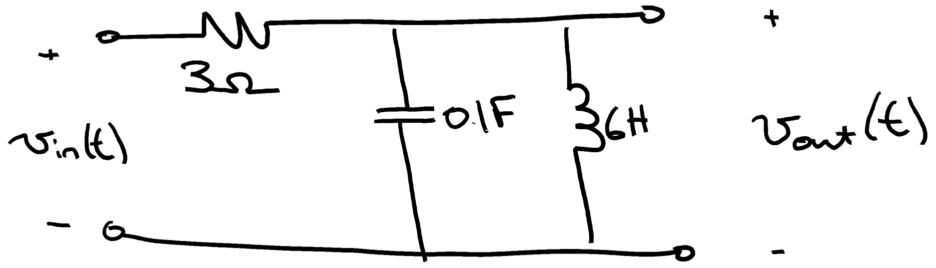
$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

H(s) has poles and zeros; these poles and zeros can be used to determine the stability of the system and the shape of the system response.

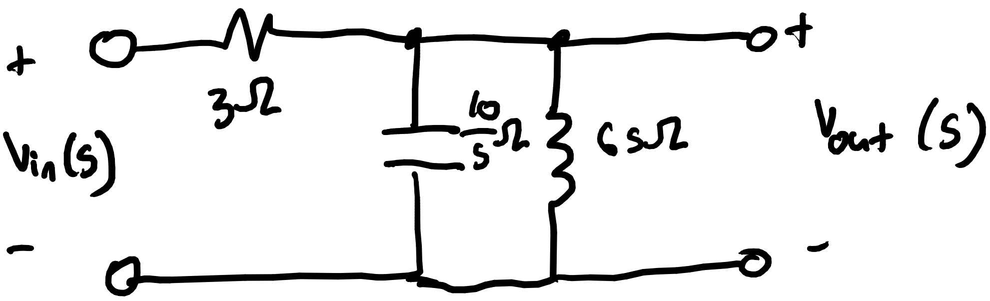
Remember: zeroes are calculated from the numerator of **H(s)**, while poles are calculated from the denominator.

Example

Find **H(s)** for the circuit below, and determine its poles and zeros.



⇓



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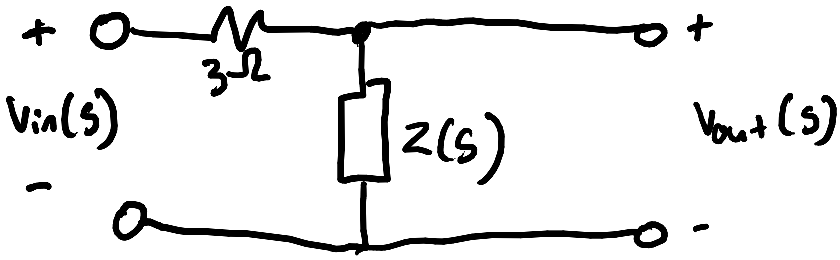


$$Y(s) = \frac{s}{10} + \frac{1}{6s} = \frac{6s^2+10}{60s}$$

$$Z(s) = \frac{60s}{6s^2+10}$$

$$V_{out} = V_{in}\left(\frac{Z(s)}{3+Z(s)}\right)$$ (voltage divider)

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{\frac{60s}{6s^2+10}}{3 + \frac{60s}{6s^2+10}}$$

$$H(s) = \frac{60s}{18s^2 + 30 + 60s}$$

$$= \frac{20s}{6s^2 + 20s + 10}$$

Zeros $\Rightarrow$ $20s = 0$

$s = 0$

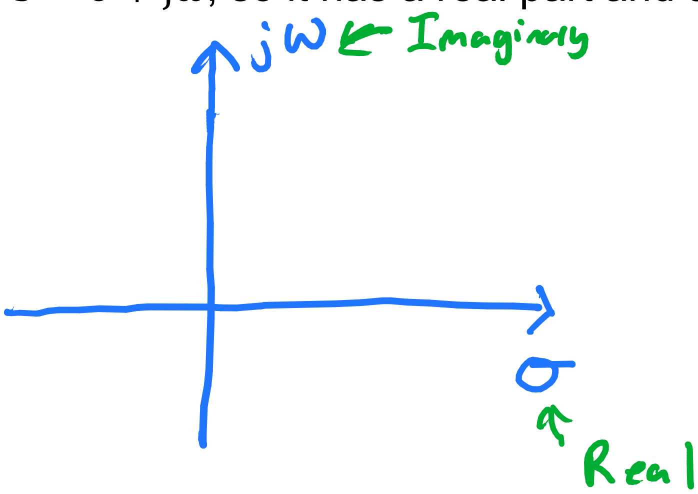
poles $\Rightarrow$ $6s^2 + 20s + 10 = 0$

$(s + 2.7)(s + 0.6) = 0$

$s = -2.7, -0.6$

Pole-zero Diagrams

Pole-zero diagrams are a graphical representation of a circuit response or transfer function - we plot the poles and zeros on the **s** plane (remember, $s = \sigma + j\omega$, so it has a real part and an imaginary part).



$s = \sigma + j\omega$

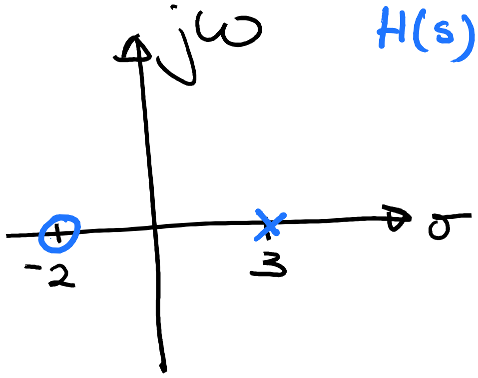
Poles are marked by ‘x’, zeros by ‘o’. It can be useful to think of the **s** plane as a sheet of rubber; the poles are poles holding the sheet up, while the zeros are tacks or pins holding the sheet down (at that point the response is zero, and the height of the sheet must be zero).

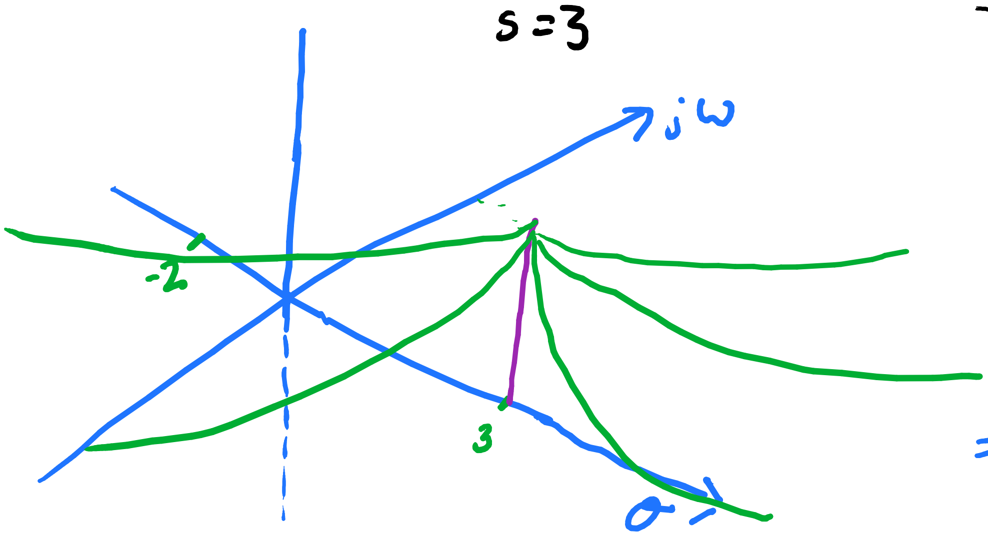
Example:

Plot the pole-zero diagram of $H(s) = \frac{s+2}{s-3}$. ← zero's ← poles

• zero at $s+2=0$ $s=-2$ "o"

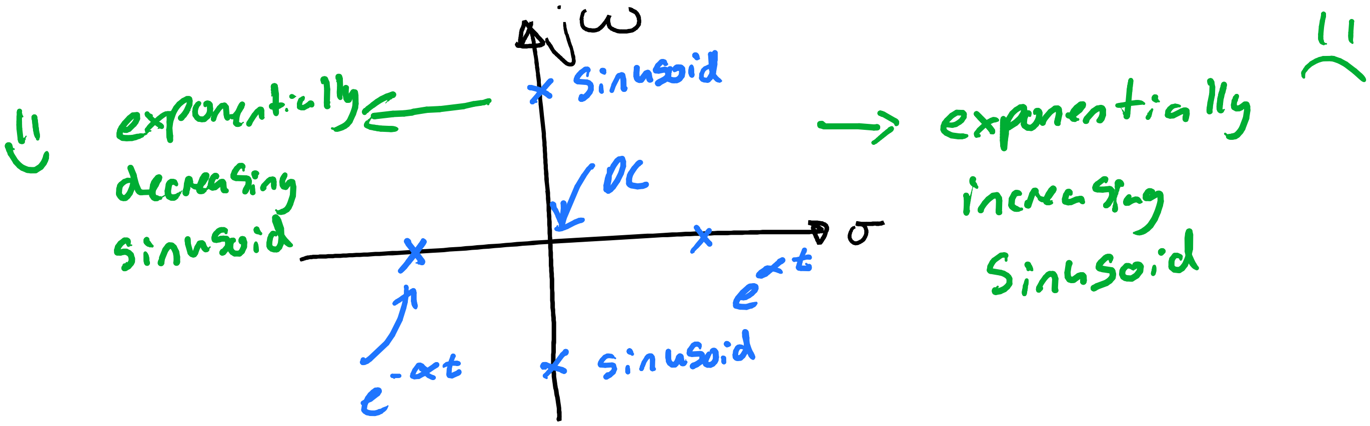
• pole at $s-3=0$ "x"
$s=3$

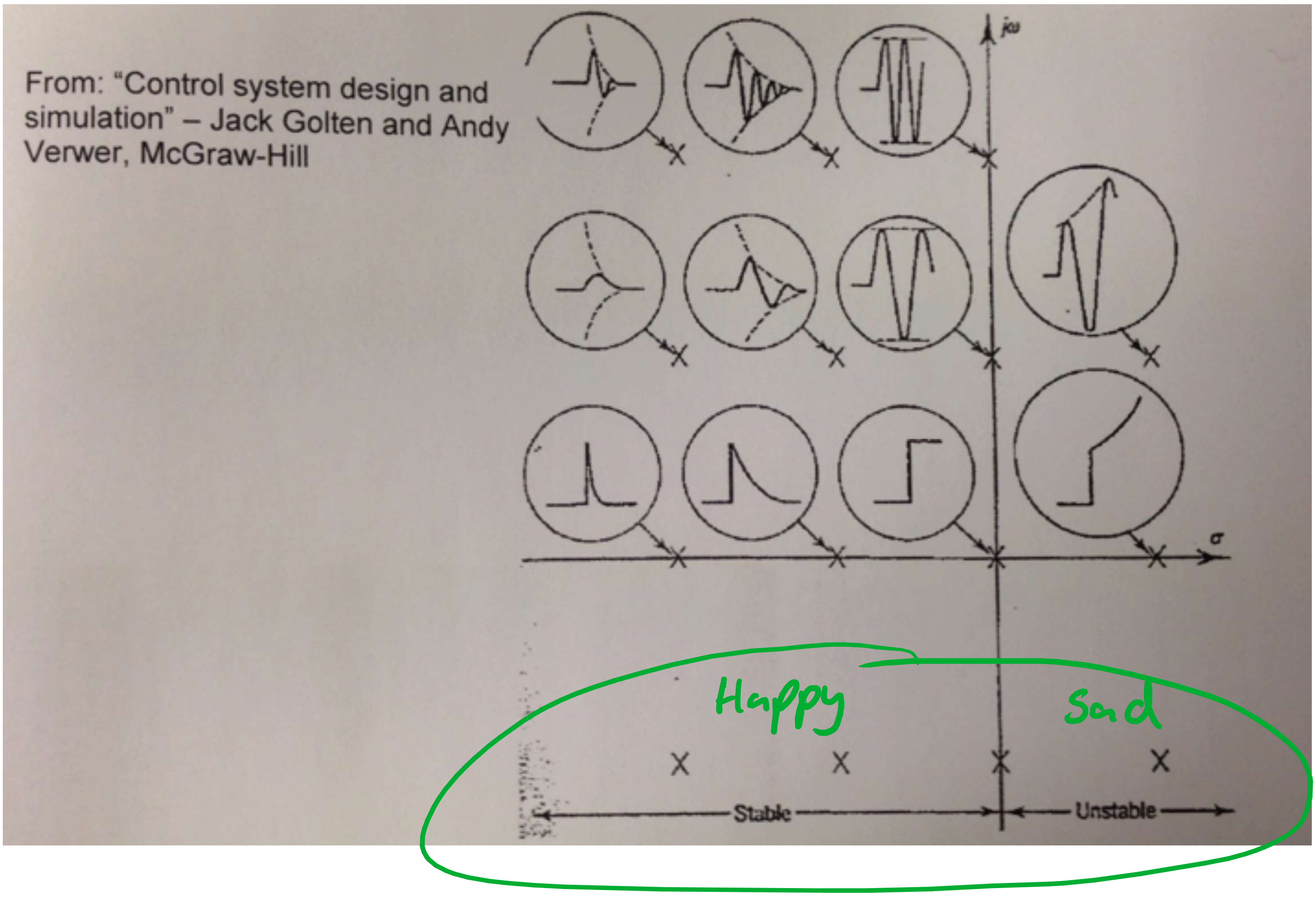




mountain (one) & one crater

The location of the poles tells you the shape of the system response (very useful for determining if you have a stable system or not).



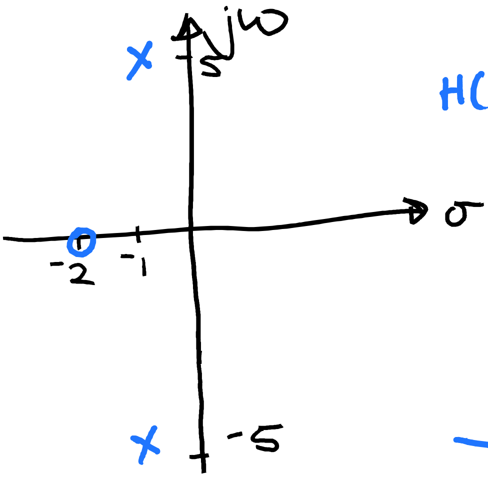


Example

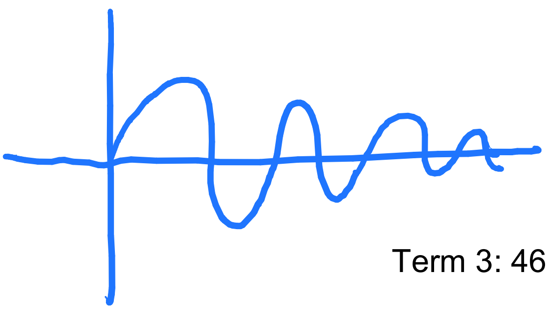
Plot the pole-zero diagram of $\mathbf{H}(\mathbf{s}) = \frac{13(s+2)}{s^2+2s+26}$, and from it determine the shape of the system response.

“o” zeros = -2

“x” poles = $-1 \pm j5$

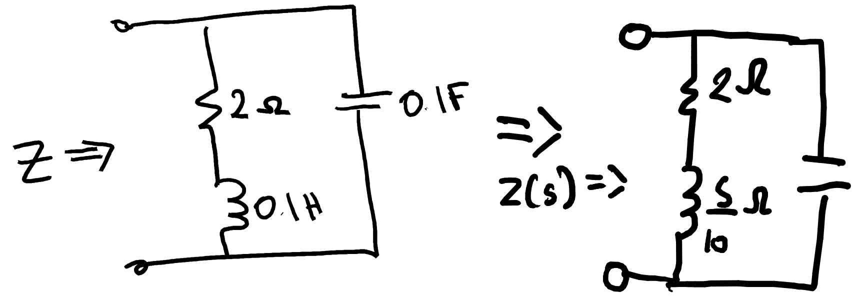


H(s) => system response has an



Example:

For the circuit below, sketch the magnitude of the impedance **Z(s)** and the magnitude of the admittance **Y(s)** as a function of σ and jω.



$z = \frac{s}{L}$

$z = sC$

$$Z(s) = \left(2 + \frac{s}{10}\right) \,\Big\|\, \frac{10}{s}$$

$$= \frac{\left(2 + \frac{s}{10}\right)\frac{10}{s}}{2 + \frac{s}{10} + \frac{10}{s}} \quad \left(\frac{Z_1 Z_2}{Z_1 + Z_2} \text{ for two impedances in } \|\right)$$

$$= \frac{\left(\frac{20}{s} + 1\right)}{\left(\frac{s^2 + 20s + 100}{10s}\right)}$$

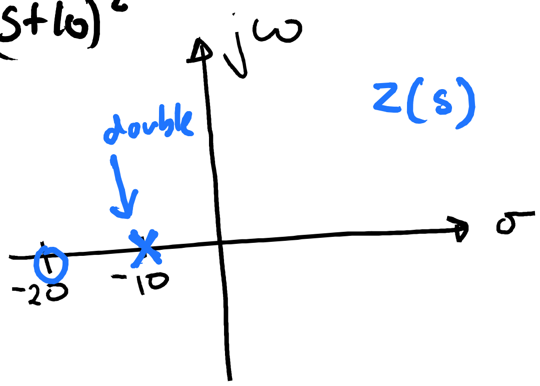
$$= \frac{20 + s}{\cancel{s}} \times \frac{10\cancel{s}}{s^2 + 20s + 100}$$

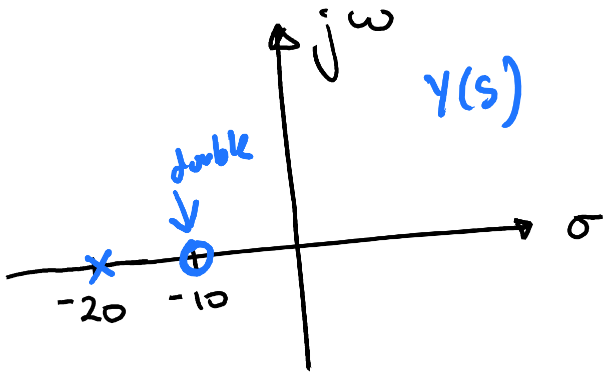
$$= \frac{10(s+20)}{(s+10)^2}$$

Zeroes = $s = -20$

polses = $s = -10$

(double pole)





$$Y(s) = \frac{1}{Z(s)}$$

$$= \frac{(s+10)^2}{10(s+20)}$$

- poles become zeroes & zeroes become poles

Convolution

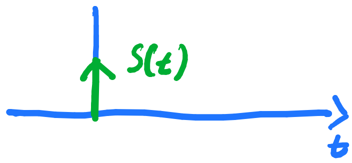
<u>Readings</u>: Section 14.11

The basic idea of convolution is that two waves are interacting, and one is reflected. An example of this is an echo. In the time domain, the maths is not very nice, but it's very useful in the **s**-domain, as it turns into multiplication. The convolution integral uses the impulse response, so we will talk about that first.

Impulse Response

If we're interested in a circuit with initial conditions = 0, and a transfer function **H(s)**, and the input is a unit impulse, $\delta(t)$, then we have:

$V_{in}(t) = \delta(t)$



$V_{in}(s) = 1$

$H(s) = \dfrac{V_{out}(s)}{V_{in}(s)}$ $\Rightarrow$ assuming interested in voltages for input & output

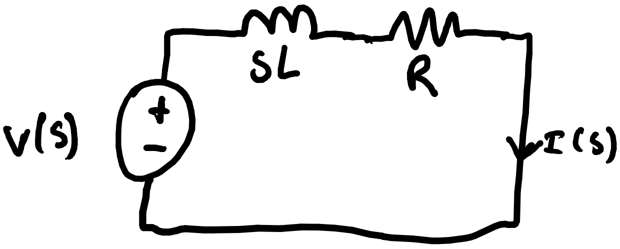
$\mathcal{L}^{-1}[H(s)] = V_{out}(t)$

$= h(t) \leftarrow$ impulse response

$\Rightarrow$ what happens when we have an impulse?

Example:

Determine the impulse response for the circuit below.



$$H(s) = \frac{I(s)}{V(s)}$$

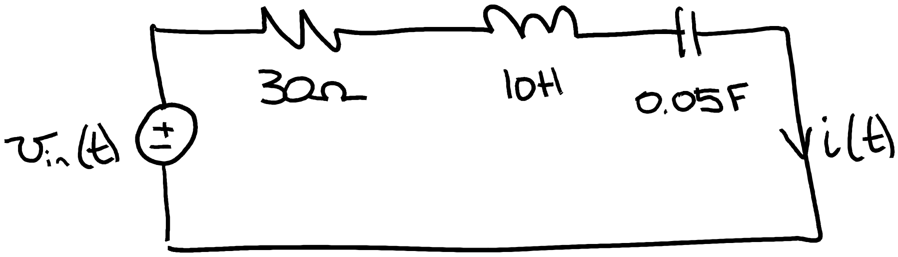
$$= \frac{1}{R+sL}$$

$$= \frac{1}{L}\left(\frac{1}{s+R/L}\right)$$

$$h(t) = \frac{1}{L}e^{-R/L \times t}\,u(t)$$

Example

Find the impulse response of the circuit below.



$\Downarrow$

$$H(s) = \frac{I(s)}{V(s)}$$

$V = IZ \Rightarrow \frac{I}{Z} = \frac{1}{Z}$

$$= \frac{1}{Z(s)}$$

$$= \frac{1}{30 + 10s + \frac{20}{s}}$$

$$= \frac{s}{30s + 10s^2 + 20}$$

$$= \frac{1}{10}\left(\frac{s}{s^2+3s+2}\right)$$

$$= \frac{1}{10}\left(\frac{s}{(s+2)(s+1)}\right)$$

$$= \frac{1}{10}\left(\frac{A}{s+2} + \frac{B}{s+1}\right)$$

$$A(s+1) + B(s+2) = s$$

$s = -1$
$B = -1$

$s = -2$
$-A = -2$
$A = 2$

$$H(s) = \frac{1}{10}\left(\frac{2}{s+2} - \frac{1}{s+1}\right)$$

(inverse)

$$h(t) = \frac{1}{10}\left(2e^{-2t} - e^{-t}\right)u(t)$$

Convolution
The mathematical definition is:

symbol for convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z)h(t-z)\,dz$$

In the **s**-domain however it is:

$$Y(s) = \underbrace{X(s)H(s)}_{s\text{ domain}} \Longleftrightarrow y(t) = x(t) * h(t)$$

Remember, **H(s)** is the transfer function, and h(t) is the impulse response.

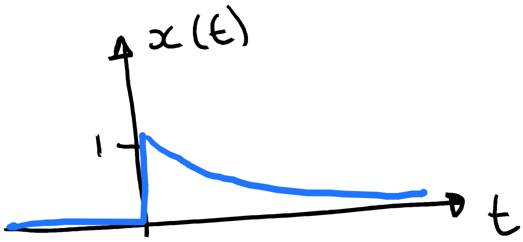
Note that using the transfer function and convolution we can find the output for any given input:

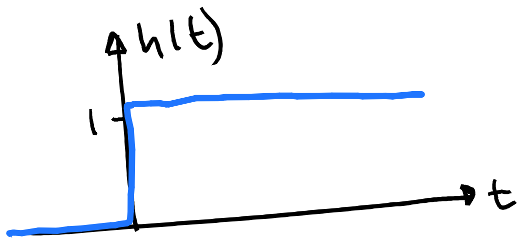
$$V_{in}(s)H(s) = V_{out}(s)$$

(Assuming we are intersted in both input & output voltages)

Example

If $x(t) = e^{-\alpha t}u(t)$ and $h(t) = u(t)$, what is $y(t)$?





Putting everything into the **s**-domain, we can work it out:

$$X(s) = \frac{1}{s+\alpha} \qquad H(s) = \frac{1}{s}$$

$$Y(s) = X(s)H(s)$$
$$= \frac{1}{s+\alpha} \cdot \frac{1}{s}$$
$$= \frac{A}{s} + \frac{B}{s+\alpha}$$

$$A(s+\alpha) + Bs = 1$$

$\underline{s=0}$

$\alpha A = 1$

$A = 1/\alpha$

$\underline{s=-\alpha}$

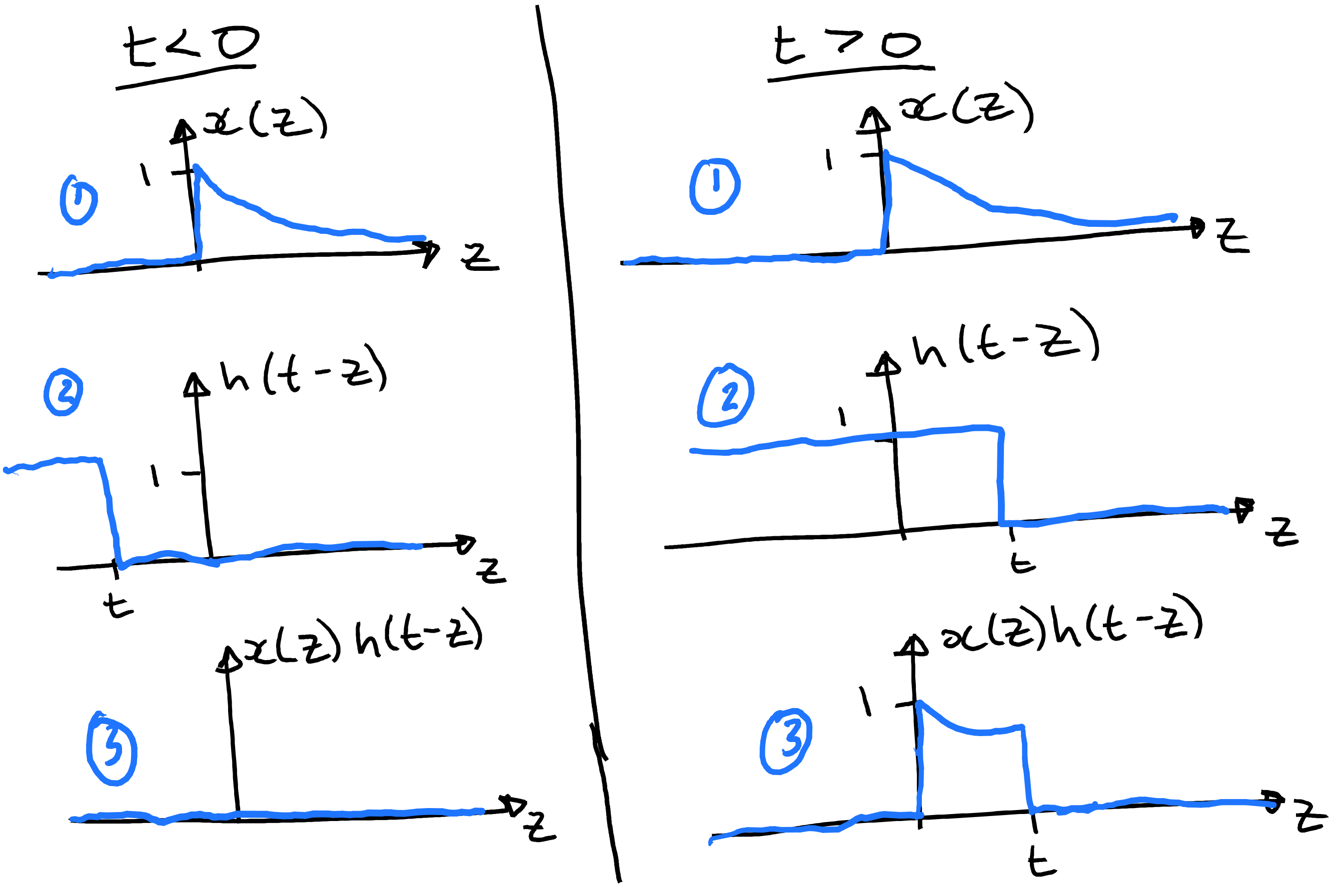
$-\alpha B = 1$

$B = \frac{-1}{\alpha}$

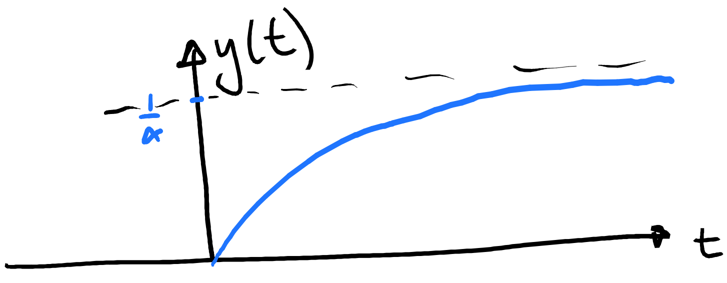
$$Y(s) = \frac{1/\alpha}{s} - \frac{\frac{1}{\alpha}}{s+\alpha}$$

$$y(t) = \frac{1}{\alpha}h(t) - \frac{1}{\alpha}e^{-\alpha t}u(t)$$

Graphically:



$$y(t) = \begin{cases} 0 & t<0 \\ \int_0^t e^{-4z}\,dz & t>0 \end{cases}$$



$$= \frac{1}{4}\left(1-e^{-4t}\right)u(t)$$

Properties of Convolution

1. $f * g = g * f$
2. $f * (g * h) = (f * g) * h$
3. $f * (g + h) = (f * g) + (f * h)$
4. $f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h)$

doesnt matter about the order

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

If $x(t) = 3u(t)$ and $h(t) = 4u(t) - 2u(t-2)$, what is $y(t)$? What is $y(t)$ if $x(t) = 3u(t) + 3u(t-10)$?

$$X(s) = \frac{3}{s} \qquad H(s) = \frac{4}{s} - \frac{2e^{-2s}}{s}$$

$$Y(s) = X(s)H(s)$$
$$= \frac{3}{s}\left(\frac{4}{s} - \frac{2e^{-2s}}{s}\right)$$
$$= \frac{12}{s^2} - \frac{6e^{-2s}}{s^2}$$
$$y(t) = 12tu(t) - 6(t-2)u(t-2)$$

time shift:
$$f(t-a)u(t-a) \Longleftrightarrow e^{-as}F(s)$$
$$f(t-a) = 3u(t-10)$$
$$\therefore f(t) = 3u(t)$$
$$F(s) = \frac{3}{s}$$

b) $$X(s) = \frac{3}{s} + \frac{3}{s}e^{-10s}$$

$$Y(s) = X(s)H(s)$$
$$= \left(\frac{3}{s} + \frac{3}{s}e^{-10s}\right)\left(\frac{4}{3} - \frac{2e^{-2s}}{s}\right)$$
$$= \frac{12}{s^2} - \frac{6e^{-2s}}{s^2} + \frac{12e^{-10s}}{s^2} - \frac{6e^{-12s}}{s^2}$$

To make it work, multiply 3 by by $u(t-10)$ Note: $u(t-10)u(t-10) = u(t-10)$

$$y(t) = 12tu(t) - 6(t-2)u(t-2) + 12(t-10)u(t-10) - 6(t-12)u(t-12)$$

Example

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

If $x(t) = 7tu(t) + 8e^{-9t}u(t) + 6u(3t)$ and $h(t) = e^{-7t}u(t)$, what is $y(t)$?

$$X(s) = \frac{7}{s^2} + \frac{8}{s+9} + \underbrace{\frac{6}{s} \cdot \frac{1}{s/3}}$$

$\Rightarrow$ scaling in tables:

$f(at) \Leftrightarrow \frac{1}{a} F\left(\frac{s}{a}\right)$

$+ \; u(t) \Leftrightarrow \frac{1}{s}$

$6u(3t)$:

$f(at) = u(3t) \;\therefore\; a = 3$

$\Leftrightarrow \frac{1}{3} F\left(\frac{s}{3}\right)$

$f(t) = u(t)$

$F(s) = \frac{1}{s}$

$\therefore \frac{1}{3}\left(\frac{1}{s/3}\right)$

$$= \frac{14s^2 + 61s + 63}{s^2(s+9)}$$

$$H(s) = \frac{1}{s+7}$$

$$Y(s) = X(s)H(s)$$

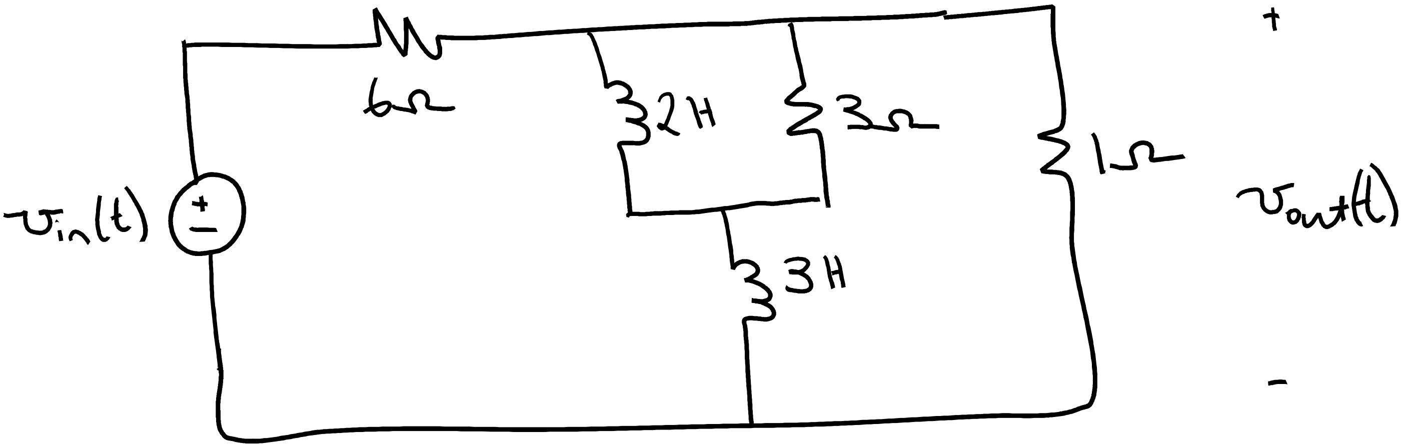
$$= \left(\frac{14s^2 + 61s + 63}{s^2(s+9)}\right)\left(\frac{1}{s+7}\right)$$

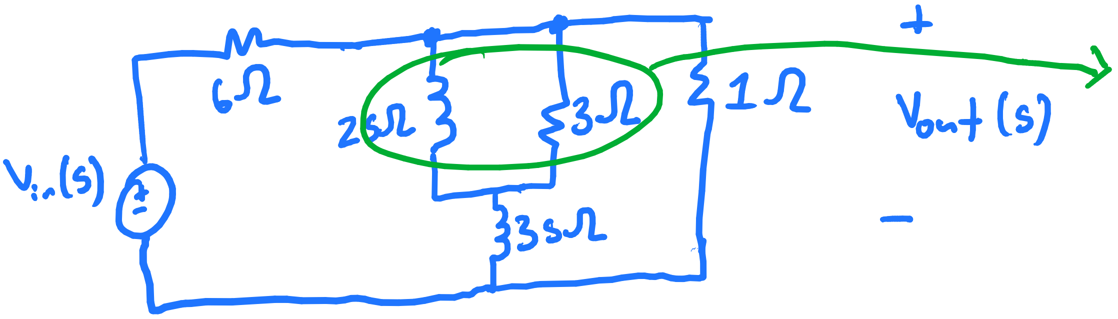
$$= \frac{1}{s^2} + \frac{5/7}{s} - \frac{4}{s+9} + \frac{23/7}{s+7}$$

$$y(t) = \left(t + \frac{5}{7} - 4e^{-9t} + \frac{23}{7}e^{-7t}\right)u(t)$$

Example

For the circuit below, if $v_{in}(t) = 6u(t) + e^{-3t}u(t)$ V, what is the transfer function, **H(s)**, and the output of the circuit, $v_{out}(t)$? Plot the pole-zero diagram for **H(s)**, and from it determine the shape of the system response.

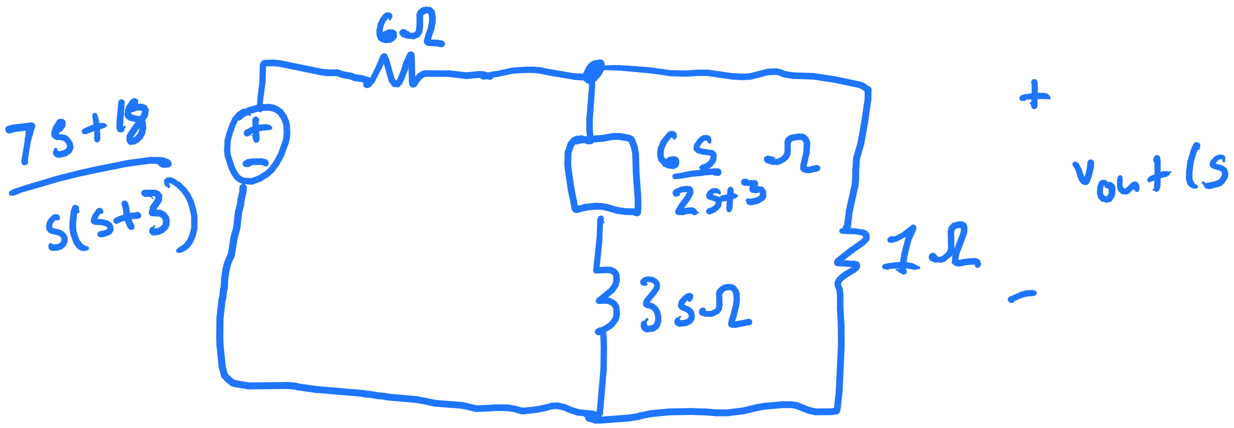




$$Y(s) = \frac{1}{2s} + \frac{1}{3} = \frac{3+2s}{6s}$$

$$Z(s) = \frac{6s}{3+2s}$$

$$V_{in}(s) = \frac{6}{s} + \frac{1}{s+3} = \frac{7s+18}{s(s+3)}$$



$$H(s) = \frac{V_{out}}{V_{in}(s)} \qquad \sum I_{in} = \sum I_{out} \quad (KCL)$$

$$\frac{V_{in} - V_{out}}{6} = \frac{V_{out}}{3s + \frac{6s}{2s+3}} + \frac{V_{out}}{1}$$

$$\frac{V_{in}}{6} = V_{out}\left(\frac{2s+3}{6s^2+9s+6s} + 1 + \frac{1}{6}\right)$$

$$V_{in} = \cancel{6}V_{out}\left(\frac{12s + 18 + 42s^2 + 65s}{\cancel{6}(6s^2+15s)}\right)$$

$$= \frac{42s^2 + 117s + 18}{6s^2 + 15s}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{6s^2 + 15s}{42s^2 + 117s + 18}$$

$$= \frac{6s^2 + 15s}{42(s+0.16)(s+2.6)}$$

$$V_{out}(s) = V_{in}(s)\,H(s)$$

$$= \frac{7s+18}{\cancel{s}(s+3)} \cdot \cancel{s}\left(\frac{6s+15}{42(s+0.16)(s+2.6)}\right)$$

$$= \frac{(7s+18)(6s+15)}{42(s+3)(s+0.16)(s+2.6)}$$

$$= \frac{0.19}{s+3} + \frac{0.81}{s+0.16} + \frac{0.002}{s+2.6}$$

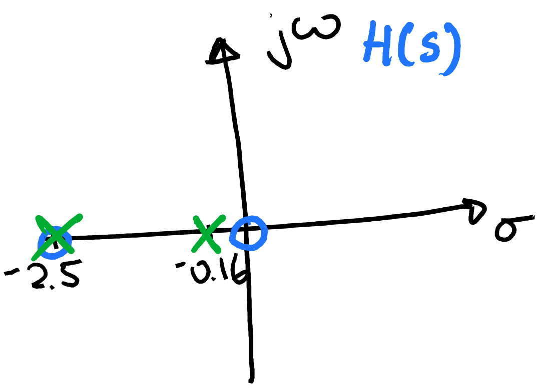
$$v_{out}(t) = \left(0.19e^{-3t} + 0.81e^{-0.16t} + 0.002e^{-2.6t}\right)u(t)\,V$$

Zeros of $H(s) \Rightarrow s(6s+15)=0$

$$s = 0, -2.5$$

poles of $H(s) \Rightarrow (s+0.16)(s+2.6)=0$

$$s = -0.16, -2.6$$



$\Rightarrow$ shape of system response is a negative exponential

* poles on LHS

☺