EMTH211–Tutorial 6

Attempt the following problem before the tutorial.
Questions 1 and 2 should be solved by hand, for the other questions, the use of Matlab is recommended/required.

1. Find the solution to the following system of differential equations:

$$\begin{aligned} x_1'(t) &= -x_1(t) - x_2(t) + 3x_3(t) \\ x_2'(t) &= x_1(t) + x_2(t) - x_3(t) \\ x_3'(t) &= -x_1(t) - x_2(t) + 3x_3(t) \end{aligned}$$

   where $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$.

2. Let A be the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}$. Use the theorem of Cayley-Hamilton to

   (i) write A^3 as a linear combination of I, A, A^2

   (ii) write A^{-1} as a linear combination of I, A, A^2.

In-tutorial problems

3. A study of pine nut crops in the American southwest from 1940 to 1947 hypothesized that nut production followed a Markov chain. The data suggested that if one year's crop was good, then the probabilities that the following year's crop would be good, fair, or poor were 0.08, 0.07, and 0.85, respectively; if one year's crop was fair, then the probabilities that the following year's crop would be good, fair, or poor were 0.09, 0.11, and 0.80, respectively; if one year's crop was poor, then the probabilities that the following year's crop would be good, fair, or poor were 0.11, 0.05, and 0.84, respectively.

   (a) Write down the transition matrix for this Markov chain.

   (b) If the pine nut crop was good in 1940, find the probabilities of a good crop in the years 1941 through 1945.

   (c) In the long run, what proportion of the crops will be good, fair, and poor?

4. Data have been accumulated on the heights of children relative to their parents. Suppose that the probabilities that a tall parent will have a tall, medium- height, or short child are 0.6, 0.2, and 0.2, respectively; the probabilities that a medium-height parent will have a tall, medium-height, or short child are 0.1, 0.7, and 0.2, respectively; and the probabilities that a short parent will have a tall, medium-height, or short child are 0.2, 0.4, and 0.4, respectively.

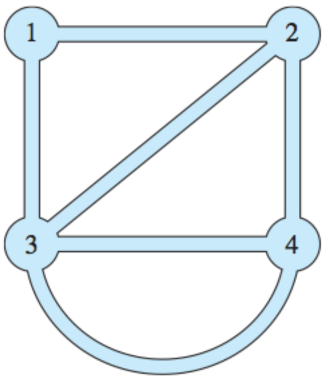
   (a) Write down the transition matrix for this Markov chain.

   (b) What is the probability that a tall person will have a short grandchild?

   (c) If 20% of the current population is tall, 50% is of medium height, and 30% is short, what will the distribution be in three generations?

   (d) What proportion of the population will be tall, of medium height, and short in the long run?

5. Robots have been programmed to traverse the maze shown below and at each junction randomly choose which way to go.



   (a) Construct the transition matrix for the Markov chain that models this situation.

   (b) Suppose we start with 15 robots at each junction. Find the steady state distribution of robots. (Assume that it takes each robot the same amount of time to travel between two adjacent junctions.)

6. A grasshopper has three life stages: egg, nymph, adult. We focus on the female grasshoppers. The population satisfies the following properties:

   - each adult produces 1000 eggs per year;
   - 2% of the eggs survives to be nymphs;
   - 5% of the nymphs survives to adulthood.

   (a) Write down the Leslie model.

   (b) How does the population evolve if there are initially 50 adults and no eggs or nymphs? What if we begin with 50 adults, 50 eggs and 100 nymphs? Give a table representing the number of eggs, nymphs, adults in 25 years.

   (c) Could you explain why this happens looking at the eigenvalues of L?

Extra question

7. Suppose that a population is divided into only two classes: children and adults. Let c_n denote the number of children at time step n and a_n the number of adults. The population evolves according to the following rules:

$$\begin{aligned} c_{n+1} &= \frac{1}{8}c_n + 6a_n \\ a_{n+1} &= \frac{1}{5}c_n \end{aligned}$$

   (a) Write down the Leslie model for this population.

   (b) What is the population starting from an initial population of 10 children and 10 adults after 3 timesteps (you can think of the population being measured in thousands to make these numbers more realistic.) ?

   (c) Make a plot that represents the number of children and the number of adults at 25 (consecutive) timesteps. Plot both graphs in the same picture.

   (d) Determine the long-time behaviour: in the long run, what percentage of the population would be adults?