UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

University of Canterbury

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Prescription Number(s):	**EMTH211-14S2**
Paper Title:	**Engineering Linear Algebra and Statistics**

Time allowed:	THREE HOURS
Number of Pages:	8

Read these instructions carefully.

- Answer ALL questions.
- All questions are of EQUAL weight.
- CLEARLY label each of your answers with the section letter and question number.
- Partial credit will be given, so INCLUDE ALL WORKING. You will be given credit for what you did right.

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1. (a) Briefly describe the techniques of *partial pivoting* and *full pivoting*, and state their purpose when numerically solving linear systems.

(b) (i) Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbf{R}^2$. Using the ordinary definition of the Euclidean norm in $\mathbf{R}^2$, show that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(ii) Draw a diagram showing $\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}$ and use (i) to deduce a geometric result about parallelograms.

(c) The plane $\mathcal{P}_0$ has equation $3x + y - 4z = -1$.

For each of the following planes $\mathcal{P}$, explain briefly whether $\mathcal{P}_0$ and $\mathcal{P}$ are parallel, perpendicular, or neither.

(i) $x + 5y + 2z = 3$, (ii) $2x - 3y + z = 4$, (iii) $-6x - 2y + 8z = 2$.

(d) Let $A = \begin{bmatrix} 2 & -1 & 4 \\ 4 & -1 & 5 \\ -2 & -2 & 5 \end{bmatrix}$.

(i) Find the LU decomposition of A.

(ii) Use this LU decomposition to check whether the system

$$\begin{aligned} 2x - y + 4z &= 0 \\ 4x - y + 5z &= 1 \\ -2x - 2y + 5z &= 0 \end{aligned}$$

is consistent. If it is consistent, find the solutions.

2. (a) (i) Let

$$A = \begin{bmatrix} 1 & -2 & 2 & 3 \\ -3 & 6 & -1 & 1 \\ 2 & -4 & 5 & 8 \end{bmatrix}.$$

Give bases for row(A), col(A), and null(A). Find, also, the rank and nullity of A.

(ii) If B is a general 3×4 matrix, what are the possible values of the nullity of B? (Give a brief reason for your answer; an equation may help.)

(b) Let V be the set of all functions from $\mathbb{R}$ into $\mathbb{R}$ with the usual definition of addition and multiplications by a constant. If E is the subset of even functions (that is, functions for which $f(-x) = f(x)$), and O is the subset of odd functions (for which $f(-x) = -f(x)$), prove that:

(i) E and O are subspaces of V; and

(ii) $E \cap O = \{\mathbf{0}\}$.

(c) Let

$$A = \begin{bmatrix} 1 & -t \\ -1 & 1 \end{bmatrix}, \text{ with } 0 < t < 1.$$

(i) Find the condition number $k(A)$ using the ∞-norm.

(ii) What happens to $k(A)$ as $t \to 1$? Describe col(A) in the two cases $t \neq 1$ and $t = 1$.

(iii) Consider the equation $A\mathbf{x} = \mathbf{b}$. Explain *briefly* the role of $k(A)$ in solving this equation when $\mathbf{b}$ may be subject to error. (No proofs are required; an equation may help.)

(iv) Solve the equation with $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (and $t \neq 1$). Using (c)(ii) above, explain why you would not expect a solution when $t = 1$.

3. (a) Find all eigenvalues of the following matrix and compute a corresponding eigenvector for *one* of them.

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

(b) A type of flower comes in three different colours: red (R), pink (P), and white (W). Crossing a pink flower with another flower results in the following distribution of colours.

red with pink results in	50% red, 50% pink, and 0% white
pink with pink results in	25% red, 50% pink, and 25% white
white with pink results in	0% red, 50% pink, and 50% white

A controlled field experiment starts with 1000 red flowers, 500 pink flowers, and 0 white flowers and crosses them, each year, with pink flowers. The procedure is repeated each year with one seed of each of the resulting 1500 flowers.

(i) Give the transition matrix, and the initial state vector of the Markov chain of this experiment.

(ii) What is the distribution of colours after the 1st year, and after 2 years?

(iii) Is the transition matrix regular?

(iv) What will the distribution of colours look like in the long run?

4. (a) The following vectors form a basis of $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}, and \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Apply the Gram-Schmidt process to the vectors to obtain an orthogonal basis.

(b) Find the coordinates of the vector

$$\mathbf{w} = \begin{bmatrix} 4 \\ -4 \\ -7 \end{bmatrix}$$

with respect to the orthogonal basis found in (a).

5. (a) Let X be a matrix and suppose that X^TX is invertible. The *Moore-Penrose pseudoinverse* of X is defined by

$$X^+ = (X^TX)^{-1}X^T$$

Note that the least squares solution of

$$\mathbf{y} = X\mathbf{b},$$

if it exists, is

$$\mathbf{b} = X^+\mathbf{y}$$

(i) Show that XX^+ is always a symmetric matrix.

(ii) Show that $(XX^+)^2 = XX^+$.

(iii) If X is invertible, show that $X^+ = X^{-1}$.

(iv) Can a 2×3 matrix have a Moore-Penrose pseudoinverse? Explain why or why not.

(v) Give an example of a matrix X such that X is not invertible but X^TX is invertible.

(b) Let $\mathbf{x} = (0, 1, 1)$, $\mathbf{y} = (0, -1, 1)$.

(i) Find the least squares solution to $\mathbf{y} = b_0 + b_1\mathbf{x}$.

(ii) Show that there is more than one choice of (b_0, b_1) which minimises

$$\sum_{i=1}^{3} |y_i - (b_0 + b_1x_i)|.$$

(Hint: drawing a picture will probably help. In fact, there are infinitely many solutions. The uniqueness of the least squares solution is one reason for preferring least squares to other approaches.)

6. (a) Give the definition of the Pearson correlation between two vectors $\mathbf{x}$ and $\mathbf{y}$.

(b) When more sheets of flypaper are hung up in a butcher's shop, more flies are caught. Is the correlation between the number of sheets of flypaper and the number of flies positive, negative or zero? Explain your answer.

(c) The magnitude F (in N) of the gravitational force between two 1 kg masses at distance r metres is given by

$$F = \frac{G}{r^2}$$

where $G > 0$ is a constant. Suppose F is measured for several different choices of r. Will the correlation between the values of F and r be positive, negative or zero? Explain your answer.

(d) It is observed that the correlation between crime rate (in crimes per person per year) and GDP per capita (in US dollars per capita) for 100 countries is -0.21. What would the correlation be if crime rate was expressed in crimes per 100 persons per year instead?

(e) In an experiment, the numbers 1 to 100 are written on identical pieces of paper and placed in a hat. The experimenter repeatedly draws a number, returns it to the hat, mixes the numbers, and then draws another number. If this is done a very large number of times, would you expect the correlation between the first and second numbers to be positive, negative or zero? Explain your answer.

(f) In another experiment, a pack of 13 cards labelled 1 to 13 is shuffled, a card is drawn and set aside, and then another card is drawn. If this was done a very large number of times, would you expect the correlation between the numbers on the first and second cards to be positive, negative or zero? Explain your answer. (Hint: the answer is *not* the same as for the hat experiment.)

7. (a) Let $\mathbf{x} = (1, 2, 3, 4)$ and $\mathbf{y} = (1, 2, 4, 8)$. Let

$$\mathbf{y} = \widehat{b}_0 + \widehat{b}_1\mathbf{x}.$$

be the least squares regression line of $\mathbf{y}$ on $\mathbf{x}$.

(i) Show that the slope of the least squares regression line is $\widehat{b}_1 = 2.3$.

(ii) Find the intercept $\widehat{b}_0$.

(iii) Using the value $t_2^*(0.975) = 4.30$, give a 95% confidence interval for b_1 from $\widehat{b}_1$.

(b) Let x_1 be the length (in m) of an elephant, x_2 be the height and x_3 be the width. A researcher is investigating the weight y (in kg) of elephants based on x_1, x_2 and x_3. The values of x_1, x_2, x_3 and y for 47 elephants are fed into regression software and it outputs the equation

$$\widehat{y} = 2000 + 500x_1 + 500x_2 + x_3.$$

(i) What is the predicted weight of an elephant with $x_1 = 2$, $x_2 = 2$ and $x_3 = 2$?

(ii) The researcher is happy because the regression has a high R^2 of 0.8, and wants to publish a paper entitled: "Whither width?: an analysis proving that the weight of an elephant has almost nothing to do with its width". Do you have any reservations about this conclusion? What common problem with multiple regression might be responsible?

(iii) The researcher has also recorded the sex (male/female) of each of the elephants, but is confused because this variable is not a number. How can the sex variable nevertheless be included in the regression?