UNIVERSITY OF CANTERBURY

Test 1

Prescription Number: **EMTH211-16S2**

Time allowed: 50 minutes.

Attempt ALL 5 questions.

Write your answers in the spaces provided.

There is a *total* of 50 points.

Use black or blue ink. Do not use pencil except for diagrams.

Only UC approved calculators are allowed.

Show all working. Write neatly. Marks will be lost for poorly presented answers.

Family name:	
Given names:	
Student ID:	

MARKS Office Use Only	
Q1	
Q2	
Q3	
Q4	
Q5	
Total	

Question 1 [8 points]

Consider the system of linear equations

$$\begin{aligned} x + ky &= 1 \\ kx + y &= 1\ . \end{aligned}$$

How many solutions does this system have depending on k?

Solution Consider the augmented matrix, and use row reduction to get it into row echelon form:

$$\begin{bmatrix} 1 & k & 1 \\ k & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{bmatrix}\ .$$

If $k = 1$ there is a free parameter and we get the general (so there are infinitely many in this case) solution

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \end{bmatrix}\ .$$

If $k = -1$ the system is inconsistent, since then the augmented matrix contains the row $\begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$

If $k \neq \pm 1$ we get the unique solution $\mathbf{x}_2 = \frac{1}{1+k}$ and $\mathbf{x}_1 = 1 - \frac{k}{1+k} = \frac{1+k-k}{1+k} = \frac{1}{1+k}$. That is

$$\mathbf{x} = \begin{bmatrix} \frac{1}{1+k} \\ \frac{1}{1+k} \end{bmatrix}\ .$$

Note that it was not specifically required to compute these solutions.

Question 2 [12 points]

Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 8 & 3 & 5 \end{bmatrix}.$$

1. Compute a LU-decomposition of A.

2. Use the above decomposition to solve

$$A\mathbf{x} = \begin{bmatrix} -4 \\ 4 \\ 16 \end{bmatrix}$$

Solution

1. We first use row reductions to get A into row echelon form.

$$A \overset{R_2-R_1}{\rightsquigarrow} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 8 & 3 & 4 \end{bmatrix} \overset{R_3-2R_1}{\rightsquigarrow} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \overset{R_3-R_2}{\rightsquigarrow} \begin{bmatrix} 4 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = U$$

We can write this in terms of elementary matrixes as $E_3E_2E_1A = U$. Let now

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}.$$

2. We want to first solve $Ly = b$, which yields $y = \begin{bmatrix} -4 \\ 8 \\ 32 \end{bmatrix}$. Next we want to solve $Ux = y$, which yields the final answer $x = \begin{bmatrix} -11 \\ 8 \\ 16 \end{bmatrix}$.

Question 3 [10 points]

Remember that a subspace of $W \subset \mathbb{R}^n$ is a set that is closed under vector addition and scalar multiplication. (That is if $u, v \in W$ and $k \in \mathbb{R}$ then $u + v \in W$ and $ku \in W$).
Let A be a $m \times n$ matrix.

1. Show that $W = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$ is a subspace
2. Why is, for $V_b = \mathbf{b} \in \mathbb{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, the set $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{b}\}$ not a subspace?

Solution

1. Let $u, v \in W$. That means that $Au = 0$ and $Av = 0$. But then

$$A(u + v) = Au + Av = 0 + 0 = 0 \ ,$$

   and therefore $(u + v) \in W$. Now let $u \in W$ and $k \in \mathbb{R}$. Then $Aku = kAu = k0 = 0$. Thus also $ku \in W$.

2. V_b is neither closed under addition nor under scalar multiplication (one is enough to show that it is not a subspace). For example even if $u \in V_b$ then $A(u + u) = 2Au = 2b \neq b$, which means $u + u \notin V_b$ (exactly the same reasoning shows that it is not closed under scalar multiplication with 2).

Question 4 [8 points]

Find a basis for the column space, the row space, and the null space for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

Solution The row echolon form is

$$A \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} .$$

That means that these three rows are linearly independent, and therefore form a basis for the row-space. We have pivots in column $1, 2, 4$, so the columns $1, 2, 4$ of A form a basis for the column space. (One can actually easily see that the third column is the first minus the second one, and that the others are linearly indpendent). To find the null-space we can also continue from the row echelon form and use back-substitution to solve the homogenous system: $x_4 = 0$, $x_3 = s$ (free parameter), $x_2 - s + 0 = 0$, so $x_2 = s$, and $x_1 + s = 0$, so $x_1 = -s$. Thus the general solution is

$$s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} ,$$

with that vector being a basis for the null-space.

Question 5 [12 points]

1. Give an example of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ such that $\|\mathbf{a}\|_1 < \|\mathbf{b}\|_1$, but $\|\mathbf{a}\|_\infty > \|\mathbf{b}\|_\infty$.
2. Are the matrix norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ independent of row swaps?

Solution

1. Take
$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$
Then $\|\mathbf{a}\|_1 = 3 < 4\|\mathbf{b}\|_1$, but $\|\mathbf{a}\|_\infty = 3 > 2\|\mathbf{b}\|_\infty$.
2. Yes both are. The 1 norm is the maximum of the column-sums, and the column sums don't change with row swaps. Similarly the ∞ norm is the maximum of the row-sums, and it doesn't matter which order the rows are in.

Page for rough working ...

END OF PAPER