1 Norms and iterative methods

Poole Ch. 7.2, 2.5

1.1 Vector norms

$\underset{\sim}{v} = \{v_1, \dots, v_n\}$

Example 1.1. Recall the Euclidean norm for vectors in $\mathbf{R}^n$:

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \dots\ v_n^2} = \left(\sum_{i=1}^{n} v_i^2\right)^{\frac{1}{2}}$$

$\boxed{\text{SRSS}}$

$$\|v\|^2 = v \cdot v$$

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Example 1.2. If you travel around central Christchurch, with its grid-style road network, then the distance travelled will be made up of a series of straight line segments as you go east or west and then north or south, and so on. The total distance travelled is the sum of the lengths of all these segments.

one norm $\|\underset{\sim}{v}\|_1$ or
- manhattan norm
- taxicab norm

If the segments are given by a set of values $\{v_i : i = 1, \dots, n\}$ then the total distance travelled is $\sum_{i=1}^{n} |v_i|$. We could write

$$\|\mathbf{v}\|_1 = \sum_{i=1}^{n} |v_i| = \left(\sum_{i=1}^{n} |v_i|^1\right)^1$$

> **Definition 1.3.** The general p-norm is defined by
>
> $$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$$

Example 1.4. For $p = 1$ and $p = 2$ we find two norms we have already met. Now let $p \to \infty$ in the above formula:

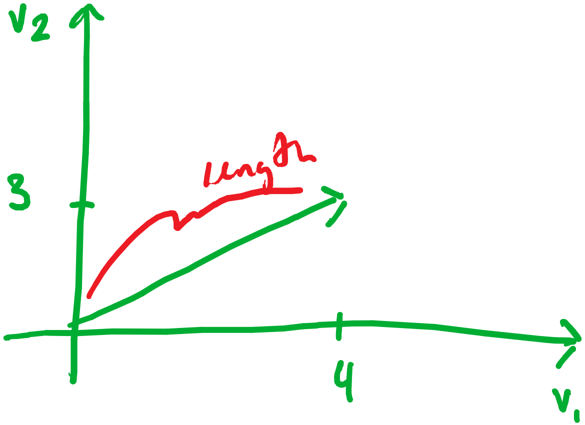
∞ norm
max norm
uniform norm

$$\|\mathbf{v}\|_\infty = \lim_{p\to\infty} \|\mathbf{v}\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}} = \max_{1 \le i \le n} |v_i|$$

$\|v\|_1$
$\|v\|_2$
$\|v\|_\infty$

Exercise 1.5. Find the 1, 2, and ∞-norm of

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}$$



$\mathbb{R}^2$

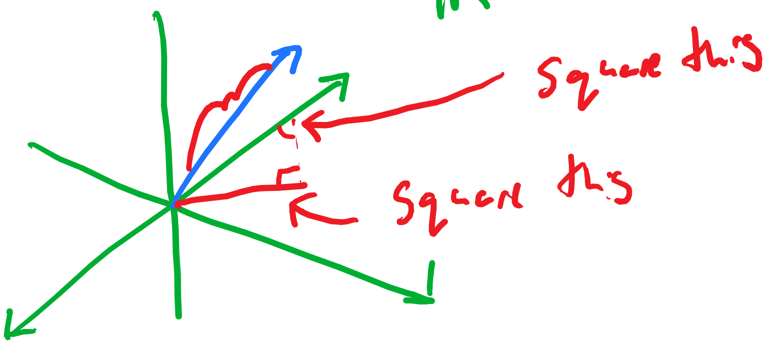
Distance Euclidean norm

$$x = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\|\underset{\sim}{v}\| = \sqrt{4^2+3^2}$$ pythag

$$= \sqrt{16+9} = \sqrt{25} = 5$$

$\mathbb{R}^3$



$$\underset{\sim}{v} = \begin{pmatrix} 4 \\ 3 \\ 12 \end{pmatrix}$$

$$\|\underset{\sim}{v}\| = \sqrt{4^2+3^2+12^2}$$

$$= \dots$$

Social norm

philosophical norms

"ought

Norms

Mathematical norm

"How close are two vectors"

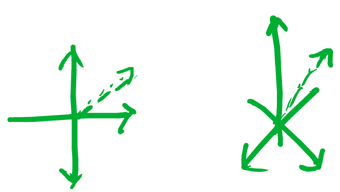
"How close are two matrices

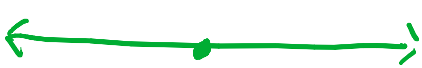
"distance & length"

Absolute value

$\|x\| = |x|$

is a norm on the 1-dimensional vector space formed by the real numbers





Solution:

$$\|v\|_1 = |1| + |-1| + |2| + |3| + |-5|$$
$$= 12$$

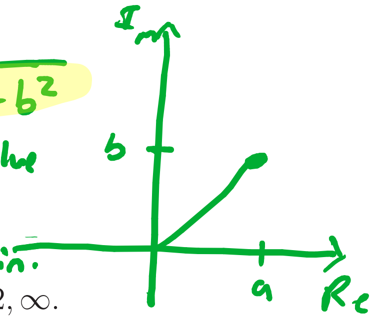
$$\|v\|_2 = \sqrt{(1)^2 + (-1)^2 + (2)^2 + (3)^2 + (-5)^2}$$
$$= \sqrt{40}$$

$$\|v\|_\infty = \max\{|1|, |-1|, |2|, |3|, |-5|\}$$
$$= 5$$

Quick reminder complex number ~~well graded~~ $a + ib$

$$|a+ib| = \sqrt{a^2+b^2}$$

- we take the absolute value of a complex number distance from the origin.



Exercise 1.6.

Let $\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1+i \\ 2i \\ 3-2i \end{bmatrix}$. Find $\|\mathbf{x}\|_p$ and $\|\mathbf{y}\|_p$ for $p = 1, 2, \infty$.

Solution:

$$\|\underset{\sim}{x}\|_1 = |2| + |-4| + |3| = 9$$

$$\|\underset{\sim}{x}\|_2 = \sqrt{|2|^2 + |-4|^2 + |3|^2} = \sqrt{29}$$

$$\|\underset{\sim}{x}\|_\infty = \max\{|2|, |-4|, |3|\} = 4$$

$$\|y\|_1 = |1+i| + |2i| + |3-2i| = \sqrt{1^2+1^2} + \sqrt{4} + \sqrt{3^2+(-2)^2}$$
$$= \sqrt{2} + 2 + \sqrt{13}$$

$$\|y\|_2 = \sqrt{19}$$

$$\|y\|_\infty = \max\{\sqrt{2}, 2, \sqrt{13}\}$$
$$= \sqrt{13}$$

Zero s
Scalarbility
Additive

Properties of all vector norms

1. For any vector $\mathbf{v}$ we have $\|\mathbf{v}\| \geqslant 0$, and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. For any vector $\mathbf{v}$ and any scalar k, we have $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.
3. (Triangle inequality) For any vectors $\mathbf{u}$ and $\mathbf{v}$, we have $\|\mathbf{u}+\mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$.

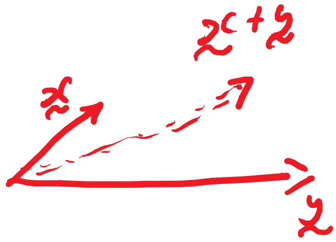
The general definition of a norm function takes the three properties of all vector norms above as the defining axioms.

Definition 1.7. Let V be a (real or complex) vector space. A *norm* or length function on V is a real valued function, $\|\mathbf{x}\|$, defined for all vectors $\mathbf{x}$ in V and which satisfies the following axioms:

(i) $\|\mathbf{x}\| \geqslant 0$, and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(ii) If k is a scalar, then $\|k\mathbf{x}\| = |k|\|\mathbf{x}\|$.

(iii) $\|\mathbf{x}+\mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$. (the *triangle inequality*)



Exercise 1.8. Check that the p-norm satisfies (i) and (ii).

Solution: (!)

1.2 Matrix Norms

Let A be an element of $V = M_{m,n}$.

(i) $\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{i,j}| = \max\{\|\mathbf{c}_1\|_1, \|\mathbf{c}_2\|_1, \ldots, \|\mathbf{c}_n\|_1\}$

(the maximum absolute column sum),

(ii) $\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{i,j}| = \max\{\|\mathbf{r}_1\|_1, \|\mathbf{r}_2\|_1, \ldots, \|\mathbf{r}_m\|_1\}$

(the maximum absolute row sum),

(iii) $\|A\|_{Fr} = \sqrt{\mathrm{Tr}\,(A^*A)} = \sqrt{\sum_{i,j=1}^{n} |a_{i,j}|^2}$ (the Frobenius norm).

Here A^* denotes the conjugate transpose of A, i.e. $A^*_{i,j} = \overline{A_{j,i}}$. The trace of an $n \times n$ matrix $B = (b_{ij})$ is the sum of the diagonal, $\mathrm{Tr}(B) = \sum_{i=1}^{n} b_{ii}$.

Exercise 1.9. In the case when A is $m \times 1$—i.e. a column vector—what do these matrix norms correspond to?

Solution:

Frobenius norm

$\|A\|_{Fr}$

→ String out all the entries of the matrix into a vector & take the Euclidean norm

$$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} (a_{ij})^2}$$

Example

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

$$\|A\|_F = \sqrt{3^2 + (-1)^2 + (2)^2 + (4)^2}$$

$= \ldots$

$= \sqrt{30}$

Extra example's for matrix norms

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix}$$

a_{ij} row col

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix}$$

$\|A\|_1$ = "the largest absolute column sum"

$$= \max_j \sum_{i=1}^{m} |a_{ij}| = 10$$

column vector norms!

$$\|c_1\|_1 = |1| + |4| + |-5| = 10$$

$$\|c_2\|_1 = |-3| + |-1| + |1| = 5$$

$$\|c_3\|_1 = |2| + |-2| + |3| = 7$$

$\|A\|_\infty$ = largest absolute row sum

$$= \max_i \sum_{j=1}^{n} |a_{ij}|$$

$$= |-5| + |1| + |3|$$

$$= 9$$