EMTH-211 Formula Sheet

Flop speeds

- Row Reduction $= \Theta(\frac{2}{3}n^3)$
- back substitution $= \Theta(n^2)$
- Gauss elim $= \Theta(\frac{2}{3}n^3)$
- partial pivot $= \frac{1}{2}n^2$
- complete $\frac{1}{3}n^3$

- Gauss-Jordan $= n^3$ $= 2n^3$
- inverse $= 2n^3$
- $n \times n$ matrix $= n$
- diagonal
- upper $= n^2$
- lower $= n^2$

• Gaussian Elimination:

$Ax = b$

To solve ② steps $\Rightarrow$ $\boxed{\text{① Row reduction; ② Back substitution}}$

e.g. ~~find~~ determine the system for k.

$k=1$, $k=-1$ & $k \neq 0$

relative error ~~correction~~ Analysis (changing the rows) upper bounded

$$\frac{\|\delta x\|}{\|x\|} \leq k(E)\frac{\|\delta b\|}{\|b\|}$$

↓ error solution; ↑ error b.

If $k(E)$ is very large the accuracy of the solution is bad

- $[0\ 0\ 0] =$ infinitely many solutions
- $[0\ 0\ 2] =$ inconsistent row } non trivial homogenous
- $[0\ 1\ 1] =$ unique solution
- $[0\ 1\ 0] =$ trivial homogenous solution

• Matrix Algebra:

~~Flop speeds~~ .. matrix multiplication (gets the same results)

e.g. $A_1^T \cdot b_1 + A_n^T \cdot b_1 =$ sum

- outer product expansion

$a_1 \cdot B_1^T + a_2 \cdot B_2^T + a_n B_n^T =$ sum

• LU Factorisation:

post-multiply = column swap

pre = row.

- 2 system

$\boxed{A = LU}$ to solve a system $\boxed{Ax = b}$

① calculate L & U — flopcount $= \frac{2}{3}n^3$

②-1) $Ly = b \equiv [L|b] = y$

2) $Ux = y \equiv [L|y] = x$

Permutation matrix P $P^TP = I$ thus $P^{-1} = P^T$ is found by row swaps

3-System

① $Ay = b$ via $L\tilde{y} = b$ followed by $Uy = \tilde{y}$

② $Az = y$ via $L\tilde{z} = y$ followed by $Uz = \tilde{z}$

③ $Ax = z$ via $L\tilde{x} = z$ followed by $Ux = \tilde{x}$

Finding L & U

$U = E_3 \cdot E_2 \cdot E_1 \cdot A$ $\quad L = E_1^{-1} \cdot E_2^{-1} \cdot E_3^{-1}$

• Linear combinations ~~→~~ subspace

- If we get unique solutions (vector is a linear combination)
- Set of linear combinations of a set is called a span
- linearly independent when $c_1\underset{\sim}{v_1} + c_2\underset{\sim}{v_2} + c_3\underset{\sim}{v_3} = \underset{\sim}{0}$
- If r.r matrix contains a free variable (linearly dependent)
- Basis = (Spanning set and linearly independant)
- subspace = subset of $\mathbb{R}^n$
- nullspace $= A\underset{\sim}{x} = \underset{\sim}{0}$
- column space $= A\underset{\sim}{x} = \underset{\sim}{b}$ ~~linear com~~ pivot column combinations
- row space = linearly independent rows — rref must not contains all zeros.
- Rank = number of pivots
- dimension = number of vectors
- Rank & nullity Formula

form → basis $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$

$\underset{\sim}{x} \in \left\{t\begin{bmatrix}-1\\1\\0\end{bmatrix} \middle| s \in \mathbb{R}\right\}$

if null basis $\left\{t\begin{bmatrix}-1\\1\\0\end{bmatrix}, s\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ then the nullity is 2

matrix Symetric when $A^T = A$

size of cols of matrix

$$\boxed{\begin{aligned} \text{rank}(A) + \text{nullity}(A) &= n \\ \text{rank}(A^T) + \text{nullity}(A^T) &= m \end{aligned}}$$

$\boxed{\text{rank}(A^T) = \text{rank}(A)}$

If they just want the formula rank + nullity A^T nullity $= n$

Norms:

$x = \begin{bmatrix}2\\-4\\3\end{bmatrix}$

linear $\|x\|_1 = |2| + |-4| + |3|$

$\|x\|_2 = \sqrt{|2|^2 + |-4|^2 + |3|^2}$

$\|x\|_\infty = \max\{|2|, |-4|, |3|\} = 4$

$k(E) = \|E^{-1}\|_\infty \cdot \|E\|_\infty$

• matrix

$\|A\|_1 = \max\{\|c_1\|, \|c_2\|\}$ largest absolute column sum

$\|A\|_\infty = \max\{\|r_1\|, \|r_2\|\}$ abs row sum

$\|A\|_{Fr} = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$

- **Eigen values & vectors:**

- Strategy
  1. Solve $\det(A-\lambda I)=0$ for eigen values
  2. Solve $(A-\lambda I)\underset{\sim}{x}=0$ for eigenvectors
- Algebraic multiplicity = total amount of each eigen value.
  e.g $\lambda = \underbrace{1, 1}_{\substack{2 \\ \text{mult}}}, \underbrace{-1}_{1}$
- geometric multiplicity is the dimension of the eigen value
- Diagonalisation = If we have enough vectors and eigen values $\neq 0$

$A = PDP^{-1}$ $A^n = P^{-1}D^nP^{-1}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $P = \begin{bmatrix} \underset{\sim}{v_1} & \underset{\sim}{v_2} \end{bmatrix}$

calculate $A^{2017}y$ $\quad y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$y = \underset{\sim}{x_1} + x_2$

$= A\underset{\sim}{x_1}^{2017} + Ax_2^{2017}$ swap with eigenvalues

$= \lambda_1 \underset{\sim}{x_1}^{2017} + \lambda_2 \underset{\sim}{x_2}^{2017}$

$= \underbrace{}\ + \underbrace{}\ = \begin{bmatrix} \vdots \end{bmatrix}$

$C^{2018} = PD^{2018}P^{-1}$

$D^{2018} = D\begin{bmatrix} i^{2018} & 0 \\ 0 & 2^{2018} \end{bmatrix}$

- **Cayley-Hamilton Thrm**
  1. verify
  2. find the inverse

1) $P_A(\lambda) = \det(A-\lambda I)$

$= \lambda^2 - 4\lambda + 5 = 0$

$A^2 - 4A + 5$

2) $A^2 + aA + bI = 0$

$A(A+a) = -bI$

manipulate to get

$A^{-1} = -\frac{1}{b}A - \frac{a}{b}I$

Rules $\quad I$ = Identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- $I^{-1} = I$
- $A \cdot I = A$

$B^4 = 5B^2 - 4I$

$(\lambda-1)(\lambda-2)(\lambda+1)(\lambda+2)$

$= \lambda^4 - 5\lambda^2 + 4$ matrix

$B^2 \cancel{x} A^{-1}$

B satisfies its own characteristic equation

- inverse matrix

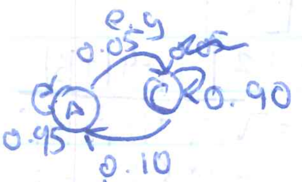
$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underset{\substack{\uparrow \\ \det}}{\frac{1}{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\frac{1}{\det} * \operatorname{Adj}(A)$

- **Markov chains**

put probability into a augmented matrix

e.g



initial

$\begin{bmatrix} 0.95 & 0.10 \\ 0.05 & 0.90 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ c_i \end{bmatrix}$

long term $\underset{\sim}{x}_k = A^k \underset{\sim}{x_0} = (PDP^{-1})^k$

$= \underset{\sim}{x_0} PD^kP^{-1} \underset{\sim}{x_0}$

- quadratic Formula:

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- imaginary unit

$i^2 = -1$ & $-i^2 = 1$

- If columns of P are linearly dependent then $\det P = 0$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - iR_1} \left[\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ P^{-1} doesn't exist.

complex eigen values & vectors

$p(\lambda) = \lambda^2 + 1$ (quadratic formula)

- The dim of a characteristic matrix is $p(t)$

$p(t) = (2-\lambda)^2(3-\lambda)\cdot(1-\lambda)$

$= 2+1+1$

$= 4\times4$ matrix

- The determinate

$\lambda_1 \cdot \lambda_2 \cdot \lambda_n = \det(A)$

- The trace

$\lambda_1 + \lambda_2 + \dots \lambda_n = \operatorname{trace}(A)$

- If matrix is invertible the eigen values $\neq 0$

2 is an eigen value and all roots of $p(\lambda)$ are eigenvalues

$M_{3,3}$ dimension = 9

give a basis $3+3$

$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ others aswell

$\frac{n \times (n+1)}{2}$

$k\begin{bmatrix} 0 & a_1 & b_1 \\ a_1 & 0 & c_1 \\ b_1 & c_1 & 0 \end{bmatrix}$

U is a subspace

If $\underset{\sim}{v} \in U$ then $k\underset{\sim}{v} \in U$