ENEL220 Circuits and Signals Term 4

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Office hours: Mon 1.30 – 4pm, Thurs 11am – 4pm (term time only). I will have a break for lunch on Thurs though ☺. Email me if you want to meet outside of this time.

Note that I am a contract lecturer, and not full time staff, so I am not usually at the university. I may be in my office outside my office hours, so feel free to come and check, but no guarantees!

References for these notes:

- W. H. Hayt, Jr., J. E Kemmerly and S. M. Durbin, “Engineering Circuit Analysis”, 7th, 8th, or 9th Editions, McGraw-Hill.

Please reference the textbook directly rather than these notes.

I strongly recommend purchasing a copy of the text book. Either Edition is fine, although the readings in this study guide refer to the 9th edition. Second hand is a good option. Library also has copies.

ENEL220 Term 4 Checklist 2019

Chapter 15

By the end of the Chapter 15 notes you should be able to:
Resonance

- ☐ Derive the transfer function of a circuit.
- ☐ Draw pole-zero diagrams when given a transfer function, H(s), or a circuit.
- ☐ Calculate the key parameters (e.g. resonant frequency, exponential damping coefficient, natural resonant frequency, quality factor, bandwidth, component values) for a parallel or series resonant circuit.
- ☐ Calculate the magnitude and phase of a transfer function. This requires knowledge of polar and Cartesian coordinates.
- ☐ Calculate the magnitude and frequency scaling constants, K_m and K_f.

Bode

- ☐ Draw bode plots (magnitude and phase) given a transfer function.
- ☐ Draw bode plots for transfer functions that have complex conjugate pairs and understand the impact of the damping factor.
- ☐ Identify the transfer function of a system, given the Bode plots (i.e. system identification).
- ☐ Be able to identify parameter values from a Bode plot (e.g. resonant frequency, damping factor and bandwidth).

Filters

- ☐ Describe/ define the characteristics of low, high, bandpass and bandstop filters.
- ☐ Design passive filters with specific corner or cutoff frequencies.

Chapter 17

By the end of the Chapter 17 notes you should be able to:

- ☐ Determine the period and fundamental frequency of a signal from a sketch of the periodic waveform.
- ☐ Calculate the Trigonometric Fourier Series coefficients of a signal.
- ☐ Calculate the Complex Fourier Series coefficients of a signal.
- ☐ Calculate and draw the line and phase spectra (note that phase spectra will not be in the exam).
- ☐ Identify if a signal has even or odd symmetry.
- ☐ Calculate the Fourier Transform of a signal.
- ☐ Apply Fourier transform theory to find the output of a system with a given impulse response and input.

Exam Content

Remember you can look up old exams on the UC library website. These are a very good guide to the type of questions you are likely to get! Basic things to remember:

- ☐ Always show all working, even if you're doing something in your head, or if you think it's obvious (for example, write "by inspection"). This makes it easy for me to give you carried error marks if you make a silly mistake.
- ☐ Always put units on your answers!
- ☐ There is no shortage of paper in the exam, so spread your answers out. Start each new question on a new page. Use an entire page for each Bode plot.

Exam Formulas for Term 4 Material

Resonance

For parallel RLC circuits: $\omega_0 = \frac{1}{\sqrt{LC}}, \quad \alpha = \frac{1}{2RC}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}, \quad Q_0 = R\sqrt{\frac{C}{L}}$

For series RLC circuits: $\omega_0 = \frac{1}{\sqrt{LC}}, \quad \alpha = \frac{R}{2L}, \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}, \quad Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}}$

Bode plots

$$\mathbf{H}(\mathrm{s}) = 1 + \frac{\mathrm{s}}{a} \rightarrow H_{dB} = 20 \log_{10}\sqrt{1 + (\frac{\omega}{a})^2}, \quad \angle(H(j\omega) = \tan^{-1}\frac{\omega}{a}$$

$$\mathbf{H}(\mathrm{s}) = \frac{1}{1 + \frac{\mathrm{s}}{a}} \rightarrow H_{dB} = -20 \log_{10}\sqrt{1 + (\frac{\omega}{a})^2}, \quad \angle(H(j\omega) = -\tan^{-1}\frac{\omega}{a}$$

$$\mathbf{H}(\mathrm{s}) = 1 + 2\xi(\frac{\mathrm{s}}{\omega_0}) + (\frac{\mathrm{s}}{\omega_0})^2 \rightarrow H_{dB} = 20 \log_{10}|1 + j2\xi(\frac{\omega}{\omega_0}) - (\frac{\omega}{\omega_0})^2|,$$

$$\angle(H(j\omega) = \tan^{-1}\frac{2\xi(\frac{\omega}{\omega_0})}{1 - (\frac{\omega}{\omega_0})^2}$$

Fourier

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

$$a_0 = \frac{1}{T}\int_0^T f(t)dt, \quad a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt, \quad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}, \quad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}F(j\omega)d\omega, \quad F(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}f(t)dt$$

f(t)	**F(jω)**	**f(t)**	**F(jω)**
$\delta(t - t_0)$	$e^{-j\omega t_0}$	$sgn(t)$	$\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$	$e^{-\alpha t}u(t)$	$\frac{1}{\alpha + j\omega}$
1	$2\pi\delta(\omega)$	$[e^{-\alpha t}\cos(\omega_d t)]u(t)$	$\frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_d^2}$

Transfer Function; Parallel Resonance; Bandwidth and High-Q Circuits; Series Resonance; Scaling

<u>Readings</u>: Sections 15.1, 15.3, 15.4, 15.5, 15.7

Frequency Response

In Terms 1 & 2 we were interested in the response of a circuit in the time domain. In Term 3 we were still interested in the time domain response, but we used the **s**-domain to help us do the analysis. Now we're going to look at the frequency domain response – i.e. what a circuit will do at different frequencies. We are only interested in the forced response to these circuits, ignoring the natural response.

Resonance

Definition: Resonance occurs when a fixed-amplitude sinusoidal forcing function produces a response of maximum amplitude.

NB: Often talk about resonance happening even for non-sinusoidal forcing functions.

When: May occur in circuits containing both inductors and capacitors.

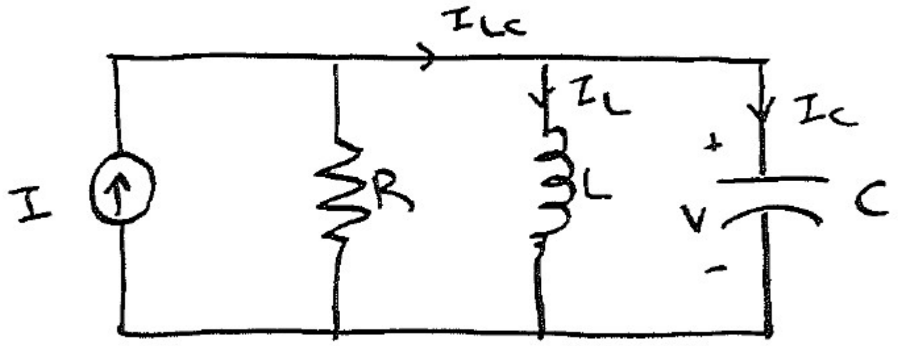
Definition: In a 2-terminal electrical network, containing at least one inductor and one capacitor, the network is **resonant when the input impedance is purely resistive**.

- V and I at input terminals are in-phase.
- Produces maximum amplitude response.

Applications: Filtering out unwanted frequencies, amplifying desired frequencies.

e.g. tuning TV, radio

Parallel RLC (Parallel Resonant Circuit)



Easiest to look at admittance

$$Y = \frac{1}{R} + j\omega C + \frac{1}{j\omega L} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

want imaginary to $= 0$

For resonance the admittance needs to be real, i.e. $\omega C - \frac{1}{\omega L} = 0$.

Therefore, resonant frequency is:

$$\omega C = \frac{1}{\omega L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(rad/s)}$$

$$\text{or } f = \frac{1}{2\pi\sqrt{LC}} \quad \text{(Hz)}$$

- We can alter ω of resonance by changing C and L.
- A practical inductor (i.e. L with series R) will lower the resonant frequency and increase $Y(j\omega_0)$.

$s = j\omega$, see next page

We can also write

- Admittance: $Y(s) = \frac{1}{R} + sC + \frac{1}{sL} = \frac{C(s^2+s/RC+1/LC)}{s}$
- Impedance: $Z(s) = \frac{s/C}{s^2+s/RC+1/LC} = \frac{s/C}{(s+\alpha-j\omega_d)(s+\alpha+j\omega_d)} = \frac{s/C}{(s+p_1)(s+p_2)}$

zero @ origin

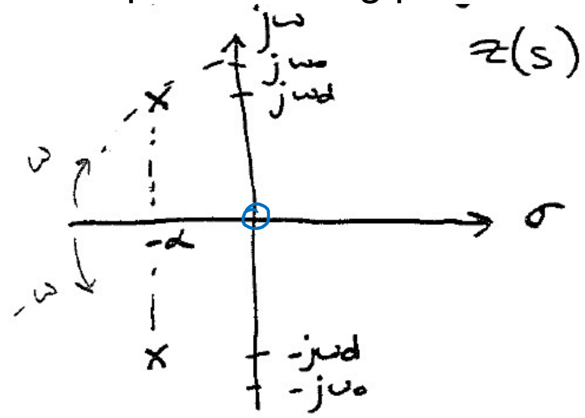
two poles at $\alpha \pm j\omega_d$

with $\omega_0 = \frac{1}{\sqrt{LC}}$ (resonant frequency)

$\alpha = \frac{1}{2RC}$ (exponential damping coefficient)

$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$ (natural damping resonant frequency)

Can represent using pole-zero diagram

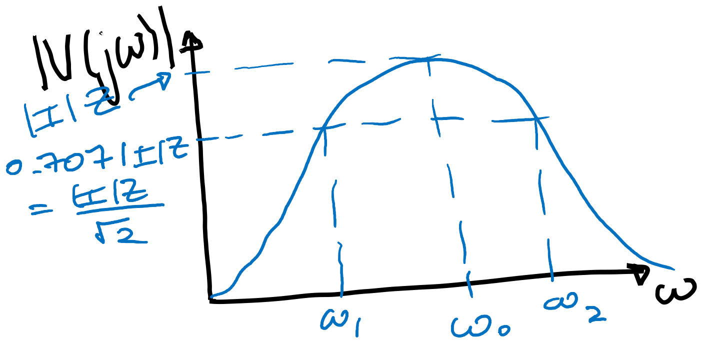


Why $s = j\omega$?
$s = \sigma + j\omega$ from last term.
But σ is associated with an exponential response (see T3 notes). This is transient, and we're not interested in it for frequency response.

Voltage Response

Remember: $s = j\omega$. Therefore:

- $\omega = 0$, response = 0
- $\omega = \infty$, response = 0
- Maximum value at resonant frequency, ω_0.



ω_1 and ω_2 can be used as a measure of width of the response curve.

At peak we know $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$. Therefore, $Y = \frac{1}{R} + j\left(\omega_0 C - \frac{1}{\omega_0 L}\right) = \frac{1}{R}$ or Z = R.

Therefore, $|V| = |I|R$, i.e. height is a function of $|I|$ and R.

Note: Maximum impedance has magnitude R at resonance for parallel RLC. It can be hard to find the maximum value of the response magnitude and frequency ω_0 for other resonant circuits.

Quality factor, Q

Measures sharpness of response curve

$$Q = 2\pi \frac{\text{maximum energy stored}}{\text{total energy lost per period}}$$

$$Q = 2\pi \frac{[w_L(t) + w_C(t)]_{\text{max}}}{P_R T}$$

For parallel RLC circuit, choose $i(t) = I_m \cos \omega_0 t$ (current at the terminals)

- Therefore, $v(t) = Ri(t) = I_m R \cos \omega_0 t$

The energy stored in the capacitor is

$$w_C(t) = \frac{1}{2} C v^2 = \frac{I_m^2 R^2 C}{2} \cos^2 \omega_0 t$$

The energy stored in the inductor is

$$w_L(t) = \frac{1}{2} L i_L^2$$

with

$$i_L(t) = \frac{1}{L} \int v \, dt = \frac{I_m R}{L\omega_0} \sin \omega_0 t$$

Therefore by substituting $i_L(t)$ into $w_L(t)$ we get:

$$w_L(t) = \frac{I_m^2 R^2 C}{2} \sin^2 \omega_0 t, \quad \text{where } \omega_0^2 = \frac{1}{LC}$$

The total stored energy is

$$w(t) = w_C(t) + w_L(t) = \frac{I_m^2 R^2 C}{2}$$

using $\cos^2 + \sin^2 = 1$

which is a constant.

The resistor is the only component that dissipates energy (in this idealised circuit). If we consider the average power

$$P_R = \frac{1}{2} I_m^2 R$$

then over one cycle the energy dissipated is

$$P_R T = \frac{I_m^2 R}{2f_0}$$

remember, period $T = \frac{1}{f_0}$

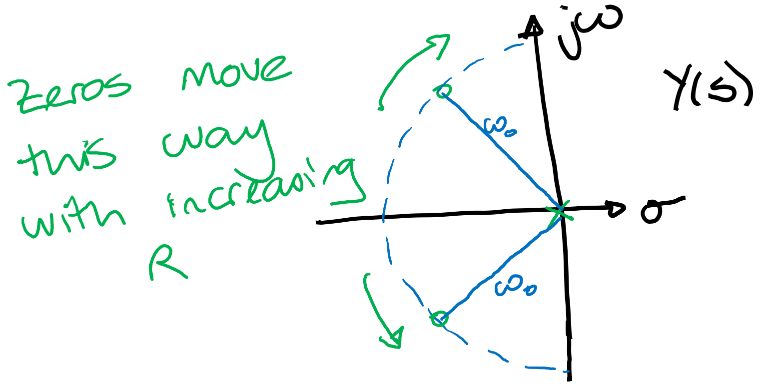
Thus at resonance,

$$Q_0 = 2\pi \frac{I_m^2 R^2 C f_0}{I_m^2 R} = \omega_0 RC = R\sqrt{\frac{C}{L}} = \frac{R}{\omega_0 L}$$

Resonance is fundamentally associated with the forced response. The two most important parameters of a resonant circuit are ω_0 and Q_0. Can be related to exponential damping coefficient, α, and natural damped resonant frequency, ω_d:

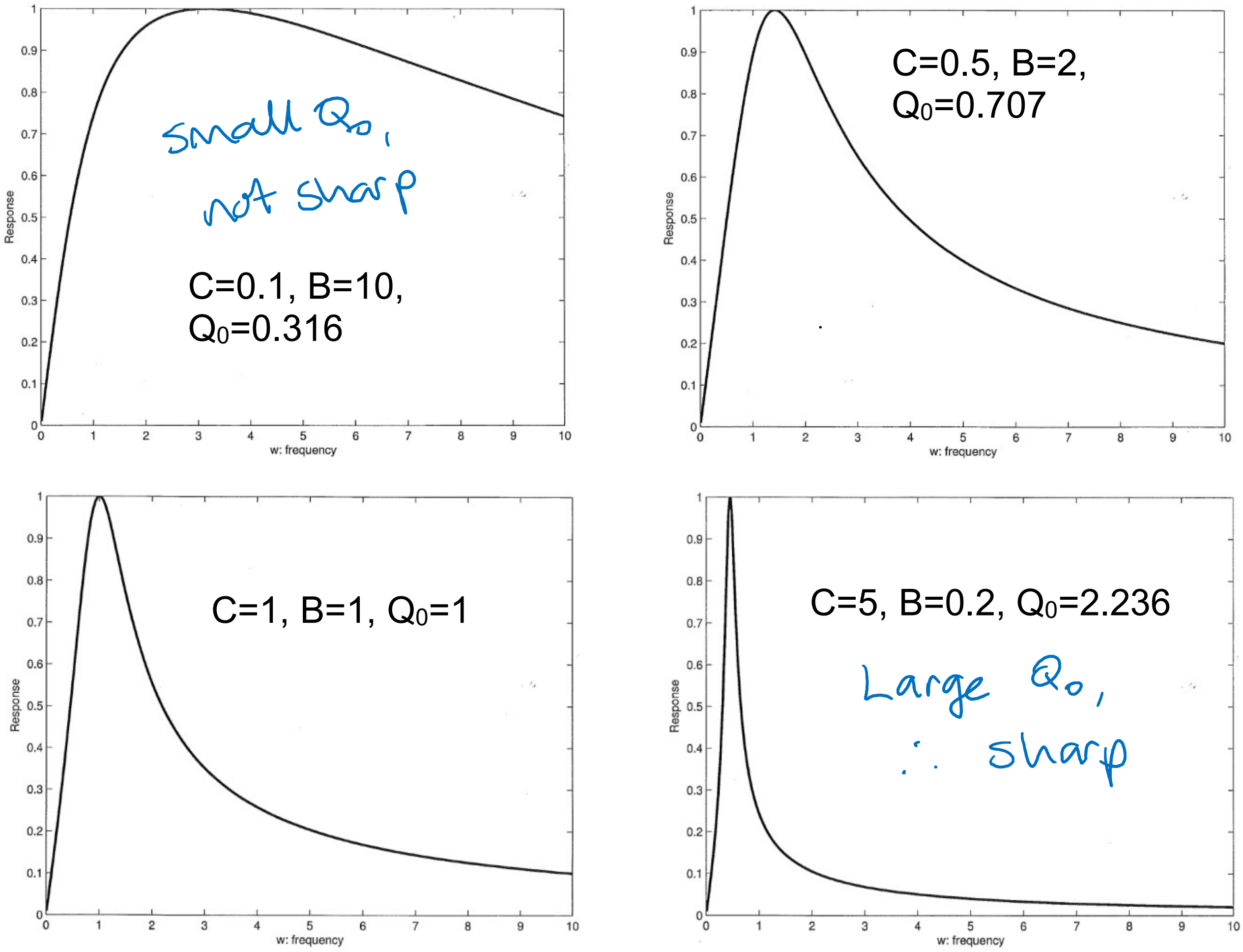
$$\alpha = \frac{1}{2RC} = \frac{\omega_0}{2Q_0} \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0\sqrt{1 - \left(\frac{1}{2Q_0}\right)^2}$$

Note: $s^2 + \left(\frac{1}{RC}\right)s + \frac{1}{LC} = s^2 + 2\alpha s + \omega_0^2$. If we plot the pole-zero plot of Y(s):

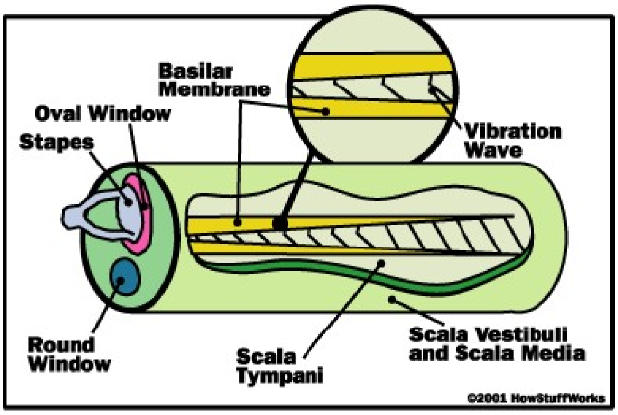


If get double zero on real axis, $R = \frac{1}{2}\sqrt{\frac{L}{C}}$ or $Q_0 = \frac{1}{2}$ (critical damping)

Recall Q measures sharpness of response curve.



In the cochlea, the small fibres on the basilar membrane resonate at different frequencies.



The ear is known to have "constant-Q" with Q ≈ 10. However it is also true that 441 Hz can be distinguished from 440 Hz, which suggests a much higher Q.

Because of the way the human ear perceives sounds, musical instruments are also designed to have the constant Q property. With the 88 keys of the piano numbered as n = 1 to 88, *

$$f(n) = 440 \times 2^{(n-49)/12}$$
$$f(n+1) = f(n) \times 2^{1/12}$$

So if we assume

$$B(n) = f(n+1) - f(n-1)$$
$$B(n) = f(n)(2^{1/12} - 2^{-1/12})$$
$$Q = 1/(2^{1/12} - 2^{-1/12})$$
$$Q = 8.65$$

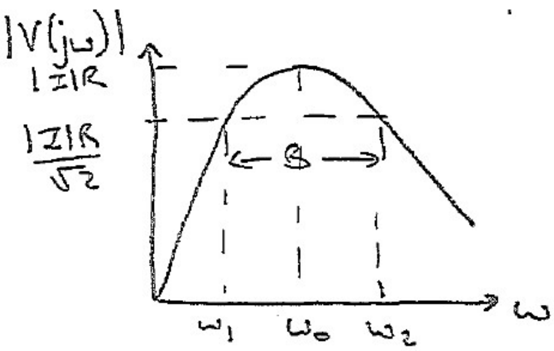
* Middle-c is 261.626 Hz

Constant Q circuits are used in audio mixers and equalizers. An equalizer is used to boost or suppress frequencies. A common format has 25 to 31 bands, each of which covers 1/3 octave (an octave is a doubling of frequency).

Bandwidth (Half-power)

Half power frequency is the frequency at which the magnitude of the input admittance of the parallel resonant circuit is greater than the magnitude at resonance by factor of $\sqrt{2}$ $(\mathrm{Y}_{\mathrm{HPF}} = \sqrt{2}\mathrm{Y}_{\mathrm{RES}})$.

- Or voltage response is $1/\sqrt{2}$ times its maximum value.



ω_1= lower half power frequency
ω_2= upper half power frequency

The half power bandwidth is given by $B = \omega_2 - \omega_1 = \frac{\omega_0}{Q_0}$

Also in textbook

ASIDE: Derivation.

$$Y(j\omega) = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

$$= \frac{1}{R} + \frac{j}{R}\left(\frac{\omega\omega_0 RC}{\omega_0} - \frac{\omega_0 R}{\omega\omega_0 L}\right) \qquad Q_0 \qquad \frac{\omega_0 R}{\omega_0 R}$$

$$= \frac{1}{R}\left[1 + jQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right]$$

and $Q_0 = \omega_0 RC = \frac{R}{\omega_0 L}$

For half power, $|Y| = \frac{\sqrt{2}}{R}$, which occurs when $Q_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = \pm 1$

or $Q_0\left(\frac{\omega_2}{\omega_0} - \frac{\omega_0}{\omega_2}\right) = +1$, $Q_0\left(\frac{\omega_1}{\omega_0} - \frac{\omega_0}{\omega_1}\right) = -1$

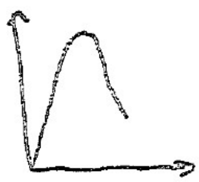
$$|Y| = \frac{\sqrt{2}}{R} = \frac{1}{R}\sqrt{1 + Q_0^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}$$

$$2 - 1 = Q_0^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2$$

$$1 = \pm Q_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)$$

$$\omega_1 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0}\right], \quad \omega_2 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0}\right]$$

- Note: $\omega_0 = \sqrt{\omega_1\omega_2}$, i.e. geometric mean of the ½ power frequency.
- Circuits with higher Q_0 have narrower bandwidth/sharper response curve, i.e. greater frequency selectivity or higher quality factor.



high Q_0

low Q_0

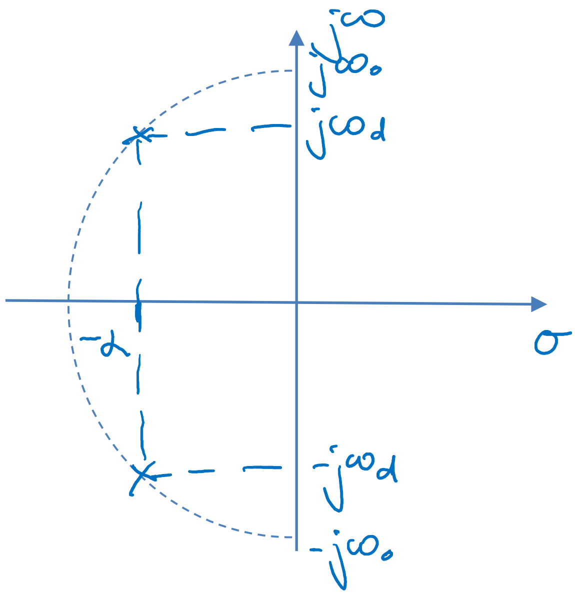
- Note that $Q_0 = \frac{\omega_0}{B}$. For a fixed value of Q, B scales with ω_0 (e.g. ears are known to have a "constant-Q" response).

Example:

Consider a parallel RLC system with |I| = 1 A, L = 1 H, R = 1 Ω and peak voltage is v = 1 V. We don't know C and so we will write equations as a function of C.

Exponential damping coefficient, α

$$\alpha = \frac{1}{2RC} = \frac{1}{2C}$$



Resonant frequency, ω_0

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{C}}$$

Natural resonant frequency, ω_d

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{\frac{1}{C} - \frac{1}{4C^2}}$$

Voltage magnitude is

$$|V(j\omega)| = |I||Z| = \left| \frac{j\omega / C}{(j\omega + \alpha - j\omega_d)(j\omega + \alpha + j\omega_d)} \right|$$

Since // RLC

Now everything is a function of C.

Quality factor

$$Q_0 = R\sqrt{\frac{C}{L}} = \sqrt{C}$$

Remember:

Low Q → large BW

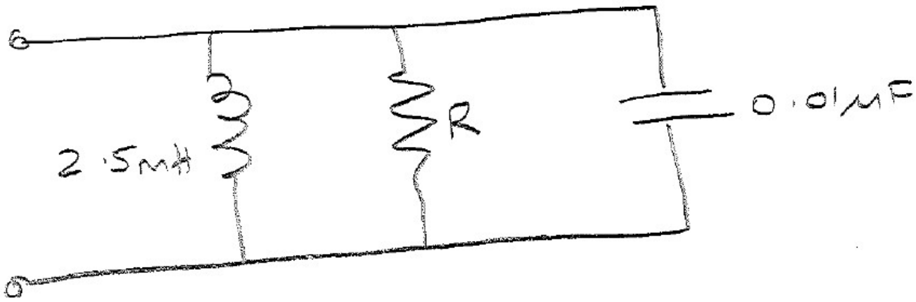
High Q → smaller BW (narrower & steeper)

Bandwidth

$$B = \frac{\omega_0}{Q_0} = \frac{1}{C}$$

Example: Parallel resonance

Calculate ω_0, α, ω_d and R for a parallel resonant circuit having L = 2.5mH, Q_0 = 5 and C = 0.01μF.



Resonant frequency, ω_0

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2.5\times10^{-3}\times 0.01\times10^{-6}}} = 200k \text{ rad/s}$$

Exponential damping coefficient, α

$$\alpha = \frac{\omega_0}{2Q_0} = \frac{200\times10^3}{2\times5} = 20{,}000 \text{ s}^{-1}$$

Natural resonant frequency, ω_d

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{(200{,}000)^2 - (20{,}000)^2}$$

$$= 199k \text{ rad/s}$$

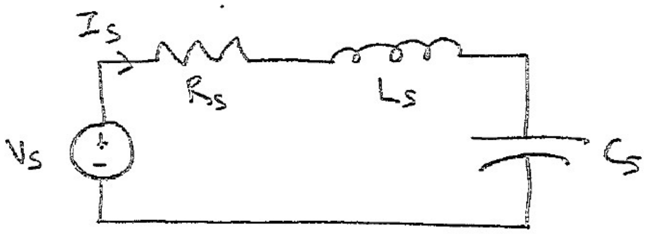
Resistance, R

Recall, $Q_0 = \omega_0 RC$

$$5 = 200\times10^3 \times R \times 0.01\times10^{-6}$$

$$\therefore R = 2.5k\Omega$$

Series resonance



- Not used as much as parallel RLC circuit.
- Network has purely resistive input impedance at resonance.

$$Z(j\omega) = R_s + j\left(\omega L_s - \frac{1}{\omega C_s}\right)$$

$\frac{1}{j\omega C} = \frac{-j}{\omega C}$

$Z(s) = R + sL + \frac{1}{sC}$

$$= \frac{L}{s}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)$$

$$Z(s) = L\frac{(s + \alpha - j\omega_d)(s + \alpha + j\omega_d)}{s}$$

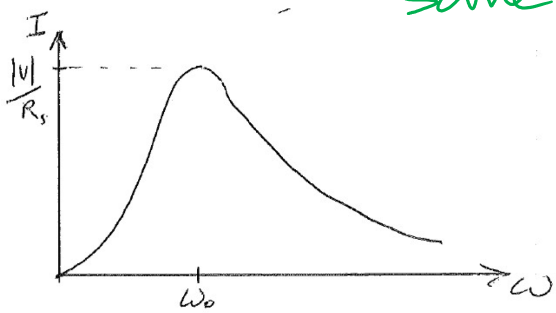
← 2 zeros

← pole @ origin

$$\underbrace{\omega_0 = \frac{1}{\sqrt{L_s C_s}}}_{\text{same as } //}, \qquad \alpha = \frac{R_s}{2L_s}, \qquad \underbrace{\omega_d = \sqrt{\omega_0^2 - \alpha^2}}_{\text{same as } //}$$

For the series RLC network,

$$Q_0 = \frac{\omega_0 L_s}{R_s}$$



Half-power frequency definitions are the same as for parallel RLC

$$\omega_1 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} - \frac{1}{2Q_0}\right], \quad \omega_2 = \omega_0\left[\sqrt{1 + \left(\frac{1}{2Q_0}\right)^2} + \frac{1}{2Q_0}\right]$$