Question 1 [7 points]

Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 2 \end{bmatrix}.$$

The reduced row echelon form for A is given by

$$RREF = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Give a basis for the row space of A.

$\{[1\ 1\ 0\ 1], [0\ 1\ -1\ 1]\}$

(b) Give a basis for the column space of A

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(c) What is the rank of A? 2

(d) What is the nullity of A? 2

(e) Give a formula that, for a general $m \times n$-matrix, relates its rank and its nullity.

rank(A) + nullity(A) = n

(f) What is the nullity of A^T? *You should not calculate the null space of A^T in order to solve this question !*

$\text{rank}(A^T) + \text{nullity}(A^T) = m$ for an $m \times n$ matrix A

$\text{rank}(A^T) = \text{rank}(A)$

so $\text{nullity}(A^T) = m - \text{rank}(A)$
$= 3 - 2 = 1$

Question 2 [7 points]

The matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 6 \\ -1 & 2 & 3 \end{bmatrix}$$

can be reduced to the echelon form

$$EF = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

by executing the row operations
$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + R_1$
$R_3 \to R_3 - R_2$.

(a) Write down the LU-decomposition for A. (*Hint: use the multiplier method*)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left(\text{since} \quad \begin{matrix} m_{21} = 3 \\ m_{31} = -1 \\ m_{32} = 1 \end{matrix} \right)$$

(b) Solve the system $A\mathbf{x} = \begin{bmatrix}1\\4\\1\end{bmatrix}$ by using the LU decomposition for A.

$$LU\bar{x} = \begin{bmatrix}1\\4\\1\end{bmatrix}$$

① $$L\bar{y} = \begin{bmatrix}1\\4\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\3 & 1 & 0\\-1 & 1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}1\\4\\1\end{bmatrix}$$

$y_1 = 1$
$y_2 = 1$
$y_3 = 1$

② $$U\bar{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 2 & 3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$x_3 = 1$
$2x_2 + 3x_3 = 1$
$\Rightarrow x_2 = -1$

$x_1 + x_3 = 1$
$\Rightarrow x_1 = 0$

solution: $\bar{x} = \begin{bmatrix}0\\-1\\1\end{bmatrix}$

Question 3 [6 points]

Let A be a 3×3 matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. Let $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an eigenvector of A with associated eigenvalue 1, let $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ be an eigenvector of A with associated eigenvalue 2 and let $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ be an eigenvector of A with associated eigenvalue 3.

(a) Calculate $A^{2017}\mathbf{y}$, where $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. *(You can of course leave expressions of the form k^l, where k is a number and l is a large number, in your answer)*

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \overline{x_1} + \overline{x_2}$$

$$A^{2017}(\overline{x_1} + \overline{x_2}) = A^{2017}\overline{x_1} + A^{2017}\overline{x_2}$$

$$= \overline{x_1} + 2^{2017}\overline{x_2}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2^{2017} \\ 2^{2017} \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2^{2017} \\ 2^{2017} \\ 0 \end{bmatrix}$$

(b) Diagonalise A (i.e. write down matrices P and D such that $A = PDP^{-1}$)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Question 4 [8 points]

Suppose that the characteristic polynomial of a matrix B is given by

$$(2-\lambda)^2(3-\lambda)(1-\lambda).$$

(a) What are the dimensions of B? (i.e. B is a $\underline{4}\times\underline{4}$ matrix)

(b) List all eigenvalues of B and their algebraic multiplicities.

$\lambda_1 = 2$ algebraic multiplicity 2

$\lambda_2 = 3$ 1

$\lambda_3 = 1$ 1

(c) What is the determinant of B? $\lambda_1^2\lambda_2\lambda_3 = 2^2 . 3 . 1 = 12$

(d) What is the trace of B? $\lambda_1 + \lambda_1 + \lambda_2 + \lambda_3 = 2+2+3+1 = 8$

(e) Is B invertible? Why (not)? yes since zero is not an eigenvalue

(f) Suppose that B has an eigenspace of dimension 2. Explain why B is diagonalisable.

The eigenspaces of λ_2 and λ_3 have dimension 1 (since their algebraic multiplicity is 1). That means that the eigenspace of dimension 2 belongs to λ_1. Since λ_1 has alg. multiplicity 2, we have that for all eigenvalues, the algebraic & geometric multiplicity coincide. This means that B is diagonalizable.

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Question 5 [6 points]

Remember that a subspace of W of a vectorspace V is a set that is closed under taking linear combinations (That is if $u, v \in W$ and $k, l \in \mathbb{R}$ then $ku + lv \in W$).
Let P_2 be the vector space of all polynomials in the variable x of degree at most 2.

(a) Show that the set $\{1-x, x^2-1, 3x\}$ spans P_2.

Let ax^2+bx+c be an arbitrary vector in P_2.

Then $k_1(1-x)+k_2(x^2-1)+k_3 \cdot 3x = ax^2+bx+c$

means that

$$\begin{cases} k_1 - k_2 = c \\ -k_1+3k_3 = b \\ k_2 = a \end{cases} \Rightarrow \begin{cases} k_1 = a+c \\ k_2 = a \\ k_3 = \frac{a+b+c}{3} \end{cases}$$

Hence, we can write an arbitrary vector in P_2, ax^2+bx+c as a linear combination of the vectors $1-x, x^2-1, 3x$. That means $\{1-x, x^2-1, 3x\}$ spans P_2.

Alternatively, you may argue that P_2 has dimension 3, and so that if $1-x, x^2-1, 3x$ are linearly independent, they certainly span P_2.
To check that the vectors are linearly independent, we check that the only

solution to
$k_1(1-x)+k_2(x^2-1)+k_3 \cdot 3x = 0$ is
$k_1 = k_2 = k_3 = 0$.

(b) Let W be the set of all polynomials of the form $a + bx^2$, where $a, b \in \mathbb{R}$. Show that W is a subspace of P_2.

Let $\overline{w_1} = a_1 + b_1x^2$
$\overline{w_2} = a_2 + b_2x^2$ be elements of W. Then we see
that $k_1(a_1+b_1x^2)+k_2(a_2+b_2x^2)$
$= k_1a_1 + k_2a_2 + (k_1b_1+k_2b_2)x^2$
$= a' + b'x^2$ with $a' = k_1a_1 + k_2a_2$
$b' = k_1b_1 + k_2b_2$
which is contained in W.

(c) Why is the set $W' = \{a + x^2 | a \in \mathbb{R}\}$ not a subspace of P_2?

•) the zero vector (= the zero polynomial in this case) does not belong to W'

or: •) W' is not closed under addition:
take e.g. $\overline{w_1} = a_1 + x^2$
$\overline{w_2} = a_2 + x^2$ $\Rightarrow \overline{w_1} + \overline{w_2} = a_1 + a_2 + 2x^2 \notin W'$

or: W' is not closed under taking scalar multiples.
if $\lambda \neq 1$, then $\lambda\overline{w_1} = \lambda a_1 + \lambda x^2 \notin W'$.

Question 6 [6 points]

Let $E = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$. The inverse of E is given by $E^{-1} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$.

(a) Calculate $\|E\|_1$, $\|E\|_\infty$ and the condition number $k(E)$ using the ∞-norm.

$\|E\|_1 = \max\{7, 10\} = 10$

$\|E\|_\infty = \max\{5, 12\} = 12$

$\|E^{-1}\|_\infty = \max\{10, 7\} = 10$

$k(E) = 12 \cdot 10 = 120$

(b) Describe **briefly** how the condition number of a matrix E may affect the accuracy of a solution to $E\mathbf{x} = \mathbf{b}$. *A formula relating the condition number to the error of a solution might be relevant.*

We know that $\underbrace{\frac{\|\bar{e}\|}{\|\bar{x}\|}}_{\text{relative error in the solution}} \leq k(E) \underbrace{\frac{\|\bar{\delta}\|}{\|\bar{b}\|}}_{\text{relative error in } \bar{b}}$

So the relative error in the solution is upper bounded by the condition number × the relative error in $\bar{b}$.

If $k(E)$ is very large, the accuracy of the solution can be very bad.