EMTH211–Tutorial 5

Attempt the following problems before the tutorial.

1. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

   (a) Find $\|A\|_1, \|A\|_\infty, \|A\|_{Fr}$ by hand and check your solution with MatLab.

2. For each of the following matrices:

   - determine all eigenvalues
   - determine for every eigenvalue its eigenspace
   - write down the algebraic and geometric multiplicity for each eigenvalue.

   (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -5 \\ 0 & 3 & -6 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 5 & 0 & 6 \\ 0 & -3 & -1 & -3 \\ 3 & -3 & 0 & -4 \end{bmatrix}$

   You can check your answer using MatLab.

In-tutorial problems

3. Recall the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

where every is the sum of the previous two terms.

We can write this sequence as $F(1)$, $F(2)$, $F(3)$, $\ldots$, where $F(1) = 1, F(2) = 1, \ldots,$ and

$$F(n) = F(n-1) + F(n-2),$$

for $n \geq 3$. We can write

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} F(n+1) + F(n) \\ F(n+1) \end{bmatrix}$$

and

$$\begin{bmatrix} F(2) \\ F(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We see that

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F(n+1) \\ F(n) \end{bmatrix},$$

so that

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F(n) \\ F(n-1) \end{bmatrix},$$

and continuing like this

$$\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} F(2) \\ F(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Calculate the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and call them λ_1 and λ_2 (take λ_1 to be the largest of the two).

(b) Show that $\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$ is an eigenvector for the eigenvalue λ_1 and that $\begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$ is an eigenvector for λ_2.

(c) Show that $\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$.

(d) Calculate $A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ by diagonalising the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.*It is a good idea work with the symbols λ_1 and λ_2 instead of their actual values.*

(e) Use the fact that $\begin{bmatrix} F(n+2) \\ F(n+1) \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to show that

$$F(n+2) = \frac{1}{\sqrt{5}} \left(\lambda_1^{n+2} - \lambda_2^{n+2} \right).$$

This implies

$$F(n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

This explicit formula for the n-th term of the Fibonacci sequence that you just derived is called *Binet's formula.*

4. Diagonalize $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, i.e. find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Extra questions

5. Let A be a 2×2-matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = 2$ respectively. Put $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Use this to find $A^{10}\mathbf{x}$ and (a formula) for $A^k\mathbf{x}$. What happens if $k \to \infty$?

6. Find a 2×2 matrix A such that:
   (a) A has two distinct real eigenvalues.
   (b) A has exactly one real eigenvalue, with a geometric multiplicity of 2.
   (c) A has exactly one real eigenvalue, with a geometric multiplicity of 1.
   (d) A has no real eigenvalue.