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Question 1:

We'll prove that convolution is time/space invariant, where x marks the time/space variable in $f(x)$, and f_o marks the signal defined as $f_o(x) = f(o + x)$ the signal f with an offset o.

$$(g * f_o)(x) = \sum_y g(x - y) \cdot f_o(y)$$

$$= \sum_y g(x - y) \cdot f(y + o) = \sum_{y+z=x+o} g(y) \cdot f(z)$$

$$= \sum_y g((x + 0) - y) \cdot f(y) = (g * f)(x + o)$$

i.e, a shift in the input causes the same shift in the output.

Therefore, by definition, convolution is time/space invariant.

Question 2:

a)

Time space invariant, as if we mark this layer L such that for any signal f we have $L[f](x) = f(x) + C$, we know of course that if we define an offset signal $f_t(x) = f(t + x)$, we get that:

$$L[f_t](x) = f_t(x) + C = f(t + x) + C = L[f](t + x)$$

i.e, a shift in the input causese the same shift in the output.

b)

Time space invariant, as if we mark this layer L such that for any signal f we have $L\left[f\right]\left(x\right) = \begin{pmatrix} ReLu\left(f\left(x\right)_1\right) \\ \vdots \\ ReLu\left(f\left(x\right)_n\right) \end{pmatrix}$, we know of course that if we define an offset signal $f_t\left(x\right) = f\left(t+x\right)$, we get that:

$$L\left[f_t\right]\left(x\right) = \begin{pmatrix} ReLu\left(f_t\left(x\right)_1\right) \\ \vdots \\ ReLu\left(f_t\left(x\right)_n\right) \end{pmatrix} = \begin{pmatrix} ReLu\left(f\left(t+x\right)_1\right) \\ \vdots \\ ReLu\left(f\left(t+x\right)_n\right) \end{pmatrix} = L\left[f\right]\left(t+x\right)$$

i.e, a shift in the input causese the same shift in the output.

c)

Not time space invariant.

We can see by defintion that a layer L is time space invariant iff it commutes with the shift signal operator S_t, defined as $S_t\left[f\right]\left(x\right) = f\left(x+t\right)$. However, for the signal vector $f = (1,2,3,4)$, strided max pooling with a factor of 2 which we'll mark L, and S_1, we can see that:

$$L\left[S_1\left[\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right]\right] = L\left[\begin{pmatrix}4\\1\\2\\3\end{pmatrix}\right] = \begin{pmatrix}4\\3\end{pmatrix}$$

And:

$$S_1\left[L\left[\begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right]\right] = S_1\left[\begin{pmatrix}2\\4\end{pmatrix}\right] = \begin{pmatrix}4\\2\end{pmatrix}$$

Therfore, we know that $S_1\left[L\left[f\right]\right] \neq L\left[S_1\left[f\right]\right]$. Not only that, but no shift of $L\left[f\right]$ will make it equal to $L\left[S_1\left[f\right]\right]$.

d)

Not time space invariant.

We'll mark the shift operator S_t as before, the convolution layre C, the additive constant operator to be B, the ReLu layer to be R and the poooling layer to be P. Therefore, we saw in the previous section that $S_1\left[P\left[f\right]\right] \neq P\left[S_1\left[f\right]\right]$. And since we saw that B, R, C are LTI, such that for any vector signal f and operator $L \in \{B, R, C\}$, we nkow that $S_1\left[L\left[f\right]\right] = P\left[L\left[f\right]\right]$. Therefore:

$$P\left[R\left[B\left[C\left[S_1\left[f\right]\right]\right]\right]\right] = P\left[R\left[B\left[S_1\left[C\left[f\right]\right]\right]\right]\right]$$

$$= P\left[R\left[S_1\left[B\left[C\left[f\right]\right]\right]\right]\right] = P\left[S_1\left[R\left[B\left[C\left[f\right]\right]\right]\right]\right]$$
$$\neq S_1\left[P\left[R\left[B\left[C\left[f\right]\right]\right]\right]\right]$$

Therefore, all in all these layers together do not commute with the shift operator, therefore the system is not LTI.

Question 3:

For all of these we'll look at the input signal of the layer as a vector $v \in \mathbb{R}^n$

a)

This layer can be described as the function $f(v) = v + C$, such that:

$$(Df)_v = Id_{n \times n}$$

b)

This layeer can be descirbed as the function $f(v) = Av$, where A is a matrix $A \in \mathbb{R}^{m \times n}$, such that:

$$(Df)_v = A_{m \times n}$$

c)

From what we saw in class,