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June 30, 2024

Question 1:

We'll prove that convolution is time/space invariant, where x marks the time/space variable in $f(x)$, and f_o marks the signal defined as $f_o(x) = f(o+x)$ the signal f with an offset o.

$$(g * f_o)(x) = \sum_y g(x-y) \cdot f_o(y)$$

$$= \sum_y g(x-y) \cdot f(y+o) = \sum_{y+z=x+o} g(y) \cdot f(z)$$

$$= \sum_y g((x+0)-y) \cdot f(y) = (g * f)(x+o)$$

i.e, a shift in the input causes the same shift in the output.

Therefore, by definition, convolution is time/space invariant.

Question 2:

a)

Time space invariant, as if we mark this layer L such that for any signal f we have $L[f](x) = f(x) + C$, we know of course that if we define an offset signal $f_t(x) = f(t+x)$, we get that:

$$L[f_t](x) = f_t(x) + C = f(t+x) + C = L[f](t+x)$$

i.e, a shift in the input causese the same shift in the output.

b)

Time space invariant, as if we mark this layer L such that for any signal f we have $L\left[f\right]\left(x\right)=\begin{pmatrix}P_{RL}\left(f\left(x\right)_{1}\right)\\ \vdots\\ P_{RL}\left(f\left(x\right)_{n}\right)\end{pmatrix}$, where P_{RL} is any pointwise non-linearity function. We know that if we define an offset signal $f_{t}\left(x\right)=f\left(t+x\right)$, we get that:

$$L\left[f_{t}\right]\left(x\right)=\begin{pmatrix}P_{RL}\left(f_{t}\left(x\right)_{1}\right)\\ \vdots\\ P_{RL}\left(f_{t}\left(x\right)_{n}\right)\end{pmatrix}=\begin{pmatrix}P_{RL}\left(f\left(t+x\right)_{1}\right)\\ \vdots\\ P_{RL}\left(f\left(t+x\right)_{n}\right)\end{pmatrix}=L\left[f\right]\left(t+x\right)$$

i.e, a shift in the input causese the same shift in the output.

c)

Not time space invariant.

We can see by defintion that a layer L is time space invariant iff it commutes with the shift signal operator S_{t}, defined as $S_{t}\left[f\right]\left(x\right)=f\left(x+t\right)$. However, for the signal vector $f=\left(1,2,3,4\right)$, strided max pooling with a factor of 2 which we'll mark L, and S_{1}, we can see that:

$$L\left[S_{1}\left[\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix}\right]\right]=L\left[\begin{pmatrix}4\\ 1\\ 2\\ 3\end{pmatrix}\right]=\begin{pmatrix}4\\ 3\end{pmatrix}$$

And:

$$S_{1}\left[L\left[\begin{pmatrix}1\\ 2\\ 3\\ 4\end{pmatrix}\right]\right]=S_{1}\left[\begin{pmatrix}2\\ 4\end{pmatrix}\right]=\begin{pmatrix}4\\ 2\end{pmatrix}$$

Therfore, we know that $S_{1}\left[L\left[f\right]\right]\neq L\left[S_{1}\left[f\right]\right]$. Not only that, but no shift of $L\left[f\right]$ will make it equal to $L\left[S_{1}\left[f\right]\right]$.

d)

Not time space invariant.

We'll mark the shift operator S_{t} as before, the convolution layre C, the additive constant operator to be B, the ReLu layer to be R and the poooling layer to be P. Therefore, we saw in the previous section that $S_{1}\left[P\left[f\right]\right]\neq$

$P\left[S_1\left[f\right]\right]$. And since we saw that B, R, C are LTI, such that for any vector signal f and operator $L \in \{B, R, C\}$, we nkow that $S_1\left[L\left[f\right]\right] = P\left[L\left[f\right]\right]$. Therefore:

$$P\left[R\left[B\left[C\left[S_1\left[f\right]\right]\right]\right]\right] = P\left[R\left[B\left[S_1\left[C\left[f\right]\right]\right]\right]\right]$$

$$= P\left[R\left[S_1\left[B\left[C\left[f\right]\right]\right]\right]\right] = P\left[S_1\left[R\left[B\left[C\left[f\right]\right]\right]\right]\right]$$
$$\neq S_1\left[P\left[R\left[B\left[C\left[f\right]\right]\right]\right]\right]$$

Therefore, all in all these layers together do not commute with the shift operator, therefore the system is not LTI.

Question 3:

For all of these we'll look at the input signal of the layer as a vector $v \in \mathbb{R}^n$

a)

This layer can be described as the function $f(v) = v + C$, such that:

$$(Df)_v = Id_{n\times n}$$

b)

This layeer can be descirbed as the function $f(v) = Av$, where A is a matrix $A \in \mathbb{R}^{m\times n}$, such that:

$$(Df)_v = A_{m\times n}$$

c)

From what we saw in class, if we look at the input signal as a 1D vector $v \in \mathbb{R}^n$, then any convolution from the signal space $\mathbb{R}^n$ to $\mathbb{R}^n$ can be defined by a circulant matrix, such that every row of the matrix is the previous row shifted.

For any signal $u \in \mathbb{R}^n$, we can define the convolution $v * u$ as:

$$(u * v)_i = \sum_{k=0}^{n-1} u_{i-k \mod n} \cdot v_k = \left(\begin{pmatrix} u_n & u_{n-1} & \cdots & u_1 \\ u_1 & u_n & \cdots & u_2 \\ \vdots & & & \vdots \\ u_{n-1} & \cdots & u_1 & u_n \end{pmatrix} v\right)_i$$

Such that:

$$\frac{d}{dv}(u*v) = \frac{d}{dv}\left(\begin{pmatrix} u_n & u_{n-1} & \cdots & u_1 \\ u_1 & u_n & \cdots & u_2 \\ \vdots & & & \vdots \\ u_{n-1} & \cdots & u_1 & u_n \end{pmatrix} v\right) = \begin{pmatrix} u_n & u_{n-1} & \cdots & u_1 \\ u_1 & u_n & \cdots & u_2 \\ \vdots & & & \vdots \\ u_{n-1} & \cdots & u_1 & u_n \end{pmatrix}$$

Similarly in the general case,