Examination
Mathematical Foundations for Software Engineering

Course codes: DIT022 / DIT023

Date:	2022-10-27
Time:	14:00-18:00
Place:	Lindholmen
Teachers:	Professor Dr. Christian Berger Assistant Professor Dr. Lucas Gren Dr. Beatriz Cabrero-Daniel
Visit to e*xam hall:*	14:30 and 16:30
Questions:	6
Results:	Will be posted by 2022-11-17.
Grade Limits (DIT022):	Pass (3) 50% Pass with honors (VG) 90%
Grade Limits (DIT023):	Pass (3): 50% Pass with credit (4): 70% Pass with distinction (5): 90%
Allowed aids:	Calculators: Casio FX-82..., Texas TI-30... and Sharp EL-W531... Attached appendix with formulas and tables.

Please observe the following:

- DO NOT write your name on any answer sheet or exam sheet – write the anonymized code instead.
- Write in legible English (unreadable responses mean no points!).
- Motivate your answers and clearly state any assumptions made.
- Start each part of the exam on a new sheet!
- Write only on one side of the paper!
- Only answers written on the answer sheets will be graded, do not write on the exam sheets!
- Before handing in your exam, number and sort the sheets in task order!

NOTE:

Not following these instructions may result in the deduction of points!

Question 1 (1 + 1 + 1 + 3 = 6 pt)
"Languages and Grammars"

1.1 What language is generated by the given grammar G1? (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.

G1 = (V, T, S, P) and V = {S, R}, where S is the start variable, T = {'a', 'b', 'MN', 'G', 'XYZ'} set of terminals and rules:

S → R
R → Ra | Rb | G | MN | XYZ

a) {w∈ {a, b, MN, G, XZY}* | w contains words beginning with one of the uppercase terminals 'MN','G','XZY' followed by a string of 'a's, or followed by a string of 'b's.}
b) {w∈ {a, b, MN, G, XYZ}* | w contains words beginning with one or more uppercase terminals 'MN','G','XYZ' followed by a string of lowercase terminal symbols 'a','b' of even length.}
c) {w∈ {a, b, MN, G, XYZ}* | w possibly contains any combination of all terminals and ends with either 'a' or 'b'.}
d) {w∈ {a, b, MN, G, XYZ}* | w contains words beginning with one of the uppercase terminals 'MN,'G','XYZ' followed by a random string of lowercase terminal symbols 'a','b'.}
e) {w∈ {a, b, MN, G, XYZ}* | w contains words of even length, with any combination of all terminals and ends with either 'a' or 'b'.}

1.2 What language is generated by the given grammar G2? (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.

G2 = (V, T, S, P) and V = {S, A, B}, where S is the start variable, T = {0,1, λ} set of terminals and rules:

S → 0A1 | 1B0
B→ 0A1 | λ
A→ 1B0 | λ

a) $\mathcal{L}(G2)$ ={w∈ {0,1}* | the length of w is even and the middle symbol is 1}
b) $\mathcal{L}(G2)$ ={w∈ {0,1}* | w is a random series of 1s and 0s}
c) $\mathcal{L}(G2)$ ={w∈ {0,1}* | w contains 0s followed by the same amount of 1s, or vice-versa} (Example: 000111, 111000, …)
d) $\mathcal{L}(G2)$ ={w∈ {0,1}* | w contains the same amount of 0s and 1s in alternating order} (Example: 010101, 101010, …)
e) $\mathcal{L}(G2)$ ={w∈ {0,1}* | the length of w is >= 0 and the middle symbol is 0}
f) $\mathcal{L}(G2)$ ={w∈ {0,1}* | the length of w is >= 0 and the middle symbol is 1}

1.3 What language is generated by the given grammar G3? (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.

G3 = (V, T, S, P) and V = {S, A}, where S is the start variable, T = {0,1} set of terminals and rules:

S → 1S
S → 00A
A → 0A
A → 0

a) L(G3) = {1^m0000^n | where $m, n > 0$}
b) L(G3) = {1^n0000^n | where $n > 0$}
c) L(G3) = {11^m0000^n | where $m, n \geq 0$}
d) L(G3) = {1^m0000^n | where $m, n \geq 0$}
e) L(G3) = {1^m000^n | where $m, n > 0$}

1.4 Provide **three** different non-empty words of length 5 that are generated by the given grammar G4. (3pt)

G4 is given in the Backus-Naur form as (starting symbol is ⟨lastname⟩):

```
⟨lastname⟩ ::= ⟨ucletter⟩|
               ⟨ucletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩|
               ⟨ucletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩⟨lcletter⟩

⟨ucletter⟩ ::= A|B|C| . . . |Z
⟨lcletter⟩ ::= a|b|c| . . . |z
```

Question 2 (1 + 1 + 6 + 14 = 22 pt)
"Automata"

2.1 What does automaton A1 do? (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.

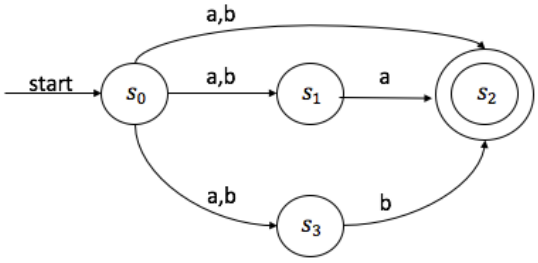


Figure 1: A1

a) It describes the language defined by grammar G = (V, S, T, P), where V = {a, b, S, T, U}, S is the start non-terminal, T = {a, b} and P = {S → a | b | aT | aU | bU, T → a, U → b}.
b) It describes the language defined by grammar G = (V, S, T, P), where V = {a, b, S, T, U}, S is the start non-terminal, T = {a, b} and P = {S → a | b | aT | aU | bT | bU, T → a, U → b}
c) It describes the language defined by grammar G = (V, S, T, P), where V = {a, b, S, T, U}, S is the start non-terminal, T = {a, b} and P = {S → a | b | aT | aUa | bU | bT, T → a, U → b}.
d) It describes the language defined by grammar G = (V, S, T, P), where V = {a, b, S, T, U}, S is the start non-terminal, T = {a, b} and P = {S → a | b | aUT | aU | bU | bT, T → a, U → b}.
e) It describes the language defined by grammar G = (V, S, T, P), where V = {a, b, S, T, U}, S is the start non-terminal, T = {a, b} and P = {S → a | b | aT | aU | bTU | λ, T → a, U → b}.

2.2 What does automaton A2 do?
Hint: This automaton recognizes a word upon generating a '1' on the output. (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark your choice(s) on your answer sheets, not this exam sheet.

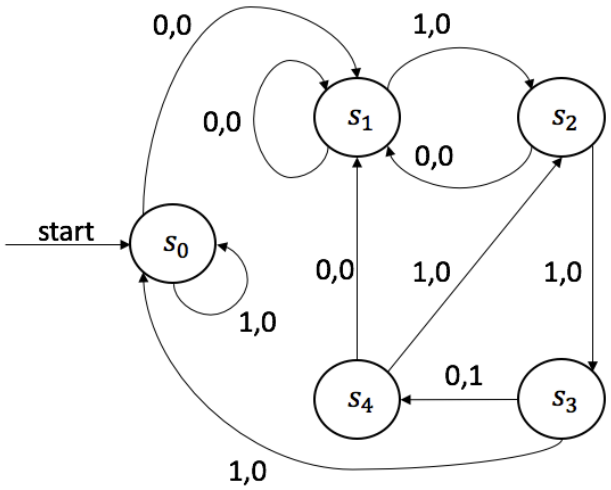


Figure 2: A2

a) It only recognizes a sequence that starts with '0' and ends with '0'.
b) It recognizes the sequence '0110' anywhere in a bit string.
c) It outputs '1' upon reading a string that ends with '10'.
d) It recognizes a sequence of even length, that ends with a '0'.
e) It recognizes a sequence of odd length, that ends with '110'.

2.3 Repair the deterministic finite-state automaton A3 with no output so that it only accepts strings with an odd number of '1's **and** an odd number of '0's. (6pt)

Note that we are not asking for partially correct solution(s) but for the fully correct one(s). Unreadable drawings will be awarded with 0 points.

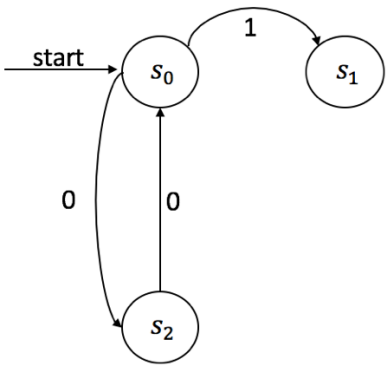


Figure 3: A3

2.4 Draw a finite-state machine with no output so that it accepts the set: $a(a \cup b)^* a \cup b(a \cup b)^* b \cup a \cup b$ (14 pt)
Note that unreadable drawings will be awarded with 0 points.
Hint: The symbol "∪" means that your finite-state machine with no output must accept ANY of the parts connected by that symbol.

Question 3 (8 + 2 + 1 + 8 = 19 pt)
"Logic"

3.1 Provide the complete truth table for the given compound proposition. (8pt)

$$(((r \vee q) \wedge ((r \rightarrow q) \wedge p)) \rightarrow (p \wedge q))$$

3.2 To what logical expression can the compound proposition from 3.1 be simplified to (ie., is the compound proposition logically equivalent) and what does that mean? (2pt)

3.3 Express this statement using predicates and quantifiers. (1pt)
Select the fully correct solution(s) only. Choosing multiple solutions when only one is correct or choosing multiple partially correct solutions will not be assessed as correct. Mark clearly the correct solution.

"There are exactly two apples in the basket."

a) $\exists x \exists y((x = y) \wedge A(x) \wedge A(y) \wedge \forall z(A(z) \rightarrow (z = x \wedge z = y)))$ where *A(n)* denotes a statement as "n is apple" and the domain for x, y and z are all fruits in the basket.
b) $\exists x \exists y((x \neq y) \wedge A(x) \wedge A(y) \wedge \forall z(A(z) \rightarrow (z = x \vee z = y)))$ where *A(n)* denotes a statement as "n is apple" and the domain for x, y and z are all fruits in the basket.
c) $\forall x \forall y((x = y) \wedge A(x) \wedge A(y) \wedge \forall z(A(z) \rightarrow (z = x \wedge z = \neg y)))$ where *A(n)* denotes a statement as "n is apple" and the domain for x, y and z are all fruits in the basket.
d) $\forall x \forall y((x \neq y) \wedge A(x) \wedge A(y) \wedge \forall z(A(z) \rightarrow (z = x \vee z = y)))$ where *A(n)* denotes a statement as "n is apple" and the domain for x, y and z are all fruits in the basket.
e) $\forall x \forall y((x = y) \wedge A(x) \vee A(y) \wedge \forall z(A(z) \rightarrow (z = x \wedge z = y)))$ where *A(n)* denotes a statement as "n is apple" and the domain for x, y and z are all fruits in the basket.

3.4 Look at the following compound proposition and answer the questions below. (8pt in total)

$$(((r \rightarrow q) \wedge ((r \rightarrow q) \wedge p)) \rightarrow (p \wedge q))$$

i) When is a compound proposition a contingency? (2pt)
ii) When is a compound proposition satisfiable? (1pt)
iii) Report the values for p, q, and r from the compound proposition above for which it is not satisfiable (ie., in what combination they have to be **True** or **False**). (5pt)

Question 4 (4 + 4 = 8 pt)
"Proofs"

4.1 Prove that $\sum_{i=1}^{n} i = 1 + \cdots + n = \frac{5n+5n^2}{10}$ for n > 0. (4pt)

4.2 Prove that $n^3 + 2n$ is divisible by 3 for every non-negative integer n. (4pt)

Question 5 (1 + 1 + 1 + 8 + 5 + 4 = 20 pt)
"Complexity and Graphs"

5.1 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
void myFunction(int n){
   int result = 0;
   for (int i=1; i<=n; i++){
      for (int j=1; j<n; j++){
         if (j >= n){
            for (int k=1; k<=j; k++) {
              result++;
            }
         }
      }
   }
}
```

5.2 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
void myFunction2(int n){
   for (int i=1; i<n; i = i+2){
      for (int j=1; j<n; j--){
         System.out.println("Hej!"); // This is O(1)
      }
   }
}
```

5.3 What is the complexity of the following code snippet (in terms of Big-O notation)? (1pt)

```
bool myFunction(int n){
   // a[..] is an array defined outside this function with at least n elements
   for(int i = 1; i < n; i++) {
      bool found = false;
      for(int j = 1; j < n; j++) {
           if (a[i] == a[j]) {
                break;
           }
      }
      if (found) {
           System.out.println("found");
           break;
      }
   }
   return true;
```

5.4 Trace the execution of the **selection sort** algorithm over the array below. Note down each pass of the algorithm until the array is sorted. Then, use your trace and match the state using the indexed list below; write down the matching upper-case letter for every pass of your algorithm (ie., the state after one complete iteration of the selection sort algorithm). Finally, write down the matching upper-case letters as your answer to this question. (8pt)

Input array: 3 15 -8 5 13 -1 9 19 -4

Indexed list to match your trace against to find the letters for your final answer:

A: -8 -4 -1 3 5 9 13 15 19
B: -8 -4 -1 5 13 3 9 19 15
C: -8 4 -1 5 13 3 9 19 15
D: -8 -4 -1 3 -5 9 13 19 15
E: -8 -4 -1 3 13 5 9 19 15
F: -8 -4 -1 3 5 13 -9 19 15
G: -8 -4 3 5 13 -1 9 19 14
H: -8 -4 3 5 13 1 9 19 15
I: -8 -4 3 5 13 -1 9 19 15
J: -8 -4 1 3 5 9 12 15 19
K: -8 -4 -1 5 13 2 9 19 15
L: -8 -4 -1 3 5 13 9 19 15
M: -8 -4 -1 3 5 13 9 18 15
N: -8 4 -1 3 13 5 9 19 15
O: -8 -4 -1 3 5 8 13 19 15
P: -8 -4 -1 3 5 9 13 19 15
Q: -8 -4 -1 3 5 7 13 19 15
R: -8 -4 -1 3 5 9 12 15 19
S: -8 -4 -1 3 5 13 -9 19 16
T: -8 15 3 5 13 -1 9 19 -4

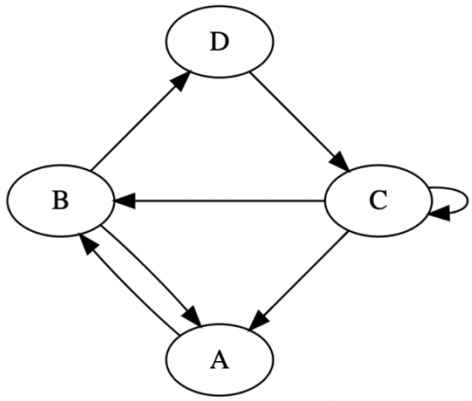
5.5 Assume you have three algorithms A, B, C with the respective complexities O(log(n)), $O(n^2)$, and $O(n^3)$. Each algorithm spends 5s to process 10,000 items.

5.5.1) What are the processing times for each algorithm for 20,000 items? Please provide the results of your calculations with a rounded accuracy of two decimal places (the decimal place accuracy of a number is the number of digits to the right of the decimal point). (3pt)

5.5.2) For how many items is it better to choose algorithm B over C? (2pt)

Note: State the number of items and show the processing time (including as many decimals needed!) when B is faster than C.

5.6 Look at the directed graph G5 = (V, E) below, where V is the set of vertices V={A,B,C,D} and E is the set of edges, and answer the questions below.



5.6.1) Determine $result = \sum_{v \in V} deg^-(v) \,/\, \sum_{v \in V} deg^+(v)$ (1pt)
Please note that "/" means "divided by".

5.6.2) How many different Euler paths do exist in G5? (1pt)

5.6.3) Provide one possible Euler path from G5 if there is one. (1pt)

5.6.4) Is the following statement "true" or "false": $\neg\exists v \in V$: there is a Hamilton circuit in G5 for *v*. (1pt)

Question 6 (8 + 5 + 12 = 25pts)
"Statistics"

"Statistics is the grammar of science."
— Karl Pearson

Question 6.1 (8pt)

In a set of experiments conducted by Assistant Professor Dr. Lucas Gren, we wanted to assess if obsolete requirements that are explicitly stated to be excluded from the effort estimates, have any impact on the estimates for the remaining requirements.

We randomly assigned the following three requirement specifications to three independent groups of students, i.e. a student was either given Task A, B, or C:

Task A

Your task is to estimate how long (in terms of weeks) it will take to implement the following requirements.

R1: The system shall receive uncompressed data and shall compress and save the data to desired JPEG size

R2: The maximum delay from a call answer is pressed to opened audio paths is XX ms

R3: The system shall have support for Time Shift (playback with delay)

R4: The system shall have a login function that consists of a username and a password.

The total estimated effort is: _____ week(s)

Task B

Your task is to estimate how long (in terms of weeks) it will take to implement the following requirements.

R1: The system shall receive uncompressed data and shall compress and save the data to desired JPEG size

R2: The maximum delay from a call answer is pressed to opened audio paths is XX ms

R3: The system shall have support for Time Shift (playback with delay)

R4: The system shall have a login function that consists of a username and a password.

R5: It shall be possible to dedicate a host buffer in RAM that is configurable between XX to XX MB for HDD

The total estimated effort is: _____ week(s)

Task C

Your task is to estimate how long (in terms of weeks) it will take to implement the following requirements.

R1: The system shall receive uncompressed data and shall compress and save the data to desired JPEG size

R2: The maximum delay from a call answer is pressed to opened audio paths is XX ms

R3: The system shall have support for Time Shift (playback with delay)

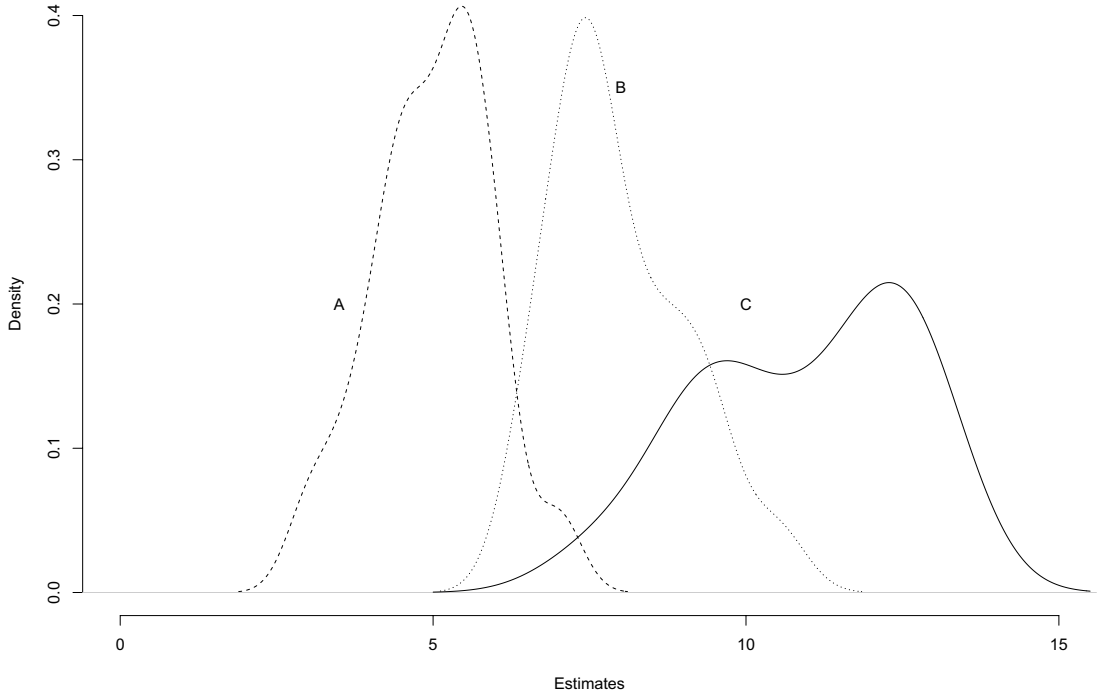
R4: The system shall have a login function that consists of a username and a password.

R5: It shall be possible to dedicate a host buffer in RAM that is configurable between XX to XX MB for HDD

Please note that **R5** should **NOT** be implemented

The total estimated effort is: _____ week(s)

We plotted raw data:



Please note that Question 6.1 is continued on the next page.

a) Just by eyeballing the plot, do you think we have hetero- or homoscedasticity? (Briefly explain why) (2pt)

b) Could we run multiple t-tests comparing the three different groups? Why or why not? (2pt)

c) State the null and the alternative hypothesis if you would choose to run an ANOVA (2pt, 1 for each).

d) Describe how you would analyze and decide if the task group is explaining estimation variance (assuming all the assumptions you need are met). What type of test would run in order to see which task groups that are significantly different from each other? (2pt)

Question 6.2 (5pt)

Backtracking a bit, we need to make sure the assumptions of our tests are met. In the case of the study presented in Question 1, the Kolmogorov-Smirnov tests of normality for the groups Task A and B were:

- Test Statistic for group Task A=0.128, p=0.040,
- Test Statistic for group Task B=0.209, p=0.000, and
- Test Statistic for group Task C=0.108, p=0.198.

a) What is the null hypothesis of the Kolmogorov-Smirnov test? (It's interpreted in the same way as the Shapiro-Wilk test) (1pt)

b) What can be said about normality (using a confidence level of 95%) for the p-values in the Kolmogorov-Smirnov tests for group A and B above? (1pt)

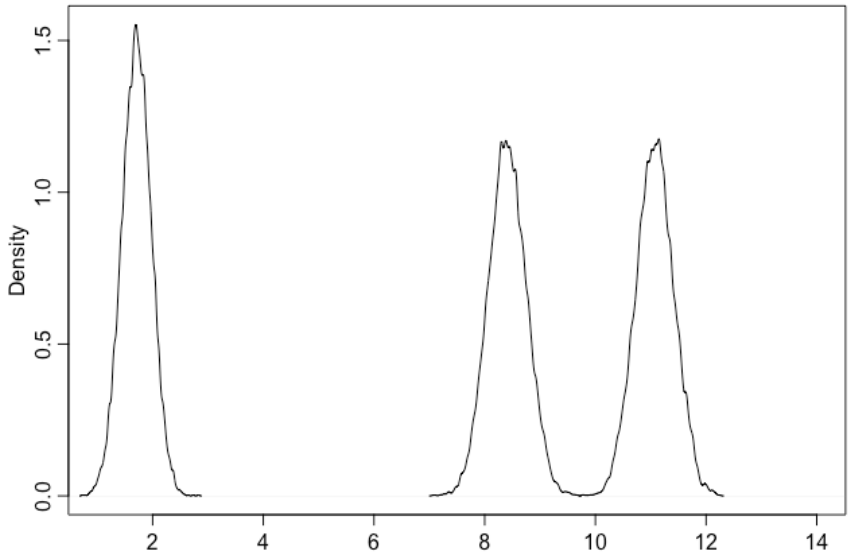
c) What can be said about normality (using a confidence level of 95%) in the case above, for group C? (1pt)

d) What can we do if we run into problems with our normality assumption? Name two things a researcher can do (2pt, 1pt for each suggestion).

Question 6.3 (12pt)

We replicated the study presented in the previous two Questions in an industry setting. Instead of checking the model for assumption the way we did before, we used a Bayesian way of defining the parameters. You don't need to know any details about Bayesian statistics, just that we then can obtain the *true* distributions of our parameters given how we believe it behaves backed up by our data.

The plot obtained for task groups A (left), B (middle), and C (right):



	Mean	StdDev	|0.89	0.89|
Task A	1.71	0.27	1.29	2.14
Task B	8.41	0.34	7.87	8.97
Task C	11.08	0.34	10.55	11.66

a) Based on the plot, do you believe that three different groups are normally distributed? Why? (1pt)

b) Why would you expect (according to what theorem) that these sampled mean values would be normally distributed? (1pt).

Please note that Question 6.3 is continued on the next page.

In order to assess the effect sizes, we still created a linear model based on the parametric assumption and obtained the following result:

```
Call:
lm(formula = estimate ~ 1 + as.factor(task), data = y5)

Residuals:
   Min      1Q  Median      3Q     Max
-5.1200 -0.7308  0.2083  0.7692  2.3800


Coefficients:
                    Estimate Std. Error t value Pr(>|t|)
Task A                4.7308     0.2581  18.329  < 2e-16 ***
Task B                2.0609     0.3725   5.532 4.83e-07 ***
Task C                3.8892     0.3687  10.550 2.93e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.316 on 72 degrees of freedom
Multiple R-squared:  0.6077,    Adjusted R-squared:  0.5968
F-statistic: 55.76 on 2 and 72 DF,  p-value: 2.35e-15
```

a) Is there a significant overall effect? Why? (2pt)

b) Are there significant differences between the treatment (task) groups? Why? (Hint: the coefficients of the model do not tell you that, look at the plot above and argue for your case) (3pt)

c) What is effect size? (2pt)

d) What is the effect size above? What does that mean in relation to our research? (3pt).

APPENDIX A

Standard deviation of the sample mean:

$$\frac{\sigma}{\sqrt{n}}$$

Variance of a sample:

$$s^2 = \frac{1}{n-1}\sum(X_i - \bar{X})^2$$

Pooled variance of a sample:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

Z-test statistic:
$\boldsymbol{z = (\overline{X} - \mu)\ /\sigma}$

T test statistic:

One-sample t-test

$$\boldsymbol{T = \frac{\overline{x} - \mu_0}{s/\sqrt{n}}}$$

$df = n - 1$

Two-sample t-test

$$\boldsymbol{T = \frac{\overline{X}_1 - \overline{X}_2 - diff}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}}$$

$df = n_1 + n_2 - 2$

One-way analysis of variance structuring help

The commonly used normal linear models for a completely randomized experiment are:

$y_{i,j} = \mu_j + \varepsilon_{i,j}$ (the means model)

or

$y_{i,j} = \mu + \tau_j + \varepsilon_{i,j}$ (the effects model)

where

$i = 1, \ldots, I$ is an index over experimental units
$j = 1, \ldots, J$ is an index over treatment groups
I_j is the number of experimental units in the jth treatment group
$I = \sum_j I_j$ is the total number of experimental units
$y_{i,j}$ are observations
μ_j is the mean of the observations for the jth treatment group
μ is the grand mean of the observations
τ_j is the jth treatment effect, a deviation from the grand mean
$\sum \tau_j = 0$
$\mu_j = \mu + \tau_j$
$\varepsilon \sim N(0, \sigma^2)$, $\varepsilon_{i,j}$ are normally distributed zero-mean random errors.

	Lists of Group Observations					
	1	2	⋯	j	⋯	J
1	y_{11}	y_{12}				y_{1J}
2	y_{21}	y_{22}				y_{2J}
3						
⋮						
i				y_{ij}		
⋮						
	$y_{I_1 1}$	⋯				
		$y_{I_2 2}$	⋯			

	Group Summary Statistics						Grand Summary Statistics	
# Observed	I_1	I_2	⋯	I_j	⋯	I_J	# Observed	$I = \sum I_j$
Sum				$\sum_i y_{ij}$			Sum	$\sum_j \sum_i y_{ij}$
Sum Sq				$\sum_i (y_{ij})^2$			Sum Sq	$\sum_j \sum_i (y_{ij})^2$
Mean	m_1	⋯		m_j	⋯	m_J	Mean	m
Variance	s_1^2	⋯		s_j^2	⋯	s_J^2	Variance	s^2

Comparing model to summaries: $\mu = m$ and $\mu_j = m_j$. The grand mean and grand variance are computed from the grand sums, not from group means and variances.

One-way ANOVA

Source of variation	Sums of squares Explanatory SS[4]	Sums of squares Computational SS[5]	Degrees of freedom DF	Mean square MS	F
Treatments	$\sum_{Treatments} I_j(m_j - m)^2$	$\sum_j \frac{(\sum_i y_{ij})^2}{I_j} - \frac{(\sum_j \sum_i y_{ij})^2}{I}$	$J - 1$	$\frac{SS_{Treatment}}{DF_{Treatment}}$	$\frac{MS_{Treatment}}{MS_{Error}}$
Error	$\sum_{Treatments} (I_j - 1)s_j^2$	$\sum_j \sum_i y_{ij}^2 - \sum_j \frac{(\sum_i y_{ij})^2}{I_j}$	$I - J$	$\frac{SS_{Error}}{DF_{Error}}$	
Total	$\sum_{Observations} (y_{ij} - m)^2$	$\sum_j \sum_i y_{ij}^2 - \frac{(\sum_j \sum_i y_{ij})^2}{I}$	$I - 1$		

MS_{Error} is the estimate of variance corresponding to σ^2 of the model.

APPENDIX B (*t* Distribution)

One Sided	75%	80%	85%	90%	95%	97.5%	99%	99.5%	99.75%	99.9%	99.95%
Two Sided	50%	60%	70%	80%	90%	95%	98%	99%	99.5%	99.8%	99.9%
1	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	2.871	3.174	3.390
120	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

APPENDIX C (*F* Distribution)

TABLE C.5
Upper Percentage Points of the *F* Distribution

df for denominator	α	df for numerator 1	2	3	4	5	6	7	8	9	10	11	12
1	.25	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.36	9.41
	.10	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	60.5	60.7
	.05	161	200	216	225	230	234	237	239	241	242	243	244
2	.25	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.39
	.10	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41
	.05	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4
	.01	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4
3	.25	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.45
	.10	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22
	.05	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74
	.01	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1
4	.25	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08
	.10	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.91	3.90
	.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.94	5.91
	.01	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.5	14.4	14.4
5	.25	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89
	.10	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.28	3.27
	.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.71	4.68
	.01	16.3	13.3	12.1	11.4	11.0	10.7	10.5	10.3	10.2	10.1	9.96	9.89
6	.25	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.77
	.10	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.92	2.90
	.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00
	.01	13.7	10.9	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.79	7.72
7	.25	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.69	1.69	1.69	1.68
	.10	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.68	2.67
	.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.60	3.57
	.01	12.2	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.54	6.47
8	.25	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.63	1.63	1.63	1.62
	.10	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.52	2.50
	.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.31	3.28
	.01	11.3	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.73	5.67
9	.25	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	.10	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.40	2.38
	.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.10	3.07
	.01	10.6	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.18	5.11
10	.25	1.49	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.55	1.54
	.10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.30	2.28
	.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.94	2.91
	.01	10.0	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.77	4.71

TABLE C.5 (continued)

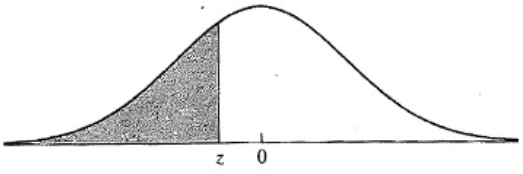
df for denominator	α	*df* for numerator											
		1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
	.25	1.47	1.58	1.58	1.57	1.56	1.55	1.54	1.53	1.53	1.52	1.52	1.51
11	.10	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.23	2.21
	.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.82	2.79
	.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.46	4.40
	.25	1.46	1.56	1.56	1.55	1.54	1.53	1.52	1.51	1.51	1.50	1.50	1.49
12	.10	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.17	2.15
	.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.72	2.69
	.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.22	4.16
	.25	1.45	1.55	1.55	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.47
13	.10	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.12	2.10
	.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.63	2.60
	.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	4.02	3.96
	.25	1.44	1.53	1.53	1.52	1.51	1.50	1.49	1.48	1.47	1.46	1.46	1.45
14	.10	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.08	2.05
	.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.57	2.53
	.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.86	3.80
	.25	1.43	1.52	1.52	1.51	1.49	1.48	1.47	1.46	1.46	1.45	1.44	1.44
15	.10	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.04	2.02
	.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.51	2.48
	.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.73	3.67
	.25	1.42	1.51	1.51	1.50	1.48	1.47	1.46	1.45	1.44	1.44	1.44	1.43
16	.10	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	2.01	1.99
	.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.46	2.42
	.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.62	3.55
	.25	1.42	1.51	1.50	1.49	1.47	1.46	1.45	1.44	1.43	1.43	1.42	1.41
17	.10	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.98	1.96
	.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.41	2.38
	.01	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.52	3.46
	.25	1.41	1.50	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.42	1.41	1.40
18	.10	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.96	1.93
	.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34
	.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.43	3.37
	.25	1.41	1.49	1.49	1.47	1.46	1.44	1.43	1.42	1.41	1.41	1.40	1.40
19	.10	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.94	1.91
	.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.34	2.31
	.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.36	3.30
	.25	1.40	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.41	1.40	1.39	1.39
20	.10	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.92	1.89
	.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.31	2.28
	.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.29	3.23

TABLE C.5 (continued)

df for denominator	α	df for numerator 1	2	3	4	5	6	7	8	9	10	11	12
22	.25	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.38	1.37
	.10	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.88	1.86
	.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.26	2.23
	.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.18	3.12
24	.25	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.37	1.36
	.10	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.85	1.83
	.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.21	2.18
	.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.09	3.03
26	.25	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.36	1.35
	.10	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.84	1.81
	.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15
	.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	3.02	2.96
28	.25	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	.10	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.81	1.79
	.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.15	2.12
	.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.96	2.90
30	.25	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.35	1.34
	.10	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.79	1.77
	.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.13	2.09
	.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.91	2.84
40	.25	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31
	.10	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.73	1.71
	.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.04	2.00
	.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.73	2.66
60	.25	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.29
	.10	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.68	1.66
	.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.95	1.92
	.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.56	2.50
120	.25	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.27	1.26
	.10	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.62	1.60
	.05	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.87	1.83
	.01	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.40	2.34
200	.25	1.33	1.39	1.38	1.36	1.34	1.32	1.31	1.29	1.28	1.27	1.26	1.25
	.10	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.60	1.57
	.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.84	1.80
	.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.34	2.27
∞	.25	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.24
	.10	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.57	1.55
	.05	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.79	1.75
	.01	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.25	2.18

APPENDIX D (*Z* Distribution)

TABLE A.2 Cumulative normal distribution (*z* table)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
−3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

1