

Analytic and Gauss integration of Stiffness Matrix

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The analytic matrix is given by

$$K = \int_{-1}^1 \int_{-1}^1 B^T(r, s) C(r, s) B(r, s) dr ds$$

$$K = \frac{E}{(1 + \nu)(1 - 2\nu)} \begin{pmatrix} -\frac{4\nu-3}{6} & \frac{1}{8} & \frac{2\nu-3}{12} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\nu-1}{8} \\ \frac{1}{8} & -\frac{4\nu-3}{6} & -\frac{4\nu-1}{12} & \frac{\nu}{8} & -\frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{2\nu-3}{12} \\ \frac{2\nu-3}{12} & -\frac{4\nu-1}{6} & -\frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{8} & \frac{4\nu-1}{12} & \frac{4\nu-3}{8} & \frac{1}{12} \\ \frac{4\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\nu-3}{6} & -\frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{8} & \frac{4\nu-3}{12} \\ \frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\nu-1}{8} & -\frac{4\nu-3}{6} & \frac{1}{12} & \frac{2\nu-3}{8} & \frac{4\nu-1}{12} \\ -\frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{6} & -\frac{4\nu-3}{12} & -\frac{4\nu-1}{8} & \frac{\nu}{8} \\ \frac{\nu}{6} & \frac{4\nu-1}{12} & \frac{4\nu-3}{8} & \frac{1}{12} & \frac{2\nu-3}{8} & -\frac{4\nu-1}{6} & -\frac{4\nu-3}{8} & -\frac{1}{6} \\ -\frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\nu-3}{6} \end{pmatrix}$$

Numerically integrated matrices are in the form

$$K_N = \sum_i^N \sum_j^N w_i w_j B^T(r_i, s_j) C(r_i, s_j) B(r_i, s_j) ,$$

where i and j are the indexes to sweep along the r and s axis, r_i and s_j represent the Gauss

coordinates, and N the number of points in each direction. For $N = 1$ and $N = 2$ we have

$$\begin{aligned}
K_1 &= \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} -\frac{4\nu-3}{16} & \frac{1}{16} & -\frac{1}{16} & \frac{4\nu-1}{16} & \frac{4\nu-3}{16} & -\frac{1}{16} & \frac{1}{16} & -\frac{4\nu-1}{16} \\ \frac{1}{16} & -\frac{4\nu-3}{16} & -\frac{1}{16} & \frac{1}{16} & -\frac{1}{16} & \frac{4\nu-3}{16} & \frac{4\nu-1}{16} & -\frac{1}{16} \\ \frac{16}{16} & -\frac{16}{16} & -\frac{16}{16} & \frac{16}{16} & -\frac{16}{16} & \frac{16}{16} & \frac{16}{16} & -\frac{16}{16} \\ \frac{4\nu-1}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\nu-3}{16} & -\frac{4\nu-1}{16} & \frac{16}{16} & \frac{1}{16} & \frac{4\nu-3}{16} \\ \frac{16}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\nu-3}{16} & -\frac{4\nu-1}{16} & -\frac{16}{16} & \frac{16}{16} & -\frac{4\nu-1}{16} \\ \frac{16}{16} & -\frac{1}{16} & \frac{4\nu-3}{16} & \frac{4\nu-1}{16} & -\frac{1}{16} & -\frac{4\nu-3}{16} & -\frac{4\nu-1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{4\nu-1}{16} & \frac{4\nu-3}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\nu-1}{16} & -\frac{4\nu-3}{16} & -\frac{1}{16} \\ -\frac{4\nu-1}{16} & -\frac{1}{16} & \frac{16}{16} & \frac{4\nu-3}{16} & \frac{4\nu-1}{16} & \frac{1}{16} & -\frac{16}{16} & -\frac{4\nu-3}{16} \\ -\frac{16}{16} & -\frac{16}{16} & \frac{16}{16} & \frac{16}{16} & \frac{16}{16} & \frac{16}{16} & -\frac{16}{16} & -\frac{16}{16} \end{pmatrix} \\
K_2 &= \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} -\frac{4\nu-3}{6} & \frac{1}{8} & \frac{2\nu-3}{12} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\nu-1}{8} \\ \frac{1}{6} & -\frac{4\nu-3}{8} & -\frac{12}{4\nu-1} & \frac{8}{\nu} & \frac{12}{1} & \frac{4\nu-3}{8} & \frac{4\nu-1}{6} & \frac{2\nu-3}{8} \\ \frac{2\nu-3}{8} & -\frac{6}{4\nu-1} & -\frac{8}{4\nu-3} & \frac{6}{1} & \frac{\nu}{8} & \frac{12}{4\nu-1} & \frac{8}{4\nu-3} & \frac{12}{1} \\ \frac{12}{4\nu-1} & \frac{\nu}{8} & -\frac{6}{1} & -\frac{8}{4\nu-3} & -\frac{6}{4\nu-1} & \frac{8}{2\nu-3} & \frac{12}{1} & \frac{4\nu-3}{8} \\ \frac{8}{4\nu-3} & \frac{6}{1} & -\frac{1}{8} & -\frac{6}{4\nu-1} & -\frac{8}{4\nu-3} & \frac{12}{1} & \frac{2\nu-3}{8} & \frac{12}{4\nu-1} \\ \frac{12}{8} & -\frac{8}{4\nu-3} & \frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{6} & -\frac{4\nu-3}{8} & -\frac{4\nu-1}{12} & \frac{8}{\nu} \\ -\frac{1}{8} & \frac{12}{4\nu-1} & \frac{8}{4\nu-3} & \frac{12}{1} & \frac{2\nu-3}{8} & -\frac{4\nu-1}{6} & -\frac{4\nu-3}{8} & -\frac{1}{6} \\ \frac{\nu}{6} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & \frac{1}{8} & \frac{2\nu-3}{12} & -\frac{4\nu-1}{8} & -\frac{4\nu-3}{6} & -\frac{1}{8} \\ -\frac{6}{4\nu-1} & \frac{2\nu-3}{12} & \frac{12}{8} & \frac{8}{12} & \frac{12}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{8}{6} \end{pmatrix},
\end{aligned}$$

we can see that for the nondistorted elements, 2 Gauss points are sufficient to get the exact integral.