Analytic integration of 4-node element

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The material tensor (in Voigt notation) is given by

$$C = \begin{pmatrix} \frac{(1-\nu)\,E}{(1-2\,\nu)\,(\nu+1)} & \frac{\nu\,E}{(1-2\,\nu)\,(\nu+1)} & 0 \\ \frac{\nu\,E}{(1-2\,\nu)\,(\nu+1)} & \frac{(1-\nu)\,E}{(1-2\,\nu)\,(\nu+1)} & 0 \\ 0 & 0 & \frac{E}{2\,(\nu+1)} \end{pmatrix} \quad .$$

The interpolation functions are

$$N = \left[\frac{(r+1)\;(s+1)}{4}, \frac{(1-r)\;(s+1)}{4}, \frac{(1-r)\;(1-s)}{4}, \frac{(r+1)\;(1-s)}{4}\right] \quad ,$$

thus, the interpolation matrix is

$$H = \begin{pmatrix} \frac{(r+1)\,(s+1)}{4} & 0 & \frac{(1-r)\,(s+1)}{4} & 0 & \frac{(1-r)\,(1-s)}{4} & 0 & \frac{(r+1)\,(1-s)}{4} & 0 \\ 0 & \frac{(r+1)\,(s+1)}{4} & 0 & \frac{(1-r)\,(s+1)}{4} & 0 & \frac{(1-r)\,(1-s)}{4} & 0 & \frac{(r+1)\,(1-s)}{4} \end{pmatrix} \quad .$$

The mass matrix is computed as

$$M = \int\limits_{-1}^{1}\int\limits_{-1}^{1} \rho H^T H\, dr\, ds$$

$$M = \begin{pmatrix} \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 \\ 0 & \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} \\ \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} & 0 \\ 0 & \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} \\ \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 \\ 0 & \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} & 0 & \frac{2\,\rho}{9} \\ \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} & 0 \\ 0 & \frac{2\,\rho}{9} & 0 & \frac{\rho}{9} & 0 & \frac{2\,\rho}{9} & 0 & \frac{4\,\rho}{9} \end{pmatrix} \quad .$$

The matrix of displacement-to-strains (B in Bathe's book) is given by

$$B = \begin{pmatrix} \frac{s+1}{4} & 0 & -\frac{s+1}{4} & 0 & -\frac{1-s}{4} & 0 & \frac{1-s}{4} & 0 \\ 0 & \frac{r+1}{4} & 0 & \frac{1-r}{4} & 0 & -\frac{1-r}{4} & 0 & -\frac{r+1}{4} \\ \frac{r+1}{4} & \frac{s+1}{4} & \frac{1-r}{4} & -\frac{s+1}{4} & -\frac{1-r}{4} & -\frac{1-s}{4} & -\frac{r+1}{4} & \frac{1-s}{4} \end{pmatrix} \quad ,$$

the stiffness matrix is computed as

$$K = \int_{-1}^{1}\int_{-1}^{1} B^T C\, B\, dr\, ds$$

$$K = \frac{E}{(\nu+1)\,(2\,\nu-1)} \begin{pmatrix} \frac{4\,\nu-3}{6} & -\frac{1}{8} & -\frac{2\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{12} & \frac{1}{8} & -\frac{\nu}{6} & \frac{4\,\nu-1}{8} \\ -\frac{1}{8} & \frac{4\,\nu-3}{6} & \frac{4\,\nu-1}{8} & -\frac{\nu}{6} & \frac{1}{8} & -\frac{4\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{2\,\nu-3}{12} \\ -\frac{2\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{6} & \frac{1}{8} & -\frac{\nu}{6} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{12} & -\frac{1}{8} \\ -\frac{4\,\nu-1}{8} & -\frac{\nu}{6} & \frac{1}{8} & \frac{4\,\nu-3}{6} & \frac{4\,\nu-1}{8} & -\frac{2\,\nu-3}{12} & -\frac{1}{8} & -\frac{4\,\nu-3}{12} \\ -\frac{4\,\nu-3}{12} & \frac{1}{8} & -\frac{\nu}{6} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{6} & -\frac{1}{8} & -\frac{2\,\nu-3}{12} & -\frac{4\,\nu-1}{8} \\ \frac{1}{8} & -\frac{4\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{2\,\nu-3}{12} & -\frac{1}{8} & \frac{4\,\nu-3}{6} & \frac{4\,\nu-1}{8} & -\frac{\nu}{6} \\ -\frac{\nu}{6} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{12} & -\frac{1}{8} & -\frac{2\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{6} & \frac{1}{8} \\ \frac{4\,\nu-1}{8} & -\frac{2\,\nu-3}{12} & -\frac{1}{8} & -\frac{4\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{\nu}{6} & \frac{1}{8} & \frac{4\,\nu-3}{6} \end{pmatrix}.$$

If we take $E = 8/3$, $\nu = 1/3$ and $\rho = 1$, we get

$$M = \begin{pmatrix} \frac{4}{9} & 0 & \frac{2}{9} & 0 & \frac{1}{9} & 0 & \frac{2}{9} & 0 \\ 0 & \frac{4}{9} & 0 & \frac{2}{9} & 0 & \frac{1}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & 0 & \frac{4}{9} & 0 & \frac{2}{9} & 0 & \frac{1}{9} & 0 \\ 0 & \frac{2}{9} & 0 & \frac{4}{9} & 0 & \frac{2}{9} & 0 & \frac{1}{9} \\ \frac{1}{9} & 0 & \frac{2}{9} & 0 & \frac{4}{9} & 0 & \frac{2}{9} & 0 \\ 0 & \frac{1}{9} & 0 & \frac{2}{9} & 0 & \frac{4}{9} & 0 & \frac{2}{9} \\ \frac{2}{9} & 0 & \frac{1}{9} & 0 & \frac{2}{9} & 0 & \frac{4}{9} & 0 \\ 0 & \frac{2}{9} & 0 & \frac{1}{9} & 0 & \frac{2}{9} & 0 & \frac{4}{9} \end{pmatrix}$$

$$K = \begin{pmatrix} \frac{5}{3} & \frac{3}{4} & -\frac{7}{6} & \frac{1}{4} & -\frac{5}{6} & -\frac{3}{4} & \frac{1}{3} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{3} & -\frac{1}{4} & \frac{1}{3} & -\frac{3}{4} & -\frac{5}{6} & \frac{1}{4} & -\frac{7}{6} \\ -\frac{7}{6} & -\frac{1}{4} & \frac{5}{3} & -\frac{3}{4} & \frac{1}{3} & \frac{1}{4} & -\frac{5}{6} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{3} & -\frac{3}{4} & \frac{5}{3} & -\frac{1}{4} & -\frac{7}{6} & \frac{3}{4} & -\frac{5}{6} \\ -\frac{5}{6} & -\frac{3}{4} & \frac{1}{3} & -\frac{1}{4} & \frac{5}{3} & \frac{3}{4} & -\frac{7}{6} & \frac{1}{4} \\ -\frac{3}{4} & -\frac{5}{6} & \frac{1}{4} & -\frac{7}{6} & \frac{3}{4} & \frac{5}{3} & -\frac{1}{4} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} & -\frac{5}{6} & \frac{3}{4} & -\frac{7}{6} & -\frac{1}{4} & \frac{5}{3} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{7}{6} & \frac{3}{4} & -\frac{5}{6} & \frac{1}{4} & \frac{1}{3} & -\frac{3}{4} & \frac{5}{3} \end{pmatrix}.$$