Analytic and Gauss integration of Stiffness Matrix

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The analytic matrix is given by

$$K = \int_{-1}^{1}\int_{-1}^{1} B^T(r,s)C(r,s)B(r,s)\,dr\,ds$$

$$K = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix}
-\frac{4\nu-3}{6} & \frac{1}{8} & \frac{2\nu-3}{12} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\nu-1}{8}\\
\frac{1}{8} & -\frac{4\nu-3}{6} & -\frac{4\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{2\nu-3}{12}\\
\frac{2\nu-3}{12} & -\frac{4\nu-1}{8} & -\frac{4\nu-3}{6} & -\frac{1}{8} & \frac{\nu}{6} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & \frac{1}{8}\\
\frac{4\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\nu-3}{6} & -\frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{8} & \frac{4\nu-3}{12}\\
\frac{4\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\nu-1}{8} & -\frac{4\nu-3}{6} & \frac{1}{8} & \frac{2\nu-3}{12} & \frac{4\nu-1}{8}\\
-\frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{8} & -\frac{4\nu-3}{6} & -\frac{4\nu-1}{8} & \frac{\nu}{6}\\
\frac{\nu}{6} & \frac{4\nu-1}{8} & \frac{4\nu-3}{12} & \frac{1}{8} & \frac{2\nu-3}{12} & -\frac{4\nu-1}{8} & -\frac{4\nu-3}{6} & -\frac{1}{8}\\
-\frac{4\nu-1}{8} & \frac{2\nu-3}{12} & \frac{1}{8} & \frac{4\nu-3}{12} & \frac{4\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\nu-3}{6}
\end{pmatrix}$$

Numerically integrated matrices are in the form

$$K_N = \sum_i^N \sum_j^N w_i w_j B^T(r_i,s_j)C(r_i,s_j)B(r_i,s_j)\ \ ,$$

where i and j are the indexes to sweep along the r and s axis, r_i and s_j represent the Gauss coordinates, and N the number of points in each direction. For $N = 1$ and $N = 2$ we have

$$K_1 = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} -\frac{4\,\nu-3}{16} & \frac{1}{16} & -\frac{1}{16} & \frac{4\,\nu-1}{16} & \frac{4\,\nu-3}{16} & -\frac{1}{16} & \frac{1}{16} & -\frac{4\,\nu-1}{16} \\ \frac{1}{16} & -\frac{4\,\nu-3}{16} & -\frac{4\,\nu-1}{16} & \frac{1}{16} & -\frac{1}{16} & \frac{4\,\nu-3}{16} & \frac{4\,\nu-1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & -\frac{4\,\nu-1}{16} & -\frac{4\,\nu-3}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{4\,\nu-1}{16} & \frac{4\,\nu-3}{16} & \frac{1}{16} \\ \frac{4\,\nu-1}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\,\nu-3}{16} & -\frac{4\,\nu-1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{4\,\nu-3}{16} \\ \frac{4\,\nu-3}{16} & -\frac{1}{16} & \frac{1}{16} & -\frac{4\,\nu-1}{16} & -\frac{4\,\nu-3}{16} & \frac{1}{16} & -\frac{1}{16} & \frac{4\,\nu-1}{16} \\ -\frac{1}{16} & \frac{4\,\nu-3}{16} & \frac{4\,\nu-1}{16} & -\frac{1}{16} & \frac{1}{16} & -\frac{4\,\nu-3}{16} & -\frac{4\,\nu-1}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{4\,\nu-1}{16} & \frac{4\,\nu-3}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\,\nu-1}{16} & -\frac{4\,\nu-3}{16} & -\frac{1}{16} \\ -\frac{4\,\nu-1}{16} & -\frac{1}{16} & \frac{1}{16} & \frac{4\,\nu-3}{16} & \frac{4\,\nu-1}{16} & \frac{1}{16} & -\frac{1}{16} & -\frac{4\,\nu-3}{16} \end{pmatrix}$$

$$K_2 = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} -\frac{4\,\nu-3}{6} & \frac{1}{8} & \frac{2\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\,\nu-1}{8} \\ \frac{1}{8} & -\frac{4\,\nu-3}{6} & -\frac{4\,\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & \frac{4\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{2\,\nu-3}{12} \\ \frac{2\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{6} & -\frac{1}{8} & \frac{\nu}{6} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{12} & \frac{1}{8} \\ \frac{4\,\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\,\nu-3}{6} & -\frac{4\,\nu-1}{8} & \frac{2\,\nu-3}{12} & \frac{1}{8} & \frac{4\,\nu-3}{12} \\ \frac{4\,\nu-3}{12} & -\frac{1}{8} & \frac{\nu}{6} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{6} & \frac{1}{8} & \frac{2\,\nu-3}{12} & \frac{4\,\nu-1}{8} \\ -\frac{1}{8} & \frac{4\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{2\,\nu-3}{12} & \frac{1}{8} & -\frac{4\,\nu-3}{6} & -\frac{4\,\nu-1}{8} & \frac{\nu}{6} \\ \frac{\nu}{6} & \frac{4\,\nu-1}{8} & \frac{4\,\nu-3}{12} & \frac{1}{8} & \frac{2\,\nu-3}{12} & -\frac{4\,\nu-1}{8} & -\frac{4\,\nu-3}{6} & -\frac{1}{8} \\ -\frac{4\,\nu-1}{8} & \frac{2\,\nu-3}{12} & \frac{1}{8} & \frac{4\,\nu-3}{12} & \frac{4\,\nu-1}{8} & \frac{\nu}{6} & -\frac{1}{8} & -\frac{4\,\nu-3}{6} \end{pmatrix},$$

we can see that for the nondistorted elements, 2 Gauss points are sufficient to get the exact integral.