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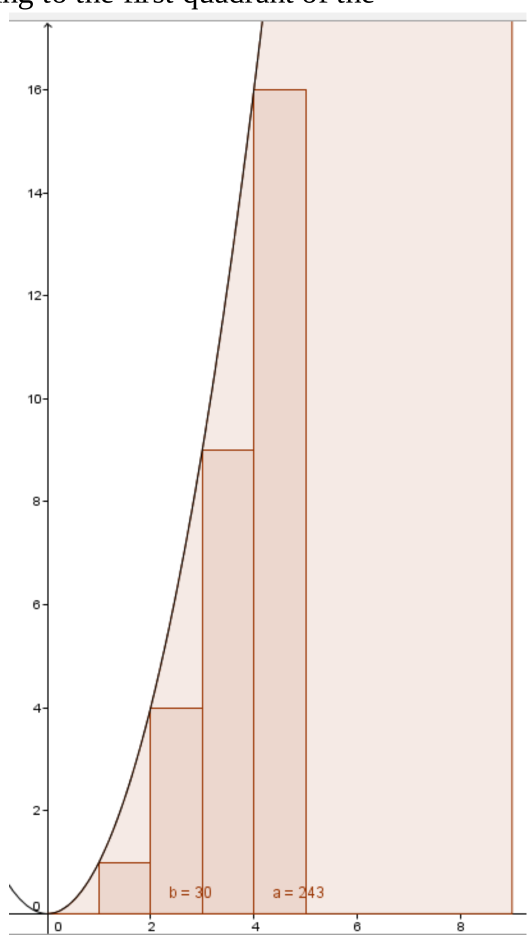
Approximations in Calculus

Legend says that basic calculus was discovered in a day by Sir Isaac Newton. While this legend is obviously untrue, the legend illustrates the simplicity of "discovering" calculus with basic integers.

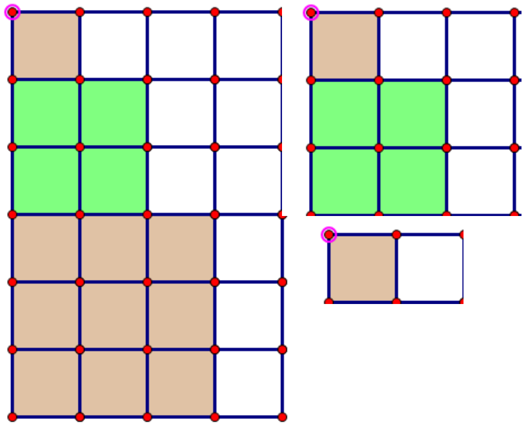
Originally, the very beginning of calculus was finding the slope of a curve at a specific point. This proves a challenge as two points are used to determine a line. Finding the slope given an x-coordinate and a function is what this question boils down to. To solve this, two points on the curve should be selected. As the first point, called point A, has a point B move closer to point A on a graphed function, the slope of the two points seems to converge on a number. The point where the two points become just point A will determine the slope of a single point. Consider the most basic curve on a Cartesian plane, $y=x^2$. Now imagine point A is at (3,9) on the line. For point B, select (6,36). The slope between A and B is now 9. For point B, select point (5,25) on the parabola. Now the slope between these two points would be 8. If point B was at (4,16), the slope would be 7. The slope only decreases by 1 as the x coordinate decreases by 1 on the parabola. However, no integer exists between a 3 and 4, so one might start approximating with decimals. Setting point B as (3.5, 12.25) , the slope is now 6.5. If point B is (3.01, 9.0601). the slope is 6.01. Someone could assume that the slope is 6. He or she would not be wrong. The slope of any point on $y=x^2$ is actually twice the value of x.

Actually proving that the slope of a coordinate on $y=x^2$ is similar to the process above, just with algebra. The slope of point A and B of these points is generally represented by the slope formula $\frac{\Delta y}{\Delta x}$ or the change in y over the change in x. Give point A the coordinates (x,y), then coordinates of point B must be (x+Δx , y +Δy). Since y= x^2, it can also be said that the y coordinate of point A is (x, x^2) and (x+Δx , $(x+\Delta x)^2$). Substitute these values into the slope formula, and the result is $\frac{(x+\Delta x)^2 \ -x^2}{\Delta x}$. Expanding the numerator results in $\frac{x^2+2x\Delta x+\ x\Delta^2-x^2}{\Delta x}$, which simplifies to $\frac{2x\Delta x+\ x\Delta^2}{\Delta x}$, and then simplifies further into the slope being $2x + \ x\Delta$. As with the problem in the paragraph above, the slope of a single point means that there is no change in the x coordinate. If xΔ = 0, then the slope of a point is simply 2x + 0, or just 2x.

While, Newton started with the problem of derivatives, Gottfried Wilhelm Leibniz was independently working on the concept of integrals. The basic problem of integrals is the area of the space under a curved function, generally sticking to the first quadrant of the Cartesian plane. Approximating the area under y=x^2 can be done with rectangles with a width of 1. The first rectangle has a width of 1, and a height of 1. The second rectangle is at an x coordinate of 2, and thus has a width of 1 but a height of 2^2=4. For the third rectangle at an x coordinate, the height is 3^2=9. The height of a rectangle multiplied by its width would mean that the area of each rectangle is its x-coordinate squared. As a result, the approximate area under the curve can be found as consecutive sum of squares. The sum of squares can be represented by $\sum_{n=1}^{\infty} n^2$ or $1^2+2^2+3^2+4^2+5^2$...

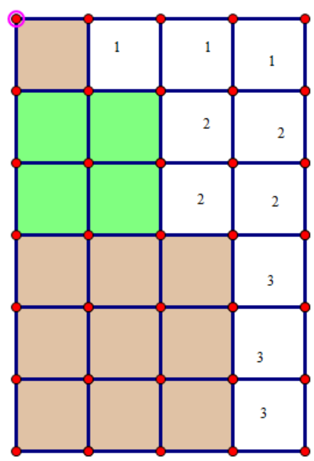


Finding the sum of squares requires understanding that the sum of a sequence of integers starting from 1 to n is $\frac{n(n+1)}{2}$. Finding a formula for the sum of squares is slightly more



complicated than this. Generally, using three dimensional graphics to derive the sum of squares is used, but there is a much simpler way.

If one draws squares with the next square placed under it with a side length equal to the term length in the sequence, the colored squares in the picture above are created. If a rectangle is created where the width is one more than the side length of the largest square and with a length equal to the combined lengths of all squares, then the rectangle's area can be represented as (n+1)(the sum of all positive integers to the n^{th} term).



Since the sum of all positive integers up to the n^{th} term is $\frac{n(n+1)}{2}$, the area of the rectangle is (n+1) ($\frac{n(n+1)}{2}$). Logically speaking, subtracting the amount of uncolored squares from the area of the rectangle should yield the sum of squares, meaning $\sum_{n=1}^{\infty} n^2$ = the uncolored area subtracted from (n+1) ($\frac{n(n+1)}{2}$). In the picture on the right, the uncolored area is composed of sum of positive integers. One could say that the uncolored area is (1+2+3)+(1+2)+1 and the area of the rectangle is (3+1)(1+2+3).

This makes the sum of squares up to the third term. (3+1)(1+2+3) - (1+2+3)+(1+2)+1, which equals 14. This way of calculating the sum of squares is certainly helpful since the sum of positive integers can be substituted in. The problem here is using the sum of positive integers to express the area of the uncolored area. The uncolored area is the sum of the sum of positive integers to the first term + the sum of positive integers to the second term all the way to the sum of positive integers of the n^{th} term. This can be represented as $\sum_{i=1}^{n}(\sum_{j=1}^{n} i)$. Overall, the sum of squares can be represented by $\sum_{n=1}^{n} n^2$=(n+1) ($\frac{n(n+1)}{2}$)- $\sum_{i=1}^{n}(\sum_{i=1}^{n} i)$.

If we substitute into the righter most sigma notation the formula for the sum of positive integers, then the equation is now $\sum_{n=1}^{n} n^2$=(n+1) ($\frac{n(n+1)}{2}$)- $\sum_{i=1}^{n}(\frac{i(i+1)}{2})$. Say we multiply both sides of the equation by 2 to get rid of the fractions, the result is $2(\sum_{n=1}^{\infty} n^2)$=(n+1) $(n(n+1))$- $\sum_{i=1}^{n}(i(i+1))$. Now, here is the creative part. If we multiply out the last two terms, the result is $2(\sum_{n=1}^{\infty} n^2)$=(n+1) $(n(n+1))$- $\sum_{i=1}^{n}(i^2+i)$. $\sum_{i=1}^{n}(i^2+i)$ is another way of saying that the sum of consecutive positive integers squared plus the sum of consecutive positive integers. If we substitute in the summation notation for the sum of squares and the sum of consecutive positive integers formula, the result is $2(\sum_{n=1}^{\infty} n^2)$=(n+1) $(n(n+1))$ - $\sum_{n=1}^{\infty} n^2$ - $\frac{n(n+1)}{2}$. If we add $\sum_{n=1}^{\infty} n^2$ to both sides, the result is $3(\sum_{n=1}^{\infty} n^2)$=(n+1) $(n(n+1))$ - $\frac{n(n+1)}{2}$. Now we multiply out the right sides of the equation to get $3(\sum_{n=1}^{\infty} n^2)$= n^3+2n^2+n - $\frac{n(n+1)}{2}$. This all boils down to $3(\sum_{n=1}^{\infty} n^2)=n^3+\frac{3n^2}{2}+\frac{n}{2}$. Dividing both sides by 3 yields the sum of consecutive positive squares, $(\sum_{n=1}^{\infty} n^2) = \frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$. This simplifies to the more popular format of $(\sum_{n=1}^{\infty} n^2) = \frac{2n^3+3n^2+n}{6}$.

http://tutorial.math.lamar.edu/Classes/CalcII/ApproximatingDefIntegrals.aspx

Approximating Definite Integrals

In this chapter we've spent quite a bit of time on computing the values of integrals. However, not all integrals can be computed. A perfect example is the following definite integral.

$$\int_0^2 \mathbf{e}^{x^2}\,dx \qquad \int_0^2 \mathbf{e}^{x^2}\,dx$$

We now need to talk a little bit about estimating values of definite integrals. We will look at three different methods, although one should already be familiar to you from your Calculus I days. We will develop all three methods for estimating

$$\int_a^b f(x)\,dx \qquad \int_a^b f(x)\,dx$$

by thinking of the integral as an area problem and using known shapes to estimate the area under the curve.

Let's get first develop the methods and then we'll try to estimate the integral shown above.

Midpoint Rule

This is the rule that should be somewhat familiar to you. We will divide the interval $[a,b]$ $[a,b]$ into *n* subintervals of equal width,

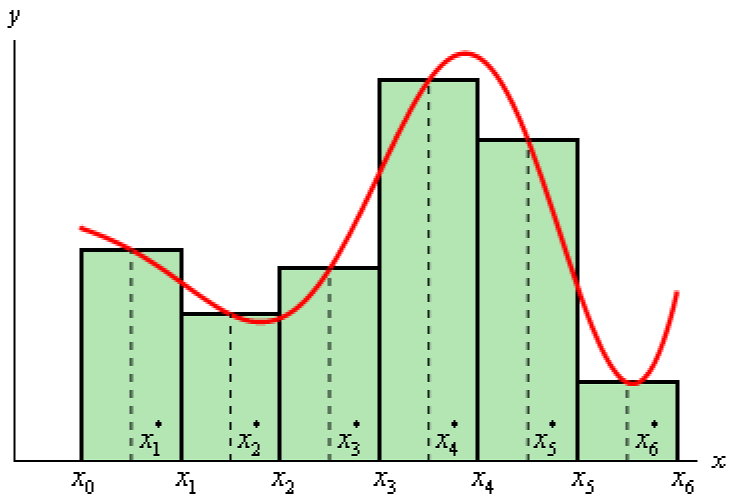
$$\Delta x = \frac{b-a}{n} \qquad \Delta x = \frac{b-a}{n}$$

We will denote each of the intervals as follows,

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$$

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \quad \text{where } x_0 = a$$

Then for each interval let x_i^* be the midpoint of the interval. We then sketch in rectangles for each subinterval with a height of $f\left(x_i^*\right)$. Here is a graph showing the set up using $n = 6$.



We can easily find the area for each of these rectangles and so for a general *n* we get that,

$$\int_a^b f(x)\,dx \approx \Delta x\, f\left(x_1^*\right) + \Delta x\, f\left(x_2^*\right) + \cdots +$$

Or, upon factoring out a Δx we get the general Midpoint Rule.

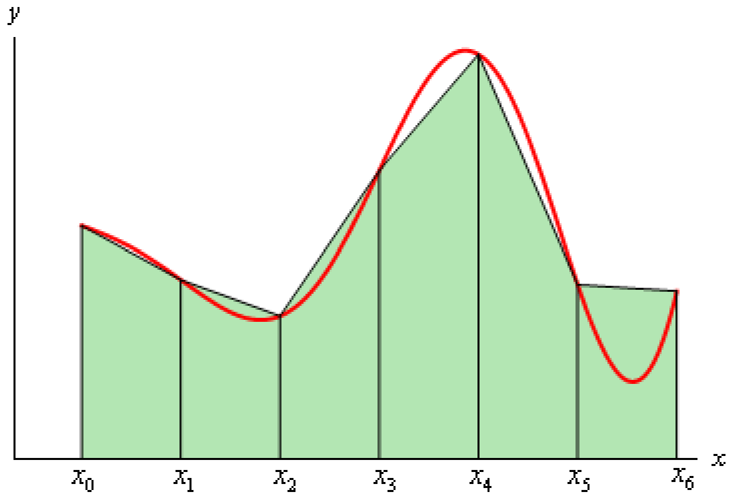
$$\boxed{\int_a^b f(x)\,dx \approx \Delta x\left[f\left(x_1^*\right) + f\left(x_2^*\right) + \cdots + \right.}$$

Trapezoid Rule

For this rule we will do the same set up as for the Midpoint Rule. We will break up the interval $\left[a,b\right]$ into *n* subintervals of width,

$$\Delta x = \frac{b-a}{n}$$

Then on each subinterval we will approximate the function with a straight line that is equal to the function values at either endpoint of the interval. Here is a sketch of this case for $n=6$ $n=6$.



Each of these objects is a trapezoid (hence the rule's name…) and as we can see some of them do a very good job of approximating the actual area under the curve and others don't do such a good job.

The area of the trapezoid in the interval $\left[x_{i-1}, x_i\right]$ $\left[x_{i-1}, x_i\right]$ is given by,

$$A_i = \frac{\Delta x}{2}\left(f\left(x_{i-1}\right)+f\left(x_i\right)\right)$$

$$A_i = \frac{\Delta x}{2}\left(f\left(x_{i-1}\right)+f\left(x_i\right)\right)$$

So, if we use *n* subintervals the integral is approximately,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f\left(x_0\right)+f\left(x_1\right)\right)+\frac{\Delta x}{2}$$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left(f\left(x_0\right)+f\left(x_1\right)\right)+\frac{\Delta x}{2}\left(f\left(x_1\right)+f\left(x_2\right)\right)+$$

Upon doing a little simplification we arrive at the general Trapezoid Rule.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f\left(x_0\right)+2f\left(x_1\right.\right.}$$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0)+2f(x_1)+2f(x_2)+\cdots+$$

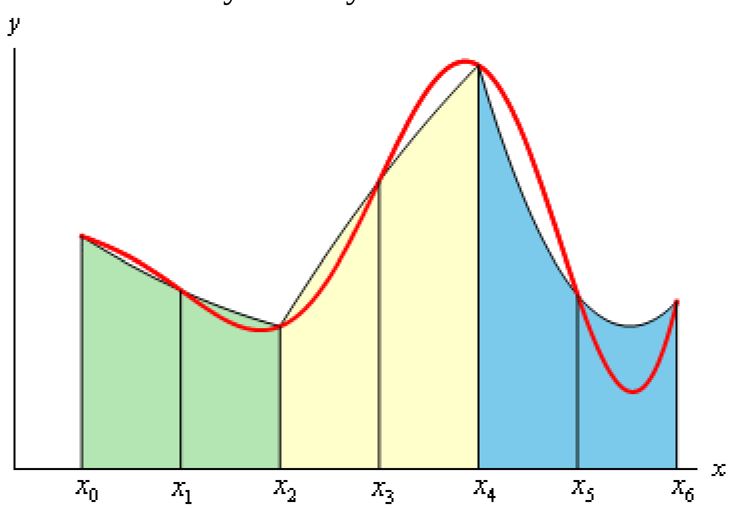
Note that all the function evaluations, with the exception of the first and last, are multiplied by 2.

Simpson's Rule

This is the final method we're going to take a look at and in this case we will again divide up the interval $\left[a,b\right]$ $[a,b]$ into n subintervals. However unlike the previous two methods we need to require that n be even. The reason for this will be evident in a bit. The width of each subinterval is,

$$\Delta x = \frac{b-a}{n} \qquad \Delta x = \frac{b-a}{n}$$

In the Trapezoid Rule we approximated the curve with a straight line. For Simpson's Rule we are going to approximate the function with a quadratic and we're going to require that the quadratic agree with three of the points from our subintervals. Below is a sketch of this using $n=6$ $n=6$. Each of the approximations is colored differently so we can see how they actually work.



Notice that each approximation actually covers two of the subintervals. This is the reason for requiring n to be even. Some of the approximations look more like a line than a quadratic, but they really are quadratics. Also note that some of the approximations do a better job than others. It can be shown that the area under the approximation on the intervals $[x_{i-1}, x_i]$ $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ $[x_i, x_{i+1}]$ is,

$$A_i = \frac{\Delta x}{3}\left(f\left(x_{i-1}\right) + 4f\left(x_i\right) \right)$$

$$A_i = \frac{\Delta x}{3}\left(f\left(x_{i-1}\right) + 4f\left(x_i\right) + f\left(x_{i+1}\right) \right)$$

If we use *n* subintervals the integral is then approximately,

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f\left(x_0\right) + 4f\left(x_1\right) + f\left(\right.\right.$$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(f\left(x_0\right) + 4f\left(x_1\right) + f\left(x_2\right) \right) + \frac{\Delta x}{3}\left(f\left(x_2\right) + \right.$$

$$+ \cdots + \frac{\Delta x}{3}\left(f \right.$$

Upon simplifying we arrive at the general Simpson’s Rule.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + 2 \right.$$

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f\left(x_0\right) + 4f\left(x_1\right) + 2f\left(x_2\right) + \cdots + 2f\left(x_n\right.\right.$$

In this case notice that all the function evaluations at points with odd subscripts are multiplied by 4 and all the function evaluations at points with even subscripts (except for the first and last) are multiplied by 2. If you can remember this, this is a fairly easy rule to remember.

Okay, it's time to work an example and see how these rules work.

Example 1 Using $n=4$ $n=4$ and all three rules to approximate the value of the following integral.

$$\int_0^2 \mathbf{e}^{x^2}\,dx$$

$$\int_0^2 \mathbf{e}^{x^2}\,dx$$

Solution

First, for reference purposes, Maple gives the following value for this integral.

$$\int_0^2 \mathbf{e}^{x^2}\,dx = 16.45262776$$

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In each case the width of the subintervals will be,

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

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and so the subintervals will be,

$$[0,\ 0.5],\ [0.5,\ 1],\ [1,\ 1.5],\ [1.5,\ 2]$$

$$[0,\ 0.5],\ [0.5,\ 1],\ [1,\ 1.5],\ [1.5,\ 2]$$

Let's go through each of the methods.

Midpoint Rule

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1}{2}\left(\mathbf{e}^{(0.25)^2} + \mathbf{e}^{(0.75)^2} + \mathbf{e}^{(1.25)^2} + \mathbf{e}^{(1.75)^2}\right) = 14.48561253$$

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1}{2}\left(\mathbf{e}^{(0.25)^2} + \mathbf{e}^{(0.75)^2} + \mathbf{e}^{(1.25)^2} + \mathbf{e}^{(1.75)^2}\right) = 14.48561253$$

Remember that we evaluate at the midpoints of each of the subintervals here! The Midpoint Rule has an error of 1.96701523.

Trapezoid Rule

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1/2}{2}\left(\mathbf{e}^{(0)^2} + 2\mathbf{e}^{(0.5)^2} + 2\mathbf{e}^{(1)^2} + 2\mathbf{e}^{(1.5)^2} + \mathbf{e}^{(2)^2}\right) = 20.64455905$$

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1/2}{2}\left(\mathbf{e}^{(0)^2} + 2\mathbf{e}^{(0.5)^2} + 2\mathbf{e}^{(1)^2} + 2\mathbf{e}^{(1.5)^2} + \mathbf{e}^{(2)^2}\right) = 20.64455905$$

The Trapezoid Rule has an error of 4.19193129

Simpson's Rule

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1/2}{3}\left(\mathbf{e}^{(0)^2} + 4\mathbf{e}^{(0.5)^2} + 2\mathbf{e}^{(1)^2} + 4\mathbf{e}^{(1.5)^2} + \mathbf{e}^{(2)^2}\right) = 17.35362645$$

$$\int_0^2 \mathbf{e}^{x^2}\,dx \approx \frac{1/2}{3}\left(\mathbf{e}^{(0)^2} + 4\mathbf{e}^{(0.5)^2} + 2\mathbf{e}^{(1)^2} + 4\mathbf{e}^{(1.5)^2} + \mathbf{e}^{(2)^2}\right) = 17.35362645$$

The Simpson's Rule has an error of 0.90099869.

None of the estimations in the previous example are all that good. The best approximation in this case is from the Simpson's Rule and yet it still had an error of almost 1. To get a better estimation we would need to use a larger *n*. So, for completeness sake here are the estimates for some larger value of *n*.

	Midpoint		Trapezoid		Simpson's	
n	Approx.	Error	Approx.	Error	Approx.	Error
8	15.9056767	0.5469511	17.5650858	1.1124580	16.5385947	0.0859669
16	16.3118539	0.1407739	16.7353812	0.2827535	16.4588131	0.0061853
32	16.4171709	0.0354568	16.5236176	0.0709898	16.4530297	0.0004019
64	16.4437469	0.0088809	16.4703942	0.0177665	16.4526531	0.0000254
128	16.4504065	0.0022212	16.4570706	0.0044428	16.4526294	0.0000016

In this case we were able to determine the error for each estimate because we could get our hands on the exact value. Often this won't be the case and so we'd next like to look at error bounds for each estimate.

These bounds will give the largest possible error in the estimate, but it should also be pointed out that the actual error may be significantly smaller than the bound. The bound is only there so we can say that we know the actual error will be less than the bound.

So, suppose that $\left|f''(x)\right| \le K$ $\left|f''(x)\right| \le K$ and $\left|f^{(4)}(x)\right| \le M$ $\left|f^{(4)}(x)\right| \le M$ for $a \le x \le b$ $a \le x \le b$ then if E_M, E_T, and E_S are the actual errors for the Midpoint, Trapezoid and Simpson's Rule we have the following bounds,

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad |E_T| \le \frac{K}{}$$

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \qquad |E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |$$

Example 2 Determine the error bounds for the estimations in the last example.

Solution

We already know that $n = 4$ $n = 4$, $a = 0$ $a = 0$, and $b = 2$ $b = 2$ so we just need to compute *K* (the largest value of the second derivative) and *M* (the largest value of the fourth derivative). This means that we'll need the second and fourth derivative of *f(x)*.

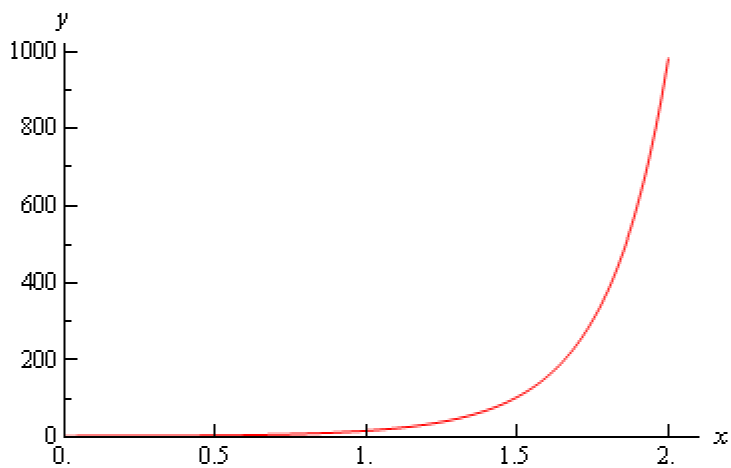
$$f''(x) = 2\mathbf{e}^{x^2}\left(1 + 2x^2\right)$$

$$f^{(4)}(x) = 4\mathbf{e}^{x^2}\left(3 + 12x^2 + 4x^4\right)$$

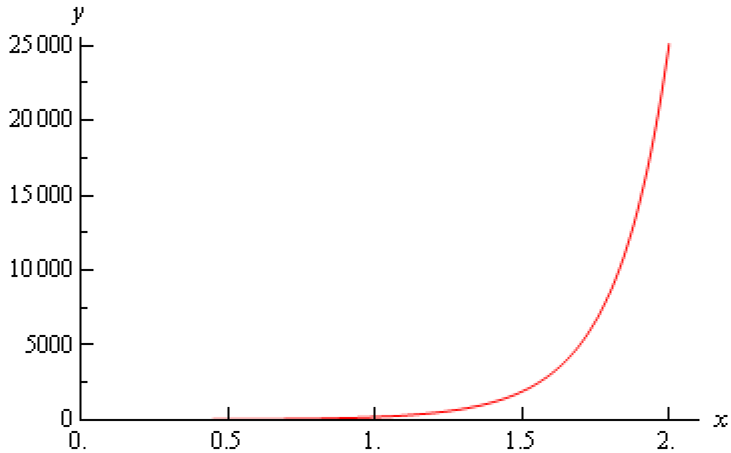
$$f''(x) = 2\mathbf{e}^{x^2}\left(1 + 2x^2\right)$$

$$f^{(4)}(x) = 4\mathbf{e}^{x^2}\left(3 + 12x^2 + 4x^4\right)$$

Here is a graph of the second derivative.



Here is a graph of the fourth derivative.



So, from these graphs it's clear that the largest value of both of these are at $x=2$ $x=2$. So,

$$f''(2) = 982.76667 \quad \Rightarrow \quad K = 983$$

$$f^{(4)}(2) = 25115.14901 \quad \Rightarrow \quad M = 25116$$

$$f''(2) = 982.76667 \quad \Rightarrow \quad K = 983$$

$$f^{(4)}(2) = 25115.14901 \quad \Rightarrow \quad M = 25116$$

We rounded to make the computations simpler.

Here are the bounds for each rule.

$$|E_M| \le \frac{983(2-0)^3}{24(4)^2} = 20.4791666667$$

$$|E_M| \le \frac{983(2-0)^3}{24(4)^2} = 20.4791666667$$

$$|E_T| \le \frac{983(2-0)^3}{12(4)^2} = 40.9583333333$$

$$|E_T| \le \frac{983(2-0)^3}{12(4)^2} = 40.9583333333$$

$$|E_S| \le \frac{25116(2-0)^5}{180(4)^4} = 17.441666667$$

$$|E_S| \le \frac{25116(2-0)^5}{180(4)^4} = 17.4416666667$$

In each case we can see that the errors are significantly smaller than the actual bounds.