$\square$ represents any statement where it appears

(Universal elimination)

// means replaces y

$\forall$ Elim

$(\forall x)\ \boxed{x}$

$\therefore\ \boxed{C // x}$

constant

for every x where $\boxed{x}$ is true, C is true

$(\forall x)\ \boxed{x}$

$\therefore\ \boxed{y // x}$

for every x where $\boxed{x}$ is true, y is true

all real #'s have cube roots

$\therefore$ -3 has a cube root

(existential introduction)

$\exists$ Intro

$\boxed{C}$

$\therefore\ (\exists x)\ \boxed{x // C}$

2 is prime

$\therefore$ there exists a prime (there exists an x, such that x is a prime)

$\forall$ Intro:

$\exists$ elimination:

$(\exists x)\ \boxed{x}$

$\therefore\ \boxed{y // x}$

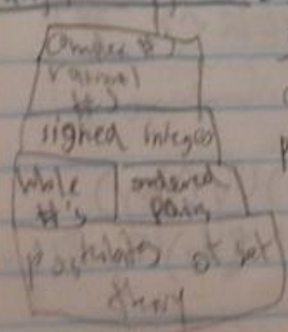
provided y is completely new to the proof

9/13/12

Aim: Foundations (continued) - The mathematical universe (sets, Numbers, Lists)

"Purist" approach: * set theory is axiomatized

define all other kinds of abstract objects in terms sets - prove their properties based on properties of sets



$0 := \emptyset$ empty set

$1 := \{0\}$

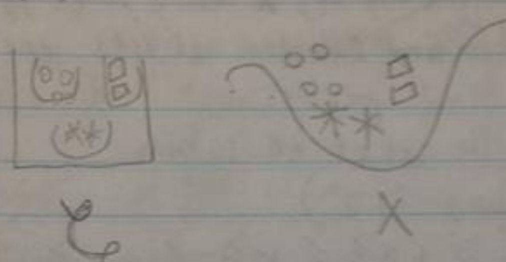
$2 := \{0, 1\}$

$3 := \{0, 1, 2\}$

$(\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\})$

Jonathan Quing

whose members are exactly those objects belonging to sets in $\mathscr{C}$



The unique set X is designated by $X = \bigcup A \quad A \in \mathscr{C}$
the union of all sets A, such that A belongs to collection $\mathscr{C}$

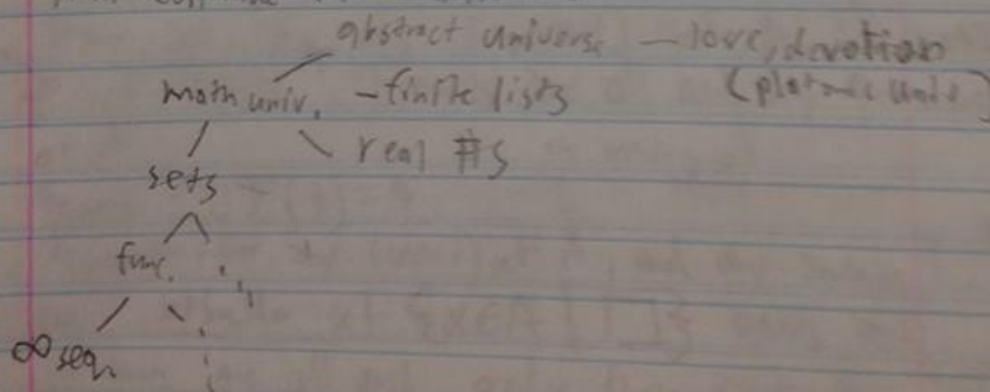
5) Postulate of Separation

$Y = \{ x \in X \mid \ldots x \ldots \}$

For any set X, there is a set Y consisting of 'just' those elem. of X satisfying a given property

9/19/16

Aim: Continue with foundations



$\{x \in \mathbb{R} \mid 1=1\} = \mathbb{R}$

$\{x \in \mathbb{R} \mid 1=0\}\ \emptyset$

$\{x \in \mathbb{R} \mid x^2 = x\} = \{0, 1\}$

$\{x \in \mathbb{R} \mid x \in \mathbb{N}\} = \mathbb{N}$

$\{x \in \mathbb{R} \mid x \notin \mathbb{Q}\} = \mathbb{R} \setminus \mathbb{Q}$

[illegible] free, [illegible] not free

$\{x \in \mathbb{R} \mid (\exists y)(y \in \mathbb{Z} \wedge x = 2y)\}$ = all even integers — true substitution

$\{y \in \mathbb{R} \mid x = 2y\}$ does not denote any set

$\{x\ \mathbb{R} \mid (\exists y)(x = 2y)\} = \mathbb{R}$

$(\forall y)$ $= \emptyset$

9/21/16

$U \neq \emptyset$

$A \subseteq U$

$A^c :\!= U \setminus A$

$:=$ means to define equal

$(x < y) :\Leftrightarrow P(y - x)$

$A \setminus \emptyset = A$ $\quad A \setminus U = \emptyset$

$\emptyset \setminus A = \emptyset$ $\quad U \setminus A = A^c$

$A \cup (A \cap B) = A$ Absorption laws

$A \cap (A \cup B) = A$

The dual of a formula is one where all $\cup$ and $\cap$ are swapped

De Morgan's Laws, $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

$[A \cap (B \cup C^c)^c]^c = A^c \cup \{B \cup C^c\}^{cc} = A^c \cup (B \cup C^c) = \boxed{A^c \cup B \cup C^c}$

Aim! Postulates of numbers

primitive predicates: $Nx :\!\cdot$ (x is a number)

$Px :\!\cdot$ x is prime

9/28/16

Ex: Prove $(\forall n)\left[n \in \mathbb{N} \Rightarrow \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}\right]$

Proof:

Base case $(n=1)$: $\sum_{k=1}^{1} \frac{1}{k(k+1)} \stackrel{?}{=} \frac{1}{1+1}$

The left hand side (LHS) is $\frac{1}{1(1+1)} = \frac{1}{2}$, the RHS is $\frac{1}{1+1} = \frac{1}{2}$, $\frac{1}{2}$

these are equal

Induction step: We will assume that $n \in \mathbb{N}$ and $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

From this, we must derive that ~~the sum~~ $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} \stackrel{?}{=} \frac{n+1}{n+2}$

Note that the $\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \left(\sum_{k=1}^{n} \frac{1}{k(k+1)}\right) + \frac{1}{(n+1)(n+2)}$

$$= \left(\frac{n}{n+1}\right) + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)}$$

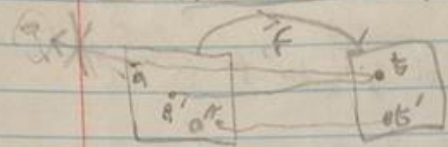
$$= \frac{n^2+2n+1}{n^2+3n+2}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

This completes the induction step, and with it, the proof. $\square$

10/13/16 Jonathan Quang

Aim: Some Generality About Functions



$A \neq \emptyset$ B

$\boxed{f: A \to B}$ f is a function from A to B

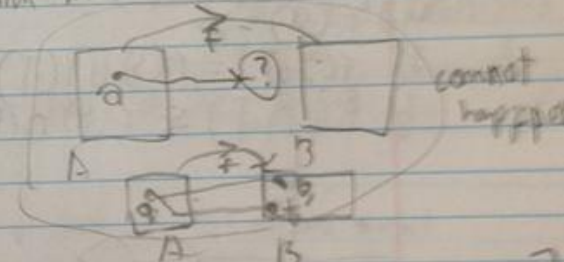
$(a, b) \in f$

$(a', b) \in f$

$(a'', b') \in f$

$(a, b') \in f$

1) Every $a \in A$ gets paired with some element of B
2) an element $a \in A$ cannot be paired with 2 or more distinct elements of B.



Functionality condition: $(\forall a)[a \in A \Rightarrow (\exists! b)(b \in B \land (a, b) \in f)]$

Def. Function Notation: given $a \in A$, we write $f(a)$ for the unique $b \in B$ such that $(a, b) \in f$.

Ex. $f: \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$ for any $x \in \mathbb{R}$
$-1 \in B = \mathbb{R}$
but
$-1 \neq f(x)$
$-1 \notin \underset{\text{range}}{\mathrm{ran}}(f)$

Vertical lines

$(b = 0 \Rightarrow a \neq 0)$

$ax + c = 0$

$\Rightarrow x = -\frac{c}{a} = k$

$\boxed{x = k}$ $(k \in \mathbb{R})$

k is unique

k is known as the x-intercept of L

Non vertical lines

$(b \neq 0)$

$y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right)$

$\boxed{y = mx + d}$

m, d are unique

m: slope

d: y-intercept

10/6/16

Aim: General [incl. Point-Point], Vertical, Slope-Intercept, Point-Slope, intercept-intercept, parametric

$\boxed{\text{General form: } \boxed{ax + by + c = 0 \quad (a^2 + b^2 > 0)}}$

- Every line has at least one such equation, by definition of "line"
  - "L has the equation $ax + by + c = 0$" means $L = \{(x, y) \mid ax + by + c = 0\}$
  - "C has the equation $x^2 + y^2 = 1$" means $C = \{(x, y) \mid x^2 + y^2 = 1\}$ (the unit circle)
- Every line has ∞ many such eqn's, but if one of them is $ax + by + c = 0$, then all the others are in the form $(\lambda a)x + (\lambda b)y + (\lambda c) = 0$ for any $\lambda \neq 0$
- The numbers a, b, c are not unique for this line, but $\frac{a}{b}, \frac{a}{c}, \frac{b}{c}, \ldots$, $\frac{b}{a}, \frac{c}{a}, \frac{c}{b}$ are unique (valued in $\mathbb{R} \cup \{?\}$ ← a non-real number)

$\boxed{\text{Point-Point form}}$

$(x_0, y_0) \neq (x_1, y_1) \Rightarrow$

$\langle (x_0, y_0), (x_1, y_1) \rangle = \{(x, y) \mid (y_1 - y_0)x - (x_1 - x_0)y + (x_0 y_1 - x_1 y_0) = 0\}$

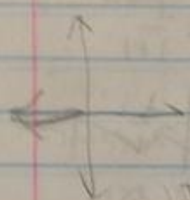
$\neq 0$

$x_0 y_1 - y_0 x_1$ = discriminant

A tm: Equation of a line

Def: 1) The plane, denoted by $\mathbb{R}^2$, is the set
$\{(x,y) \mid x \in \mathbb{R} \wedge y \in \mathbb{R}\} = \{\vec{u} \in \mathcal{L}(\mathbb{R}) \mid \#\vec{u} = 2\}$

the set of all positive length with real entry



The elements of $\mathbb{R}^2$ are called points

2) A line in $\mathbb{R}^2$ is a set of the form $L_{a,b,c} := \{x,y \mid ax+by+c=0\}$
Where, $a, b, c \in \mathbb{R}$ such that $a^2+b^2>0$ (i.e., $a \neq 0$ or $b \neq 0$)
[$L_{0,0,c}$ = undefined]

Theorem: $L_{a,b,c} = L_{ta,tb,tc}$ provided $a^2+b^2>0$
and $t \neq 0$

Theorem: If $\vec{p}, \vec{q} \in \mathbb{R}^2$ such that $\vec{p} \neq \vec{q}$ then
$\exists !$ line L such that $\vec{p}, \vec{q} \in L$. We
call it "$\langle \vec{p}, \vec{q} \rangle$".

9/29/16

Am: System of 2 first degree Equations in two Variables

Given $p, q, r, s, t, u \in \mathbb{R}$, Consider the system of equations (i.e. conjunction)

$$* \begin{cases} px+qy=t \\ rx+sy=u \end{cases} \qquad [\text{i.e., } (px+qy=t) \wedge (rx+sy=u)]$$

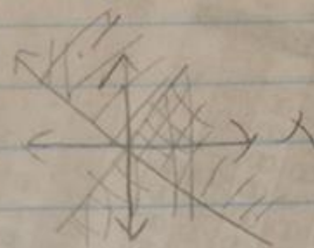
1) Suppose $ps \neq qr$, There is exactly one pair $(x,y) \in \mathbb{R}^2$ satisfying (*), In fact $(x,y) = \left(\frac{ts-qu}{ps-qr}, \frac{pu-tr}{ps-qr}\right)$

transformation:

$\vec{F}(x,y) := (\sqrt{x+y}, \sqrt{x-y})$

$x+y \geq 0 \quad y \geq -x$

$x-y \geq 0 \quad y \leq x$



limit attention to total transformations:

$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$

limit attention to one-to-one total transformations

$(x,y) \neq (\tilde{x},\tilde{y}) \Rightarrow \vec{F}(x,y) \neq \vec{F}(\tilde{x},\tilde{y})$

$$\vec{F}(x,y) = \begin{cases} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0) \end{cases}$$

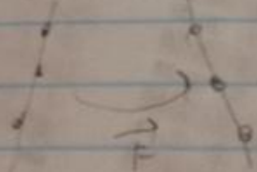
$(1,1) \mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$(2,2) \mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

limit attention to one-to-one, total, exhaustive transformation

$\forall (a,b) ; \exists (x,y) \in \mathbb{R}^2 : \vec{F}($

collinearity-perserving



$\vec{F}(x,y) = (ax+by+p, cx+dy+q)$

affine transformations

(Mazur-Ulam Thm)