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HW#2

1) Prove $(\forall n)[n \in \mathbb{N} \Rightarrow 1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\dots+n\cdot 2^{n-1} = (n-1)2^n+1]$

Proof:

Base Case (n=1): $1\cdot 2^{1-1} \overset{?}{=} (1-1)2^1+1$

The LHS is $1\cdot 2^{1-1} = 1$. $\quad 1\cdot 2^0 \quad | \quad (0)(2)+1$

The RHS is $(1-1)(2^1)+1 = 1$. $\quad 1\cdot 1 \quad | \quad 0+1$

These are equal $\quad 1 \Leftrightarrow 1$

Induction Step: We will assume that $n \in \mathbb{N}$ and $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\dots+n\cdot 2^{n-1} =$ $(n-1)2^n+1$. Note that $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\dots+n\cdot 2^{n-1} = \sum_{k=1}^{n} k\cdot 2^{k-1}$.

So part of the assumption becomes $\sum_{k=1}^{n} k\cdot 2^{k-1} = (n-1)2^n+1$.

From this, we derive $\sum_{k=1}^{n+1} k\cdot 2^{k-1} \overset{?}{=} (n)2^{n+1}+1$

Note that $\sum_{k=1}^{n+1} k\cdot 2^{k-1} = \sum_{k=1}^{n} k\cdot 2^{k-1} + (n+1)(2^n)$

$= (n-1)(2^n)+1+n(2^n)$

$= (n-1)(2^n)+n(2^n)+1$

$= (n-1)2^n+1$

This completes the induction step, and with it, the proof. QED

19. Prove $\forall n \, [n \in \mathbb{N} \Rightarrow n \leq 2^{n-1}]$

Proof:

Base Case ($n=1$): $1 \overset{?}{\leq} 2^{1-1}$

The LHS is 1. The RHS is also 1. $1 \leq 1$.

$$\begin{array}{c|c} 1 & 2^0 \\ 1 & 1 \\ \end{array}$$

$1 \leq 1$ ✓

Induction Step: Assume $n \in \mathbb{N}$ and $n \leq 2^{n-1}$

From this, we derive that $n+1 \leq 2^n$

$$n \leq n+1 \leq 2^n$$

$$n \neq 1 \Rightarrow \frac{2}{n} \leq n \leq 2^n$$

$$\frac{1}{2}\left(\frac{2}{n}\right) \leq (2^n)\frac{1}{2}$$

$$n \leq 2^{n-1}$$

This concludes the induction step and with it the proof. □

22. Prove $(\forall n)[n \in \mathbb{N} \implies 3 \mid n^3+2n]$

Proof:

Base case $(n=1)$: $3 \overset{?}{\mid} 1^3+2(1)$

The RHS $1^3+2(1)$ is 3. 3 is a factor of 3

$3 \mid 1+2$

$3 \mid 3$ ✓

Induction step: Assume that $n \in \mathbb{N}$ and $3 \mid n^3+2n$

From this, we can derive that $3 \mid (n+1)^3+2(n+1)$

$$(n+1)^3+2(n+1) = n^3+3n^2+3n+1+2n+2$$
$$= n^3+3n^2+3n+3+2n$$
$$= \underbrace{n^3+2n}_{3l \text{ for some } l \in \mathbb{Z}} + 3n^2+3n+3$$
$$= 3l+3n^2+3n+3$$
$$= 3(\underbrace{l+n^2+n+1}_{k})$$
$$= 3k$$

k is the sum of integers and natural numbers. k will be an integer. If k is an integer, then 3k contains the factor 3.

This concludes the induction step, and with it, the proof. QED

36. Prove $(\forall n)[n \in \mathbb{N} \Rightarrow 4 \mid 5^n + 3]$

Proof:

Base Case ($n=1$): $4 \mid 5^1 + 3$

The RHS is $5^1 + 3 = 8$ $\quad 4 \mid 5 + 3$

4 is a factor of 8 $\quad 4 \mid 8$ ✓

Induction Step: We will assume that $n \in \mathbb{N}$ and $4 \mid 5^n + 3$,

It follows that $4 \mid 5^{n+1} + 3$ can be derived

$$5^{n+1} + 3 = 5^{n+1} + 5(3) - 5(3) + 3$$

$$= 5(\underbrace{5^n + 3}_{4l}) - 3(5+1)$$

where $l \in \mathbb{Z}$. By the assumption, we know that 4 is a factor of $5^n + 3$, which allows $5^n + 3$ to be expressed as $4l$

$$= 5(4)(l) - 3(4)$$

$$= 4(5l - 3)$$

$5l - 3 \in \mathbb{Z}$ because the product of an integer and and integer $(5l)$ is an integer. The difference between 2 integers is also an integer.

Anything that is 4 multiplied by an integer has a factor of 4. QED

39. Prove $(\forall n)[n \in \mathbb{Z}^{*} \Rightarrow 3 \mid 2^{n+1} + (-1)^n$

Proof: Base case = 0 $\quad 3 \mid 2^{0+1} + (-1)^0$

The RHS $2^{0+1} + (-1)^0$ $\quad 3 \mid 2+1$

is 3. 3 is a factor $\quad 3 \mid 3$ ✓

of 3

Base case = 1 $\quad 3 \mid 2^{1+1} + (-1)^1$

The RHS $2^{1+1} + (-1)^0$ $\quad 3 \mid 2^2 + (-1)^1$

is 3. 3 is $\quad 3 \mid 4-1$

a factor of 3 $\quad 3 \mid 3$ ✓

The set of nonnegative numbers is $0 \cup \mathbb{N}$. Now, $m \in n \in \mathbb{Z}^{*}$ and $m \in \mathbb{N}$

Induction step: assume that $m \in \mathbb{N}$ and $3 \mid 2^{n+1} + (-1)^n$

we derive that $3 \mid 2^{n+2} + (-1)^{n+1}$

$$\begin{aligned}
2^{n+2} + (-1)^{n+1} &= 2(2^{n+1}) + (-1)(-1)^n \\
&= 2(2^{n+1}) + 2(-1)^n - 2(-1)^n + (-1)(-1)^n \\
&= \underbrace{2(2^{n+1} + (-1)^n)}_{3l \text{ for some } l \in \mathbb{Z}} + (-2)(-1)^n + (-1)(-1)^n \\
&= 2(3l) + (-3)(-1)^n \\
&= 3(2l + (-1)(-1)^n \\
&= 3(2l + (-1)^n)
\end{aligned}$$

In the case of $2l + (-1)^n$, $2l$ will always be an integer because the product of 2 integers is an integer. $(-1)^n$ will always be an integer because (-1) to the power of a positive natural number is always -1 or 1. 3 times an integer is always an integer with a factor of 3. This completes the induction step, and with it, the proof.

4) Prove: $\forall n (n \in \mathbb{N} \Rightarrow x-y \mid x^n - y^n)$

Proof:

Base case ($n=1$) $\quad x-y \overset{?}{\mid} x^1 - y^1$

$x-y \mid x-y$ ✓

The RHS $x^1 - y^1 = x-y$,

$x-y$ is a factor of $x-y$.

Induction step: We will assume that $n \in \mathbb{N}$ and $x-y \mid x^n - y^n$

From this, we derive that $x-y \mid x^{n+1} - y^{n+1}$

$$\begin{aligned} x^{n+1} - y^{n+1} &= x^n(x) - y^n(y) \\ &= x^n(x) - (x^n)(y) + (x^n)(y) - y^n(y) \\ &= x^n(x-y) + y\underbrace{(x^n - y^n)}_{l(x-y) \text{ where } l \in \mathbb{Z}.} \end{aligned}$$

This can be done because $x-y$ is a factor of $x^n - y^n$ by assumption

$$\begin{aligned} &= x^n(x-y) + yl(x-y) \\ &= (x-y)(x^n + yl) \end{aligned}$$

Since $x^{n+1} - y^{n+1}$ can be reduced to $(x-y)(x^n + yl)$ where l is some integer, $(x-y)$ is proven to be a factor of $x^{n+1} - y^{n+1}$, which completes the induction step, and with it, the proof. □

43. Prove $(\forall n)(n \in \mathbb{N} \wedge p > -1 \Rightarrow (1+p)^n \geq 1+np$

Proof:

Base case $(n=1)$: $(1+p)^1 \overset{?}{\geq} 1+1p$

$1+p \geq 1+p$ ✓

Induction Step: Assume $n \in \mathbb{N}$ and $p > -1$ and $(1+p)^n \geq 1+np$

We derive $(1+p)^{n+1} \geq 1+(n+1)p$

$$(1+p)^{n+1} \geq 1+(n+1)p$$

$$(1+p)(1+p)^n \geq 1+np+p$$

$$(1+p)(1+p)^n \geq (1+p)+np$$

$$(1+p)^n \geq np$$

$$(1+p)^n \geq np$$

$$(1+p^n) \geq np+1$$

This completes the induction step, and with it, the proof $\square$