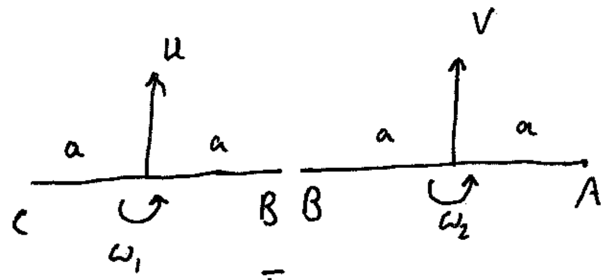
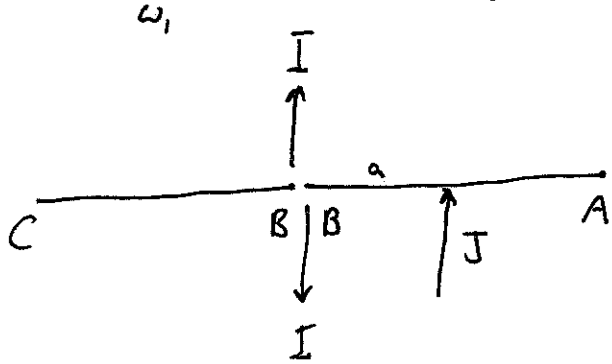
Impulsive motion of a rigid body

1.





Now the speeds at B must be equal

$$\therefore \quad u + a\omega_1 = V - a\omega_2$$

Linear momentum

BC : $\quad I = mu$

AB : $\quad J - I = mV$

Angular momentum

BC : $\quad Ia = \frac{1}{3} m a^2 \omega_1$

AB : $\quad J \cdot 0 + Ia = \frac{1}{3} m a^2 \omega_2 \quad \therefore \quad Ia = \frac{1}{3} m a^2 \omega_2$

$$\therefore \quad \omega_1 = \omega_2$$

Now $\quad mu + ma\omega_1 = mV - ma\omega_2$

$$\therefore \quad I + 3I = J - I - 3I$$

$$\therefore \quad J = 8I$$

$$\therefore \quad u = \frac{J}{8m}, \quad v = \frac{7J}{8m}$$

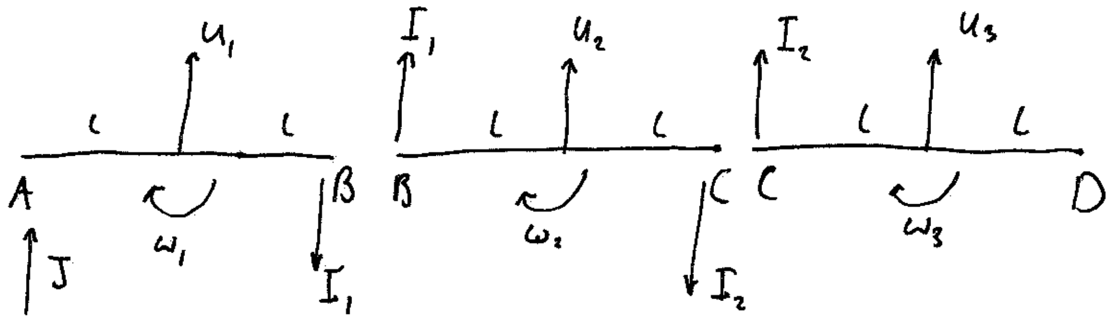
$$\omega_1 = \omega_2 = \frac{3J}{8ma}$$

Speed of A : $v + a\omega_2$

$$= \frac{7J}{8m} + \frac{3J}{8m} = \frac{5J}{4m}$$

Speed of C : $u - a\omega_1$

$$= \frac{J}{8m} - \frac{3J}{8m} = -\frac{J}{4m}$$

2. 

AB : LM : $J - I_1 = mu_1$

AM : $(J + I_1)l = \frac{1}{3}ml^2\omega_1$

BC : LM : $(I_1 - I_2) = mu_2$

AM : $(I_1 + I_2)l = \frac{1}{3}ml^2\omega_2$

Common speed at B:

$$u_1 - l\omega_1 = u_2 + l\omega_2$$

Substitute the previous equations into this to get

$$J - I_1 - 3(J + I_1) = I_1 - I_2 + 3(I_1 + I_2)$$

$$\text{ie.} \quad -J = 4I_1 + I_2$$

CD: LM: $I_2 = m u_3$

AM: $I_2 l = \frac{1}{3} m l^2 \omega_3$

Common speed at C:

$$u_2 - l\omega_2 = u_3 + l\omega_3$$

By a similar process as to before,

$$I_1 - I_2 - 3(I_1 + I_2) = I_2 + 3I_2$$

$$\therefore \quad I_1 = -4I_2$$

$$\therefore \quad -J = -16I_2 + I_2 \quad \therefore \; I_2 = \frac{J}{15}$$

$$\text{and} \quad I_1 = -\frac{4J}{15}$$

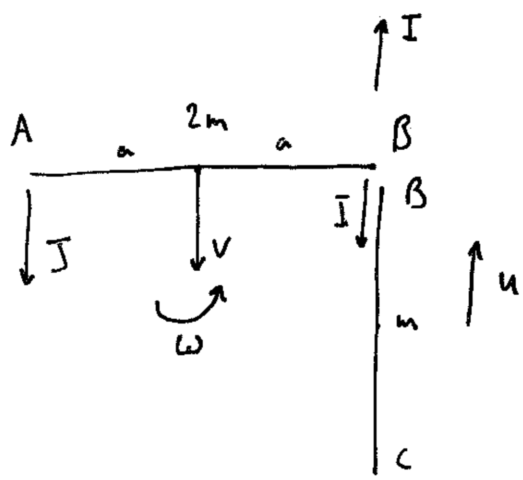
Speed of A: $\quad u_1 + l\omega_1 = \dfrac{J - I_1}{m} + \dfrac{3(J+I_1)}{m}$

$$= \frac{19J}{15m} + \frac{33J}{15m} = \frac{52J}{15m}$$

Speed of C: $\quad u_3 - l\omega_3 = \dfrac{I_2}{m} - \dfrac{3I_2}{m}$

$$= -\frac{2I_2}{m} = -\frac{2J}{15m}$$

3.



AB: $\quad$ LM: $\quad 2mv = J - I$

AM: $\quad \frac{1}{3}\cdot 2ma^2\omega = Ja + Ia$

$$\therefore \frac{2}{3} ma\omega = I + J$$

BC: LM: $I = -mu$

Common speed at B:

$$-V + a\omega = u$$

By performing suitable substitutions

$$\frac{3}{2}(J+I) - \left(\frac{J-I}{2}\right) = -I$$

$$\therefore \quad J + 2I = -I \qquad \therefore \quad I = -\frac{J}{3}$$

$\therefore \quad u = \dfrac{J}{3m}$ is the initial speed of BC

It also follows that

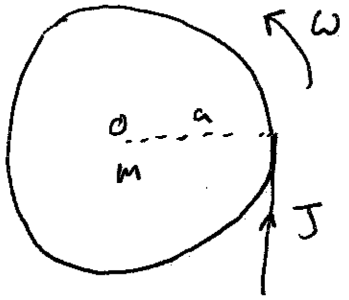
$$V = \frac{2J}{3m} \quad \text{and} \quad a\omega = \frac{J}{m}$$

$$\therefore \quad KE = \frac{1}{2}\left(\frac{1}{3}\cdot 2ma^2\right)\omega^2 + \frac{1}{2}\cdot 2mV^2 + \frac{1}{2}mu^2$$

$$= \frac{m}{3}\frac{J^2}{m^2} + m\cdot\frac{4J^2}{9m^2} + \frac{1}{2}m\frac{J^2}{9m^2}$$

$$= \frac{5J^2}{6m}$$

4.

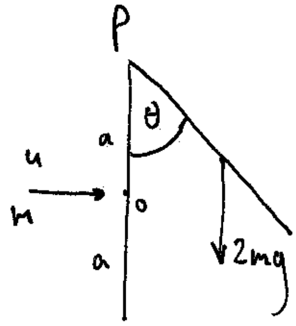


Conservation of angular momentum about O:

$$Ja = \frac{1}{2} m a^2 \omega$$

$$\therefore \omega = \frac{2J}{ma}$$

5.



Conservation of angular momentum

$$mua = \underbrace{\frac{5}{4} m a^2 \omega}_{\text{disc}} + \underbrace{m a^2 \omega}_{\text{particle}}$$

ω is the angular speed just after impact.

Use energy conservation just after impact, ie.

energy after impact = energy at 'top' of motion

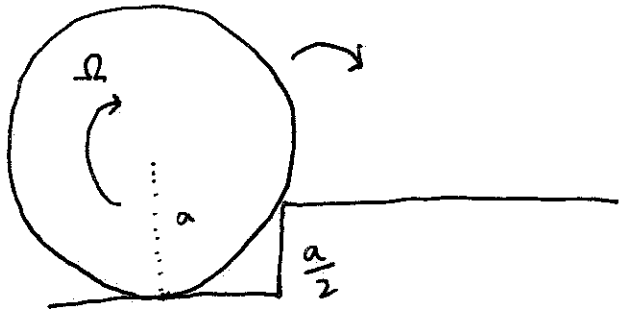
$$\therefore \frac{1}{2}\left(\frac{5}{4} m a^2\right)\omega^2 + \frac{1}{2} m (a\omega)^2 = 2mg(1-\cos\theta)$$

$$\therefore \frac{9}{8} a^2 \omega^2 = 2ga(1-\cos\theta)$$

$$\therefore \frac{9}{8}\left(\frac{4u}{9}\right)^2 = 2ga(1-\cos\theta)$$

$$\therefore \cos\theta = 1 - \frac{u^2}{9ga}$$

6.



ω is the angular speed of the disc about the point of contact just after impact.

Conservation of angular momentum

$$\frac{1}{2} m a^2 \Omega + m a \Omega \frac{a}{2} = \frac{3}{2} m a^2 \omega$$

$$\therefore \quad 2\Omega = 3\omega \quad \therefore \quad \omega = \frac{2\Omega}{3}$$

Let ω_1 be the angular speed of the disc just after ~~impact~~ it mounts the 1st step.
Using energy conservation after impact:

$$\frac{1}{2} \cdot \frac{3}{2} m a^2 \omega^2 = \frac{1}{2} \cdot \frac{3}{2} m a^2 \omega_1^2 + m g \frac{a}{2}$$

$$\therefore \quad \omega_1^2 = \omega^2 - \frac{2g}{3a}$$

If ω_2 is the angular speed of the disc just after mounting the 2nd step, then

$$\omega_2^2 = \left(\frac{2}{3}\omega_1\right)^2 - \frac{2g}{3a}$$, from A.M. & energy equations above.

$$\therefore \quad \omega_2^2 = \left(\frac{2}{3}\right)^2\left(\omega^2 - \frac{2g}{3a}\right) - \frac{2g}{3a}$$

$$= \left(\frac{4}{9}\right)^2 \Omega^2 - \left[\left(\frac{4}{9}\right)\frac{2g}{3a} + \frac{2g}{3a}\right]$$

Continuing gives

$$\omega_3^2 = \frac{4}{9}\,\omega_2^2 - \frac{2g}{3a}$$

$$= \left(\frac{4}{9}\right)^3 \Omega^2 - \frac{2g}{3a}\left[\left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right) + 1\right]$$

$$\therefore \quad \omega_n^2 = \left(\frac{4}{9}\right)^n \Omega^2 - \frac{2g}{3a}\left[\left(\frac{4}{9}\right)^{n-1} + \left(\frac{4}{9}\right)^{n-2} + \dots + 1\right]$$

$$= \left(\frac{4}{9}\right)^n \Omega^2 - \frac{2g}{3a}\left[\frac{1-(4/9)^n}{1-4/9}\right]$$

$$\therefore \quad \Omega^2 = \frac{6g}{5a}\left[\left(\frac{9}{4}\right)^n - 1\right]$$

$$= \frac{6g}{5a}\left[\left(\frac{9}{4}\right)^{2n} - 1\right]$$