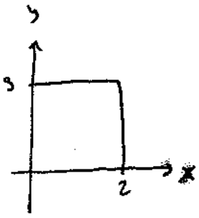
1. $$\int_0^1 dx \int_x^1 30x^2y\,dy = \int_0^1 \left[15x^2y^2\right]_x^1 dx$$

$$= \int_0^1 \left(15x^2 - 15x^4\right) dx$$

$$= 15\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 15 \cdot \frac{2}{15} = 2$$

2. a)



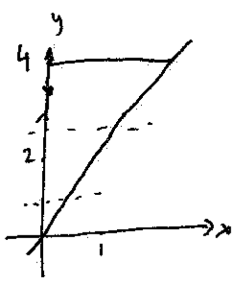
$$\int_0^2 dx \int_0^3 (x^2 + 2y)\,dy = \int_0^2 \left[x^2y + y^2\right]_0^3 dx$$

$$= \int_0^2 \left(3x^2 + 9\right) dx$$

$$= \left[x^3 + 9x\right]_0^2$$

$$= 26$$

b)



$$\int_0^4 dy \int_0^{\frac{y}{2}} xy\,dx = \int_0^4 \left[\frac{x^2}{2}y\right]_0^{\frac{y}{2}} dy$$

$$= \int_0^4 \frac{y^3}{8}\,dy = \frac{y^4}{32}\Big|_0^4$$

$$= 8$$

3. $$\int_1^2 dx \int_1^3 \left(\frac{2}{x} + \frac{4}{y}\right) dy = \int_1^2 \left[\frac{2y}{x} + 4\log y\right]_1^3 dx$$

$$= \int_1^2 \left(\frac{6}{x} + 4\log 3 - \frac{2}{x} - 0\right) dx$$

$$= \int_1^2 \left(\frac{4}{x} + 4\log 3\right) dx = \left[4\log x + (4\log 3)x\right]_1^2$$

$$= 4\log 2 + 8\log 3 - 4\log 3$$

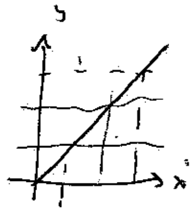
$$= 4\left(\log 2 + \log 3\right)$$

$$= 4\log 6$$

4. $$\int_0^2\int_0^{\sqrt{4-y^2}} xy\,dx\,dy = \int_0^2 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{4-y^2}} dy = \int_0^2 \frac{y}{2}(4-y^2)\,dy$$

$$= \left[y^2 - \frac{y^4}{8}\right]_0^2 = 2$$

5. a)



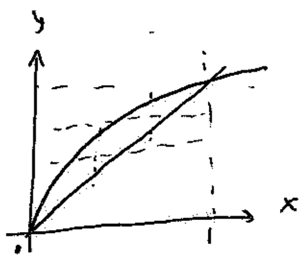
$$\int_0^1 dy \int_0^y \frac{x}{1+y^3}\,dx = \int_0^1 \left[\frac{x^2}{2(1+y^3)}\right]_0^y dy = \frac12\int_0^1 \frac{y^2}{1+y^3}\,dy$$

$$= \frac16\int_0^1 \frac{1}{1+u}\,du \quad (u=y^3)$$

$$= \frac16 \log(1+u)\Big|_0^1$$

$$= \frac16\log 2$$

b)



$$\int_0^1 dy\int_{y^2}^{y} \frac{x}{\sqrt{x^2+y^2}}\,dx \qquad u = x^2+y^2 \therefore du = 2x\,dx$$

$$= \int_0^1 dy \int_{y^2+y^4}^{2y^2} \frac12 u^{-1/2}\,du$$

$$= \int_0^1 \left[u^{1/2}\right]_{y^2+y^4}^{2y^2} dy$$

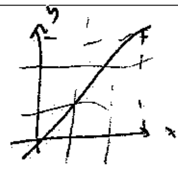
$$= \int_0^1\left(\sqrt{2y^2} - \sqrt{y^4+y^2}\right)dy$$

$$= \frac{\sqrt2}{2}y^2\Big|_0^1 - \int_0^1 y\sqrt{1+y^2}\,dy$$

$$= \frac{\sqrt2}{2} - \int_1^2 \frac12 u^{1/2}\,du \qquad (u = 1+y^2)$$

$$= \frac{\sqrt2}{2} - \left[\frac13 u^{3/2}\right]_1^2$$

$$= \frac{\sqrt2}{2} - \frac{(\sqrt2)^3}{3} + \frac13 = \frac{(2-\sqrt2)}{6}$$

5. c) 

$$\int_0^1 dx \int_0^x 3y^2 e^{-x^2}\,dy = \int_0^1 y^3 e^{-x^2}\Big|_0^x dx = \int_0^1 x^3 e^{-x^2}\,dx$$

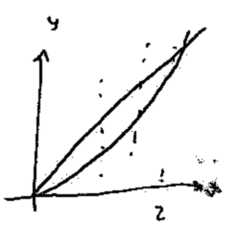
$$= \int_0^1 \frac{1}{2} u e^{-u}\,du \qquad u = x^2,\quad du = 2x\,dx$$

(Integration by parts)

$$= \frac{1}{2}\left[-ue^{-u}\Big|_0^1 + \int_0^1 e^{-u}\,du \right]$$

$$= \frac{1}{2}\left[-\frac{1}{e} + \left[-e^{-u}\right]_0^1 \right]$$

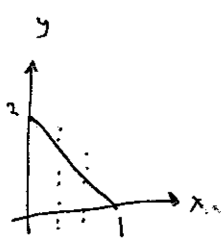
$$= \frac{1}{2}\left[1 - \frac{2}{e} \right]$$

6. 

$$\int_0^2 dx \int_{x^2}^{2x} xy\,dy = \int_0^2 \left[\frac{xy^2}{2}\right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(2x^3 - \frac{x^5}{2}\right) dx$$

$$= \left[\frac{x^4}{2} - \frac{x^6}{12} \right]_0^2 = \frac{8}{3}$$

7. 

$y = -2x + 2$

$$\int_0^1 dx \int_0^{-2x+2} (10 - 3x - 2y)\,dy$$

$$= \int_0^1 \left[10y - 3xy - y^2 \right]_0^{2-2x} dx$$

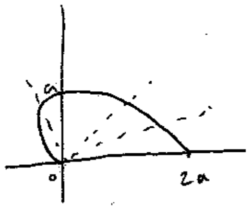
$$= \int_0^1 (20 - 20x - 6x + 6x^2 - 4 + 8x - 4x^2)\,dx$$

$$= \left[20x - 10x^2 - 3x^2 + 2x^3 - 4x + 4x^2 - \frac{4x^3}{3} \right]_0^1$$

$$= 20 - 10 - 3 + 2 - 4 + 4 - \frac{4}{3}$$

$$= 9 - \frac{4}{3} = \frac{23}{3}$$

8.



$$\int_0^{\pi} d\theta \int_0^{a(1+\cos\theta)} r\,dr$$

$$= \int_0^{\pi} \frac{a^2}{2}(1+\cos\theta)^2\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

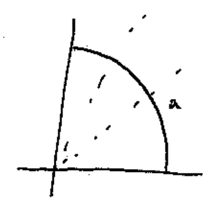
$$= \frac{a^2}{2}\int_0^{\pi}\left(1+2\cos\theta+\tfrac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= \frac{a^2}{2}\left[\theta+2\sin\theta+\tfrac{1}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)\right]_0^{\pi}$$

$$= \frac{3\pi a^2}{4}$$

9: (i)

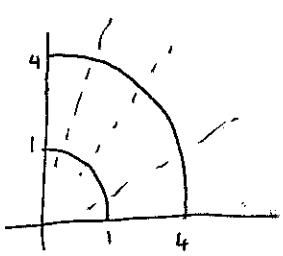
$$\iint \frac{y}{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{a} \frac{r\sin\theta}{r^2}\cdot r\,dr\,d\theta$$



$$= \int_0^{\frac{\pi}{2}} a\sin\theta\,d\theta$$

$$= -a\cos\theta\Big|_0^{\frac{\pi}{2}} = a$$

(ii)



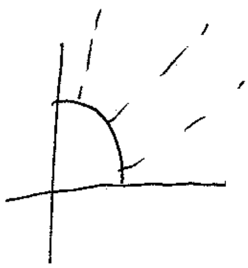
$$\int_0^{\frac{\pi}{2}} d\theta \int_1^{4} r\cos\theta\, r^2\sin^2\theta\cdot r\,dr$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\sin^2\theta\,d\theta \int_1^{4} r^4\,dr$$

$$= \frac{1}{3\cdot 1}\cdot 1\cdot\left[\frac{r^5}{5}\right]_1^4$$

$$= \frac{1}{3}\left[\frac{4^5}{5}-\frac{1}{5}\right] = \frac{341}{5}$$

9. (iii)



$$\int_0^{\frac{\pi}{2}} d\theta \int_1^{\infty} r^2 e^{-r^2} \cdot r\, dr$$

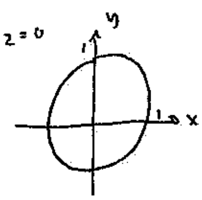
$$= \frac{\pi}{2} \cdot \int_1^{\infty} \frac{1}{2} u e^{-u}\, du \qquad \begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array}$$

$$= \frac{\pi}{4} \cdot \left[-ue^{-u}\Big|_1^{\infty} + \int_1^{\infty} e^{-u} du \right]$$

$$= \frac{\pi}{4} \left[\frac{1}{e} + \left[-e^{-u}\right]_1^{\infty} \right]$$

$$= \frac{\pi}{2e}$$

10. At $z=0$



$$V = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= 2\pi \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1$$

$$= \frac{\pi}{2}$$