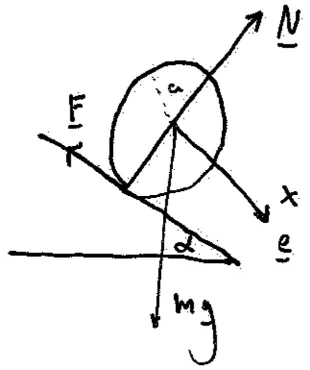
Rolling and sliding motion

1.



In the direction of $\underline{e}$

$x = a\theta$

$m\ddot{x} = -F + mg\sin\alpha$

$I\ddot{\theta} = aF$

$\therefore \; F = \dfrac{I\ddot{x}}{a^2} = \dfrac{m}{2}\ddot{x}$ (for disc)

$\therefore \; m\ddot{x} = -\frac{1}{2}m\ddot{x} + mg\sin\alpha$

$\therefore \; \frac{3}{2}m\ddot{x} = mg\sin\alpha$

i.e. $\ddot{x} = \frac{2}{3}g\sin\alpha$

$\therefore \; |F| = \frac{1}{3}mg\sin\alpha$

From geometry, $|N| = mg\cos\alpha$

$$\therefore \; |R| = \sqrt{F^2 + N^2} = \sqrt{m^2g^2\cos^2\alpha + \tfrac{1}{9}m^2g^2\sin^2\alpha}$$

$$= \frac{mg}{3}\sqrt{9\cos^2\alpha + 1 - \cos^2\alpha}$$

$$= \frac{mg}{3}\sqrt{1 + 8\cos^2\alpha}$$

Rolling and sliding motion. (1½)

2. The eqn. of motion for c.o.m

$$m\ddot{x} = F - \mu m g$$

Rotational : $\frac{2}{5} m a^2 \ddot{\theta} = \mu m g a$

$\therefore \frac{2}{5} m \ddot{x} = \mu m g$ when rolling starts

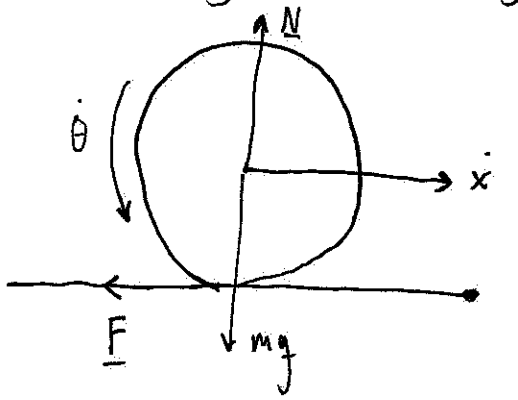
$\therefore \frac{5}{2} m g \mu = F - \mu m g$ upon rolling

ie. $F = \frac{7}{2} \mu m g$

For rolling without slipping

$$\dot{x} \leqslant a\dot{\theta}$$

Hence, $F \leqslant \frac{7}{2} \mu m g$

3. 

Let the forces be as shown

Horizontally : $m\ddot{x} = -F \quad \therefore \quad m\dot{x} = -Ft + mu \quad -(1)$

$(\dot{x} = u \text{ at } t = 0)$

Rotation : $\frac{2}{5}ma^2\ddot{\theta} = -Fa \quad \therefore \quad \frac{2ma}{5}\dot{\theta} = -Ft + \frac{2ma}{5}\omega \quad -(2)$

$(\dot{\theta} = \omega \text{ at } t = 0)$

(1) − (2) gives

$$m\dot{x} - \frac{2ma}{5}\dot{\theta} = mu - \frac{2ma}{5}\omega$$

$$\text{ie.} \quad \dot{x} = u - \frac{2a}{5}(\omega - \dot{\theta})$$

Sphere does not turn back if $\dot{x} > 0$ at this stage, ie.

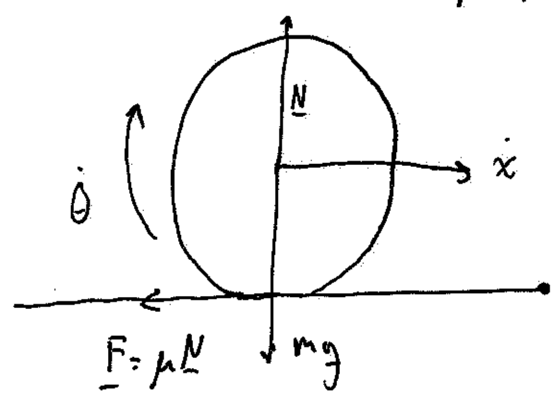
$$u > \frac{2a}{5}(\omega - \dot{\theta})$$

As $\dot{\theta}$ takes values from 0 to ω, then we must have

$$u > \frac{2a}{5}\omega \quad \therefore \quad 5u > 2a\omega \quad \text{as required}$$

Further, from (2), we see that $\ddot{\theta} < 0$ in the first stage of the motion. Hence $\dot{\theta}$ decreases from ω to 0 and so eventually will change sign as $\dot{x} > 0$.

In the 2nd phase of motion, sphere slips the same way as $\dot{x} - a\dot{\theta} > 0$ (rotation of sphere the other way)



Eqn. (1) still holds but rotational eqn. is

$$\frac{2ma^2}{5}\ddot{\theta} = Fa \quad \therefore \quad \frac{2ma}{5}\dot{\theta} = Ft$$ (starts at $\dot{\theta} = 0$ in this phase)

As $N = mg$, $F = \mu mg$

$$\therefore \quad \frac{2a}{5}\dot{\theta} = \mu g t \qquad -(3)$$

Now, when $\dot{\theta} = 0$ in (2) we get

$$t_1 = \frac{2ma\omega}{5F} = \frac{2a\omega}{5\mu g}$$ (time taken for backspin to stop)

and the speed of the sphere at this time is given by using t_1 in (1), i.e.

$$m\dot{x} = -\mu mg \cdot \frac{2a\omega}{5\mu g} + mu \quad \text{i.e.} \quad \dot{x} = \frac{5u - 2a\omega}{5}$$

Use this value in the equation of motion (1) for the 2nd phase to give:

$$m\dot{x} = -\mu m g t + \frac{m(5u - 2a\omega)}{5} \qquad -(4)$$

When rolling starts, we have $\dot{x} = a\dot{\theta}$ and equs. (3) and (4) give

$$\frac{5\mu g t}{2} = -\mu g t + \frac{5u - 2a\omega}{5}$$

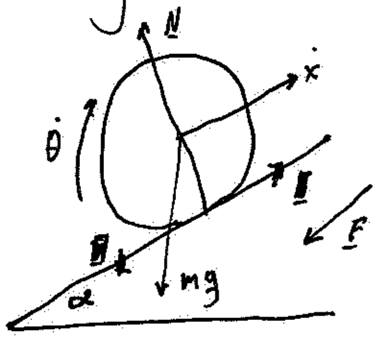
i.e. $$\frac{7\mu g t}{2} = \frac{5u - 2a\omega}{5}$$

$$\therefore \quad t = t_2 = \frac{2(5u - 2a\omega)}{35\mu g}$$

which is the time of slipping for the 2nd phase. Hence the total time taken from the start of the motion until rolling occurs is

$$t_1 + t_2 = \frac{2\omega a}{5\mu g} + \frac{2(5u - 2a\omega)}{35\mu g} = \frac{2(a\omega + u)}{7\mu g}$$

Q.4.



Let the forces be as shown.

Sphere rolls if $\tan\alpha \leqslant \mu\left(1+\frac{r^2}{k^2}\right)$

Now, $\mu = \frac{1}{7}\tan\alpha$ and so $\tan\alpha \leqslant \frac{1}{7}\tan\alpha\left(1+\frac{5}{2}\right)$

$$\leqslant \frac{1}{2}\tan\alpha$$

This is impossible!

$\therefore$ sphere always slides.

First phase of motion - slides up the plane $(\dot{x} > r\dot\theta)$

$F=\mu N$ and $\perp$ plane we have $N = mg\cos\alpha$

$$\therefore \quad F = \frac{1}{7}\tan\alpha \cdot mg\cos\alpha = \frac{1}{7}mg\sin\alpha$$

Equation of motion up the plane gives:

$$m\ddot{x} = -F - mg\sin\alpha = -\frac{8}{7}mg\sin\alpha$$

$$\therefore \quad \dot{x} = -\frac{8}{7}gt\sin\alpha + V \quad -(1)$$

(after integration and using $\dot{x} = V$ at $t=0$)

Rotational : $\frac{2}{5} mr^2 \ddot{\theta} = Fr$

which, upon integration gives,

$$\frac{2}{5} mr\dot{\theta} = Ft \qquad (\dot{\theta} = 0 \text{ at } t = 0)$$

$$\therefore \quad r\dot{\theta} = \frac{5}{14} gt\sin\alpha \quad -(2)$$

Time taken before $\dot{x} = r\dot{\theta}$ instantaneously is found from (1) and (2) above, ie.

$$-\frac{8}{7} gt_1 \sin\alpha + V = \frac{5}{14} gt_1 \sin\alpha$$

$$\therefore \quad t_1 = \frac{2V}{3g\sin\alpha}$$

Second phase of motion - slides down plane $(\dot{x} < r\dot{\theta})$
Forces in diagram still apply, but F is reversed (acts up the plane)

Eqn. of motion along the plane :

$$m\ddot{x} = F - mg\sin\alpha$$
$$= \frac{1}{7} mg\sin\alpha - mg\sin\alpha$$
$$= -\frac{6}{7} mg\sin\alpha$$

$\therefore \dot{x} = -\frac{6}{7} gt\sin\alpha + V_1$, where V_1 is the speed at $t = t_1$

ie. from (1) and (3), $V_1 = -\frac{8}{7} g\sin\alpha . \frac{2V}{3g\sin\alpha} + V = \frac{5V}{21}$

Hence, in the 2nd phase we have

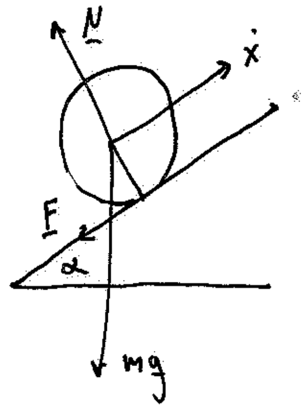
$$\dot{x} = \frac{5}{21}V - \frac{6}{7}gt\sin\alpha \qquad -(4)$$

Time taken to reach instantaneous rest is given by $\dot{x} = 0$ in (4), i.e. $t_2 = \dfrac{5V}{18g\sin\alpha}$ $\qquad -(5)$

Hence, the total time taken from the start of motion is given by (3) and (5) as

$$t_1 + t_2 = \frac{2V}{3g\sin\alpha} + \frac{5V}{18g\sin\alpha} = \frac{17V}{18g\sin\alpha}$$

5.



$$m\ddot{x} = -F - \mu N$$

$$= -mg\sin\alpha - \mu mg\cos\alpha$$

$$\therefore \; m\dot{x} = -mg\sin\alpha \, t - \mu mg\cos\alpha \, t + mU$$

$$(\dot{x} = U \text{ at } t = 0)$$

Also $\quad ma^2\ddot{\theta} = Fa = \mu mga\cos\alpha$

$$\therefore \; ma^2\dot{\theta} = \mu mga\cos\alpha \qquad (\dot{\theta} = 0 \text{ at } t = 0)$$

For rolling $\quad a\dot{\theta} \geq \dot{x}$

$$\therefore \text{ Take } \; \mu mg\cos\alpha \, t = -mg\sin\alpha \, t - \mu mg\cos\alpha \, t + mU$$

$$\text{ie. } \; g(2\mu\cos\alpha + \sin\alpha)t = U$$

$$\text{ie. } \; t = \frac{U}{g(2\mu\cos\alpha + \sin\alpha)}$$

Condition for rolling up the plane is

$$\tan\alpha \leq \mu\left(1 + \frac{a^2}{k^2}\right)$$

$$\leq \mu\left(1 + \frac{a^2}{a^2}\right) \quad \text{for hollow cylinder}$$

$$\text{ie. } \; \mu \geq \tan\alpha$$