1. $\nabla\phi = (2z^4 - 2xy, -x^2, 8xz^3)$

At $(2,-2,-1)$ $\nabla\phi = (2\cdot(-1)^4 - 2\cdot 2\cdot(-2), -(2)^2, 8\cdot 2\cdot(-1)^3)$

$$= (10, -4, -16)$$

$$|\nabla\phi|^2 = 10^2 + (-4)^2 + (-16)^2$$

$$= 372$$

$$\therefore |\nabla\phi| = \sqrt{372} = 2\sqrt{93}$$

2. a) $\nabla(\phi+\psi) = \nabla(x^2 z + \exp(y/x) + 2x^2y - xy^2)$

$$= \left(2xz - \frac{y}{x^2}\exp(y/x) + 4xy - y^2,\right.$$
$$\frac{1}{x}\exp\left(\frac{y}{x}\right) + 2x^2 - 2xy,$$
$$\left.x^2\right)$$

At $(1,0,-2)$, $\nabla(\phi+\psi) = (-4, 3, 1)$

b) $\nabla(\phi\psi) = \nabla\left(2x^4yz - x^3y^2z + 2x^2y\exp\left(\frac{y}{x}\right) - xy^2\exp\left(\frac{y}{x}\right)\right)$

$$= \left(8x^3yz - 3x^2y^2z + 4xy\exp\left(\frac{y}{x}\right) - 2y^2\exp\left(\frac{y}{x}\right) - y^2\exp\left(\frac{y}{x}\right)\right.$$
$$+ \frac{y^3}{x}\exp\left(\frac{y}{x}\right),$$
$$2x^4z - 2x^3yz + 2x^2\exp\left(\frac{y}{x}\right) + 2xy\exp\left(\frac{y}{x}\right) - 2xy\exp\left(\frac{y}{x}\right)$$
$$- y^2\exp\left(\frac{y}{x}\right),$$
$$\left.2x^4y - x^3y^2\right)$$

At $(1,0,-2)$, $\nabla(\phi\psi) = (0, -2, 0)$

3. Let $\underline{a} = (a_1, a_2, a_3)$ , $\underline{b} = (b_1, b_2, b_3)$.

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2, \; a_3 b_1 - a_1 b_3, \; a_1 b_2 - a_2 b_1)$$

$$\nabla \cdot (\underline{a} \times \underline{b}) = \frac{\partial}{\partial x}(a_2 b_3 - a_3 b_2) + \frac{\partial}{\partial y}(a_3 b_1 - a_1 b_3) + \frac{\partial}{\partial z}(a_1 b_2 - a_2 b_1)$$

$$\nabla \times \underline{a} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ a_1 & a_2 & a_3 \end{vmatrix} = \left( \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}, \; \frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}, \; \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right)$$

$$\nabla \times \underline{b} = \left( \frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}, \; \frac{\partial b_1}{\partial z} - \frac{\partial b_3}{\partial x}, \; \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} \right)$$

$$\begin{aligned}
\underline{b} \cdot \nabla \times \underline{a} - \underline{a} \cdot \nabla \times \underline{b} &= b_1 \left( \frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z} \right) + b_2 \left( \frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x} \right) + b_3 \left( \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y} \right) \\
&\quad - a_1 \left( \frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z} \right) - a_2 \left( \frac{\partial b_1}{\partial z} - \frac{\partial b_3}{\partial x} \right) - a_3 \left( \frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y} \right) \\
&= \left[ - b_2 \frac{\partial a_3}{\partial x} + b_3 \frac{\partial a_2}{\partial x} + a_2 \frac{\partial b_3}{\partial x} - a_3 \frac{\partial b_2}{\partial x} \right] \\
&\quad + \left[ b_1 \frac{\partial a_3}{\partial y} - b_3 \frac{\partial a_1}{\partial y} - a_1 \frac{\partial b_3}{\partial y} + a_3 \frac{\partial b_1}{\partial y} \right] \\
&\quad + \left[ - b_1 \frac{\partial a_2}{\partial z} + b_2 \frac{\partial a_1}{\partial z} + a_1 \frac{\partial b_2}{\partial z} - a_2 \frac{\partial b_1}{\partial z} \right] \\
&= \frac{\partial}{\partial x}(a_2 b_3 - a_3 b_2) + \frac{\partial}{\partial y}(a_3 b_1 - a_1 b_3) + \frac{\partial}{\partial z}(a_1 b_2 - a_2 b_1) \\
&= \nabla \cdot (\underline{a} \times \underline{b})
\end{aligned}$$

4. a) $\nabla \cdot \underline{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

b) $\nabla \times \underline{r} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = (0, 0, 0) = \underline{0}$

c) $r = (x^2+y^2+z^2)^{1/2}$

$$\nabla r = \left( \tfrac{1}{2}\cdot 2x\,(x^2+y^2+z^2)^{-1/2},\ \tfrac{1}{2}\cdot 2y\,(x^2+y^2+z^2)^{-1/2},\ \tfrac{1}{2}\cdot 2z\,(x^2+y^2+z^2)^{-1/2} \right)$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\,(x, y, z) = \frac{\underline{r}}{r}$$

d) $$\nabla r^3 = \left( \tfrac{3}{2}\cdot 2x\,(x^2+y^2+z^2)^{1/2},\ \tfrac{3}{2}\cdot 2y\,(x^2+y^2+z^2)^{1/2},\ \tfrac{3}{2}\cdot 2z\,(x^2+y^2+z^2)^{1/2} \right)$$

$$= 3\,(x^2+y^2+z^2)^{1/2}(x, y, z) = 3r\underline{r}$$

5. Let $\phi = z - x^2 + y^2$

$\nabla\phi = (-2x, -2y, 1)$ At $(1,2,5)$, $\nabla\phi = (-2, -4, 1)$

Unit normal : $\pm \frac{\nabla\phi}{|\nabla\phi|} = \pm \frac{(-2, -4, 1)}{\sqrt{21}}$

6. $\nabla\phi = (4z^3 - 6xy^2z,\ -6x^2yz,\ 12xz^2 - 3x^2y^2)$

At $(2,-1,2)$, $\nabla\phi = (32 - 6\cdot 2\cdot(-1)^2\cdot 2,\ -6\cdot 4\cdot(-1)\cdot 2,\ 12\cdot 2\cdot 4 - 3\cdot 4\cdot(-1)^2)$

$$= (8, 48, 84)$$

$$\nabla\phi \cdot \frac{(2,-3,6)}{7} = (2\cdot 8 - 3\cdot 48 + 6\cdot 84)/7$$

$$= \frac{376}{7}$$

7. $\nabla \cdot \underline{a} = \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$

$= 0 \quad \therefore \underline{a}$ is solenoidal

8. $\nabla \times \underline{b} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 6xy+z^3 & 3x^2-z & 3xz^2-y \end{vmatrix} = (-1-(-1),\ 3z^2-3z^2,\ 6x-6x)$

$= \underline{0}$

$\therefore \underline{b}$ is irrotational

9. $\underline{u}, \underline{v}$ are irrotational

$$\therefore \left.\begin{aligned} \nabla \times \underline{u} &= \underline{0} \\ \nabla \times \underline{v} &= \underline{0} \end{aligned}\right\} (*)$$

$\nabla \cdot (\underline{u} \times \underline{v}) = \underline{v} \cdot \nabla \times \underline{u} - \underline{u} \cdot \nabla \times \underline{v}$ (from Q3)

$= 0$ since $(*)$

$\therefore \underline{u} \times \underline{v}$ is solenoidal

10. $\nabla \times \underline{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ B_0 \cos\frac{\pi x}{l} e^{-\frac{\pi z}{l}} & \gamma B_0 \cos\frac{\pi x}{l} e^{-\frac{\pi z}{l}} & -B_0 \sin\frac{\pi x}{l} e^{-\frac{\pi z}{l}} \end{vmatrix}$

$= \Big( 0 + \gamma B_0 \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}},\ -B_0 \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}} + B_0 \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}},$

$-\gamma B_0 \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}} - 0 \Big)$

$= \left( \gamma B_0 \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}},\ 0,\ -\gamma B_0 \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}} \right)$

$\nabla \times \underline{B} \times \underline{B} = \begin{vmatrix} i & j & k \\ \gamma B_0 \frac{\pi}{l} \cos\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}} & 0 & -\gamma B_0 \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right) e^{-\frac{\pi z}{l}} \\ B_0 \cos\frac{\pi x}{l} e^{-\frac{\pi z}{l}} & \gamma B_0 \cos\frac{\pi x}{l} e^{-\frac{\pi z}{l}} & -B_0 \sin\frac{\pi x}{l} e^{-\frac{\pi z}{l}} \end{vmatrix}$

$= \left( \gamma^2 B_0^2 \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi x}{l}\right) e^{-\frac{2\pi z}{l}},\ 0,\ \gamma^2 B_0^2 \frac{\pi}{l} \cos^2\left(\frac{\pi x}{l}\right) e^{-\frac{2\pi z}{l}} \right)$

10. Take the curl of the equilibrium eqn.

$$\nabla \times \nabla p = \nabla \times \left[ \mu^{-1} (\nabla \times \underline{B}) \times \underline{B} + \underline{F} \right]$$

Now $\nabla \times \nabla p = 0 \quad \forall p$

$$\therefore \quad \mu^{-1} \nabla \times (\nabla \times \underline{B} \times \underline{B}) + \nabla \times \underline{F} = \underline{0} \qquad -(*)$$

$$\nabla \times (\nabla \times \underline{B} \times \underline{B}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \gamma^2 B_0^2 \frac{\pi}{l} \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right) e^{-\frac{2\pi z}{l}} & 0 & \gamma^2 B_0^2 \frac{\pi}{l} \cos^2\left(\frac{\pi x}{l}\right) e^{-\frac{2\pi z}{l}} \end{vmatrix}$$

$$= \left( 0, \; -2\frac{\pi^2}{l^2} \gamma^2 B_0^2 \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right) + 2\gamma^2 B_0^2 \frac{\pi^2}{l^2} \cos\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi x}{l}\right) e^{-\frac{2\pi z}{l}}, \; 0 \right)$$

$$= \underline{0}$$

From $(*)$ it follows that $\nabla \times \underline{F} = \underline{0}$

$$\therefore \quad \underline{F} = \nabla \phi$$ for some arbitrary function $\phi$