1. a) $\frac{\partial f}{\partial x} = xy + 2 \quad \therefore \quad f = \frac{x^2}{2}y + 2x + g(x)$

b) $\frac{\partial u}{\partial y} + 3u = 1 + 2x \quad \therefore \quad e^{3y}u = \int (1+2x)e^{3y}dy$

$$= \frac{(1+2x)}{3}e^{3y} + g(x)$$

$$\therefore \quad u = \frac{(1+2x)}{3} + g(x)e^{-3y}$$

c) $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = f \qquad \frac{dx}{1} = \frac{dy}{1} = \frac{df}{f}$

$\therefore \int dx = \int dy$ ie. $y = x + c_1$

$\int dy = \int \frac{df}{f}$ ie. $y + c = \log f$

or $f = c_2 e^y$

$\phi(c_1, c_2) = 0$

ie $\phi\left(x-y, \frac{f}{e^y}\right) = 0$

$\therefore \quad fe^{-y} = g(x-y)$ or $f = e^y g(x-y)$

d) $2x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3 \qquad \frac{dx}{2x} = \frac{dy}{y} = \frac{du}{3}$

$\int \frac{dx}{2x} = \int \frac{dy}{y} \quad \therefore \quad \frac{1}{2}\log x = \log y + c \quad \therefore \quad y = c_1\sqrt{x}$

$\int \frac{dy}{y} = \int \frac{du}{3} \quad \therefore \quad \log y = \frac{1}{3}u + c_2$

$\phi(c_1, c_2) = \phi\left(\frac{y}{\sqrt{x}}, \log y - \frac{1}{3}u\right) = 0 \quad \therefore \quad \frac{1}{3}u - \log y = g\left(\frac{y}{\sqrt{x}}\right)$

ie. $u = 3\left(\log y + g\left(\frac{y}{\sqrt{x}}\right)\right)$

e) $\frac{2}{x}\frac{\partial f}{\partial x} + \frac{1}{y}\frac{\partial f}{\partial y} = 0$

$$\frac{dx}{d\tau} = \frac{2}{x}, \quad \frac{dy}{d\tau} = \frac{1}{y}, \quad \frac{df}{d\tau} = 0$$

$\therefore$ f is a const. w.r.t. τ

$$\frac{x^2}{4} = \tau + \tau_0, \quad \frac{y^2}{2} = \tau + \tau_1$$

$$\therefore \quad \frac{y^2}{2} = \frac{x^2}{4} - \tau_0 + \tau_1$$

$$\therefore \quad x^2 - 2y^2 = \tau_3, \text{ say}$$

$\therefore$ Take $f = g(x^2 - 2y^2)$

f) $x^2\frac{\partial g}{\partial x} - y^2\frac{\partial g}{\partial y} = x \qquad \frac{dx}{x^2} = -\frac{dy}{y^2} = \frac{dg}{x}$

$$\int \frac{dx}{x^2} = -\int \frac{dy}{y^2} \quad \therefore \quad -\frac{1}{x} = \frac{1}{y} + c_1$$

$$\int \frac{dx}{x^2} = \int \frac{dg}{x} \quad \therefore \quad \log x = g + c_2$$

$$\phi(c_1, c_2) = \phi\left(g - \log x, \frac{1}{x} + \frac{1}{y}\right) = 0$$

$$\therefore g = \log x + f\left(\frac{1}{x} + \frac{1}{y}\right)$$

2. $\frac{\partial u}{\partial t} + xu\frac{\partial u}{\partial x} = u$

$$\frac{dt}{d\tau} = 1, \quad \frac{dx}{d\tau} = xu, \quad \frac{du}{d\tau} = u$$

The initial data may be parameterised by $t=0,\ x=s,\ u=s$
$0<s<1$.

The solution in parametric from is

$$x = s\exp(s(e^t-1)), \quad u = se^t, \quad 0<s<1$$

Eliminating s, the ~~explicit~~ implicit solution is

$$x = u\exp(u - t - ue^{-t}).$$

3. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^3 \qquad \frac{dx}{d\tau} = 1, \ \frac{dy}{d\tau} = 1, \ \frac{du}{d\tau} = u^3$

The initial data: $x=0,\ y=s,\ u=s$ on $\tau=0$, $0<s<3$

$$x = \tau, \quad y = s+\tau, \quad u = \frac{s}{\sqrt{1-2s^2\tau}}, \quad 0<s<3$$

The explicit solution is

$$u = \frac{y-x}{\sqrt{1-2x(y-x)^2}}$$

The solution blows up when the numerator becomes zero. i.e. when

$$1-2x(y-x)^2 = 0 \quad \text{or} \quad y = x + \frac{1}{\sqrt{2x}}$$

4. a) $\frac{dx}{d\tau} = \frac{1}{x+2}$, $\frac{dy}{d\tau} = \frac{z}{y+3}$, $\frac{dz}{d\tau} = 1$

$\therefore$ $z = \tau$

$z = x^2 + 2x + c$

$\frac{y^2}{2} + 3y = \frac{z^2}{2} + C$

$\therefore$ $\frac{9}{2} - 9 = \frac{4}{2} + C$

$\therefore$ $-13 = 2C$

$\therefore$ $y^2 + 6y \pm z^2 + 2C$

$\therefore$ $(y+3)^2 + 4 = z^2$

i.e. $z = \sqrt{4 + (y+3)^2}$

b) $\frac{dx}{y^2} = -\frac{dy}{x^2} = \frac{f\,df}{x^2 y}$ $\therefore$ $\int \frac{dx}{y^2} = -\int \frac{dy}{x^2}$ $\therefore$ $\frac{x^3}{3} = C_1 - \frac{y^3}{3}$

$-\int \frac{dy}{x^2} = \int \frac{f\,df}{x^2 y}$ $\therefore$ $C_2 = f^2 + y^2$

$\phi(c_1, c_2) = \phi\left(\frac{x^3}{3} + \frac{y^3}{3}, f^2 + y^2\right)$

$\therefore$ $f^2 = g\left(\frac{x^3}{3} + \frac{y^3}{3}\right) - y^2$

ICs : $x^3 = g\left(\frac{x^3}{3}\right)$ $\therefore$ $g(x) = 3x$

$\therefore$ $f = \sqrt{(x^3 + y^3) - y^2}$

4. c) $x\,dx = y\,dy = f\,df$

$$\therefore \quad c_1 + x^2 = y^2$$
$$c_2 + y^2 = f^2$$

$$\phi(c_1, c_2) = \phi(y^2 - x^2,\ f^2 - y^2) = 0$$

$$\therefore \quad f^2 = y^2 + g(y^2 - x^2)$$

Now $\quad (y+1)^2 = y^2 + g(y^2)$

$$\therefore \quad y^2 + 2y + 1 = y^2 + g(y^2)$$

$$\therefore \quad g(y) = 2\sqrt{y} + 1$$

$$\therefore \quad f^2 = y^2 + 2\sqrt{y^2 - x^2} + 1$$