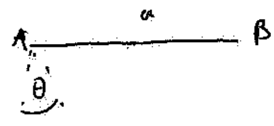
2D rigid body motion - fixed axis

1. 

$$m\ddot{\underline{r}} = m\underline{g} + \underline{R}$$

i.e. $m\left(-\frac{a}{2}\dot{\theta}^2 \underline{e}_r + \frac{a}{2}\ddot{\theta}\,\underline{e}_\theta\right) = -mg\sin\theta\,\underline{e}_\theta + mg\cos\theta\,\underline{e}_r + X\underline{e}_r + Y\underline{e}_\theta$

$$\therefore\ -\frac{ma}{2}\dot{\theta}^2 = mg\cos\theta + X$$

$$\frac{ma}{2}\ddot{\theta} = -mg\sin\theta + Y$$

Rotational eqn. of motion: $I\ddot{\theta} = -mga\sin\theta$

$$\therefore\ \ddot{\theta} = -\frac{mga\sin\theta}{I}$$

I through c.o.m. is $\frac{1}{12}ma^2$

$\therefore$ for this problem, $I = I_{com} + m\frac{a^2}{4}$

$$= \frac{4ma^2}{12} = \frac{ma^2}{3}$$

$$\therefore\ \ddot{\theta} = -3\frac{g\sin\theta}{a}$$

Now $\ddot{\theta} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$ $\qquad \therefore \int \dot{\theta}\,d\dot{\theta} = \int -\frac{3g\sin\theta}{a}\,d\theta$

$$\therefore\ \frac{\dot{\theta}^2}{2} = \frac{3g\cos\theta}{a} + C$$

When $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$

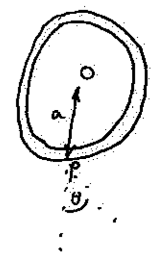
$$\therefore\ 0 = C$$

$$\therefore\ \dot{\theta}^2 = 6\frac{g\cos\theta}{a}$$

$$\therefore\ X = -\frac{ma}{2}\dot{\theta}^2 - mg\cos\theta = -3mg\cos\theta - mg\cos\theta = -4mg\cos\theta$$

$$Y = \frac{ma}{2}\ddot{\theta} + mg\sin\theta = -\frac{3}{2}mg\sin\theta + mg\sin\theta = -\frac{mg}{2}\sin\theta$$

Reaction on hinge is $-X = 4mg\cos\theta$, $-Y = \frac{mg}{2}\sin\theta$

2. 

$$M\ddot{\underline{r}} = m\underline{g} + \underline{R}$$

$$\therefore \quad m\left(-a\dot{\theta}^2\underline{e}_r + a\ddot{\theta}\underline{e}_\theta\right) = -mg\sin\theta\,\underline{e}_\theta + mg\cos\theta\,\underline{e}_r + X\underline{e}_r + Y\underline{e}_\theta$$

Now $\quad I\ddot{\theta} = -mga\sin\theta$

$$\therefore \quad \ddot{\theta} = -\frac{mga\sin\theta}{I}$$

Now $\quad I = \underbrace{ma^2}_{I_{com}} + ma^2 = 2ma^2$

$$\therefore \quad \ddot{\theta} = -\frac{g\sin\theta}{2a}$$

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = -\frac{g\sin\theta}{2a}$$

$$\therefore \quad \frac{\dot{\theta}^2}{2} = \frac{g\cos\theta}{2a} + c$$

At $\theta = \pi$, $\dot{\theta} = 0$

$$\therefore \quad 0 = c - \frac{g}{2a} \quad \therefore \quad c = \frac{g}{2a}$$

$$\therefore \quad \frac{\dot{\theta}^2}{2} = \frac{g}{2a}\left(\cos\theta + 1\right)$$

Force on axis when O is below P:

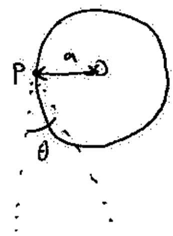
$$-X = +a\dot{\theta}^2\Big|_{\theta=0} + mg\cos\theta\Big|_{\theta=0}$$

$$= +2mg + mg = +3mg$$

$$Y = 0$$

$\therefore$ Magnitude of force on axis is $3mg$.

3.



$$m\ddot{\underline{r}} = m\underline{g} + \underline{R}$$

$$m(-a\dot{\theta}^2\underline{e}_r + a\ddot{\theta}\underline{e}_\theta) = -mg\sin\theta\,\underline{e}_\theta + mg\cos\theta\,\underline{e}_r + X\underline{e}_r + Y\underline{e}_\theta$$

$$I\ddot{\theta} = -mga\sin\theta$$

$$\therefore \ddot{\theta} = -\frac{mga\sin\theta}{I}$$

Integrating gives $\dfrac{\dot{\theta}^2}{2} = \dfrac{mga\cos\theta}{I} + C$

When $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$

$$\therefore C = 0$$

$$\therefore \dot{\theta}^2 = \frac{2mga\cos\theta}{I}$$

For this problem, $I = \frac{3}{2}ma^2$

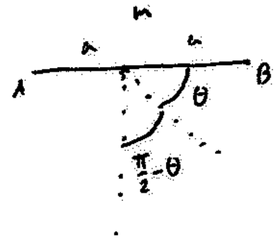
When vertical, force on axis is

$$-X = +ma\dot{\theta}^2\big|_{\theta=0} + mg\cos\theta\big|_{\theta=0}$$

$$= +\frac{4mg}{3} + mg = +\frac{7mg}{3}$$

$$Y = 0$$

$\therefore$ magnitude of force is $\dfrac{7mg}{3}$

4. 

The easiest way to proceed is to use the coordinate system of previous questions but replace θ with $\frac{\pi}{2}-\theta$

For ring $\qquad m\ddot{\underline{r}} = m\underline{g} + \underline{R}$

$$m\left(-\frac{a}{2}\left(\dot{\overline{\tfrac{\pi}{2}-\theta}}\right)^2 \underline{e}_r + \frac{a}{2}\left(\ddot{\overline{\tfrac{\pi}{2}-\theta}}\right)\underline{e}_\theta\right) = -mg\sin\left(\tfrac{\pi}{2}-\theta\right)\underline{e}_\theta + mg\cos\left(\tfrac{\pi}{2}-\theta\right)\underline{e}_r + X\underline{e}_r + Y\underline{e}_\theta$$

$$\therefore \quad -\frac{ma}{2}\dot{\theta}^2 = mg\sin\theta + X$$

$$-\frac{ma}{2}\ddot{\theta} = -mg\cos\theta + Y$$

Rotation: $\quad I\left(\ddot{\overline{\tfrac{\pi}{2}-\theta}}\right) = a\underline{e}_r \times m\underline{g} + \frac{a}{2}\underline{e}_r \times (-\underline{R})$

ie. $\quad -I\ddot{\theta} = -mga\cos\theta + \frac{ma^2}{4}\ddot{\theta} + \frac{mga}{2}\cos\theta$

Now $I = \frac{1}{3}ma^2 \quad \therefore \quad -\frac{7}{12}ma^2\ddot{\theta} = -\frac{1}{2}mga\cos\theta$

$$\therefore \quad \frac{7a}{12}\ddot{\theta} = \frac{1}{2}g\cos\theta \qquad \left[\ddot{\theta} = \frac{6}{7a}g\cos\theta\right]$$

Now $\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \ddot{\theta}$

$\therefore$ upon integration

$$\frac{7a}{12}\cdot\frac{\dot{\theta}^2}{2} = \frac{1}{2}g\sin\theta + C$$

$\dot{\theta} = 0$ at $\theta = 0 \quad \therefore \quad C = 0$

$$\therefore \quad 7a\dot{\theta}^2 = 12g\sin\theta$$

Now $Y = mg\cos\theta - \frac{ma}{2}\cdot\frac{6}{7a}g\cos\theta = \frac{4}{7}mg\cos\theta$

$X = -mg\sin\theta - \frac{ma}{2}\cdot\frac{12}{7a}g\sin\theta = -\frac{13}{7}mg\sin\theta$

$|F| = \mu|Y| = \mu\frac{4}{7}mg\cos\theta$

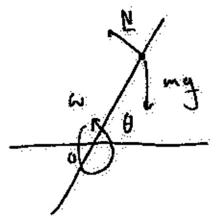
For slipping to occur

$|X| > \mu|N| \ (= \mu|Y|)$

ie. $\frac{13}{7}mg\sin\theta > \mu\frac{4}{7}\cos\theta\cdot mg$

ie. $\theta > \tan^{-1}\left(\frac{4\mu}{13}\right)$

5\.



No friction $\therefore$ $\underline{R} = \underline{N}$

$$m\ddot{\underline{r}} = m\underline{g} + N$$

$$\therefore \quad m\left((\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\underline{e}_\theta \right) = -mg(\sin\theta\,\underline{e}_r + \cos\theta\,\underline{e}_\theta) + N\underline{e}_\theta$$

Now $\theta = \omega t$, $\dot{\theta} = \omega$, $\ddot{\theta} = 0$

$$\therefore \quad m(\ddot{r} - r\omega^2) = -mg\sin(\omega t) \qquad -(1)$$

$$m(2\dot{r}\omega) = -mg\cos(\omega t) + N \qquad -(2)$$

For (1)

CF: $\quad r = Ae^{\omega t} + Be^{-\omega t}$ for consts. A, B

PI: $\quad$ Suppose $r = C\sin\omega t + D\cos\omega t$

$$\dot{r} = \omega C\cos\omega t - \omega D\sin\omega t, \quad \ddot{r} = \omega(-\omega C\sin\omega t - \omega D\cos\omega t)$$

$$= -\omega^2 r$$

$$\therefore \quad -2\omega^2 r = -g\sin\omega t$$

$$\therefore \quad C = \frac{g}{2\omega^2}, \quad D = 0$$

$$\therefore \quad r = \frac{g}{2\omega^2}\sin\omega t + Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{r} = \frac{g}{2\omega}\cos\omega t + A\omega e^{\omega t} - B\omega e^{-\omega t}$$

$$\therefore \quad N = mg\cos\omega t + mg\cos\omega t + 2m\omega^2 Ae^{-\omega t} - 2m\omega^2 Be^{-\omega t}$$

$$= 2m\left[g\cos\omega t + \omega^2 Ae^{-\omega t} - \omega^2 Be^{-\omega t} \right]$$