EXAMINATION	AMS....	SETTER	.....
FOR DIET	May 2006	CHECKER	TNL ...PAGE NO....

COMMENTS	SOLUTION	MARK

QUESTION NO. 1

(a) $0 \le x \le 1 \quad y' + 2y = 1 \qquad$ Integrating factor $= e^{2x}$

$(e^{2x}y)' = e^{2x}$ so that $e^{2x}y = \frac{1}{2}e^{2x} + A$

i.e. $y = \frac{1}{2} + Ae^{-2x}$ — 3

Since $y(0) = 0$ we have $0 = \frac{1}{2} + A \quad \therefore A = -\frac{1}{2}$ — 1

In $0 \le x \le 1 \quad y = \frac{1}{2} - \frac{1}{2}e^{-2x}$

$x > 1 \quad y' + 2y = 0 \qquad$ i.e. $y = Be^{-2x}$ — 2

Matching at $x = 1$ we find $\frac{1}{2} - \frac{1}{2}e^{-2} = Be^{-2} \quad \therefore B = \frac{1}{2}(e^{2} - 1)$

$\therefore$ Solution is $y = \begin{cases} \frac{1}{2} - \frac{1}{2}e^{-2x} & 0 \le x \le 1 \\ \frac{1}{2}(e^{2}-1)e^{-2x} & x > 1 \end{cases}$. — 2

(b) Auxiliary equation $m^2 + m - 2 = (m-1)(m+2) = 0$

i.e. $m = 1$ and -2

$\therefore$ General solution is $y = Ae^{x} + Be^{-2x}$. — 3

(c) $\ddot{y} + 2\dot{y} + 10y = 50t \qquad y(0) = -1, \dot{y}(0) = 6$

Auxiliary equation is $m^2 + 2m + 10 = 0 \quad m \Rightarrow -1 \pm 3i$

CF is $e^{-t}(A\cos 3t + B\sin 3t)$ — 3

For the PI try $y = Ct + D$ to yield

$2C + 10(Ct + D) = 50t$

i.e. $C = 5$ and $2C + 10D = 0 \quad \therefore D = -1$ — 3

General solution is $y = e^{-t}(A\cos 3t + B\sin 3t) + 5t - 1$ — 1

$\dot{y} = -e^{-t}(A\cos 3t + B\sin 3t) + e^{-t}(-3A\sin 3t + 3B\cos 3t) + 5$

Then $y(0) = -1 = A - 1 \qquad \therefore A = 0$

$\dot{y}(0) = 6 = -A + 3B + 5 \qquad \therefore B = \frac{1}{3}$ — 2

Particular solution is $y = \frac{1}{3}e^{-t}\sin 3t + 5t - 1$.

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QUESTION NO. 2

(a) $4u_{r+2} - 4u_{r+1} + u_r = 2^r, \quad u_0 = \frac{4}{9}, u_1 = \frac{2}{9}$

Characteristic equation is $4\lambda^2 - 4\lambda + 1 = 0 = (2\lambda - 1)^2$

Repeated root $\lambda = \frac{1}{2}$. CF is $(Ar+B)(\frac{1}{2})^r$. — 4

For the inhomogeneous equation try $u_r = C\,2^r$.

$\Rightarrow 4C\,2^{r+2} - 4C\,2^{r+1} + C\,2^r = 2^r$

i.e. $2^r(16C - 8C + C) = 2^r$ or $9C = 1$ — 4

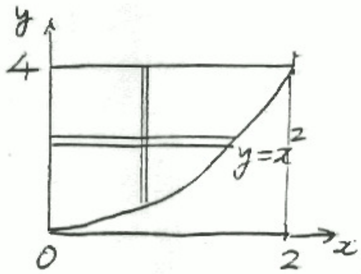
Hence general solution is $u_r = (Ar+B)(\frac{1}{2})^r + \frac{1}{9}2^r$ — 1

$u_0 = \frac{4}{9} = B + \frac{1}{9} \quad \therefore B = \frac{1}{3}$

$u_1 = \frac{2}{9} = (A+B)\frac{1}{2} + \frac{2}{9} \quad \therefore A = -\frac{1}{3}$

$\therefore$ Particular solution is $u_r = \frac{1}{3}(1-r)(\frac{1}{2})^r + \frac{2^r}{9}$. — 3

(b) $\displaystyle\int_0^2 dx \int_{x^2}^4 x(x^2+y^2)\,dy$ — 2



$= \displaystyle\int_0^4 dy \int_0^{\sqrt{y}} x(x^2+y^2)\,dx$ — 3

$= \displaystyle\int_0^4 \left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right]_0^{\sqrt{y}} dy = \int_0^4 \left(\frac{y^2}{4} + \frac{y^3}{2}\right) dy$

$= \left[\frac{y^3}{12} + \frac{y^4}{8}\right]_0^4 = \frac{16}{3} + 32 = \underline{\frac{112}{3}}$. — 3

EXAMINATION AM3	SETTER RBP
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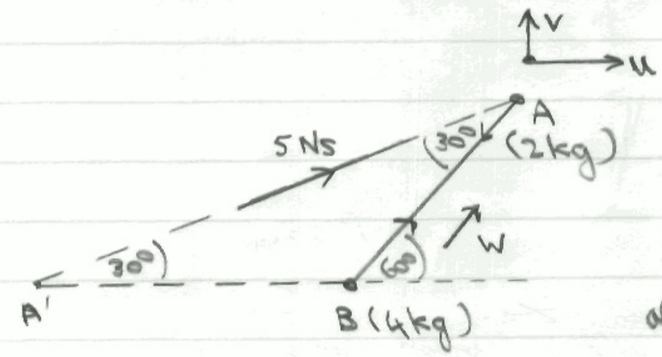
COMMENTS	SOLUTION	MARK
QUESTION NO. 3.	(a) $\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x}y & -e^{-x} & 1+3z^2 \end{vmatrix} = 0\underline{i}+0\underline{j}+(e^{-x}-e^{-x})\underline{k} = \underline{0}$ $\therefore \underline{F}$ is conservative	
	Since $\nabla\times\underline{F}\equiv\underline{0}$, then $\underline{F}=\nabla\phi = \underline{i}\frac{\partial\phi}{\partial x}+\underline{j}\frac{\partial\phi}{\partial y}+\underline{k}\frac{\partial\phi}{\partial z}$.	2
	Then $\frac{\partial\phi}{\partial x} = F_x = e^{-x}y \quad \therefore \phi = -e^{-x}y + f(y,z)$	
	$\frac{\partial\phi}{\partial y} = -e^{-x}+\frac{\partial f}{\partial y} = F_y = -e^{-x} \Rightarrow \frac{\partial f}{\partial y}=0$ and so $f=g(z)$	
	$\frac{\partial\phi}{\partial z} = g'(z) = F_z = 1+3z^2 \Rightarrow g(z) = z^3+z+C$	
	Hence $\phi = -e^{-x}y+z^3+z+C \qquad \phi(0,0,0)=1 \Rightarrow C=1$	
	$\therefore$ <u>Scalar potential $\phi = -e^{-x}y+z^3+z+1$</u>.	5
	$\int_C \underline{F}.d\underline{r} = \phi(\infty,1,2)-\phi(0,0,1) = 11-3 = \underline{8}$.	2
	(b) Green's theorem: $\oint_C Pdx+Qdy = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$	
	$P=3x^2+y, \ \frac{\partial P}{\partial y}=1$ C is boundary of rectangle taken in the positive sense $Q=3x-3y^2, \ \frac{\partial Q}{\partial x}=3$	1
	$\therefore \oint_C (3x^2+y)dx+(3x-3y^2)dy = \iint_R (3-1)\,dxdy$	
	$= 2\iint_R dxdy = \underline{12}$. Area of rectangle $=3\times2$	4
	(c) $\iint_S \underline{n}.\nabla\times\underline{F}\,dS = \oint_C \underline{F}.d\underline{r}$ where C is the circle $x^2+y^2=1$ in the positive sense	1
	$\underline{F}.d\underline{r} = (-y\underline{i}+x\underline{k}).(\underline{i}dx+\underline{j}dy) = -ydx$ on C	2
	$\oint\underline{F}.d\underline{r} = \oint_C -ydx = \int_0^{2\pi}\sin^2\theta\,d\theta \qquad x=\cos\theta, y=\sin\theta, \ 0\le\theta\le2\pi$	
	$= \underline{\pi}$.	3

Solutions to CS0980a - 2005/06

B1. (a)



Angles will be as shown. Let u, v, W be velocities of A and B after string becomes taut as shown. 2

Momentum eq^n along A'B direction:

$$5\cos 30^\circ = 2u + 4W\cos 60^\circ$$

$$\therefore \frac{5\sqrt{3}}{2} = 2u + 2W$$

$$\therefore u = \frac{5\sqrt{3}}{4} - W \quad \text{---- (1)}$$ 2

Momentum eq^n perp. to A'B:

$$5\sin 30^\circ = 2v + 4W\sin 60^\circ$$

$$\frac{5}{2} = 2v + 2\sqrt{3}W$$

$$\therefore v = \frac{5}{4} - \sqrt{3}W \quad \text{--- (2)}$$ 2

Velocities of A and B along string are equal:

$$W = u\cos 60^\circ + v\sin 60^\circ$$

$$\therefore 2W = u + \sqrt{3}v \quad \text{--- (3)}$$ 2

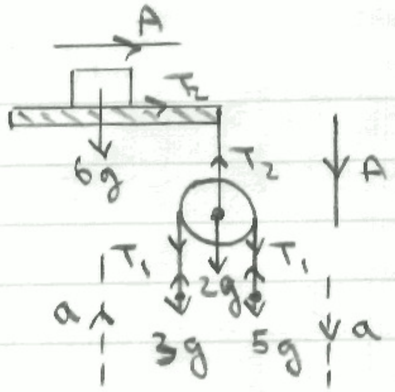
(1) & (2) into (3): $$2W = \frac{5\sqrt{3}}{4} - W + \sqrt{3}\left(\frac{5}{4} - \sqrt{3}W\right)$$

$$\therefore 6W = \frac{5\sqrt{3}}{2}$$

$$\therefore \underline{W = \frac{5\sqrt{3}}{12}\ \text{m/s}} \quad (\simeq 0.722)$$ 2

is the speed of B.

B1. (b)



A = accel^n of pulley
a = " " masses relative to pulley
T_1, T_2 = tensions in strings.

Using Newton's 2nd law:

6kg mass — $6A = T_2 \quad \cdots (1)$ 1

pulley — $2A = 2g + 2T_1 - T_2 \quad \cdots (2)$ 1

3kg mass — $3(a-A) = T_1 - 3g \quad \cdots (3)$ 2

5kg mass — $5(a+A) = 5g - T_1 \quad \cdots (4)$ 2

(3) + (4) $\Rightarrow \quad 8a + 2A = 2g \quad \cdots (5)$

(1) & (2) $\Rightarrow \quad 8A = 2g + 2T_1$

subst. into (3): $3(a-A) = 4A - g - 3g$

$\therefore \quad 3a = 7A - 4g \quad \cdots (6)$ 1

$3\times(5) - 8\times(6) \Rightarrow \quad 6A = 6g - 56A + 32g$

$\therefore \quad 62A = 38g$

$\therefore \quad A = \frac{19}{31} g \ m/s^2 \quad (\simeq 6.01)$ 1

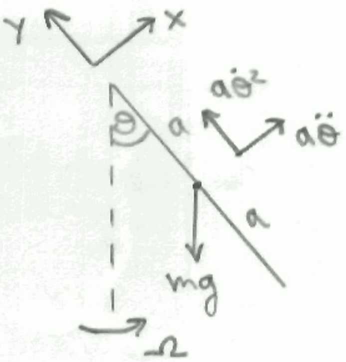
(6) $\Rightarrow \quad 3a = 7 \times \frac{19}{31} g - 4g = \frac{9}{31} g$

$\therefore \quad a = \frac{3}{31} g \ m/s^2 \quad (\simeq 0.948)$ 1

Hence accel^n of 3kg mass $= a - A = -\frac{16}{31} g \ m/s^2$ 1

$(\simeq -5.06)$

B2. (a) X and Y are components of thrust required.



1

Radial eq^n of motion:

$$ma\dot{\theta}^2 = Y - mg\cos\theta$$

$$\therefore Y = ma\dot{\theta}^2 + mg\cos\theta \quad \cdots\cdots (1)$$ 2

Transverse eq^n of motion:

$$ma\ddot{\theta} = X - mg\sin\theta$$

$$\therefore X = ma\ddot{\theta} + mg\sin\theta \quad \cdots (2)$$ 2

Rotational eq^n of motion (moments about hinge):

$$I\dot{\omega} = \sum \text{moments}$$

$$\frac{4}{3}ma^2 \cdot \ddot{\theta} = -mga\sin\theta$$

$$\therefore a\ddot{\theta} = -\frac{3}{4}g\sin\theta \quad \cdots\cdots (3)$$ 2

Integrating (3) gives $\quad \dfrac{a\dot{\theta}^2}{2} = \dfrac{3}{4}g\cos\theta + c$

$\dot{\theta} = \Omega$ at $\theta = 0 \Rightarrow a\dot{\theta}^2 = a\Omega^2 + \frac{3}{2}g(\cos\theta - 1)$

$\cdots (4)$ 3

(3) & (2) $\Rightarrow \quad X = -\frac{3}{4}mg\sin\theta + mg\sin\theta$

$$\therefore X = \frac{mg}{4}\sin\theta$$ 1

(1) & (4) $\Rightarrow \quad Y = ma\Omega^2 + \frac{3mg}{2}(\cos\theta - 1) + mg\cos\theta$

$$\therefore Y = ma\Omega^2 + \frac{mg}{2}(5\cos\theta - 3)$$ (as required) 1

B2.(b) Truck rolls without slipping so no work done by frictional forces on wheels.

$\therefore$ Work done by force F = K.E. of truck + 4 wheels

i.e. $F x = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} . 4m \dot{x}^2 + 4 . \frac{1}{2}\left(\frac{1}{2} m a^2\right)\omega^2$ 4

(ω = angular velocity of wheels)

$\therefore F x = \frac{1}{2} M \dot{x}^2 + 2m\dot{x}^2 + m a^2 \omega^2$

But $\dot{x} = a\omega \Rightarrow F x = \dot{x}^2\left(\frac{1}{2}M + 3m\right)$

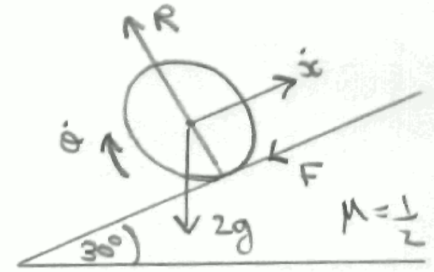
$\therefore \dot{x} = \sqrt{\frac{2Fx}{M+6m}}$ (as required) 2

From $\dot{x}^2 = \frac{2Fx}{M+6m}$, differentiation gives

$\cancel{2}\cancel{\dot{x}}\ddot{x} = \frac{\cancel{2}F\cancel{\dot{x}}}{M+6m}$

$\therefore \ddot{x} = \frac{F}{M+6m}$ is accel^n. 2

B3. (a)



First, check rolling condition that $\tan\alpha \leq \mu(1 + a^2/k^2)$

i.e. $\tan 30° \leq \frac{1}{2}\left(1 + a^2/\tfrac{1}{2}a^2\right)$ 2,1

or $\frac{1}{\sqrt{3}} \leq \frac{3}{2}$ which is true

and so disc will roll on plane.

$\dot{x} = 5$ and $\dot{\theta} = 0$ initially, so disc will slide up the plane, hence $F = \mu R = \frac{1}{2} \cdot 2g\cos 30°$ (resolving perp. to plane)

i.e. $F = g\frac{\sqrt{3}}{2}$. 1

Eq^n of motion of disc up plane:

$$2\ddot{x} = -F - 2g\sin 30°$$
$$= -g\frac{\sqrt{3}}{2} - g$$
$$\therefore \ddot{x} = -\frac{g}{4}(\sqrt{3}+2) \Rightarrow \dot{x} = -\frac{gt}{4}(\sqrt{3}+2) + 5 \quad \text{--- (1)}$$ 3

Rotational eq^n of motion:

$$\frac{1}{2}\cdot 2\left(\frac{1}{2}\right)^2\ddot{\theta} = F\cdot\frac{1}{2} = \frac{g\sqrt{3}}{4}$$
$$\therefore \frac{1}{2}\ddot{\theta} = \frac{g\sqrt{3}}{2} \Rightarrow \frac{1}{2}\dot{\theta} = gt\frac{\sqrt{3}}{2} \quad \text{--- (2)}$$ 3

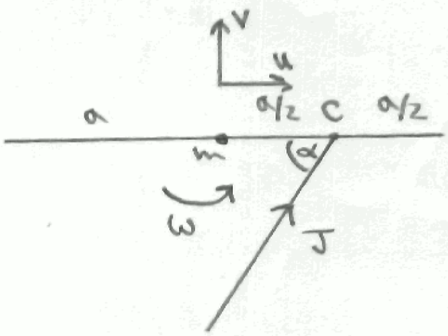
Disc stops slipping and starts rolling when $\dot{x} = \frac{1}{2}\dot{\theta}$,

(1) & (2) give $\frac{gt}{2}\sqrt{3} = -\frac{gt}{4}(\sqrt{3}+2) + 5$

$$\therefore gt\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} + \frac{1}{2}\right) = 5 \Rightarrow \frac{gt}{4}(3\sqrt{3}+2) = 5$$

$$\therefore t = \frac{20}{(3\sqrt{3}+2)g} \simeq 0.284 \text{ seconds.}$$ 2

B3.(b)



ω = ang. vel. of rod after impulse.

u, v = components of velocity of c. of g. of rod after impulse. 1

Momentum eq^n perp. to rod:

$$J \sin\alpha = mv \quad \text{---} \; (1)$$ 1

Momentum eq^n along rod:

$$J\cos\alpha = mu \quad \text{---} \; (2)$$ 1

Moments about centre of rod:

$$I\omega = \Sigma \text{ moments of impulses}$$

$$\therefore \frac{1}{3} m a^2 \omega = J \cdot \frac{a}{2} \sin\alpha$$

$$\therefore a\omega = \frac{3J}{2m}\sin\alpha \quad \text{----} \; (3)$$ 2

Velocity of point C has components of u along the rod and $v + \frac{a\omega}{2}$ perp. to rod, after impulse. 1

Now (2) $\Rightarrow$ $u = \frac{J}{m}\cos\alpha$ and (1) $\Rightarrow$ $v = \frac{J}{m}\sin\alpha$

$\therefore$ speed of C $= \sqrt{u^2 + \left(v + \frac{a\omega}{2}\right)^2}$ and using (3) gives

$$\text{speed} = \frac{J}{m}\sqrt{\cos^2\alpha + \left(\sin\alpha + \frac{3}{4}\sin\alpha\right)^2}$$

$$= \frac{J}{m}\sqrt{\cos^2\alpha + \frac{49}{16}\sin^2\alpha} = \frac{J}{4m}\sqrt{16\cos^2\alpha + 49\sin^2\alpha}$$

$$= \frac{J}{4m}\sqrt{16 + 33\sin^2\alpha} \quad \text{(as required)}$$ 2