Advanced Methods in ML 2018 - Exercise 1

1. Consider a distribution q over three random variables X_1, X_2, X_3 defined as:

$$q(x_1, x_2, x_3) = \begin{cases} 1/12 & x_1 \oplus x_2 \oplus x_3 = 0 \\ 1/6 & x_1 \oplus x_2 \oplus x_3 = 1 \end{cases} \tag{1}$$

   (a) What is $I(q)$ (namely the set of all correct conditional independence statements) in this case?

   (b) Is there a DAG G where $I_{LM}(G) = I(q)$?

   (c) Is there an undirected graph G such that $I_{sep}(G) = I(q)$

2. Consider four random variables W, X, Y, Z where the distribution $p(w, x, y, z)$ is positive (i.e., not zero for any assignment). Assume that the two following properties are known:

$$(X \perp Y|Z, W) \ , \ (X \perp W|Z, Y) \tag{2}$$

   Show that $X \perp Y, W|Z$.

3. Consider random variables $X_1, \ldots, X_n$. The Markov Blanket of X_i is the minimal subset $S \subset \{1, \ldots, n\}$ such that $X_i \perp X_{\bar{S}\setminus i}|X_S$ (here $\bar{S}$ is the complement of S). In other words, conditioned the Markov blanket, variables X_S the variable X_i is independent of all the other variables. Given a DAG G and variable X_i, find a subset S that is the Markov blanket of X_i for any Bayesian network on G. The blanket should be described in terms of graph properties such as children, parents, non-descendents etc.

4. Given a distribution $p(x)$, we say that an undirected graph G is a *minimal I-map* for p if it satisfies $I_{sep}(G) \subseteq I(p)$, and any edge removed from G will make this false. Given a *positive* distribution p, construct a graph as follows: if $(X_i \perp X_j|X_{\{1,\ldots,n\}\setminus\{i,j\}}) \notin I(p)$ add the edge (i, j) to G.

   (a) Show that the G constructed above satisfies $I_{sep}(G) \subseteq I(p)$. You may use results mentioned in the slides and scribe.

   (b) Show that this G is a minimal I-map for p.

5. Here you will show that there exist distributions that satisfy $I_{sep}(G)$ but are not Markov networks with respect to G. Consider the distribution $p(x_1, x_2, x_3, x_4)$ which has probability $1/8$ for each of the assignments $(0, 0, 0, 0), (1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)$, and probability zero for all others. Show that $I(p) \supseteq I_{sep}(G)$ where G is a square graph. But that p is not a Markov network with respect to this graph.

6. Consider a tree graph G with edges E, and a Markov network $p(x)$ on this graph. Show that p satisfies that for any assignment $x_1, \ldots, x_n$ it holds that:

$$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} p(x_i) \prod_{ij \in E} \frac{p(x_i, x_j)}{p(x_i)p(x_j)} \tag{3}$$