1 Limits

The concept of a limit $\lim_{x \to a} f(x)$ is related to the behavior of a function $f(x)$ *near* $x = a$. $f(x)$ doesn't necessarily have to exist at that point.

A one-sided limit is the value that the function approaches as the x-values approach the limit from one side only:

$$\lim_{x \to a^-} f(x)$$

Left-handed/left-sided limits. What value the function approaches as the x-values approach a from the left.

$$\lim_{x \to a^+} f(x)$$

Right-handed/right-sided limits. What value the function approaches as the x-values approach a from the right.

$$\text{If } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Then $\lim_{x \to a} f(x)$ exists.

1.1 Continuity

Functions that contain no breaks along their entire domain are continuous. A function is not continuous at a jump, hole, or VA.

A function $f(x)$ is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

1. $f(a)$ must exist
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x) = f(a)$

E.g. For a function defined by $f(x) = \begin{cases} x^2 + a, & x < 2 \\ 2a - x, & x \geq 2 \end{cases}$ find a so that $f(x)$ is continuous.

$$\lim_{x \to 2^-} x^2 + a = 4 + a \qquad \lim_{x \to 2^+} 2a - x = 2a - 2$$

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^+} f(x) \\ 4 + a &= 2a - 2 \\ a &= 6 \end{aligned}$$

1.2 Infinity

1.2.1 Limits at infinity: Horizontal Asymptotes

The behavior of a function $f(x)$ as the x-values approach $\pm\infty$. Divide both numerator and denominator by the highest power in the denominator.

1. degree numerator = degree denominator: $\lim_{x\to\pm\infty} f(x) =$ ratio of leading coefficients
2. degree numerator < degree denominator: $\lim_{x\to\pm\infty} f(x) = 0$
3. degree numerator > degree denominator: $\lim_{x\to\pm\infty} f(x) = \pm\infty$

E.g. degree numerator = degree denominator

$$\begin{aligned}\lim_{x\to\infty} \frac{4x^2-2x+1}{5x^2+7x-3} &= \lim_{x\to\infty} \frac{4-\frac{2}{x}+\frac{1}{x^2}}{5+\frac{7}{x}-\frac{3}{x^2}} \\ &= \frac{4}{5}\end{aligned}$$

$$\therefore \text{HA: } y = \frac{4}{5}$$

E.g. degree numerator < degree denominator

$$\begin{aligned}\lim_{x\to\infty} \frac{4x^2-2x+1}{5x^3+7x^2-3x} &= \lim_{x\to\infty} \frac{\frac{4}{x}-\frac{2}{x^2}+\frac{1}{x^3}}{5+\frac{7}{x}-\frac{3}{x^2}} \\ &= 0\end{aligned}$$

$$\therefore \text{HA: } y = 0$$

E.g. degree numerator > degree denominator

$$\begin{aligned}\lim_{x\to\infty} \frac{4x^3-2x^2+1x}{5x^2+7x-3} &= \lim_{x\to\infty} \frac{4x-2+\frac{1}{x}}{5+\frac{7}{x}-\frac{3}{x^2}} \\ &= \infty\end{aligned}$$

$$\therefore \text{HA: } y = \infty$$

1.2.2 Infinite limits: Vertical Asymptotes

The behavior of a function $f(x)$ near vertical asymptote $x = a$.

For a vertical asymptote $x = a$, $\lim_{x \to a^-} f(x) = \pm\infty$ and $\lim_{x \to a^+} f(x) = \pm\infty$

E.g. Determine the behavior of $f(x) = \dfrac{x+1}{x(x-4)}$ around it's vertical Asymptotes

$$\begin{aligned}\lim_{x \to 0^-} f(x) &= \left[\frac{1}{(-0.0000\ldots1)(-4)}\right] \\ &= +\infty\end{aligned}$$

$$\begin{aligned}\lim_{x \to 0^+} f(x) &= \left[\frac{1}{(+0.0000\ldots1)(-4)}\right] \\ &= -\infty\end{aligned}$$

$$\begin{aligned}\lim_{x \to 4^-} f(x) &= \left[\frac{5}{(4)(-0.0000\ldots1)}\right] \\ &= -\infty\end{aligned}$$

$$\begin{aligned}\lim_{x \to 4^+} f(x) &= \left[\frac{5}{(4)(+0.0000\ldots1)}\right] \\ &= +\infty\end{aligned}$$

1.3 Absolute Value

For a function $f(x)$ containing $|a|$, split the absolute value into it's two cases:

1. $a \geq 0$
2. $a < 0$

E.g. $f(x) = \lim_{x \to 5} \dfrac{|x-5|}{x-5}, \quad x \neq 5$

$$\begin{aligned}x - 5 > 0 &: & \lim_{x \to 5^+} \frac{x-5}{x-5} = 1 \\ x > 5 & & \end{aligned}$$

$$\begin{aligned}x - 5 < 0 &: & \lim_{x \to 5^-} \frac{-(x-5)}{x-5} = -1 \\ x < 5 & & \end{aligned}$$

$$\therefore \lim_{x \to 5} \frac{|x-5|}{x-5} = \text{DNE}$$